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Abstract

In this paper, we establish a vanishing theorem of Nadel type for the Witt multiplier ideals on threefolds over perfect fields of characteristic larger than five. As an application, if a projective normal threefold over $\mathbb{F}_q$ is not klt and its canonical divisor is anti-ample, then the number of the rational points on the klt-locus is divisible by q.

Contents

1. Introduction

One of the useful tools in complex algebraic geometry is the Kodaira vanishing theorem, which is generalised to the Kawamata–Viehweg vanishing theorem and the Nadel vanishing theorem. For instance, these vanishing theorems yield the following consequences.

$(1)_0$ If X is a smooth projective variety over $\mathbb{C}$ such that $-K_X$ is ample, then $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$.

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$(2)_0$ If (X,Δ) is a projective klt pair over $\mathbb{C}$ such that $-(K_X+\Delta)$ is ample, then $H^i(X,\mathcal{O}_X)=0$ for $i>0$.

$(3)_0$ If (X,Δ) is a projective log pair over $\mathbb{C}$ such that $-(K_X+\Delta)$ is ample, then $H^i(X,\mathcal{J}(X,\Delta)) = 0$ for $i>0$, where $\mathcal{J}(X,\Delta)$ denotes the multiplier ideal sheaf of (X,Δ).

Indeed, $(1)_0$, $(2)_0$ and $(3)_0$ follow from the Kodaira, Kawamata–Viehweg and Nadel vanishing theorems, respectively (cf. [KMM87, Theorem 1-2-5], [KM98, Corollary 2.68] and [Laz04, Corollary 9.4.15]).

Although the Kodaira vanishing is known to fail in positive characteristic (cf. [Ray78]), similar vanishing still holds in positive characteristic in terms of Witt vectors. The first result in this direction was given by Esnault in [Esn03].

$(1)_p$ If X is a smooth projective variety over a perfect field of characteristic $p>0$ such that $-K_X$ is ample, then $H^i(X,W\mathcal{O}_{X,\mathbb{Q}})=0$ for $i>0$.

Then it is natural to seek a positive-characteristic analogue of $(2)_0$. Indeed, this is partially established by Gongyo and the authors [GNT19].

$(2)_p$ If (X,Δ) is a projective klt pair over a perfect field of characteristic $p>5$ such that $-(K_X+\Delta)$ is ample and $\dim X\leqslant 3$, then $H^i(X,W\mathcal{O}_{X,\mathbb{Q}})=0$ for $i>0$.

The main theorem of this paper is a positive-characteristic analogue of $(3)_0$ for the three-dimensional case of characteristic $p>5$. Furthermore, we treat a relative setting as follows.

Theorem 1.1 (= Theorem 4.10). *Let k be a perfect field of characteristic $p>5$. Let (X,Δ) be a three-dimensional log pair over k and let $f:X\to Z$ be a projective k-morphism to a quasi-projective k-scheme Z. Assume that $-(K_X+\Delta)$ is f-nef and f-big. Then the equation*

$$R^if_*(WI_{\mathrm{Nklt}(X,\Delta),\mathbb{Q}})=0$$

holds for $i>0$, where $\mathrm{Nklt}(X,\Delta)$ denotes the reduced closed subscheme of X consisting of the non-klt points of (X,Δ) and $I_{\mathrm{Nklt}(X,\Delta)}$ is the coherent ideal sheaf on X corresponding to $\mathrm{Nklt}(X,\Delta)$ (cf. Remark 2.2).

As a consequence of Theorem 1.1, we obtain the Kollár–Shokurov connectedness theorem.

Theorem 1.2 (= Theorem 4.12). *Let k be a perfect field of characteristic $p>5$. Let (X,Δ) be a three-dimensional log pair over k and let $f:X\to Z$ be a projective k-morphism to a quasi-projective scheme Z over k such that $f_*\mathcal{O}_X=\mathcal{O}_Z$. Assume that $-(K_X+\Delta)$ is f-nef and f-big. If $\mathrm{Nklt}(X,\Delta)$ denotes the reduced closed subscheme of X consisting of the non-klt points of (X,Δ) and we let $g:\mathrm{Nklt}(X,\Delta)\to f(\mathrm{Nklt}(X,\Delta))$ be the induced morphism, then the fibre $g^{-1}(z)$ over an arbitrary point z of $f(\mathrm{Nklt}(X,\Delta))$ is geometrically connected over the residue field $k(z)$ at z.*

We note that Birkar proved a weaker version of this theorem in the case when f is birational and the coefficients of Δ are at most one [Bir16, Theorem 1.8].

Also, we have applications of Theorem 1.1 to rational points on varieties over finite fields. The starting point is the following theorem by Esnault [Esn03], which is a consequence of $(1)_p$ and a Lefschetz trace formula for $W\mathcal{O}_{X,\mathbb{Q}}$.

$(1)'_p$ If X is a geometrically connected smooth projective variety over a finite field k such that $-K_X$ is ample, then $\#X(k)\equiv 1 \pmod{\#k}$.

In [GNT19], Gongyo and the authors prove that the same formula still holds for Fano threefolds with klt singularities.

$(2)'_p$ If (X, Δ) is a three-dimensional geometrically connected projective klt pair over a finite field k of characteristic $p > 5$ such that $-(K_X + \Delta)$ is ample, then $\#X(k) \equiv 1 \pmod{\#k}$.

Then it is natural to seek an application of Theorem 1.1 to the number of the rational points on a non-klt Fano threefold. In this direction, we show that the number of the rational points on the klt-locus is divisible by $\#k$.

Theorem 1.3 (= Corollary 5.2). *Let (X, Δ) be a three-dimensional geometrically connected projective log pair over a finite field k of characteristic $p > 5$. Assume that $-(K_X + \Delta)$ is nef and big and that (X, Δ) is not klt. Then the congruence*

$$\#X(k) \equiv \#V(k) \mod \#k$$

holds, where V denotes the closed subset of X consisting of the non-klt points of (X, Δ).

An interesting point is that this theorem is not true if we drop the assumption that (X, Δ) is not klt (cf. $(2)'_p$). On the other hand, the following theorem gives a common generalisation of $(2)'_p$ and Theorem 1.3. Moreover, we treat a relative setting.

Theorem 1.4 (= Theorem 5.1). *Let (X, Δ) be a three-dimensional log pair over a finite field k of characteristic $p > 5$. Let $f : X \to Y$ be a projective k-morphism to a quasi-projective k-scheme Y such that $f_*\mathcal{O}_X = \mathcal{O}_Y$. Assume that $-(K_X + \Delta)$ is f-nef and f-big. Then the congruence*

$$\#X(k) - \#V(k) \equiv \#Y(k) - \#f(V)(k) \mod \#k$$

holds, where V denotes the closed subset of X consisting of the non-klt points of (X, Δ).

Furthermore, we shall show that some hypersurfaces D on smooth Fano threefolds contain rational points even if D is not klt. It can be seen as a variation of the Ax–Katz theorem (cf. [Ax64, Kat71]).

Theorem 1.5 (cf. Theorem 5.3). *Let X be a three-dimensional projective geometrically connected variety with klt singularities over a finite field k of characteristic $p > 5$. Let D be a non-zero effective $\mathbb{Q}$-Cartier Weil divisor on X. Assume that:*

(i) *$-K_X$ is ample; and*
(ii) *$-(K_X + D)$ is ample.*

Then the congruence

$$\#D(k) \equiv 1 \mod \#k$$

holds. In particular, there exists a k-rational point on D.

In Theorem 5.3, we work in a more general setting.

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1.1 Description of the proof

We now overview some of the ideas of the proof of Theorem 1.1. In the following, we work over a perfect field k of characteristic $p > 5$. Roughly speaking, the argument consists of two steps.

(A) We prove the $W\mathcal{O}$-vanishing for log Fano contractions, i.e. Theorem 3.11 (§ 3).

(B) Using Theorem 3.11, we prove Theorem 1.1 (§ 4).

(A) Now, let us give an overview of how to prove Theorem 3.11. Given a three-dimensional klt pair (X, Δ) and a projective morphism $f : X \to Z$ such that $-(K_X + \Delta)$ is f-ample and $f_*\mathcal{O}_X = \mathcal{O}_Z$, we want to prove that $R^i f_*(W\mathcal{O}_{X,\mathbb{Q}}) = 0$ for any $i > 0$. We further divide the proof of Theorem 3.11 into four cases depending on the dimension of Z. Since the cases $\dim Z = 0$ and $\dim Z = 3$ have been settled already in [GNT19], it is enough to assume that either:

(A1) $\dim Z = 1$ (§ 3.1); or

(A2) $\dim Z = 2$ (§ 3.2).

(A1) We first treat the case when $\dim Z = 1$. In this case, the generic fibre $X_{K(Z)}$ of f is a log del Pezzo surface over an imperfect field. One of the significant steps is to show that $(X_{K(Z)} \times_{K(Z)} \overline{K(Z)})_{\mathrm{red}}$ is a rational surface (Proposition 2.26). Indeed, this result enables us to use a result by Chatzistamatiou–Rülling (Theorem 3.2) after taking suitable purely inseparable covers of X and Z (cf. Proposition 3.6), which in turn implies what we want.

(A2) We now treat the case when $\dim Z = 2$. The crucial part of this case is to prove that Z has $W\mathcal{O}$-rational singularities (Theorem 3.8). To this end, we first reduce the problem to the case when $k = \overline{\mathbb{F}}_p$. Assume that $k = \overline{\mathbb{F}}_p$. In order to prove that Z has $W\mathcal{O}$-rational singularities, we compute, for sufficiently divisible $e \in \mathbb{Z}_{>0}$, the numbers of $\mathbb{F}_{p^e}$-rational points on the models X' and Z' over $\mathbb{F}_{p^e}$ of X and Z, respectively (cf. Step 1 in the proof of Theorem 3.8).

(B) We now overview how to prove Theorem 1.1. For simplicity, we treat only the case when $Z = \operatorname{Spec} k$ and k is an algebraically closed field. Taking a dlt modification of (X, Δ) (Proposition 2.10), we may assume that X is $\mathbb{Q}$-factorial, $(X, \Delta^{\wedge 1})$ is dlt (for the definition of $(-)^{\wedge 1}$, see § 2.1) and $-(K_X + \Delta)$ is ample (cf. Lemma 4.8). By the ampleness of $-(K_X + \Delta)$, we can find an effective $\mathbb{R}$-divisor Ω on X such that:

(i) $(X, \Omega^{\wedge 1})$ is dlt;

(ii) $K_X + \Omega \sim_{\mathbb{R}} 0$;

(iii) Ω is big;

(iv) $\operatorname{Supp} \Omega^{\geqslant 1} = \operatorname{Supp} \Omega^{>1}$; and

(v) $\operatorname{Supp} \operatorname{Nklt}(X, \Omega) = \operatorname{Supp} \operatorname{Nklt}(X, \Delta)$.

Then it suffices to prove the vanishing of $H^i(X, WI_{\operatorname{Nklt}(X,\Omega)}) = 0$ for $i > 0$. Furthermore, we may assume that Supp $\Omega^{>1} \neq \emptyset$, since the assertion is nothing but [GNT19, Theorem 1.3] when (X, Δ) is klt.

The first step is to run a $(K_X + \Omega^{\wedge 1})$-MMP in order to reduce the problem to the end result (cf. Proposition 4.1). In Proposition 4.1, it is proved that the cohomologies are preserved under this MMP.

Replacing X by the end result, let us assume that X itself is the end result of this MMP. By (ii) and Supp $\Omega^{>1} \neq \emptyset$, X has a $(K_X + \Omega^{\wedge 1})$-Mori fibre space structure $g : X \to W$. Then the problem is reduced to vanishing of cohomologies for dlt Mori fibre spaces (Lemma 4.7). By induction on the number of the irreducible components of $\llcorner \Xi \lrcorner$ for $\Xi := \Omega^{\wedge 1}$, Lemma 4.7 is proved by using the $W\mathcal{O}$-vanishing for klt Mori fibre spaces (Theorem 3.11).

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2. Preliminaries

2.1 Notation

In this subsection, we summarise the notation used in this paper.

- We will freely use the notation and terminology in [Har77] and [Kol13].
- For a scheme X, its *reduced structure* X_{red} is the reduced closed subscheme of X such that the induced closed immersion $X_{\mathrm{red}} \to X$ is surjective.
- A morphism $f : X \to Y$ of schemes *has connected fibres* if $X \times_Y \operatorname{Spec} L$ is either empty or connected for any field L and any morphism $\operatorname{Spec} L \to Y$.
- For an integral scheme X, we define the *function field* $K(X)$ of X as $\mathcal{O}_{X,\xi}$ for the generic point ξ of X.
- For a field k, we say that X is a *variety over* k or a k*-variety* if X is an integral scheme that is separated and of finite type over k. We say that X is a *curve* over k or a k*-curve* (respectively a *surface* over k or a k*-surface*, respectively a *threefold* over k) if X is a k-variety of dimension one (respectively two, respectively three).
- For a field k, let $\overline{k}$ be an algebraic closure of k. If k is of characteristic $p > 0$, then we set $k^{1/p^\infty} := \bigcup_{e=0}^{\infty} k^{1/p^e} = \bigcup_{e=0}^{\infty} \{x \in \overline{k} \mid x^{p^e} \in k\}$.
- Let $f : X \to Y$ be a projective morphism of noetherian schemes. Let M be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor M on X. We say that M is f*-ample* if we can write $M = \sum_{i=1}^{r} a_i M_i$ for some $r \geqslant 1$, positive real numbers a_i and f-ample Cartier divisors M_i. We say that M is f*-big* if we can write $M = A + E$ for some f-ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor A and effective $\mathbb{R}$-divisor E. We can define f-nef $\mathbb{R}$-divisors in the same way as in [Kol13, Definition 1.4]. We say that M is f*-numerically-trivial*, denoted by $M \equiv_f 0$, if both M and $-M$ are f-nef.
- Let $\Delta = \sum r_i D_i$ be an $\mathbb{R}$-divisor, where D_i are distinct prime divisors. We define $\Delta^{\geqslant 1} := \sum_{r_i \geqslant 1} r_i D_i$ and $\Delta^{\wedge 1} := \sum r_i' D_i$, where $r_i' := \min\{r_i, 1\}$. We also define $\Delta^{>1}$ and $\Delta^{<1}$ similarly. Moreover, we denote $\{\Delta\} = \Delta - \lfloor \Delta \rfloor$.
- A *sub-log pair* (X, Δ) over a field k consists of a normal variety X over k and an $\mathbb{R}$-divisor Δ such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. A *log pair* (X, Δ) is a sub-log pair such that $\Delta \geqslant 0$.
- For a closed subscheme V of a scheme X, we denote by I_V the quasi-coherent ideal sheaf corresponding to V. For an effective $\mathbb{R}$-divisor D on a normal variety X over a field, we denote $I_D := I_{\mathcal{D}}$, where $\mathcal{D}$ denotes the closed subscheme of X corresponding to the coherent ideal sheaf $\mathcal{O}_X(-\ulcorner D \urcorner)$ on X.
- For terminology on derived category, we refer to [Wei94]. Especially, for morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ of $\mathbb{Z}/p\mathbb{Z}$-schemes and a $W\mathcal{O}_X$-module M, we shall frequently use the isomorphism $Rg_* \circ Rf_*(M) \simeq R(g \circ f)_*(M)$ (cf. [Wei94, Corollary 10.8.10]).

2.2 Results on minimal model program

Definition 2.1. Let k be a field.

(i) We say that (X, Δ) is a *sub-klt* pair over k if (X, Δ) is a sub-log pair over k such that (X, Δ) is klt in the sense of [Kol13, Definition 2.8]. Given a point x of X, we say that (X, Δ) is *sub-klt around* x if there exists an open neighbourhood X' of $x \in X$ such that $(X', \Delta|_{X'})$ is sub-klt.

(ii) We say that (X, Δ) is *klt* (respectively *log canonical*) if (X, Δ) is a log pair such that (X, Δ) is klt (respectively log canonical) in the sense of [Kol13, Definition 2.8].

(iii) Given a point x of X, we say that x is a *non-klt point* of (X, Δ) if (X, Δ) is not sub-klt around x. We define $\operatorname{Nklt}(X, \Delta)$, called the *non-klt locus of* (X, Δ), as the subset of X

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that consists of all the non-klt points. Note that $\mathrm{Nklt}(X, \Delta)$ is a closed subset of X, as its complement is an open subset of X by definition. We equip $\mathrm{Nklt}(X, \Delta)$ with the reduced scheme structure.

Remark 2.2. For coherent ideal sheaves $I, J \subset \mathcal{O}_X$ with $\sqrt{I} = \sqrt{J}$, it follows that $WI_{\mathbb{Q}} = WJ_{\mathbb{Q}}$ (cf. [BBE07, Proposition 2.1]). Hence, we need not care about the scheme structure of $\mathrm{Nklt}(X, \Delta)$ when we consider $WI_{\mathrm{Nklt}(X,\Delta),\mathbb{Q}}$. By the same reason, if X is a non-reduced noetherian scheme and $j : X_{\mathrm{red}} \to X$ denotes the closed immersion from its reduced structure X_{red}, then the induced homomorphism $W\mathcal{O}_{X,\mathbb{Q}} \to j_* W\mathcal{O}_{X_{\mathrm{red}},\mathbb{Q}}$ is an isomorphism. See Lemma 2.17 for a generalisation.

Definition 2.3. A log pair (X, Δ) is called *dlt* if the coefficients of Δ are at most one and there exists a log resolution $g : Y \to X$ of (X, Δ) with the condition that $a_E(X, \Delta) > 0$ holds for any g-exceptional prime divisor E on Y.

Definition 2.4. Given a field k and a projective k-morphism $f : X \to Z$ from a normal k-variety X to a quasi-projective k-scheme Z, we say that X is *of Fano type over* Z if there exists an effective $\mathbb{R}$-divisor Δ on X such that (X, Δ) is klt and $-(K_X + \Delta)$ is f-nef and f-big.

Definition 2.5. Given a field k, a log pair (X, Δ) over k and projective k-morphisms $X \xrightarrow{f_1} Z_1 \to Z_2$ of quasi-projective k-schemes, we say that $f_1 : X \to Z_1$ is a $(K_X + \Delta)$-*Mori fibre space over* Z_2 if $\dim X > \dim Z_1$, $(f_1)_*\mathcal{O}_X = \mathcal{O}_{Z_1}$, $\rho(X/Z_1) = 1$ and $-(K_X + \Delta)$ is f_1-ample. If $Z_2 = \mathrm{Spec}\, k$, then $f_1 : X \to Z_1$ is simply called a $(K_X + \Delta)$-*Mori fibre space.*

Lemma 2.6. *Let k be a field and let $f : X \to Y$ be a projective birational k-morphism of normal varieties over k. Let (Y, Δ_Y) be a sub-log pair and let Δ be the $\mathbb{R}$-divisor defined by $K_X + \Delta = f^*(K_Y + \Delta_Y)$. Then the following hold.*

(i) *Let x be a closed point of X. If x is a non-klt point of (X, Δ), then $f(x)$ is a non-klt point of (Y, Δ_Y).*
(ii) *Let y be a closed point of Y. If y is a non-klt point of (Y, Δ_Y), then there exists a closed point x of X such that $f(x) = y$ and x is a non-klt point of (X, Δ).*

In particular, there exists a commutative diagram consisting of projective morphisms:

$$\begin{array}{ccc} \mathrm{Nklt}(X,\Delta) & \longrightarrow & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathrm{Nklt}(Y,\Delta_Y) & \longrightarrow & Y \end{array}$$

where the horizontal arrows are the induced closed immersions and f' is a projective surjective morphism. In particular, $f(\mathrm{Nklt}(X, \Delta)) = \mathrm{Nklt}(Y, \Delta_Y)$.

Proof. Both of the assertions follow from the fact that $(U, \Delta|_U)$ is sub-klt if and only if $(f^{-1}(U), \Delta_Y|_{f^{-1}(U)})$ is sub-klt for any open subset $U \subset X$ (cf. [KM98, Lemma 2.30]). □

Proposition 2.7. *Let (X, Δ) be a $\mathbb{Q}$-factorial sub-log pair over a field such that $(X, (\Delta^{>0})^{<1})$ is klt. Then it holds that*

$$\mathrm{Nklt}(X, \Delta) = \mathrm{Nklt}(X, \Delta^{>0}) = \mathrm{Supp}\, \Delta^{\geqslant 1}.$$

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Proof. The second equality follows from the fact that $(X, (\Delta^{>0})^{<1})$ is klt. Clearly, the inclusion $\mathrm{Nklt}(X, \Delta) \subset \mathrm{Nklt}(X, \Delta^{>0})$ holds. It suffices to prove the opposite one. Let D be a prime divisor contained in $\mathrm{Supp}\,\Delta^{\geqslant 1}$. Since $\mathrm{Nklt}(X, \Delta)$ is a closed subset containing general closed points of D, we have that $D \subset \mathrm{Nklt}(X, \Delta)$. In particular, $\mathrm{Supp}\,\Delta^{\geqslant 1} \subset \mathrm{Nklt}(X, \Delta)$. □

Lemma 2.8. *Let $\mathbb{K} \in \{\mathbb{Q}, \mathbb{R}\}$. Let k be a perfect field of characteristic $p > 0$. Let (X, Δ) be a quasi-projective dlt pair over k with $\dim X \leqslant 3$. Let A be an ample $\mathbb{K}$-Cartier $\mathbb{K}$-divisor on X. Then there exists an effective $\mathbb{K}$-Cartier $\mathbb{K}$-divisor A' on X such that $A \sim_{\mathbb{K}} A'$ and $(X, \Delta + A')$ is dlt.*

Proof. If k is an infinite field, then the proof of [Bir16, Lemma 9.2] works without any changes. Assume that k is a finite field. Thanks to [Poo04, Theorem 1.1], we can still make use of Bertini's theorem. Hence, we can apply the same argument as in [Bir16, Lemma 9.2]. □

The existence of a minimal model program is known for log canonical threefolds. For terminology appearing in the following theorem, we refer to [HNT17, § 2.4].

Theorem 2.9. *Let k be a perfect field of characteristic $p > 5$. Let (X, Δ) be a three-dimensional $\mathbb{Q}$-factorial log canonical pair over k, where Δ is an $\mathbb{R}$-divisor. Let $f : X \to Z$ be a projective k-morphism to a quasi-projective k-scheme Z. Then there exists a $(K_X + \Delta)$-MMP over Z that terminates. In other words, there is a sequence of birational maps of three-dimensional normal varieties:*

$$X =: X_0 \overset{\varphi_0}{\dashrightarrow} X_1 \overset{\varphi_1}{\dashrightarrow} \cdots \overset{\varphi_{\ell-1}}{\dashrightarrow} X_\ell$$

such that if Δ_i denotes the proper transform of Δ on X_i, then the following properties hold.

- (i) *For any $i \in \{0, \ldots, \ell\}$, (X_i, Δ_i) is a $\mathbb{Q}$-factorial log canonical pair which is projective over Z.*
- (ii) *For any $i \in \{0, \ldots, \ell - 1\}$, $\varphi_i : X_i \dashrightarrow X_{i+1}$ is either a $(K_{X_i} + \Delta_i)$-divisorial contraction over Z or a $(K_{X_i} + \Delta_i)$-flip over Z.*
- (iii) *If $K_X + \Delta$ is pseudo-effective over Z, then $K_{X_\ell} + \Delta_\ell$ is nef over Z.*
- (iv) *If $K_X + \Delta$ is not pseudo-effective over Z, then there exists a $(K_{X_\ell} + \Delta_\ell)$-Mori fibre space $X_\ell \to Y$ over Z.*

Proof. See [HNT17, Theorem 1.1]. □

Proposition 2.10. *Let (X, Δ) be a three-dimensional quasi-projective log pair over a perfect field k of characteristic $p > 5$. Then there exists a projective birational morphism $f : Y \to X$ that satisfies the following conditions:*

- (i) *$a_F(X, \Delta) \leqslant 0$ holds for any f-exceptional prime divisor F;*
- (ii) *$(Y, \Delta_Y^{\wedge 1})$ is a $\mathbb{Q}$-factorial dlt pair, where Δ_Y is the $\mathbb{R}$-divisor defined by $K_Y + \Delta_Y = f^*(K_X + \Delta)$ (see § 2.1 for the definition of $\Delta_Y^{\wedge 1}$);*
- (iii) *$\mathrm{Nklt}(Y, \Delta_Y) = f^{-1}(\mathrm{Nklt}(X, \Delta))$ holds.*

Proof. See [HNT17, Proposition 3.5]. □

For later use, we establish the following result on plt centres.

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Proposition 2.11. *Let* (X, Δ) *be a three-dimensional plt pair over a perfect field* k *of characteristic* $p > 5$. *Set* $S := \llcorner \Delta \lrcorner$. *Then the normalisation* $\nu : S^N \to S$ *of* S *is a universal homeomorphism.*

Proof. Let $f : Y \to X$ and Δ_Y be as in Proposition 2.10. Since (X, Δ) is plt, it follows from Proposition 2.10(ii) that (Y, Δ_Y) is also plt. For $S_Y := \llcorner \Delta_Y \lrcorner$, we have that $f^{-1}(S) = f^{-1}(\mathrm{Nklt}(X, \Delta)) = \mathrm{Nklt}(Y, \Delta_Y) = S_Y$, where the second equality holds by Proposition 2.10(iii). Hence, the induced morphism $S_Y \to S$ has connected fibres. Since Y is $\mathbb{Q}$-factorial and (Y, Δ_Y) is plt, S_Y is normal (cf. [GNT19, Theorem 2.11]). Therefore, $S_Y \to S$ factors through the normalisation $\nu : S^N \to S$:

$$S_Y \to S^N \xrightarrow{\nu} S.$$

Then ν is a finite morphism and has connected fibres and hence ν is a universal homeomorphism. □

2.3 Connectedness theorem for the birational case

The purpose of this subsection is to establish the three-dimensional Kollár–Shokurov connectedness theorem for the birational case (Theorem 2.15). A key result is Proposition 2.14. To prove this proposition, we show Lemma 2.13 (cf. [Bir16, Theorem 1.8]). We start with the following auxiliary result.

Lemma 2.12. *Let* X *be a noetherian topological space (for the definition, see [Har77, p. 5]). Let* F *be a closed subset of* X *and let* $\{N_i\}_{i \in I}$ *be a set of closed subsets of* X*, where* I *is a finite set. Set* $N := \bigcup_{i \in I} N_i$. *Assume that the following hold.*

(i) $N_i \cap F$ *is connected for any* $i \in I$ *(*$N_i \cap F$ *is possibly empty).*

(ii) $N \cap U$ *is connected for any sufficiently small open neighbourhood* U *in* X *of* F. *In other words, there exists an open subset* U_0 *of* X *such that* $F \subset U_0$ *and, if* U *is an open subset of* X *satisfying* $F \subset U \subset U_0$*, then* $N \cap U$ *is connected.*

Then $N \cap F$ *is connected.*

Proof. We first reduce the problem to the case when $N_i \cap F \neq \emptyset$ for any $i \in I$. Set $I' := \{i \in I \mid N_i \cap F \neq \emptyset\}$ and

$$U_0' := U_0 \Big\backslash \Bigg(\bigcup_{i \in I \setminus I'} N_i \Bigg) = U_0 \cap \Bigg(\bigcap_{i \in I \setminus I'} (X \setminus N_i) \Bigg),$$

where U_0 is as in (2). Then we have $F \subset U_0'$. If U is an open subset of X such that $F \subset U \subset U_0'$, then (2) implies that $N \cap U$ is connected. Therefore, the problem can be reduced to the case when $N_i \cap F \neq \emptyset$ for any $i \in I$. In what follows, we assume that $N_i \cap F \neq \emptyset$ for any $i \in I$.

Take the decomposition into connected components of $N \cap F$:

$$N \cap F = \coprod_{j \in J} \Gamma_j.$$

We also have $N \cap F = \bigcup_{i \in I} (N_i \cap F)$. We first show that:

(iii) for any $i \in I$, there exists an index $j_i \in J$ such that $N_i \cap \Gamma_{j_i} \neq \emptyset$ and $N_i \cap \Gamma_j = \emptyset$ for any $j \in J \setminus \{j_i\}$. In particular, it holds that $N_i \cap F = N_i \cap \Gamma_{j_i}$ for any $i \in I$.

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Fix $i \in I$. We have

$$N_i \cap F = \coprod_{j \in J} (N_i \cap \Gamma_j).$$

Since $N_i \cap F$ is non-empty and connected, (iii) holds.

For $j \in J$, set $I_j := \{i \in I \mid N_i \cap \Gamma_j \neq \emptyset\}$ and $N_{I_j} := \bigcup_{i \in I_j} N_i$. Then (iii) implies that $I = \coprod_{j \in J} I_j$. In particular, we obtain $N = \bigcup_{j \in J} N_{I_j}$. Set

$$U_1 := \bigcap_{j, j' \in J, j \neq j'} X \setminus (N_{I_j} \cap N_{I_{j'}}), \tag{2.12.1}$$

which is an open subset of X.

We now show that $F \subset U_1$. Pick $j, j' \in J$ such that $j \neq j'$. For $i \in I_j$ and $i' \in I_{j'}$, we have

$$N_i \cap N_{i'} \cap F = (N_i \cap F) \cap (N_{i'} \cap F) \subset \Gamma_{j_i} \cap \Gamma_{j_{i'}} = \Gamma_j \cap \Gamma_{j'} = \emptyset,$$

where the inclusion follows from (iii). Hence, we have

$$N_{I_j} \cap N_{I_{j'}} \cap F = \bigcup_{i \in I_j, i' \in I_{j'}} (N_i \cap N_{i'} \cap F) = \emptyset,$$

i.e. $F \subset X \setminus (N_{I_j} \cap N_{I_{j'}})$. Hence, (2.12.1) implies that $F \subset U_1$.

Set $U := U_0 \cap U_1$, where U_0 is as in (ii). Since $F \subset U_0 \cap U_1 = U$, (ii) implies that $N \cap U$ is connected. We have

$$N \cap U = \bigcup_{j \in J} (N_{I_j} \cap U) = \coprod_{j \in J} (N_{I_j} \cap U),$$

where the last equality follows from (2.12.1). For $j \in J$, we have that

$$N_{I_j} \cap U \supset N_{I_j} \cap F \neq \emptyset.$$

Since $N \cap U$ is connected, we obtain $|J| = 1$, i.e. $N \cap F$ is connected. □

Lemma 2.13. *Let k be an algebraically closed field of characteristic $p > 5$. Let (X, D) be a three-dimensional $\mathbb{Q}$-factorial dlt pair over k and let $f : X \to Y$ be a projective birational k-morphism to a normal threefold Y over k. If f is either a $(K_X + D)$-divisorial contraction or a $(K_X + D)$-flipping contraction, then the induced morphism $\mathrm{Nklt}(X, D) \to Y$ has connected fibres.*

Proof. Set $S := \llcorner D \lrcorner$ and let $S = \sum_{i \in I} S_i$ be the irreducible decomposition. We have $\mathrm{Nklt}(X, D) = \bigcup_{i \in I} S_i$.

For any sufficiently small open neighbourhood U in X of $f^{-1}(y)$, it follows from [Bir16, Theorem 1.8] that $\mathrm{Nklt}(X, D) \cap U$ is connected. We apply Lemma 2.12 to $N_i := S_i$ and $F := f^{-1}(y)$. Then it is enough to prove that $S_i \cap F$ is connected. Therefore, after perturbing coefficients of D, we may assume that $S = S_i$, i.e. (X, D) is plt.

From now on, we treat the case when $\llcorner D \lrcorner = S$ is a prime divisor. In this case, we have $\mathrm{Nklt}(X, D) = S$. If $S \subset \mathrm{Ex}(f)$, then f is a $(K_X + D)$-divisorial contraction such that $S = \mathrm{Ex}(f)$. Then the assertion is clear because f has connected fibres. Thus, we may assume that $S \not\subset \mathrm{Ex}(f)$. Since $-(K_X + D)$ is f-ample, there exists an effective $\mathbb{R}$-divisor A on X such that $A \sim_{\mathbb{R}, f} -(K_X + D)$ and $(X, D + A)$ is plt. Since $K_X + D + A \sim_{\mathbb{R}, f} 0$, we have $K_X + D + A = f^*(K_Y + D_Y + A_Y)$ for $D_Y := f_*D$ and $A_Y := f_*A$. In particular, $(Y, D_Y + A_Y)$ is plt. Then the induced morphism $g : S \to S_Y := f(S)$ has connected fibres by Proposition 2.11. Since $\mathrm{Nklt}(X, D) \cap f^{-1}(y) = S \cap f^{-1}(y) = g^{-1}(y)$ for any $y \in f(S) = S_Y$, $\mathrm{Nklt}(X, D) \to Y$ has connected fibres. □

Proposition 2.14. *Let k be an algebraically closed field of characteristic $p > 5$. Let (V, Δ) be a three-dimensional quasi-projective $\mathbb{Q}$-factorial log pair over k. Let $\varphi : U \to V$ be a log resolution of (V, Δ). Let Δ_U be the $\mathbb{R}$-divisor defined by $K_U + \Delta_U = \varphi^*(K_V + \Delta)$. Then the induced morphism $\mathrm{Nklt}(U, \Delta_U) \to V$ has connected fibres.*

Proof. Let F be the sum of the φ-exceptional prime divisors F' on U whose log discrepancies are positive: $a_{F'}(V, \Delta) > 0$. Let G be the sum of the φ-exceptional prime divisors G' on U whose log discrepancies are non-positive: $a_{G'}(V, \Delta) \leqslant 0$. We set

$$D_U := \varphi_*^{-1}\Delta^{\wedge 1} + (1-\epsilon)F + G$$

for a sufficiently small positive real number ϵ. Then it holds that:

(i) (U, D_U) is dlt; and
(ii) $\mathrm{Supp}\,\Delta_U^{\geqslant 1} = \mathrm{Supp}\,D_U^{=1}$.

By Theorem 2.9, there is a $(K_U + D_U)$-MMP over V that terminates:

$$U =: X_0 \dashrightarrow \cdots \dashrightarrow X_\ell. \tag{2.14.1}$$

For any $i \in \{0, \ldots, \ell\}$, we define Δ_{X_i}, F_{X_i}, G_{X_i} and D_{X_i} as the push-forwards of Δ_U, F, G and D_U on X_i, respectively. Then it holds that $K_{X_i} + \Delta_{X_i} = \psi_i^*(K_V + \Delta)$, where $\psi_i : X_i \to V$ denotes the induced morphism. Moreover, for any $i \in \{0, \ldots, \ell\}$, we get:

(i)′ (X_i, D_{X_i}) is dlt; and
(ii)′ $\mathrm{Supp}\,\Delta_{X_i}^{\geqslant 1} = \mathrm{Supp}\,D_{X_i}^{=1}$.

Step 1. For any $i \in \{0, \ldots, \ell\}$, it holds that

$$\mathrm{Nklt}(X_i, \Delta_{X_i}) = \mathrm{Nklt}(X_i, D_{X_i}) = \mathrm{Supp}\,\Delta_{X_i}^{\geqslant 1} = \mathrm{Supp}\,D_{X_i}^{=1}.$$

Proof of Step 1. Fix $i \in \{0, \ldots, \ell\}$. We obtain

$$\mathrm{Nklt}(X_i, D_{X_i}) = \mathrm{Supp}\,D_{X_i}^{=1} = \mathrm{Supp}\,\Delta_{X_i}^{\geqslant 1} \subset \mathrm{Nklt}(X_i, \Delta_{X_i}),$$

where the first equality holds by (i)′ and the second one follows from (ii)′. Hence, it is sufficient to show that $\mathrm{Nklt}(X_i, \Delta_{X_i}) \subset \mathrm{Supp}\,D_{X_i}^{=1}$. For sufficiently large $b > 0$, the $\mathbb{R}$-divisor

$$A_{X_i} := (\psi_i^{-1})_*\Delta + (1-\epsilon)F_{X_i} + bG_{X_i}$$

satisfies $\Delta_{X_i} \leqslant A_{X_i}$. Therefore, we get

$$\mathrm{Nklt}(X_i, \Delta_{X_i}) \subset \mathrm{Nklt}(X_i, A_{X_i}).$$

Since $A_{X_i}^{\wedge 1} = D_{X_i}$, we have $A_{X_i}^{<1} = D_{X_i}^{<1}$ and hence $(X, A_{X_i}^{<1})$ is klt by (i)′. Thus, it follows from Proposition 2.7 that

$$\mathrm{Nklt}(X_i, A_{X_i}) = \mathrm{Supp}\,A_{X_i}^{\geqslant 1} = \mathrm{Supp}\,D_{X_i}^{=1}.$$

Thus, we obtain the desired inclusion $\mathrm{Nklt}(X_i, \Delta_{X_i}) \subset \mathrm{Supp}\,D_{X_i}^{=1}$. This completes the proof of Step 1. □

Step 2. Let $g : X_i \to X_{i+1}$ be a divisorial contraction appearing in the MMP (2.14.1). If $\mathrm{Nklt}(X_{i+1}, D_{X_{i+1}}) \to V$ has connected fibres, then so does $\mathrm{Nklt}(X_i, D_{X_i}) \to V$.

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Proof of Step 2. It follows from Lemma 2.13 that $\mathrm{Nklt}(X_i, D_{X_i}) \to \mathrm{Nklt}(X_{i+1}, D_{X_{i+1}})$ has connected fibres. This completes the proof of Step 2. □

Step 3. Let $h : X_i \dashrightarrow X_{i+1}$ be a flip appearing in the MMP (2.14.1). If $\mathrm{Nklt}(X_{i+1}, D_{X_{i+1}}) \to V$ has connected fibres, then so does $\mathrm{Nklt}(X_i, D_{X_i}) \to V$.

Proof of Step 3. Assume that $\mathrm{Nklt}(X_{i+1}, D_{X_{i+1}}) \to V$ has connected fibres. Let $\varphi_i : X_i \to Y$ be the flipping contraction and let $\varphi_{i+1} : X_{i+1} \to Y$ and $\psi_Y : Y \to V$ be the induced morphisms. Set $N_Y := \varphi_i(\mathrm{Nklt}(X_i, D_{X_i}))$ and $N_V := \psi_Y(N_Y)$. It follows from Step 1 that $N_Y = \varphi_{i+1}(\mathrm{Nklt}(X_{i+1}, D_{X_{i+1}}))$. By assumption, it holds that the composite morphism

$$\mathrm{Nklt}(X_{i+1}, D_{X_{i+1}}) \to N_Y \to N_V$$

is a surjective morphism with connected fibres. In particular, $N_Y \to N_V$ has connected fibres. Since $\mathrm{Nklt}(X_i, D_{X_i}) \to N_Y$ has connected fibres by Lemma 2.13, their composition

$$\mathrm{Nklt}(X_i, D_{X_i}) \to N_Y \to N_V$$

also has connected fibres. This completes the proof of Step 3. □

Step 4. The induced morphism $\mathrm{Nklt}(X_\ell, D_\ell) \to V$ has connected fibres.

Proof of Step 4. We have

$$B_\ell := (\psi_\ell^{-1})_*\Delta^{\wedge 1} + (1-\epsilon)F_{X_\ell} + G_{X_\ell} - \Delta_{X_\ell} \sim_{V,\mathbb{R}} K_{X_\ell} + D_{X_\ell}.$$

Then B_ℓ is nef over V. The push-forward of $-B_\ell$ on V, which is nothing but the push-forward of $\Delta_{X_\ell} - (\psi_\ell^{-1})_*\Delta^{\wedge 1}$, is effective. Hence, it turns out by the negativity lemma that $-B_\ell$ itself is effective. Since ϵ is sufficiently small, it follows that $F_{X_\ell} = 0$, that is, any φ_ℓ-exceptional prime divisor E satisfies $a_E(V, \Delta) \leqslant 0$. Since V is $\mathbb{Q}$-factorial, it holds that

$$\mathrm{Ex}(\varphi_\ell) \subset \mathrm{Nklt}(X_\ell, \Delta_{X_\ell}).$$

In particular, $\mathrm{Nklt}(X_\ell, \Delta_{X_\ell}) \cap \varphi_\ell^{-1}(v) = \varphi_\ell^{-1}(v)$ holds and this is connected for any closed point v of V. This completes the proof of Step 4. □

Step 2, Step 3 and Step 4 complete the proof of Proposition 2.14. □

THEOREM 2.15. *Let k be a perfect field of characteristic $p > 5$. Let $f : X \to V$ be a projective birational k-morphism of normal quasi-projective threefolds over k. Let (X, Δ) be a sub-log pair over k such that $-(K_X + \Delta)$ is f-nef and $f_*\Delta$ is effective. Then the induced morphism $\mathrm{Nklt}(X, \Delta) \to V$ has connected fibres.*

Proof. Taking the base change to the algebraic closure of k, we may assume that k is an algebraically closed field. We now reduce the problem to the case when $K_X + \Delta \sim_{\mathbb{R},f} 0$. Since f is birational, $-(K_X + \Delta)$ is f-nef and f-big. After replacing Δ, we may assume that $-(K_X + \Delta)$ is f-ample. Then there exists an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor A such that $A \sim_{\mathbb{R},f} -(K_X + \Delta)$ and $\mathrm{Nklt}(X, \Delta) = \mathrm{Nklt}(X, \Delta + A)$. Thus, we may assume that $K_X + \Delta \sim_{\mathbb{R},f} 0$. In particular, for $\Delta_V := f_*\Delta$, it holds that (V, Δ_V) is a log pair and $K_X + \Delta = f^*(K_V + \Delta_V)$.

Let $\varphi : V_1 \to V$ be a dlt modification of (V, Δ_V) such that $\mathrm{Nklt}(V_1, \Delta_{V_1}) = \varphi^{-1}(\mathrm{Nklt}(V, \Delta_V))$ (Proposition 2.10). In particular, $\mathrm{Nklt}(V_1, \Delta_{V_1}) \to \mathrm{Nklt}(V, \Delta_V)$ has connected fibres. Let $f_1 : X_1 \to V_1$ be a log resolution of (V_1, Δ_{V_1}) that factors through X. By Proposition 2.14, $\mathrm{Nklt}(X_1, \Delta_{X_1}) \to \mathrm{Nklt}(V_1, \Delta_{V_1})$ has connected fibres. Thus, the composite morphism

$$\mathrm{Nklt}(X_1, \Delta_{X_1}) \to \mathrm{Nklt}(V_1, \Delta_{V_1}) \to \mathrm{Nklt}(V, \Delta_V)$$

has connected fibres and factors through $\mathrm{Nklt}(X, \Delta)$. In particular, also $\mathrm{Nklt}(X, \Delta) \to \mathrm{Nklt}(V, \Delta_V)$ has connected fibres. □

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Remark 2.16. When we apply Proposition 2.14 in the above proof, we only use the properties that V_1 is $\mathbb{Q}$-factorial and $\mathrm{Nklt}(V_1, \Delta_{V_1}) = \varphi^{-1}(\mathrm{Nklt}(V, \Delta_V))$, whilst we do not use the fact that $(V_1, \Delta_{V_1}^{\wedge 1})$ is dlt.

2.4 Results on the Witt vector cohomologies

For the definition of the Witt vector cohomology and its properties, we refer to [GNT19] and [CR12]. Our goal of this subsection is to show Propositions 2.22 and 2.23. As far as the authors know, it is an open problem whether $R^i f_*(W\mathcal{O}_{X,\mathbb{Q}})$ commute with base changes. Such a problem occurs because inverse limits do not commute with tensor products. We start by showing some auxiliary results.

Lemma 2.17. *Let k be a perfect field of characteristic $p > 0$. Let $f : X \to Y$ be a proper surjective k-morphism of separated schemes of finite type over k. Then the following conditions are equivalent:*

(i) *f has connected fibres;*
(ii) *the induced homomorphism $W\mathcal{O}_{Y,\mathbb{Q}} \to f_* W\mathcal{O}_{X,\mathbb{Q}}$ is an isomorphism.*

Proof. It follows from [GNT19, Lemma 2.22] that (i) implies (ii).

It is enough to show that (ii) implies (i). Assume (ii). Taking the Stein factorisation, the problem is reduced to the case when f is a finite surjective morphism. Since the problem is local on Y, we may assume that $X = \mathrm{Spec}\, B$ and $Y = \mathrm{Spec}\, A$. For the induced ring homomorphism $A \to B$, we have that $W(A)_{\mathbb{Q}} \to W(B)_{\mathbb{Q}}$ is an isomorphism. Fix a maximal ideal $\mathfrak{m}$ of A. We have the following commutative diagram of ring homomorphisms.

$$\begin{array}{ccc} W(A)_{\mathbb{Q}} & \xrightarrow{\simeq} & W(B)_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ W(A/\mathfrak{m})_{\mathbb{Q}} & \xrightarrow{\psi} & W(B/\mathfrak{m}B)_{\mathbb{Q}} \end{array}$$

By a diagram chase, $\psi : W(A/\mathfrak{m})_{\mathbb{Q}} \to W(B/\mathfrak{m}B)_{\mathbb{Q}}$ is surjective. On the other hand, $W(A/\mathfrak{m})_{\mathbb{Q}}$ is a field and hence ψ is an isomorphism. In particular, $f^{-1}(\mathfrak{m})$ consists of one point $\mathfrak{n}$. By $W(A/\mathfrak{m})_{\mathbb{Q}} \simeq W(B/\mathfrak{m}B)_{\mathbb{Q}}$ and $W(B/\mathfrak{n})_{\mathbb{Q}} \simeq W(B/\mathfrak{m}B)_{\mathbb{Q}}$, we have $W(A/\mathfrak{m})_{\mathbb{Q}} \simeq W(B/\mathfrak{n})_{\mathbb{Q}}$. Hence, the finite extension $W(A/\mathfrak{m}) \hookrightarrow W(B/\mathfrak{n})$ of discrete valuation rings is also an isomorphism. Taking modulo p reduction, we have that $A/\mathfrak{m} \to B/\mathfrak{n}$ is an isomorphism. Thus, (i) holds. □

We often use the following exact sequences, which we call the Mayer–Vietoris exact sequences.

Lemma 2.18. *Let k be a perfect field of characteristic $p > 0$. Let V be a scheme of finite type over k. Let X, X_1 and X_2 be closed subschemes of V such that the set-theoretic equation $X = X_1 \cup X_2$ holds. Let $X_1 \cap X_2$ be the scheme-theoretic intersection. Let I_X, I_{X_1}, I_{X_2} and $I_{X_1 \cap X_2}$ be the corresponding coherent ideal sheaves on V. Then there exist the exact sequences:*

(i) $0 \to W\mathcal{O}_{X,\mathbb{Q}} \to W\mathcal{O}_{X_1,\mathbb{Q}} \oplus W\mathcal{O}_{X_2,\mathbb{Q}} \to W\mathcal{O}_{X_1 \cap X_2,\mathbb{Q}} \to 0$*; and*
(ii) $0 \to WI_{X,\mathbb{Q}} \to WI_{X_1,\mathbb{Q}} \oplus WI_{X_2,\mathbb{Q}} \to WI_{X_1 \cap X_2,\mathbb{Q}} \to 0.$

Proof. By using Remark 2.2 and the fact that the functor $(-)_{\mathbb{Q}}$ is exact, we obtain the exact sequence (i) by the same argument as in [BBE07, Proposition 2.2]. The exact sequence (ii) is obtained by (i) and the snake lemma. □

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Lemma 2.19. *Let $k \subset k'$ be an extension of perfect fields of characteristic $p > 0$. Let X be a proper scheme over k and set $X' := X \times_k k'$. Then the induced $W(k')_{\mathbb{Q}}$-linear map*

$$H^0(X, W\mathcal{O}_{X,\mathbb{Q}}) \otimes_{W(k)_{\mathbb{Q}}} W(k')_{\mathbb{Q}} \to H^0(X', W\mathcal{O}_{X',\mathbb{Q}})$$

is bijective.

Proof. Taking the Stein factorisation of the structure morphism $X \to \operatorname{Spec} k$, we may assume that X is of dimension zero. Replacing X by a connected component, we may assume that $X = \operatorname{Spec} L$, where L is a finite extension of k. Then the assertion is clear. □

To prove Proposition 2.23, we first show the following weaker statement.

Lemma 2.20. *Let k be a perfect field of characteristic $p > 0$ and let X be a one-dimensional smooth projective scheme over k. Then the following are equivalent:*

(i) $H^1(X, \mathcal{O}_X) = 0$;
(ii) $H^1(X, W_n\mathcal{O}_X) = 0$ *for some* $n \in \mathbb{Z}_{>0}$;
(iii) $H^1(X, W_n\mathcal{O}_X) = 0$ *for any* $n \in \mathbb{Z}_{>0}$;
(iv) $H^1(X, W\mathcal{O}_X) = 0$;
(v) $H^1(X, W\mathcal{O}_{X,\mathbb{Q}}) = 0$.

Proof. Clearly, we may assume that X is connected. By the exact sequence

$$0 \to W_n\mathcal{O}_X \xrightarrow{V} W_{n+1}\mathcal{O}_X \to \mathcal{O}_X \to 0$$

and the fact that X is one dimensional, it holds that (i), (ii) and (iii) are equivalent. By [GNT19, Lemma 2.19], (iii) implies (iv). Moreover, by the exact sequence

$$0 \to W\mathcal{O}_X \xrightarrow{V} W\mathcal{O}_X \to \mathcal{O}_X \to 0$$

and the fact that X is one dimensional, (iv) implies (i).

The equivalence between (iv) and (v) follows from the fact that $H^1(X, W\mathcal{O}_X)$ is a free $W(k)$-module [Ill79, ch. II, Proposition 2.19]. □

Lemma 2.21. *Let $k \subset k'$ be an extension of perfect fields of characteristic $p > 0$. Let X be a proper one-dimensional scheme over k. Then the following are equivalent:*

(i) $H^1(X, W\mathcal{O}_{X,\mathbb{Q}}) = 0$;
(ii) $H^1(X \times_k k', W\mathcal{O}_{X\times_k k',\mathbb{Q}}) = 0$.

Proof. For simplicity, we denote $K = W(k)_{\mathbb{Q}}$, $K' = W(k')_{\mathbb{Q}}$ and $Y' = Y \times_k k'$ for a k-scheme Y. We may assume that X is reduced. Let

$$X^N \to X$$

be the normalisation of X. Thanks to Lemma 2.20, if one of (i) and (ii) holds, then it follows that

$$H^1(X^N, W\mathcal{O}_{X,\mathbb{Q}}) = H^1(X'^N, W\mathcal{O}_{X'^N,\mathbb{Q}}) = 0.$$

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For the conductor subschemes C and D of X and X^N, respectively, we have a commutative diagram with exact horizontal sequences:

$$\begin{array}{ccccccc} H^0(W\mathcal{O}_{X^N,\mathbb{Q}})_{K'} \oplus H^0(W\mathcal{O}_{C,\mathbb{Q}})_{K'} & \longrightarrow & H^0(W\mathcal{O}_{D,\mathbb{Q}})_{K'} & \longrightarrow & H^1(W\mathcal{O}_{X,\mathbb{Q}})_{K'} & \longrightarrow & 0 \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\ H^0(W\mathcal{O}_{X'^N,\mathbb{Q}}) \oplus H^0(W\mathcal{O}_{C',\mathbb{Q}}) & \longrightarrow & H^0(W\mathcal{O}_{D',\mathbb{Q}}) & \longrightarrow & H^1(W\mathcal{O}_{X',\mathbb{Q}}) & \longrightarrow & 0 \end{array}$$

where $(-)_{K'}$ denotes the tensor product $(-) \otimes_K K'$. As both α and β are isomorphisms by Lemma 2.19, so is γ by the five lemma, as desired. □

Proposition 2.22. *Let $k \subset k'$ be an extension of perfect fields of characteristic $p > 0$. Let X be a normal surface over k. Then the following are equivalent:*

(i) *X has $W\mathcal{O}$-rational singularities;*
(ii) *$X \times_k k'$ has $W\mathcal{O}$-rational singularities.*

Proof. We may assume that Q is a unique non-regular point of X. Let $f : Y \to X$ be a resolution of singularities such that $f(\mathrm{Ex}(f)) = Q$. For $E := \mathrm{Ex}(f)$, we have the exact sequence

$$0 \to WI_{E,\mathbb{Q}} \to W\mathcal{O}_{Y,\mathbb{Q}} \to W\mathcal{O}_{E,\mathbb{Q}} \to 0.$$

Thanks to the vanishing of $R^if_*(WI_{E,\mathbb{Q}}) = 0$ for $i > 0$ [BBE07, Theorem 2.4], it holds that

$$R^if_*(W\mathcal{O}_{Y,\mathbb{Q}}) \simeq H^i(E, W\mathcal{O}_{E,\mathbb{Q}}).$$

Therefore, it follows from Lemma 2.21 that (i) and (ii) are equivalent. □

Proposition 2.23. *Let k be a perfect field of characteristic $p > 0$ and let X be a reduced projective scheme over k such that:*

(a) *any irreducible component of X is one dimensional; and*
(b) *any non-regular point x of X is an ordinary double point.*

Then the following are equivalent:

(i) *$H^1(X, \mathcal{O}_X) = 0$;*
(ii) *$H^1(X, W_n\mathcal{O}_X) = 0$ for some $n \in \mathbb{Z}_{>0}$;*
(iii) *$H^1(X, W_n\mathcal{O}_X) = 0$ for any $n \in \mathbb{Z}_{>0}$;*
(iv) *$H^1(X, W\mathcal{O}_X) = 0$;*
(v) *$H^1(X, W\mathcal{O}_{X,\mathbb{Q}}) = 0$;*
(vi) *any connected component of $X \times_k \overline{k}$ is a tree of smooth rational curves.*

Proof. We may assume that X is connected. Moreover, replacing k by k' for the Stein factorisation $X \to \operatorname{Spec} k' \to \operatorname{Spec} k$, we may assume that X is geometrically connected.

We now show that the assertions (i), (ii), (iii) and (iv) are equivalent. By the exact sequence

$$0 \to W_n\mathcal{O}_X \xrightarrow{V} W_{n+1}\mathcal{O}_X \to \mathcal{O}_X \to 0$$

and the fact that X is one dimensional, it holds that (i), (ii) and (iii) are equivalent. We have that (iii) implies (iv) by [GNT19, Lemma 2.19]. Moreover, by the exact sequence

$$0 \to W\mathcal{O}_X \xrightarrow{V} W\mathcal{O}_X \to \mathcal{O}_X \to 0$$

and the fact that X is one dimensional, (iv) implies (i). Thus, (i), (ii), (iii) and (iv) are equivalent.

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Thanks to [Kol96, ch. II, Lemma 7.5], it holds that (vi) implies (i). Further, (iv) clearly implies (v). Thus, it suffices to show that (v) implies (vi). Lemma 2.21 allows us to replace $X \to \operatorname{Spec} k$ by the base change $X \times_k \overline{k} \to \operatorname{Spec} \overline{k}$. Then it follows from [CR12, the second last paragraph of § 4.6] that (v) implies (vi), as desired. □

2.5 Geometric rationality of del Pezzo surfaces over imperfect fields

In this subsection, we prove Proposition 2.26. To this end, we start with the following lemma.

Lemma 2.24. *Let k be a separably closed field of characteristic $p > 0$ which is not algebraic over a finite field. Let X be a projective normal $\mathbb{Q}$-factorial surface over k with $k = H^0(X, \mathcal{O}_X)$. If there is an $\mathbb{R}$-divisor Δ such that $0 \leqslant \Delta < 1$ and $-(K_X + \Delta)$ is nef and big, then $(X \times_k \overline{k})_{\mathrm{red}}$ is a rational surface.*

Proof. Replacing Δ, we may assume that $-(K_X + \Delta)$ is ample. If $X \to X'$ is a birational k-morphism of projective normal varieties with $k = H^0(X, \mathcal{O}_X) = H^0(X', \mathcal{O}_{X'})$, then also $(X \times_k \overline{k})_{\mathrm{red}} \to (X' \times_k \overline{k})_{\mathrm{red}}$ is birational. Thus, we may replace (X, Δ) by the end result of a $(K_X + \Delta)$-MMP [Tan18b, Theorem 1.1]. Hence, we may assume that one of the following conditions holds:

(a) $\rho(X) = 1$;
(b) there is a $(K_X + \Delta)$-Mori fibre space $\pi_1 : X \to B_1$ onto a regular projective curve B_1 with $(\pi_1)_* \mathcal{O}_X = \mathcal{O}_{B_1}$.

In what follows, we denote by Y the normalisation of $(X \times_k \overline{k})_{\mathrm{red}}$ and denote by $f : Y \to X$ the composite morphism. By applying [Tan18a, Theorem 1.1] to the regular locus of X, we can write

$$K_Y + D = f^* K_X$$

for some effective $\mathbb{Z}$-divisor D.

Suppose (a). Then Y is a projective normal $\mathbb{Q}$-factorial surface such that $\rho(Y) = 1$ [Tan18a, Proposition 2.4(2)]. Since $-K_Y$ is ample, Y is a ruled surface. Assume that Y is not rational; let us derive a contradiction. Let $\mu : Z \to Y$ be the minimal resolution of Y. Since Z is an irrational ruled surface, there is a projective morphism $\pi : Z \to B$ onto a smooth projective irrational curve whose general fibres are $\mathbb{P}^1$. Since $\overline{k} \neq \overline{\mathbb{F}}_p$, it follows from [Tan14, Theorem 3.20] that π factors through μ:

$$\pi : Z \xrightarrow{\mu} Y \to B.$$

This is a contradiction to $\rho(Y) = 1$. Thus, we are done for the case (a).

Suppose (b). Since $-(K_X + \Delta)$ is ample, there exists an extremal ray R of $\overline{\mathrm{NE}}(X)$ not corresponding to π_1. By [Tan18b, Theorem 4.4], the extremal ray R induces either a birational morphism or another $(K_X + \Delta)$-Mori fibre space $X \to B_2$ onto a curve B_2. If the former case occurs, then the problem is reduced to the case (a). Therefore, we may assume that there exist two Mori fibre space structures $\pi_1 : X \to B_1$ and $\pi_2 : X \to B_2$ onto curves B_1 and B_2. In particular, any fibre of π_i dominates B_{3-i}. Let $\pi'_i : Y \to B'_i$ be the Stein factorisation of the composite morphism:

$$Y \to X \times_k \overline{k} \xrightarrow{\pi_i \times_k \overline{k}} B_i \times_k \overline{k}.$$

Then any fibre of π'_i dominates B'_{3-i}. Since $-K_Y$ is big, a general fibre of each π'_i is isomorphic to $\mathbb{P}^1$. In particular, $B'_1 \simeq \mathbb{P}^1$ and Y is rational. □

Remark 2.25. The statement of Lemma 2.24 does not hold if we drop the assumption on the base field k. Indeed, if $k = \overline{\mathbb{F}}_p$, then any normal surface is $\mathbb{Q}$-factorial (e.g. see [Tan14, Theorem 4.5]). Thus, the cone X over an elliptic curve over $\overline{\mathbb{F}}_p$ is $\mathbb{Q}$-factorial and $-K_X$ is ample.

Proposition 2.26. *Let (X, Δ) be a projective two-dimensional klt pair over a field k of characteristic $p > 0$ such that $-(K_X + \Delta)$ is nef and big. Assume that $k = H^0(X, \mathcal{O}_X)$. Then $(X \times_k \overline{k})_{\mathrm{red}}$ is a rational surface. In particular, X is rationally connected over k.*

Proof. We may assume that k is separably closed. Since the assertion is well known if k is an algebraically closed field (cf. [Tan15, Fact 3.4 and Theorem 3.5]), the problem is reduced to the case when k is an imperfect field. In particular, k is not algebraic over a finite field. As X is $\mathbb{Q}$-factorial [Tan18b, Corollary 4.11], the assertion follows from Lemma 2.24. □

3. $W\mathcal{O}$-vanishing for log Fano contractions

In this section, we prove a vanishing theorem for log Fano contractions (Theorem 3.11). We shall divide the proof into cases depending on the dimension of the base scheme Z. The cases $\dim Z = 1$ and $\dim Z = 2$ are treated in § 3.1 and § 3.2, respectively. The remaining cases $\dim Z = 0$ and $\dim Z = 3$ have been already settled in [GNT19] (cf. the proof of Theorem 3.11).

Before starting the case study, we summarise some results used repeatedly in the proof of Theorem 3.11.

Theorem 3.1. *Let k be a perfect field of characteristic $p > 5$. Then the following hold.*

(i) *If (X, Δ) is a three-dimensional klt pair over k, then X has $W\mathcal{O}$-rational singularities.*
(ii) *If X is a three-dimensional projective variety of Fano type over k, then $H^i(X, W\mathcal{O}_{X,\mathbb{Q}}) = 0$ for $i > 0$.*

Proof. When Δ is a $\mathbb{Q}$-divisor, both (i) and (ii) follow from [GNT19, Theorem 1.4] and [GNT19, Theorem 1.3], respectively. Thanks to [Fuj17, Lemma 4.6.1], the general case is reduced to this case. □

Theorem 3.2. *Let $f : X \to Y$ be a projective morphism between integral schemes with $W\mathcal{O}$-rational singularities. Suppose that Y is normal and that the generic fibre $X_{K(Y)}$ of f is smooth and rationally chain connected. Then $R^i f_* W\mathcal{O}_{X,\mathbb{Q}} = 0$ holds for $i > 0$.*

Proof. This is a special case of [CR12, Theorem 4.8.1]. □

Lemma 3.3. *Let k be a field of characteristic $p > 0$. Let $f : X \to Y$ be a projective k-morphism of normal k-varieties such that $f_*\mathcal{O}_X = \mathcal{O}_Y$. Then there exists a commutative diagram*

$$\begin{array}{ccc} X' & \xrightarrow{\alpha} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{\beta} & Y \end{array}$$

of projective k-morphisms of normal k-varieties that satisfies the following properties.

(i) *Both α and β are finite universal homeomorphisms.*
(ii) *$f'_*\mathcal{O}_{X'} = \mathcal{O}_{Y'}$.*

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(iii) *The generic fibre $X'_{K(Y')}$ of f' is geometrically normal over $K(Y')$.*

(iv) *The induced morphism $X'_{K(Y')} \to (X \times_Y K(Y'))_{\mathrm{red}}$ is a finite birational morphism. In particular, this morphism coincides with the normalisation of $(X \times_Y K(Y'))_{\mathrm{red}}$.*

Proof. We set $K := K(Y)$. Let $\nu_0 : X'_0 \to (X \times_Y K^{1/p^\infty})_{\mathrm{red}}$ be the normalisation of $(X \times_Y K^{1/p^\infty})_{\mathrm{red}}$. Since ν_0 is a finite universal homeomorphism by [Tan18a, Lemma 2.2], we have that X'_0 is geometrically connected and projective over a perfect field K^{1/p^∞} and hence $K^{1/p^\infty} = H^0(X'_0, \mathcal{O}_{X'_0})$. There exist an intermediate field L between K and K^{1/p^∞} satisfying $[L : K] < \infty$ and a projective normal L-variety X'_1 such that $X'_1 \times_L K^{1/p^\infty} = X'_0$ with the following commutative diagram, where ν_1 is birational.

$$\begin{array}{ccccccc}
X'_0 & \longrightarrow & X'_1 & & & & \\
\downarrow{\scriptstyle \nu_0} & & \downarrow{\scriptstyle \nu_1} & & & & \\
(X \times_Y K^{1/p^\infty})_{\mathrm{red}} & \longrightarrow & (X \times_Y L)_{\mathrm{red}} & \longrightarrow & X \times_Y K & \longrightarrow & X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Spec}\, K^{1/p^\infty} & \longrightarrow & \mathrm{Spec}\, L & \longrightarrow & \mathrm{Spec}\, K & \longrightarrow & Y
\end{array}$$

In particular, it follows that $L = H^0(X'_1, \mathcal{O}_{X'_1})$ and ν_1 is a finite universal homeomorphism. Since ν_1 is a finite birational morphism and X'_1 is normal, ν_1 is nothing but the normalisation of $(X \times_Y L)_{\mathrm{red}}$. Furthermore, X'_1 is geometrically normal, since X'_0 is normal and K^{1/p^∞} is perfect.

Let X' (respectively Y') be the normalisation of X (respectively Y) in $K(X'_1)$ (respectively L). Then we get the commutative diagram as in the statement and the properties (i), (iii) and (iv) follow from the construction.

Let us show (ii). Since $\mathcal{O}_{Y'} \to f'_*\mathcal{O}_{X'}$ is an isomorphism on some non-empty open subset of Y', it holds that $Y'' \to Y'$ is a finite birational morphism for the Stein factorisation

$$f' : X' \to Y'' \to Y'$$

of f'. As Y' is normal, we have that $Y'' \to Y'$ is an isomorphism and hence (ii) holds. □

3.1 Del Pezzo fibrations

In this subsection, we establish the $W\mathcal{O}$-vanishing for del Pezzo fibrations (Proposition 3.6). A key result is the following.

Lemma 3.4. *Let k be a perfect field of characteristic $p > 0$. Let $f : X \to Y$ be a projective k-morphism such that:*

(i) *X is a normal threefold over k that has $W\mathcal{O}$-rational singularities;*

(ii) *Y is a smooth k-curve;*

(iii) *$f_*\mathcal{O}_X = \mathcal{O}_Y$; and*

(iv) *the geometric generic fibre $X_{\overline{K(Y)}}$ of f is a normal rational surface.*

Then $R^i f_*(W\mathcal{O}_{X,\mathbb{Q}}) = 0$ for $i > 0$.

Proof. We divide the proof into two steps.

Step 1. The assertion of Lemma 3.4 holds if there exists a projective birational $K(Y)$-morphism

$$g_0 : Z_0 \to X_{K(Y)}$$

from a smooth projective $K(Y)$-surface Z_0.

Proof of Step 1. Killing the denominators of all the elements of $K(Y)$ defining g_0, we can find a non-empty open subset Y' of Y and morphisms

$$h' : Z' \xrightarrow{g'} X' := f^{-1}(Y') \xrightarrow{f|_{f^{-1}(Y')}} Y'$$

whose base changes by $(-) \times_{Y'} \operatorname{Spec} K(Y)$ are the same as

$$Z_0 \xrightarrow{g_0} X_{K(Y)} \to \operatorname{Spec} K(Y).$$

Furthermore, we may assume that:

- Z' is an integral scheme;
- g' is a projective birational morphism; and
- the composite morphism h' is smooth.

In particular, Z' is a smooth threefold over k. Let $g : Z \to X$ be a smooth projective compactification of the induced morphism

$$Z' \xrightarrow{g_1} X' = f^{-1}(Y') \hookrightarrow X,$$

i.e. there are morphisms

$$g' : Z' \xrightarrow{j} Z \xrightarrow{g} X$$

such that j is an open immersion, g is projective and Z is an integral scheme smooth over k. In particular, Z is a smooth threefold over k which is projective over X and hence over Y. We get the composite morphism

$$h : Z \xrightarrow{g} X \xrightarrow{f} Y$$

whose geometric generic fibre $Z \times_Y \overline{K(Y)}$ satisfies the following isomorphisms:

$$Z \times_Y \overline{K(Y)} \simeq Z' \times_{Y'} \overline{K(Y)} \simeq Z_0 \times_{K(Y)} \overline{K(Y)}.$$

In particular, $Z \times_Y \overline{K(Y)}$ is a smooth projective rational surface over $\overline{K(Y)}$.

Therefore, we have that

$$Rf_*(W\mathcal{O}_{X,\mathbb{Q}}) \simeq Rf_*Rg_*(W\mathcal{O}_{Z,\mathbb{Q}}) \simeq Rh_*(W\mathcal{O}_{Z,\mathbb{Q}}) \simeq W\mathcal{O}_{Y,\mathbb{Q}},$$

where the first isomorphism follows from the assumption (i) and the third follows from Theorem 3.2. This completes the proof of Step 1. □

Step 2. The assertion of Lemma 3.4 holds without any additional assumptions.

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Proof of Step 2. Let $K(Y)^{1/p^\infty}$ be the purely inseparable closure of $K(Y)$ in the algebraic closure $\overline{K(Y)}$ of $K(Y)$. We fix a projective birational $K(Y)^{1/p^\infty}$-morphism

$$g_1 : Z_1 \to X \times_Y K(Y)^{1/p^\infty}$$

from a regular $K(Y)^{1/p^\infty}$-surface Z_1. Note that Z_1 is smooth over $K(Y)^{1/p^\infty}$, since $K(Y)^{1/p^\infty}$ is a perfect field. Then there exist a finite purely inseparable extension L of $K(Y)$ and a projective normal L-surface Z_2 with the following projective birational L_0-morphism:

$$g_2 : Z_2 \to X \times_Y L,$$

whose base change by $(-) \times_L K(Y)^{1/p^\infty}$ is isomorphic to g_1. In particular, Z_2 is a smooth projective surface over L.

Let Y_2 be the normalisation of Y in L and let X_2 be the normalisation of $(X \times_Y Y_2)_{\mathrm{red}}$. We get the following commutative diagram of normal k-varieties.

$$\begin{array}{ccc} X & \xleftarrow{\alpha} & X_2 \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_2} \\ Y & \xleftarrow{\beta} & Y_2 \end{array}$$

Claim 3.5. *The following hold.*

(a) *There exists a non-empty open subset Y_3 of Y_2 such that the induced morphism*

$$X_3 := f_2^{-1}(Y_3) \to X \times_Y Y_3$$

is an isomorphism.

(b) *The generic fibre of $f_2 : X_2 \to Y_2$ is isomorphic to $X \times_Y L$.*

(c) *α is a finite universal homeomorphism.*

(d) *β is a finite universal homeomorphism.*

(e) $(f_2)_*\mathcal{O}_{X_2} = \mathcal{O}_{Y_2}$.

Proof of Claim 3.5. Note that the generic fibre of $X \times_Y Y_2 \to Y_2$ is normal by the geometric normality of $X_{K(Y)}$ (the assumption (iv)). Therefore, (a) holds, since $X_2 \to (X \times_Y Y_2)_{\mathrm{red}}$ is the normalisation. It is clear that (b) follows from (a). As $K(Y) \subset L$ is a purely inseparable extension, the assertion (d) holds.

Let us show (c). It follows from the construction that $\alpha : X_2 \to X$ is a finite surjective morphism of normal schemes. In particular, X_2 coincides with the normalisation of X in $K(X_2)$. Therefore, it suffices to show that the field extension $K(X) \subset K(X_2)$ is purely inseparable, which in turn follows from the following equation:

$$K(X_2) = K(X_2 \times_{Y_2} K(Y_2)) = K(X \times_Y L),$$

where the second equality follows from (b). Thus, (c) holds.

Let us show (e). Let $f_2 : X_2 \to Y_2' \to Y_2$ be the Stein factorisation of f_2. By (a), there exists a non-empty open subset Y_4 of Y_2 such that the induced homomorphism

$$\mathcal{O}_{Y_2}|_{Y_4} \to (f_2)_*\mathcal{O}_{X_2}|_{Y_4}$$

is an isomorphism. In particular, $Y_2' \to Y_2$ is a finite birational morphism of integral k-varieties. As Y_2 is normal, it holds that $Y_2' \to Y_2$ is an isomorphism and hence we obtain (e). This completes the proof of Claim 3.5. □

Let us go back to the proof of Step 2. Thanks to (c), (d), (e) and (b) of Claim 3.5, also f_2 satisfies the same properties (i), (ii), (iii) and (iv) for f_2, respectively. By Step 1, it holds that $R^i(f_2)_*(W\mathcal{O}_{X_2,\mathbb{Q}}) = 0$ for $i > 0$. Thanks to (c) and (d) of Claim 3.5, we have that $R^i f_*(W\mathcal{O}_{X,\mathbb{Q}}) = 0$ for $i > 0$. This completes the proof of Step 2. □

Step 2 completes the proof of Lemma 3.4. □

PROPOSITION 3.6. *Let k be a perfect field of characteristic $p > 5$. Let (X, Δ) be a three-dimensional klt pair over k and let $f : X \to Y$ be a projective k-morphism to a smooth k-curve Y such that $f_*\mathcal{O}_X = \mathcal{O}_Y$. If $-(K_X + \Delta)$ is f-nef and f-big, then $R^i f_*(W\mathcal{O}_{X,\mathbb{Q}}) = 0$ for $i > 0$.*

Proof. Applying Lemma 3.3 to f, we obtain a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\alpha} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{\beta} & Y \end{array}$$

that satisfies the properties listed in Lemma 3.3. Since X has $W\mathcal{O}$-rational singularities and α is a finite universal homeomorphism, it holds that X' has $W\mathcal{O}$-rational singularities. Furthermore, the geometric generic fibre $X_{\overline{K(Y')}}$ of f' is normal (Lemma 3.3(iii)) and hence it is a normal rational surface by Proposition 2.26. Therefore, it follows from Lemma 3.4 that $Rf'_*(W\mathcal{O}_{X',\mathbb{Q}}) = W\mathcal{O}_{Y',\mathbb{Q}}$. Thus, we get

$$Rf_*(W\mathcal{O}_{X,\mathbb{Q}}) \simeq Rf_*R\alpha_*(W\mathcal{O}_{X',\mathbb{Q}}) \simeq R\beta_*(W\mathcal{O}_{Y',\mathbb{Q}}) \simeq W\mathcal{O}_{Y,\mathbb{Q}},$$

where the first and last isomorphisms hold because α and β are finite universal homeomorphisms (Lemma 3.3(i)) and the second isomorphism follows from $Rf'_*(W\mathcal{O}_{X',\mathbb{Q}}) = W\mathcal{O}_{Y',\mathbb{Q}}$. □

3.2 Conic bundles

In this subsection, we prove the $W\mathcal{O}$-vanishing for conic bundles (Proposition 3.9). To this end, we show that their base schemes have $W\mathcal{O}$-rational singularities (Theorem 3.8). Let us start by recalling the following basic fact.

LEMMA 3.7. *Let k be a perfect field of characteristic $p > 0$. Let $f : X \to Y$ be a proper birational k-morphism of normal k-surfaces. If Y has $W\mathcal{O}$-rational singularities, then so does X.*

Proof. Fix a resolution of singularities of X: $\varphi : V \to X$. We have the following exact sequence induced by the corresponding Grothendieck spectral sequence:

$$\begin{aligned} 0 \to R^1 f_*(W\mathcal{O}_{X,\mathbb{Q}}) \to R^1(f \circ \varphi)_*(W\mathcal{O}_{V,\mathbb{Q}}) \to f_*R^1\varphi_*(W\mathcal{O}_{V,\mathbb{Q}}) \\ \to R^2 f_*(W\mathcal{O}_{X,\mathbb{Q}}). \end{aligned}$$

We obtain $R^1(f\circ\varphi)_*(W\mathcal{O}_{V,\mathbb{Q}}) = 0$, since Y has $W\mathcal{O}$-rational singularities. Moreover, we have that $R^2 f_*(W\mathcal{O}_{X,\mathbb{Q}}) = 0$, as the fibres of f are at most one dimensional (cf. [GNT19, Lemma 2.20]). Therefore, it holds that $f_*R^1\varphi_*(W\mathcal{O}_{V,\mathbb{Q}}) = 0$. Thanks to the fact that Supp $R^1\varphi_*(W\mathcal{O}_{V,\mathbb{Q}})$ is zero dimensional, we get $R^1\varphi_*(W\mathcal{O}_{V,\mathbb{Q}}) = 0$. □

THEOREM 3.8. *Let k be a perfect field of characteristic $p > 5$. Let $f : X \to Y$ be a projective k-morphism of normal k-varieties with $f_*\mathcal{O}_X = \mathcal{O}_Y$ which satisfies the following properties:*

(i) $\dim X = 3$ *and* $\dim Y = 2$*;*
(ii) *there exists an effective* $\mathbb{R}$*-divisor* Δ *on* X *such that* (X, Δ) *is klt and* $-(K_X + \Delta)$ *is* f*-nef and* f*-big.*

Then Y *has* $W\mathcal{O}$*-rational singularities.*

Proof. Replacing Δ, we may assume that $-(K_X + \Delta)$ is f-ample.

Step 1. The assertion of Theorem 3.8 holds if $k = \overline{\mathbb{F}}_p$.

Proof of Step 1. In this proof, $X(\mathbb{F}_{p^e})$ denotes the number of $\mathbb{F}_{p^e}$-rational points on a model X_0 of X over $\mathbb{F}_{p^e}$, i.e. X_0 is a projective $\mathbb{F}_{p^e}$-scheme such that $X_0 \times_{\mathbb{F}_{p^e}} k \simeq X$. We can define $X(\mathbb{F}_{p^e})$ if e is a sufficiently divisible positive integer and we fix a model X_0 (this number possibly depends on the choice of a model X_0).

We may assume that Y has a unique singular point y. Let $g : Y' \to Y$ be a log resolution such that $g(\mathrm{Ex}(g)) = \{y\}$. Set $C := \mathrm{Ex}(g) = g^{-1}(y)$. Let $\varphi : W \to X$ be a log resolution of (X, Δ) that admits a morphism to Y'. Then $f^{-1}(y)$ is rationally chain connected [GNT19, Theorem 4.1]. Hence, by [GNT19, Theorem 4.8], also $(f \circ \varphi)^{-1}(y)$ is rationally chain connected. Therefore, its image on Y', which is nothing but C, is a union of rational curves. In order to prove that Y has $W\mathcal{O}$-rational singularities, it is suffices to show that C forms a tree by Proposition 2.23 and [CR12, Corollary 4.6.4]. Let s be the number of the vertices and let t be the number of the edges of the dual graph of C. Note that since C is connected and simple normal closing, the condition that C forms a tree is equivalent to the condition that $s = t+1$. Then, for a sufficiently divisible e,

$$C(\mathbb{F}_{p^e}) = s(p^e + 1) - t$$

holds because we may assume that each component of C and their intersection are defined over $\mathbb{F}_{p^e}$. Hence, the condition $s = t + 1$ is equivalent to the condition that

$$C(\mathbb{F}_{p^e}) \equiv 1 \mod p^e$$

for sufficiently divisible e. Therefore, it suffices to show that

$$Y'(\mathbb{F}_{p^e}) \equiv Y(\mathbb{F}_{p^e}) \mod p^e$$

for any sufficiently divisible positive integer e.

Let E be the sum of all the φ-exceptional prime divisors. We run a $(K_W + \varphi_*^{-1}\Delta + E)$-MMP over Y' that terminates. Since $K_W + \varphi_*^{-1}\Delta + E$ is generically anti-ample over Y', we end with a Mori fibre space $X_1 \to Y_1$ over Y'. Note that the induced morphism $Y_1 \to Y'$ is birational and hence Y_1 has $W\mathcal{O}$-rational singularities by Lemma 3.7.

Take an arbitrary divisible positive integer e. We obtain

$$Y_1(\mathbb{F}_{p^e}) \equiv Y'(\mathbb{F}_{p^e}) \mod p^e,$$

since Y_1 and Y' are birational and have $W\mathcal{O}$-rational singularities [CR12, Corollary 4.4.16]. Furthermore, it follows from [GNT19, Theorem 5.1] that

$$X(\mathbb{F}_{p^e}) \equiv W(\mathbb{F}_{p^e}) \equiv X_1(\mathbb{F}_{p^e}) \mod p^e.$$

On the other hand, X and X_1 are of Fano type over Y and Y_1, respectively. Hence, by [GNT19, Theorem 5.4], we obtain

$$X(\mathbb{F}_{p^e}) \equiv Y(\mathbb{F}_{p^e}), \quad X_1(\mathbb{F}_{p^e}) \equiv Y_1(\mathbb{F}_{p^e}) \mod p^e.$$

To summarise, we get

$$Y(\mathbb{F}_{p^e}) \equiv Y'(\mathbb{F}_{p^e}) \mod p^e.$$

This completes the proof of Step 1. □

Step 2. The assertion of Theorem 3.8 holds if k is algebraically closed.

Proof of Step 2. We fix a closed point $y \in Y$ and we may assume that y is a unique singularity of Y. Let $g : Z \to Y$ be a log resolution such that $g(\mathrm{Ex}(g)) = y$. By Proposition 2.23, Y has $W\mathcal{O}$-rational singularities if and only if $\mathrm{Ex}(g)$ is a tree of smooth rational curves. We take a model over some finitely generated $\overline{\mathbb{F}}_p$-algebra R of a diagram $X \to Y \leftarrow Z$, i.e. an intermediate ring $\overline{\mathbb{F}}_p \subset R \subset k$ that is a finitely generated $\overline{\mathbb{F}}_p$-algebra, and R-morphisms of projective schemes over R

$$\mathfrak{X} \to \mathfrak{Y} \leftarrow \mathfrak{Z}$$

whose base changes by $(-) \times_R k$ are the same as $X \to Y \leftarrow Z$. Then the base change $\mathfrak{X}_\mu \xrightarrow{f_\mu} \mathfrak{Y}_\mu \xleftarrow{g_\mu} \mathfrak{Z}_\mu$ by a general closed point $\mu \in \mathrm{Spec}\ R$ satisfies the same properties as $X \to Y \leftarrow Z$. Therefore, $\mathrm{Ex}(g_\mu)$ is a tree of smooth rational curves by Step 1 and hence so is $\mathrm{Ex}(g)$ by Proposition 2.23 and the upper semicontinuity of cohomologies [Har77, ch. III, Theorem 12.11]. This completes the proof of Step 2. □

Step 3. The assertion of Theorem 3.8 holds without any additional assumptions.

Proof of Step 3. Thanks to Proposition 2.22, we may assume that k is algebraically closed. Then the assertion of Theorem 3.8 follows from Step 2. □

Step 3 completes the proof of Theorem 3.8. □

PROPOSITION 3.9. *Let k be a perfect field of characteristic $p > 5$. Let $f : X \to Y$ be a projective k-morphism of normal k-varieties with $f_*\mathcal{O}_X = \mathcal{O}_Y$ which satisfies the following properties:*

(i) $\dim X = 3$ *and* $\dim Y = 2$;

(ii) *there exists an effective $\mathbb{R}$-divisor Δ on X such that (X, Δ) is klt and $-(K_X + \Delta)$ is f-nef and f-big.*

Then $R^i f_*(W\mathcal{O}_{X,\mathbb{Q}}) = 0$ for all $i > 0$.

Proof. By Lemma 3.3, there exists a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\alpha} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{\beta} & Y \end{array}$$

of projective k-morphisms of normal k-varieties which satisfies the properties listed in Lemma 3.3 (for an alternative argument, see Remark 3.10). Since (X, Δ) is klt, X has $W\mathcal{O}$-rational singularities (Theorem 3.1(i)). By Theorem 3.8, also Y has $W\mathcal{O}$-rational singularities. Since α and β are finite universal homeomorphisms (Lemma 3.3(i)), also X' and Y' have $W\mathcal{O}$-rational singularities. Since the generic fibre of f' is smooth and will be a rational curve after taking the base change to the algebraic closure, it follows from Theorem 3.2 that $Rf'_*(W\mathcal{O}_{X',\mathbb{Q}}) \simeq W\mathcal{O}_{Y',\mathbb{Q}}$. Therefore, we get

$$Rf_*(W\mathcal{O}_{X,\mathbb{Q}}) \simeq Rf_*R\alpha_*(W\mathcal{O}_{X',\mathbb{Q}}) \simeq R\beta_*(W\mathcal{O}_{Y',\mathbb{Q}}) \simeq W\mathcal{O}_{Y,\mathbb{Q}},$$

where the first and the last isomorphisms follow because α and β are finite universal homeomorphisms (Lemma 3.3(i)) and the second isomorphism follows from $Rf'(W\mathcal{O}_{X',\mathbb{Q}}) \simeq W\mathcal{O}_{Y',\mathbb{Q}}$. □

Remark 3.10. In the situation of Proposition 3.9, the generic fibre is a conic curve in $\mathbb{P}^2_{K(Y)}$ (cf. [Kol13, Lemma 10.6(3)]). Hence, the assumption $p > 2$ implies that the generic fibre of f is generically smooth. Thus, α and β in the proof can be assumed to be isomorphisms and we can avoid using Lemma 3.3. We adopt the above argument, as it is less dependent on the assumption on the characteristic p.

3.3 Proof of $W\mathcal{O}$-vanishing for log Fano contractions

We now prove the main theorem of this section.

Theorem 3.11. *Let k be a perfect field of characteristic $p > 5$. Let $f : X \to Y$ be a projective k-morphism of normal k-varieties. Assume that $\dim X \leqslant 3$ and there exists an effective $\mathbb{R}$-divisor Δ such that (X, Δ) is klt and $-(K_X + \Delta)$ is f-nef and f-big. Then $R^i f_*(W\mathcal{O}_{X,\mathbb{Q}}) = 0$ for $i > 0$.*

Proof. Taking the Stein factorisation of f, we may assume that $f_*\mathcal{O}_X = \mathcal{O}_Y$. If $\dim Y = 0$, then the assertion follows from Theorem 3.1(ii). If $\dim Y = 3$, then we have that also (Y, Δ_Y) is klt for some effective $\mathbb{R}$-divisor Δ_Y and hence the assertion holds by Theorem 3.1(i). If $\dim Y = 1$ (respectively $\dim Y = 2$), then the assertion follows from Proposition 3.6 (respectively Proposition 3.9). □

4. A Nadel vanishing theorem for Witt multiplier ideal sheaves

In this section, we prove the main theorem of this paper (Theorem 4.10). Our strategy is to run a suitable minimal model program, which enables us to replace the given variety X by the end result. In § 4.1, we study the behaviour of Witt vector cohomologies under such minimal model programs. In § 4.2, we prove Theorem 4.10 for dlt Mori fibre spaces with an extra assumption (Lemma 4.7). In § 4.3, we give a proof of Theorem 4.10. Furthermore, we also give a generalisation of Theorem 4.10 (Theorem 4.11) and the Kollár–Shokurov connectedness theorem (Theorem 4.12).

4.1 Witt vector cohomologies under MMP

The purpose of this subsection is to prove the following.

Proposition 4.1. *Let k be a perfect field of characteristic $p > 5$. Let (X, Ω) be a three-dimensional $\mathbb{Q}$-factorial log pair over k and let $h : X \to Z$ be a projective k-morphism to a quasi-projective k-scheme Z. Suppose that:*

- *$(X, \Omega^{\wedge 1})$ is dlt; and*
- *$K_X + \Omega \sim_{Z,\mathbb{R}} 0$.*

Let

$$X =: X_0 \dashrightarrow X_1 \dashrightarrow X_2 \dashrightarrow \cdots \dashrightarrow X_r \dashrightarrow \cdots$$

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be a $(K_X + \Omega^{\wedge 1})$*-MMP over* Z *with the induced morphism* $h_r : X_r \to Z$*. Set* Ω_r *to be the push-forward of* Ω *on* X_r*. Then the isomorphism*

$$R^i h_* WI_{\Omega^{\geqslant 1},\mathbb{Q}} \simeq R^i(h_r)_* WI_{\Omega_r^{\geqslant 1},\mathbb{Q}}$$

holds for any $i \geqslant 0$ *and* $r \geqslant 0$*.*

Proof. By induction, it is sufficient to prove the case when $r = 1$. We have the following properties.

(1) (X_1, Ω_1) satisfies the same conditions as in the statement, i.e. (X_1, Ω_1) is a $\mathbb{Q}$-factorial log pair such that $(X_1, \Omega_1^{\wedge 1})$ is dlt and $K_{X_1} + \Omega_1 \sim_{Z,\mathbb{R}} 0$.
(2) Given projective birational morphisms $\psi_0 : Y \to X$ and $\psi_1 : Y \to X_1$, it holds that $\psi_0^*(K_X + \Omega) \sim_{Z,\mathbb{R}} \psi_1^*(K_{X_1} + \Omega_1) \sim_{Z,\mathbb{R}} 0$.
(3) $\mathrm{Nklt}(X, \Omega) = \mathrm{Supp}\,\Omega^{\geqslant 1}$ and $\mathrm{Nklt}(X, \Omega_1) = \mathrm{Supp}\,\Omega_1^{\geqslant 1}$ (cf. Proposition 2.7).

Case 1. Suppose that $g : X \to X_1$ is a divisorial contraction.
We have the following spectral sequence:

$$E_2^{i,j} := R^i(h_1)_* R^j g_*(WI_{\Omega^{\geqslant 1},\mathbb{Q}}) \Rightarrow R^{i+j}(h_1 \circ g)_*(WI_{\Omega^{\geqslant 1},\mathbb{Q}}) =: E^{i+j}.$$

Hence, it is sufficient to show the following two equations:

$$g_* WI_{\Omega^{\geqslant 1},\mathbb{Q}} = WI_{\Omega_1^{\geqslant 1},\mathbb{Q}} \tag{4.1.1}$$

and

$$R^i g_* WI_{\Omega^{\geqslant 1},\mathbb{Q}} = 0 \quad \text{for } i > 0. \tag{4.1.2}$$

Here, the equation (4.1.1) is equivalent to the equation

$$g(\mathrm{Supp}(\Omega^{\geqslant 1})) = \mathrm{Supp}(\Omega_1^{\geqslant 1}) \tag{4.1.3}$$

as sets.

Case 1-1. Suppose that the contracted divisor E is an irreducible component of $\mathrm{Supp}(\Omega^{\geqslant 1})$.
In this case, the equation (4.1.2) follows from [BBE07, Theorem 2.4], since g is an isomorphism outside $\mathrm{Supp}(\Omega^{\geqslant 1})$. By (ii), $g(E)$ is a non-klt centre of (X_1, Ω_1). Thanks to (iii), it holds that $g(E) \subset \mathrm{Supp}(\Omega_1^{\geqslant 1})$, which implies the required equation (4.1.3).

Case 1-2. Suppose that the contracted divisor E is not contained in $\mathrm{Supp}(\Omega^{\geqslant 1})$.
The equation (4.1.3) is trivial in this case. We have the exact sequence

$$0 \to WI_{\Omega^{\geqslant 1},\mathbb{Q}} \to W\mathcal{O}_{X,\mathbb{Q}} \to W\mathcal{O}_{\mathrm{Supp}(\Omega^{\geqslant 1}),\mathbb{Q}} \to 0.$$

In order to prove (4.1.2), it is sufficient to show that:

- $0 \to g_* WI_{\Omega^{\geqslant 1},\mathbb{Q}} \to g_* W\mathcal{O}_{X,\mathbb{Q}} \to g_* W\mathcal{O}_{\mathrm{Supp}(\Omega^{\geqslant 1}),\mathbb{Q}} \to 0$ is exact; and
- $R^i g_* W\mathcal{O}_{X,\mathbb{Q}} \simeq R^i g_* W\mathcal{O}_{\mathrm{Supp}(\Omega^{\geqslant 1}),\mathbb{Q}}$ holds for $i > 0$.

Since

$$R^i g_* W\mathcal{O}_{X,\mathbb{Q}} \simeq \begin{cases} W\mathcal{O}_{X_1,\mathbb{Q}} & (i = 0), \\ 0 & (i > 0) \end{cases}$$

holds by the $W\mathcal{O}$-rationality of the klt threefolds X and X_1 (Theorem 3.1(i)), it is sufficient to show that

$$R^i g_* W\mathcal{O}_{\mathrm{Supp}(\Omega^{\geqslant 1}),\mathbb{Q}} \simeq \begin{cases} W\mathcal{O}_{\mathrm{Supp}(\Omega_1^{\geqslant 1}),\mathbb{Q}} & (i=0), \\ 0 & (i>0). \end{cases} \tag{4.1.4}$$

This follows from the Mayer–Vietoris exact sequence (Lemma 2.18)

$$0 \to W\mathcal{O}_{S_1\cup S_2,\mathbb{Q}} \to W\mathcal{O}_{S_1,\mathbb{Q}} \oplus W\mathcal{O}_{S_2,\mathbb{Q}} \to W\mathcal{O}_{S_1\cap S_2,\mathbb{Q}} \to 0$$

for each union S_i of strata of $\mathrm{Supp}(\Omega^{\geqslant 1})$ and Claim 4.2 below. To summarise, in order to prove (4.1.2), it suffices to show Claim 4.2.

Claim 4.2. *If S is a stratum of $\mathrm{Supp}(\Omega^{\geqslant 1})$, then g induces*

$$R^i g_* W\mathcal{O}_{S,\mathbb{Q}} \simeq \begin{cases} W\mathcal{O}_{g(S),\mathbb{Q}} & (i=0), \\ 0 & (i>0). \end{cases}$$

Proof of Claim 4.2. Note that S is normal, since $(X, \Omega^{\wedge 1})$ is dlt and $p > 5$ (cf. [HX15, Proposition 4.1]). We define an effective $\mathbb{R}$-divisor Λ_S by adjunction: $K_S + \Lambda_S = (K_X + \Omega^{\wedge 1})|_S$. Then $-(K_S + \Lambda_S)$ is $(g|_S)$-ample and (S, Λ_S) is dlt. Hence, $R^i g_* W\mathcal{O}_{S,\mathbb{Q}} = 0$ holds for $i > 0$ by [GNT19, Proposition 3.3].

In order to prove the required equation $g_* W\mathcal{O}_{S,\mathbb{Q}} = W\mathcal{O}_{g(S),\mathbb{Q}}$, it is sufficient to prove that $g : S \to g(S)$ has connected fibres (Lemma 2.17). When $\dim S = 2$, then this follows because $g : S \to g(S)$ is a projective birational morphism of normal varieties. When $\dim S \leqslant 1$, we can apply [GNT19, Lemma 3.10] (cf. Remark 4.4). □

Case 2. Suppose that $g : X \to Z'$ is a $(K_X + \Omega^{\wedge 1})$-flipping contraction over Z and let $g_1 : X_1 \to Z'$ be its flip.

Claim 4.3.

(4.3.1) $g_* W I_{\Omega^{\geqslant 1},\mathbb{Q}} \simeq (g_1)_* W I_{\Omega_1^{\geqslant 1},\mathbb{Q}}$ *holds; and*

(4.3.2) $R^i g_* W I_{\mathrm{Supp}(\Omega^{\geqslant 1}),\mathbb{Q}} = 0$ *and* $R^i (g_1)_* W I_{\mathrm{Supp}(\Omega_1^{\geqslant 1}),\mathbb{Q}} = 0$ *hold for any* $i > 0$.

Proof of Claim 4.3. First, (4.3.1) follows from the set-theoretical equation $g(\mathrm{Supp}(\Omega^{\geqslant 1})) = g_1(\mathrm{Supp}(\Omega_1^{\geqslant 1}))$, which is trivial.

Let us prove (4.3.2). We may assume that $i = 1$, since both g and g_1 have at most one-dimensional fibres [GNT19, Lemma 2.20]. Consider the exact sequences

$$0 \to W I_{\Omega^{\geqslant 1},\mathbb{Q}} \to W\mathcal{O}_{X,\mathbb{Q}} \to W\mathcal{O}_{\mathrm{Supp}(\Omega^{\geqslant 1}),\mathbb{Q}} \to 0$$

and

$$0 \to W I_{\Omega_1^{\geqslant 1},\mathbb{Q}} \to W\mathcal{O}_{X_1,\mathbb{Q}} \to W\mathcal{O}_{\mathrm{Supp}(\Omega_1^{\geqslant 1}),\mathbb{Q}} \to 0.$$

We have that $R^1 g_* W\mathcal{O}_{X,\mathbb{Q}} = 0$ and $R^1 (g_1)_* W\mathcal{O}_{X_1,\mathbb{Q}} = 0$, since all of X, X_1 and Z' have $W\mathcal{O}$-rational singularities (Theorem 3.1(i)). Hence, it is sufficient to show the surjectivity of

$$g_* W\mathcal{O}_{X,\mathbb{Q}} \to g_* W\mathcal{O}_{\mathrm{Supp}(\Omega^{\geqslant 1}),\mathbb{Q}}$$

and

$$(g_1)_*W\mathcal{O}_{X_1,\mathbb{Q}} \to (g_1)_*W\mathcal{O}_{\mathrm{Supp}(\Omega_1^{\geqslant 1}),\mathbb{Q}},$$

which are equivalent to

$$g_*W\mathcal{O}_{\mathrm{Supp}(\Omega^{\geqslant 1}),\mathbb{Q}} = W\mathcal{O}_{g(\mathrm{Supp}(\Omega^{\geqslant 1})),\mathbb{Q}}$$

and

$$(g_1)_*W\mathcal{O}_{\mathrm{Supp}(\Omega_1^{\geqslant 1}),\mathbb{Q}} = W\mathcal{O}_{g_1(\mathrm{Supp}(\Omega_1^{\geqslant 1})),\mathbb{Q}},$$

respectively. Therefore, by Lemma 2.17, it is enough to prove the following:

(i) $g' : \mathrm{Supp}(\Omega^{\geqslant 1}) \to Z'$ has connected fibres;
(ii) $g_1' : \mathrm{Supp}(\Omega_1^{\geqslant 1}) \to Z'$ has connected fibres.

Both (i) and (ii) follow from (1) and Theorem 2.15. This completes the proof of Claim 4.3. □

For the induced morphism $\theta : Z' \to Z$, we obtain isomorphisms

$$\begin{aligned} Rh_*WI_{\Omega^{\geqslant 1},\mathbb{Q}} &\simeq R\theta_*Rg_*(WI_{\Omega^{\geqslant 1},\mathbb{Q}}) \\ &\simeq R\theta_*R(g_1)_*(WI_{\Omega_1^{\geqslant 1},\mathbb{Q}}) \simeq R(h_1)_*(WI_{\Omega_1^{\geqslant 1},\mathbb{Q}}), \end{aligned}$$

where the second isomorphism follows from Claim 4.3. This completes the proof of Proposition 4.1. □

Remark 4.4. In the proof above, we use [GNT19, Lemma 3.10], which is a special case of the two-dimensional version of Theorem 1.1. In the proof of [GNT19, Lemma 3.10], they use [Tan16, Proposition 2.2], whose proof depends on a classification result on surfaces. Here, for the reader's convenience, we give a sketch of an alternative proof. When U in [GNT19, Lemma 3.10] has positive dimension, the assertion follows from the Nadel vanishing theorem for two-dimensional relative cases [Tan15, Theorem 2.10]. Hence, the remaining case is when (S, Δ_S) is a two-dimensional dlt pair over an algebraically closed field such that $-(K_S + \Delta_S)$ is ample, and it is sufficient to show that $\llcorner \Delta_S \lrcorner$ is connected. In this case, we may apply the idea of this paper (cf. (B) of § 1.1), and the problem can be reduced to the study on Mori fibre spaces (cf. § 4.2).

4.2 Vanishing for Mori fibre spaces

In this subsection, we prove Lemma 4.7, which is a special case of Theorem 4.10. We start with the following auxiliary result.

Lemma 4.5. *Let k be a perfect field of characteristic $p > 0$. Let (S, Δ_S) be a two-dimensional dlt pair over k and let $f : S \to Z$ be a projective k-morphism to a quasi-projective k-scheme Z. Assume that $-(K_S + \Delta_S)$ is f-ample. Let $h : \llcorner \Delta_S \lrcorner \to Z$ be the induced morphism. Then the following holds:*

(i) $R^ih_*\mathcal{O}_{\llcorner \Delta_S \lrcorner} = 0$ *for* $i > 0$;
(ii) $R^ih_*(W\mathcal{O}_{\llcorner \Delta_S \lrcorner,\mathbb{Q}}) = 0$ *for* $i > 0$.

Proof. Let us prove (i). Taking the Stein factorisation of f, we may assume that $f_*\mathcal{O}_S = \mathcal{O}_Z$. Furthermore, the problem is reduced to the case when k is algebraically closed. Thanks to [Tan15, Theorems 2.12 and 3.5], it holds that $R^i f_*\mathcal{O}_S = 0$ for $i > 0$. If $\dim Z \geqslant 1$, then we have a surjection:

$$0 = R^1 f_*\mathcal{O}_S \to R^1 h_*\mathcal{O}_{\llcorner \Delta_S \lrcorner},$$

which implies that $R^i h_*\mathcal{O}_{\llcorner \Delta_S \lrcorner} = 0$ for $i > 0$. Thus, we may assume that $\dim Z = 0$, i.e. $Z = \operatorname{Spec} k$. Then we have the exact sequence

$$0 = H^1(S, \mathcal{O}_S) \to H^1(\llcorner \Delta_S \lrcorner, \mathcal{O}_{\llcorner \Delta_S \lrcorner}) \to H^2(S, \mathcal{O}_S(-\llcorner \Delta_S \lrcorner)).$$

Hence, it suffices to prove that $H^2(S, \mathcal{O}_S(-\llcorner \Delta_S \lrcorner)) = 0$.

It follows from Serre duality that

$$h^2(S, \mathcal{O}_S(-\llcorner \Delta_S \lrcorner)) = h^0(S, \mathcal{O}_S(K_S + \llcorner \Delta_S \lrcorner)).$$

For an ample Cartier divisor A, we have that

$$(K_S + \llcorner \Delta_S \lrcorner) \cdot A = ((K_S + \Delta_S) - \{\Delta_S\}) \cdot A < 0,$$

which in turn implies that $H^0(S, \mathcal{O}_S(K_S + \llcorner \Delta_S \lrcorner)) = 0$. Hence, it holds that $H^2(S, \mathcal{O}_S(-\llcorner \Delta_S \lrcorner)) = 0$, as desired. This completes the proof of (i).

The assertion (ii) follows from (i) and [GNT19, Lemma 2.19]. □

Lemma 4.6. *Let k be a perfect field of characteristic $p > 5$. Let (X, Ξ) be a three-dimensional $\mathbb{Q}$-factorial dlt pair over k and let $f : X \to Z$ be a projective surjective k-morphism to a quasi-projective k-scheme Z such that:*

(i) $\dim X > \dim Z$;
(ii) *f has connected fibres;*
(iii) *$-(K_X + \Xi)$ is f-ample; and*
(iv) *there exists an irreducible component D_0 of $\llcorner \Xi \lrcorner$ such that D_0 is f-ample.*

Then

$$R^i f_* W\mathcal{O}_{\mathrm{Supp}\lfloor \Xi \rfloor, \mathbb{Q}} \simeq \begin{cases} W\mathcal{O}_{Z,\mathbb{Q}} & (i = 0), \\ 0 & (i > 0) \end{cases}$$

holds.

Proof. Replacing Z by Z' for the Stein factorisation $X \to Z' \to Z$ of f, we may assume that $f_*\mathcal{O}_X = \mathcal{O}_Z$. Let $g : D_0 \to Z$ be the induced morphism. Let $\llcorner \Xi \lrcorner = \sum_{i=0}^m D_i$ be the irreducible decomposition.

Step 1. The isomorphism $Rg_* W\mathcal{O}_{D_0,\mathbb{Q}} \simeq W\mathcal{O}_{Z,\mathbb{Q}}$ holds. In particular, Lemma 4.6 holds if $m = 0$.

Proof of Step 1. We define an effective $\mathbb{R}$-divisor Ξ_{D_0} by adjunction: $(K_X + \Xi)|_{D_0} = K_{D_0} + \Xi_{D_0}$. It holds that (D_0, Ξ_{D_0}) is dlt and $-(K_{D_0} + \Xi_{D_0})$ is g-ample. Hence, it follows from [GNT19, Proposition 3.3] that $R^i g_* W\mathcal{O}_{D_0,\mathbb{Q}} = 0$ for $i > 0$.

In order to prove that $g_* W\mathcal{O}_{D_0,\mathbb{Q}} \simeq W\mathcal{O}_{Z,\mathbb{Q}}$, it suffices to show that $g : D_0 \to Z$ has connected fibres (Lemma 2.17). Since Z is normal, it is enough to prove that $D_0|_F$ is geometrically connected for the generic fibre $F := X \times_Z \operatorname{Spec} K(Z)$ of f. If $\dim F \geqslant 2$, then the restriction $D_0|_F$ is geometrically connected because $D_0|_F$ is an ample $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisor on F. Thus, we may assume that $\dim F = 1$. Since $-(K_X + \Xi)$ is f-ample, the geometric generic fibre $\overline{F} := F \times_{K(Z)} \overline{K(Z)}$ is $\mathbb{P}^1_{\overline{K(Z)}}$. Moreover, it holds that

$$0 > \deg(K_X + \Xi)|_{\overline{F}} \geqslant \deg(K_{\overline{F}} + D_0|_{\overline{F}}) = -2 + \deg(D_0|_{\overline{F}}).$$

This implies that g has connected fibres. This completes the proof of Step 1. □

Step 2. For any $i \in \{1, \ldots, m\}$, it holds that:

(a) $\dim f(D_i) < \dim D_i$; and
(b) $f(D_i) = f(D_i \cap D_0)$.

Proof of Step 2. Let us prove (a). If $\dim Z \leqslant 1$, then there is nothing to show. Thus, we may assume that $\dim Z = 2$. Assuming that there is $i \in \{1, \ldots, m\}$ such that $f(D_i) = Z$, let us derive a contradiction. Since a general fibre F of f is $\mathbb{P}^1$, it holds that

$$(K_X + \Xi) \cdot F \geqslant (K_X + D_0 + D_i) \cdot F \geqslant 0.$$

This contradicts the fact that $-(K_X + \Xi)$ is f-ample. Therefore, D_i does not dominate Z for any $i \in \{1, \ldots, m\}$, which implies (a).

Let us prove (b). Fix an arbitrary closed point $x \in f(D_i)$. Since $\dim f(D_i) < \dim D_i$, there exists a curve C on X contained in $D_i \cap f^{-1}(x)$. Since D_0 is ample over Z, the contracted curve C intersects D_0. This implies that $x \in f(D_i \cap D_0)$. Thus, we get $f(D_i) = f(D_i \cap D_0)$. Hence, (b) holds. This completes the proof of Step 2. □

Step 3. For any $i \in \{1, \ldots, m\}$, it holds that the induced homomorphism

$$R^q f_* W\mathcal{O}_{D_i,\mathbb{Q}} \to R^q f_* W\mathcal{O}_{D_i \cap E,\mathbb{Q}}$$

is an isomorphism for any $q \geqslant 0$, where $E := \bigcup_{j \in \{0,\ldots,m\}\setminus\{i\}} D_j$.

Proof of Step 3. Set $C := f(D_i)$ and let $D_i \xrightarrow{f'} C' \xrightarrow{s} C$ be the Stein factorisation of $D_i \to C$. Let Ξ_{D_i} be the effective $\mathbb{R}$-divisor on D_i defined by adjunction: $(K_X + \Xi)|_{D_i} = K_{D_i} + \Xi_{D_i}$. Then the following properties hold:

(c) (D_i, Ξ_{D_i}) is dlt and $\mathrm{Supp}(\llcorner \Xi_{D_i} \lrcorner) = D_i \cap E$;
(d) $-(K_{D_i} + \Xi_{D_i})$ is f'-ample.

We have the exact sequence

$$0 \to WI_{\llcorner \Xi_{D_i} \lrcorner,\mathbb{Q}} \to W\mathcal{O}_{D_i,\mathbb{Q}} \to W\mathcal{O}_{D_i \cap E,\mathbb{Q}} \to 0.$$

By [GNT19, Proposition 3.3] and Lemma 4.5, it holds that

$$R^i f'_* W\mathcal{O}_{D_i,\mathbb{Q}} = 0 \quad \text{and} \quad R^i f'_* W\mathcal{O}_{D_i \cap E,\mathbb{Q}} = 0$$

for $i > 0$, respectively. Hence, it suffices to prove that $f'_* W\mathcal{O}_{D_i,\mathbb{Q}} \to f'_* W\mathcal{O}_{D_i \cap E,\mathbb{Q}}$ is an isomorphism. By Step 2(b), it is enough to prove that the induced morphism $D_i \cap E \to C'$ has connected fibres (Lemma 2.17), which follows from [GNT19, Lemma 3.10]. This completes the proof of Step 3. □

Step 4. The assertion of Lemma 4.6 holds.

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Proof of Step 4. We prove the assertion by induction on m. By Step 1, there is nothing to show if $m = 0$. Thus, assume that $m > 0$ and that the assertion of Lemma 4.6 holds if the number of the irreducible components of $\llcorner \Xi \lrcorner$ is less than m. Fix $i \in \{1, \ldots, m\}$. Since $(X, \Xi' := \Xi - \epsilon D_i)$ satisfies the same assumption as in Lemma 4.6 for sufficiently small $\epsilon > 0$, it follows from the induction hypothesis that

$$Rf_* W\mathcal{O}_{E,\mathbb{Q}} \simeq W\mathcal{O}_{Z,\mathbb{Q}},$$

where $E := \bigcup_{j \in \{0,\ldots,m\} \setminus \{i\}} D_j$. By the Mayer–Vietoris exact sequence (Lemma 2.18)

$$0 \to W\mathcal{O}_{D_i \cup E,\mathbb{Q}} \to W\mathcal{O}_{D_i,\mathbb{Q}} \oplus W\mathcal{O}_{E,\mathbb{Q}} \to W\mathcal{O}_{D_i \cap E,\mathbb{Q}} \to 0,$$

it is sufficient to show that the induced homomorphism

$$R^q f_* W\mathcal{O}_{D_i,\mathbb{Q}} \to R^q f_* W\mathcal{O}_{D_i \cap E,\mathbb{Q}}$$

is an isomorphism for any $q \geqslant 0$. This is nothing but the assertion of Step 3. This completes the proof of Step 4. □

Step 4 completes the proof of Lemma 4.6. □

Lemma 4.7. *Let k be a perfect field of characteristic $p > 5$. Let (X, Ξ) be a three-dimensional $\mathbb{Q}$-factorial dlt pair over k and let $f : X \to Z$ be a $(K_X + \Xi)$-Mori fibre space to a quasi-projective k-variety Z. Suppose that $f(\mathrm{Supp}\lfloor \Xi \rfloor) = Z$. Then*

$$R^i f_* W I_{\llcorner \Xi \lrcorner,\mathbb{Q}} = 0$$

for any $i \geqslant 0$.

Proof. It follows from Theorem 3.11 that

$$Rf_* W\mathcal{O}_{X,\mathbb{Q}} \simeq W\mathcal{O}_{Z,\mathbb{Q}}.$$

By $f(\mathrm{Supp}\lfloor \Xi \rfloor) = Z$, there exists an irreducible component D_0 of $\llcorner \Xi \lrcorner$ such that $f(D_0) = Z$. Since $\rho(X/Z) = 1$, we have that D_0 is f-ample. In particular, we may apply Lemma 4.6 and obtain the isomorphism

$$Rf_* W\mathcal{O}_{\mathrm{Supp}\llcorner \Xi \lrcorner,\mathbb{Q}} \simeq W\mathcal{O}_{Z,\mathbb{Q}}.$$

Therefore, by the exact sequence

$$0 \to W I_{\llcorner \Xi \lrcorner,\mathbb{Q}} \to W\mathcal{O}_{X,\mathbb{Q}} \to W\mathcal{O}_{\mathrm{Supp}\llcorner \Xi \lrcorner,\mathbb{Q}} \to 0,$$

the induced homomorphism

$$R^i f_* W\mathcal{O}_{X,\mathbb{Q}} \to R^i f_* W\mathcal{O}_{\mathrm{Supp}\llcorner \Xi \lrcorner,\mathbb{Q}}$$

is an isomorphism for any $i \geqslant 0$. Therefore, $R^i f_* W I_{\llcorner \Xi \lrcorner,\mathbb{Q}} = 0$ for any $i \geqslant 0$. □

4.3 Proof of the main theorem and related results

In this subsection, we prove the main theorem of this paper (Theorem 4.10) and a generalisation of it (Theorem 4.11). As a consequence, we obtain the Kollár–Shokurov connectedness theorem (Theorem 4.12).

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Lemma 4.8. *Let k be a perfect field of characteristic $p > 5$. Let (X, Δ) be a three-dimensional quasi-projective log pair over k. Let $f : Y \to X$ be a projective birational morphism that satisfies the properties (i)–(iii) of Proposition 2.10. Set Δ_Y to be the effective $\mathbb{R}$-divisor defined by $K_Y + \Delta_Y = f^*(K_X + \Delta)$. Then we have the following:*

(a) $f_*WI_{\mathrm{Nklt}(Y,\Delta_Y),\mathbb{Q}} = WI_{\mathrm{Nklt}(X,\Delta_X),\mathbb{Q}}$ *holds;*
(b) $R^i f_*WI_{\mathrm{Nklt}(Y,\Delta_Y),\mathbb{Q}} = 0$ *holds for* $i > 0$.

Proof. The condition (a) follows from the equation $f(\mathrm{Nklt}(Y, \Delta_Y)) = \mathrm{Nklt}(X, \Delta)$ (Lemma2.6).

Let us prove (b). By the $W\mathcal{O}$-rationality of klt threefolds (Theorem 3.1(i)), the vanishing in (b) holds outside $\mathrm{Nklt}(X, \Delta)$. The set of the non-$\mathbb{Q}$-factorial points on a three-dimensional klt pair is a zero-dimensional closed subset [GNT19, Proposition 2.15(4)]. Thus, after removing finitely many closed points of $X \setminus \mathrm{Nklt}(X, \Delta)$, we may assume that all the non-$\mathbb{Q}$-factorial points of X are contained in $\mathrm{Nklt}(X, \Delta)$. Hence, it follows that $\mathrm{Ex}(f) \subset f^{-1}(\mathrm{Nklt}(X, \Delta))$. Then we get

$$\mathrm{Ex}(f) \subset f^{-1}(\mathrm{Nklt}(X, \Delta)) = \mathrm{Nklt}(Y, \Delta_Y),$$

where the equality holds by Proposition 2.10(iii). Since f is an isomorphism outside $\mathrm{Nklt}(Y, \Delta_Y)$, it follows from [GNT19, Proposition 2.23] that $R^i f_*WI_{\mathrm{Nklt}(Y,\Delta_Y),\mathbb{Q}} = 0$ holds for $i > 0$. This completes the proof of (b). □

Proposition 4.9. *Let k be a perfect field of characteristic $p > 5$. Let (X, Ω) be a three-dimensional $\mathbb{Q}$-factorial log pair over k and let $f : X \to Z$ be a projective k-morphism to a quasi-projective k-scheme Z. Assume that:*

(i) $(X, \Omega^{\wedge 1})$ *is dlt;*
(ii) $K_X + \Omega \sim_{f,\mathbb{R}} 0$;
(iii) Ω *is f-big; and*
(iv) $\mathrm{Supp}\, \Omega^{>1} = \mathrm{Supp}\, \Omega^{\geqslant 1}$.

Then $R^i f_*(WI_{\mathrm{Nklt}(X,\Omega),\mathbb{Q}}) = 0$ *for* $i > 0$.

Proof. Taking the Stein factorisation of f, we may assume that $f_*\mathcal{O}_X = \mathcal{O}_Z$. We have that

$$\mathrm{Supp}(\Omega - \Omega^{\wedge 1}) = \mathrm{Supp}\, \Omega^{>1} = \mathrm{Supp}\, \Omega^{\geqslant 1} = \mathrm{Nklt}(X, \Omega), \tag{4.9.1}$$

where the second equality holds by (iv) and the third one follows from Proposition 2.7.

Step 1. The assertion of Proposition 4.9 holds if there is a $(K_X+\Omega^{\wedge 1})$-Mori fibre space $g : X \to Z'$ over Z.

Proof of Step 1. We have the induced morphisms

$$f : X \xrightarrow{g} Z' \xrightarrow{h} Z.$$

Since $\Omega - \Omega^{\wedge 1}$ is g-ample, it follows from (4.9.1) that $g(\mathrm{Supp}\, \Omega^{\geqslant 1}) = Z'$. Therefore, Lemma 4.7 implies that $Rg_*(WI_{\mathrm{Nklt}(X,\Omega),\mathbb{Q}}) = 0$. Hence, we have that

$$Rf_*(WI_{\mathrm{Nklt}(X,\Omega),\mathbb{Q}}) \simeq Rh_*Rg_*(WI_{\mathrm{Nklt}(X,\Omega),\mathbb{Q}}) = 0.$$

This completes the proof of Step 1. □

Step 2. In order to prove the assertion of Proposition 4.9, it is sufficient to prove the assertion under the following additional assumptions:

(i) $(K_X + \Omega^{\wedge 1})$ is f-nef;
(ii) (X, Ω) is not klt, i.e. $\mathrm{Supp}(\Omega^{\geqslant 1}) \neq \emptyset$;
(iii) the set-theoretic equation $\mathrm{Nklt}(X, \Omega) = f^{-1}(Z_1)$ holds for some closed subset Z_1 of Z;
(iv) if we set $Z_0 := Z \setminus Z_1$ and $X_0 := f^{-1}(Z_0)$, then X_0 is of Fano type over Z_0; and
(v) $\dim Z = 1$.

Proof of Step 2. By Theorem 2.9, there exists a $(K_X + \Omega^{\wedge 1})$-MMP over Z that terminates:

$$X =: X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_\ell.$$

Let $f_i : X_i \to Z$ be the induced morphism and let Ω_i be the push-forward of Ω on X_i. Then (X_ℓ, Ω_ℓ) still satisfies the conditions (i)–(iv) in Proposition 4.9. By Proposition 4.1, replacing (X, Ω) by (X_ℓ, Ω_ℓ), we may assume that $(K_X + \Omega^{\wedge 1})$ is f-nef or that there is a $(K_X + \Omega^{\wedge 1})$-Mori fibre space $g : X \to Z'$ over Z. In the latter case, the assertion of Proposition 4.9 follows from Step 1. Thus, we may assume that (i) holds. If (X, Ω) is klt, then (ii) and (iii) imply that X is of Fano type over Z. In this case, the assertion of Proposition 4.9 follows from Theorem 3.11. Thus, we may assume (ii).

Since $K_X + \Omega^{\wedge 1} \sim_{f,\mathbb{R}} -(\Omega - \Omega^{\wedge 1})$, it holds that $-(\Omega - \Omega^{\wedge 1})$ is f-nef. Since $\Omega - \Omega^{\wedge 1}$ is a non-zero effective divisor by (iv) and (ii), we have that $\dim Z \geqslant 1$. If $z \in Z$ is a closed point such that $f^{-1}(z) \cap \mathrm{Supp}(\Omega - \Omega^{\wedge 1}) \neq \emptyset$, then it follows that $f^{-1}(z) \subset \mathrm{Supp}(\Omega - \Omega^{\wedge 1})$ because $-(\Omega - \Omega^{\wedge 1})$ is f-nef. Thus, there exists a closed subset Z_1 of Z such that $Z_1 \subsetneq Z$ and

$$\mathrm{Nklt}(X, \Omega) = \mathrm{Supp}(\Omega - \Omega^{\wedge 1}) = f^{-1}(Z_1),$$

where the first equality follows from (4.9.1). Thus, (iii) holds. It is clear that (iii) implies (iv).

Therefore, assuming that $\dim Z \neq 1$, it is enough to show the assertion of Proposition 4.9. Since $\dim Z \geqslant 1$, we have $\dim Z \geqslant 2$. Then there are finitely many closed points $z_1, \ldots, z_n$ of Z_0 such that $R^i f_* \mathcal{O}_X|_{Z_0 \setminus \{z_1, \ldots, z_n\}} = 0$ for $i > 0$. Indeed, if $\mathrm{Supp}(R^i f_* \mathcal{O}_X) \cap Z_0$ contains a curve C, then it contradicts the vanishing obtained by [Tan18b, Theorem 3.3] for the morphism $X \times_Z \mathrm{Spec}\, \mathcal{O}_{Z, \xi_C} \to \mathrm{Spec}\, \mathcal{O}_{Z, \xi_C}$, where ξ_C denotes the generic point of C. By (iii), [CR12, Proposition 4.6.1] and $R^i f_* \mathcal{O}_X|_{Z_0 \setminus \{z_1, \ldots, z_n\}} = 0$, it holds that

$$R^i f_*(W I_{\mathrm{Nklt}(X,\Omega),\mathbb{Q}})|_{Z \setminus \{z_1, \ldots, z_n\}} = 0$$

for any $i > 0$. Since X_0 is of Fano type over Z_0, it follows from Theorem 3.11 that

$$R^i f_*(W I_{\mathrm{Nklt}(X,\Omega),\mathbb{Q}})|_{Z_0} = R^i (f|_{X_0})_*(W \mathcal{O}_{X_0,\mathbb{Q}}) = 0$$

holds for any $i > 0$. Since $Z = Z_0 \cup (Z \setminus \{z_1, \ldots, z_n\})$, we obtain

$$R^i f_*(W I_{\mathrm{Nklt}(X,\Omega),\mathbb{Q}}) = 0$$

for any $i > 0$. Thus, the assertion of Proposition 4.9 holds. This completes the proof of Step 2. $\square$

Step 3. Assume (i)–(v) in Step 2. There exists a commutative diagram of projective morphisms of normal varieties:

$$\begin{array}{ccc} Y' & \xrightarrow{\beta} & Y \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ X' & \xrightarrow{\alpha} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Z' & \xrightarrow{\gamma} & Z \end{array} \qquad (4.9.2)$$

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such that:

(vi) $f'_*\mathcal{O}_{X'} = \mathcal{O}_{Z'}$;
(vii) α, β and γ are finite universal homeomorphisms;
(viii) g is a log resolution of (X, Ω) and g' is birational; and
(ix) there are finitely many closed points $z_1, \dots, z_n$ of Z_0 such that

$$R^i h'_* \mathcal{O}_{Y'}|_{\gamma^{-1}(Z_0 \setminus \{z_1,\dots,z_n\})} = 0$$

for $i > 0$, where $h' := f' \circ g'$.

We set $h := f \circ g$ for later use.

Proof of Step 3. Set $\mathcal{X} := X_{K(Z)}$. There exist a finite purely inseparable extension $K(Z) \subset L$ and a projective birational L-morphism $\mathfrak{g}'_1 : \mathcal{Y}'_1 \to \mathcal{X}'$ of projective normal surfaces over L such that there is a finite universal homeomorphism $\mathfrak{a} : \mathcal{X}' \to \mathcal{X}$, $\mathcal{X}' \times_L \overline{L}$ is isomorphic to the normalisation of $(\mathcal{X} \times_{K(Z)} \overline{K(Z)})_{\mathrm{red}}$ and $\mathfrak{g}'_1 \times_L \overline{L} : \mathcal{Y}'_1 \times_L \overline{L} \to \mathcal{X}' \times_L \overline{L}$ is a resolution of singularities of $\mathcal{X}' \times_L \overline{L}$ (cf. Lemma 3.3). In particular, $\mathcal{Y}'_1$ is smooth over L. There exists a normal projective surface $\mathcal{Y}_1$ over $K(Z)$ which completes the following commutative diagram:

$$\begin{array}{ccc} \mathcal{Y}'_1 & \xrightarrow{\mathfrak{b}_1} & \mathcal{Y}_1 \\ \downarrow{\scriptstyle \mathfrak{g}'_1} & & \downarrow{\scriptstyle \mathfrak{g}_1} \\ \mathcal{X}' & \xrightarrow{\mathfrak{a}} & \mathcal{X} \\ \downarrow & & \downarrow \\ \operatorname{Spec} L & \longrightarrow & \operatorname{Spec} K(Z) \end{array} \qquad (4.9.3)$$

where $\mathfrak{g}_1$ is a projective birational morphism and $\mathfrak{b}_1$ is a finite universal homeomorphism. Indeed, we have $K(\mathcal{Y}'_1)^{p^e} = K(\mathcal{X}')^{p^e} \subset K(\mathcal{X})$ for sufficiently large $e \in \mathbb{Z}_{>0}$ and hence we can find such $\mathcal{Y}_1$ by taking the normalisation of $\mathcal{Y}''_1$ in $K(\mathcal{X})$, where $\mathcal{Y}'_1 \to \mathcal{Y}'_1 =: \mathcal{Y}''_1$ denotes the eth iterated absolute Frobenius morphism. Since $(\mathcal{X} \times_{K(Z)} \overline{K(Z)})_{\mathrm{red}}$ is a rational surface (Proposition 2.26), $\mathcal{Y}'_1 \times_L \overline{L}$ is a smooth rational surface. Therefore, we obtain $H^i(\mathcal{Y}'_1, \mathcal{O}_{\mathcal{Y}'_1}) = 0$ for $i > 0$.

Then there exists a commutative diagram of projective morphisms of normal varieties:

$$\begin{array}{ccc} Y'_1 & \xrightarrow{\beta_1} & Y_1 \\ \downarrow{\scriptstyle g'_1} & & \downarrow{\scriptstyle g_1} \\ X' & \xrightarrow{\alpha} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Z' & \xrightarrow{\gamma} & Z \end{array} \qquad (4.9.4)$$

such that the horizontal arrows are finite universal homeomorphisms and the base change of (4.9.4) by $(-) \times_Z \operatorname{Spec} K(Z)$ is (4.9.3). Let $g : Y \to X$ be a log resolution of (X, Ω) which factors through $g_1 : Y_1 \to X$:

$$g : Y \to Y_1 \xrightarrow{g_1} X.$$

Set Y' to be the normalisation of Y in $K(Y_1')$. Automatically, we obtain a commutative diagram of the induced morphisms:

$$\begin{array}{ccc} Y' & \xrightarrow{\beta} & Y \\ \downarrow{\scriptstyle g_2'} & & \downarrow{\scriptstyle g_2} \\ Y_1' & \xrightarrow{\beta_1} & Y_1 \end{array} \tag{4.9.5}$$

Combining (4.9.4) and (4.9.5), we obtain a commutative diagram (4.9.2). By the construction, the properties (vi)–(viii) hold. It suffices to show (ix). For $\mathcal{Y}' := Y' \times_Z \operatorname{Spec} K(Z)$, $\mathfrak{g}_2' : \mathcal{Y}' \to \mathcal{Y}_1'$ is a birational morphism of projective normal surfaces over $\operatorname{Spec} K(Z)$. As $\mathcal{Y}_1'$ is smooth, $\mathcal{Y}'$ has at worst rational singularities by [Lip69, Proposition 1.2(2)] or the same argument as in Lemma 3.7. Since $H^i(\mathcal{Y}_1', \mathcal{O}_{\mathcal{Y}_1'}) = 0$ for $i > 0$, we have $H^i(\mathcal{Y}', \mathcal{O}_{\mathcal{Y}'}) = 0$ for $i > 0$. Thus, (ix) holds. This completes the proof of Step 3. □

Step 4. Assume (i)–(v) in Step 2. We use the same notation as in Step 3. Then there exists an effective $\mathbb{R}$-divisor D on Y such that:

(x) (Y, D) is dlt;

(xi) the set-theoretic equation $\operatorname{Nklt}(Y, D) = h^{-1}(Z_1)$ holds; and

(xii) $Rg_*(WI_{\operatorname{Nklt}(Y,D),\mathbb{Q}}) \simeq WI_{\operatorname{Nklt}(X,\Omega),\mathbb{Q}}$.

Proof of Step 4. Let E be the sum of the g-exceptional prime divisors F such that $F \subset h^{-1}(Z_1)$. Let E' be the sum of the g-exceptional prime divisors F' such that $F' \not\subset h^{-1}(Z_1)$. Set

$$D := g_*^{-1}\Omega^{\wedge 1} + E + (1-\epsilon)E'$$

for a sufficiently small positive real number ϵ. It is clear that (x) and (xi) hold.

Let us prove that (xii) holds. We fix a closed point x of X. Since the problem is local on X, it is enough to find an open neighbourhood $\widetilde{X}$ of $x \in X$ such that $Rg_*(WI_{\operatorname{Nklt}(Y,D),\mathbb{Q}})|_{\widetilde{X}} \simeq WI_{\operatorname{Nklt}(X,\Omega),\mathbb{Q}}|_{\widetilde{X}}$.

By Theorem 2.9, there is a $(K_Y + D)$-MMP over X that terminates:

$$Y = Y_0 \overset{\varphi_0}{\dashrightarrow} Y_1 \overset{\varphi_0}{\dashrightarrow} \cdots \overset{\varphi_{\ell-1}}{\dashrightarrow} Y_\ell.$$

Let $g_j : Y_j \to X$ be the induced morphism and let D_j be the push-forward of D on Y_j. By the construction of D and the $\mathbb{Q}$-factoriality of X,

$$g_j^{-1}(\operatorname{Nklt}(X, \Omega)) = \operatorname{Nklt}(Y_j, D_j) \tag{4.9.6}$$

holds for any $j \in \{0, \ldots, \ell\}$. In particular, it holds that $g_*(WI_{\operatorname{Nklt}(Y,D),\mathbb{Q}}) = WI_{\operatorname{Nklt}(X,\Omega),\mathbb{Q}}$. Thus, it is enough to prove that

$$R^i(g_j)_*(WI_{\operatorname{Nklt}(Y_j,\Omega_{Y_j}),\mathbb{Q}}) = 0 \tag{4.9.7}$$

for any $i > 0$ and $j \in \{0, \ldots, \ell\}$. We prove this by descending induction on j.

We first prove (4.9.7) for $j = \ell$. We have

$$K_Y + D \sim_{X,\mathbb{R}} g_*^{-1}\Omega^{\wedge 1} + E + (1-\epsilon)E' - \Omega_Y,$$

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where Ω_Y is defined by $K_Y+\Omega_Y=g^*(K_X+\Omega)$. Since ϵ is sufficiently small, it follows from the negativity lemma that any g-exceptional prime divisor G with $a_G(X,\Omega)>0$ is contracted on Y_ℓ. Since X is $\mathbb{Q}$-factorial, $Y_\ell \to X$ is isomorphic over $X\setminus \mathrm{Nklt}(X,\Omega)$. Therefore, we obtain

$$\mathrm{Ex}(g_\ell)\subset g_\ell^{-1}(\mathrm{Nklt}(X,\Omega))=\mathrm{Nklt}(Y_\ell,D_\ell).$$

Then [GNT19, Proposition 2.23] implies that the equation (4.9.7) holds when $j=\ell$.

Suppose that $0\leqslant j<\ell$. Then $\varphi_j: Y_j \dashrightarrow Y_{j+1}$ is either a divisorial contraction or a flip. We first treat the case when φ_j is a divisorial contraction. In order to prove the equation (4.9.7) by induction, it is sufficient to show that

$$R^i(\varphi_j)_*(WI_{\mathrm{Nklt}(Y_j,D_j),\mathbb{Q}})=0 \tag{4.9.8}$$

for $i>0$. Let $G=\mathrm{Ex}(\varphi_j)$ be the contracted divisor. Suppose that $\dim\varphi_j(G)=0$. If $G\subset \mathrm{Nklt}(Y_j, D_j)$, then the equation (4.9.8) follows from [GNT19, Proposition 2.23]. If $G\not\subset \mathrm{Nklt}(Y_j,D_j)$, then $G\cap \mathrm{Nklt}(Y_j,D_j)=\emptyset$ holds by (4.9.6). Therefore, since g_j is an isomorphism outside G, it is sufficient to check the equation (4.9.8) outside $\mathrm{Nklt}(Y_j,D_j)$ and this follows from Theorem 3.11. Suppose that $\dim\varphi_j(G)=1$. In this case, the relative dimension of φ is one and therefore it is sufficient to show the equation (4.9.8) only for $i=1$ (cf. [GNT19, Lemma 2.20]). This follows from the exact sequence

$$0\to WI_{\mathrm{Nklt}(Y_j,D_j),\mathbb{Q}}\to W\mathcal{O}_{Y_j,\mathbb{Q}}\to W\mathcal{O}_{\mathrm{Nklt}(Y_j,D_j),\mathbb{Q}}\to 0,$$

the fact that $R^1(\varphi_j)_*W\mathcal{O}_{Y_j,\mathbb{Q}}=0$ by Theorem 3.11 and the surjectivity of $(\varphi_j)_*W\mathcal{O}_{Y_j,\mathbb{Q}}\to (\varphi_j)_*W\mathcal{O}_{\mathrm{Nklt}(Y_j,D_j),\mathbb{Q}}$ by Theorem 2.15.

Next we assume that $\varphi_j: Y_j \dashrightarrow Y_{j+1}$ is a flip. Let $\psi: Y_j\to V$ be the corresponding flipping contraction and let $\psi^+: Y_{j+1}\to V$ be the induced morphism. In order to prove the equation (4.9.7) by induction, it is sufficient to show that

$$R^i\psi_*(WI_{\mathrm{Nklt}(Y_j,\Omega_{Y_j}),\mathbb{Q}})=R^i\psi^+_*(WI_{\mathrm{Nklt}(Y_{j+1},\Omega_{Y_{j+1}}),\mathbb{Q}}) \tag{4.9.9}$$

for any $i\geqslant 0$. When $i=0$, the equation (4.9.9) can be confirmed by the set-theoretic equation $\psi(\mathrm{Nklt}(Y_j,\Omega_{Y_j}))=\psi^+(\mathrm{Nklt}(Y_{j+1},\Omega_{Y_{j+1}}))$ and this follows from the equation (4.9.6). In what follows, we prove that both sides in the equation (4.9.9) are zero for $i\geqslant 1$. Since we work around a fixed closed point $x\in X$, after replacing X by an open neighbourhood of $x\in X$, we may assume that $g_j(\mathrm{Ex}(\psi))=\{x\}$. There are the following two cases: $x\in\mathrm{Nklt}(X,\Omega)$ and $x\notin \mathrm{Nklt}(X,\Omega)$. In the case when $x\in \mathrm{Nklt}(X,\Omega)$, it follows from (4.9.6) that $\mathrm{Ex}(\psi)\subset \mathrm{Nklt}(Y_j,D_j)$ and $\mathrm{Ex}(\psi^+)\subset\mathrm{Nklt}(Y_{j+1},D_{j+1})$. Then both sides in the equation (4.9.9) are zero for $i\geqslant 1$ by [GNT19, Proposition 2.23]. In the case when $x\notin\mathrm{Nklt}(X,\Omega)$, we may assume that (X,Ω) is klt and hence so is each (Y_j,D_j). Then both sides in the equation (4.9.9) are zero for $i\geqslant 1$ by Theorem 3.11. This completes the proof of Step 4. □

Step 5. Assume (i)–(v) in Step 2. Then the assertion of Proposition 4.9 holds.

Proof. We use the same notation as in Step 3 and Step 4. By (vii), (ix), (xi) and [CR12, Proposition 4.6.1], it holds that

$$R^ih_*(WI_{\mathrm{Nklt}(Y,D),\mathbb{Q}})|_{Z\setminus\{z_1,\ldots,z_n\}}=0$$

for any $i>0$. Then (xii) implies that

$$R^if_*(WI_{\mathrm{Nklt}(X,\Omega),\mathbb{Q}})|_{Z\setminus\{z_1,\ldots,z_n\}}=0$$

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for any $i > 0$. Since X_0 is of Fano type over Z_0, it follows from Theorem 3.11 that

$$R^i f_*(WI_{\mathrm{Nklt}(X,\Omega),\mathbb{Q}})|_{Z_0} = R^i(f|_{X_0})_*(W\mathcal{O}_{X_0,\mathbb{Q}}) = 0$$

holds for any $i > 0$. Since $Z = Z_0 \cup (Z \setminus \{z_1, \ldots, z_n\})$, we obtain

$$R^i f_*(WI_{\mathrm{Nklt}(X,\Omega),\mathbb{Q}}) = 0.$$

This completes the proof of Step 5. □

Step 2 and Step 5 complete the proof of Proposition 4.9. □

Theorem 4.10. *Let k be a perfect field of characteristic $p > 5$. Let (X, Δ) be a three-dimensional log pair over k and let $f : X \to Z$ be a projective k-morphism to a quasi-projective k-scheme Z. Assume that $-(K_X + \Delta)$ is f-nef and f-big. Then $R^i f_*(WI_{\mathrm{Nklt}(X,\Delta),\mathbb{Q}}) = 0$ for $i > 0$.*

Proof. We divide the proof into two steps.

Step 1. The assertion of Theorem 4.10 holds if X is $\mathbb{Q}$-factorial and $-(K_X + \Delta)$ is f-ample.

Proof of Step 1. Let $g : Y \to X$ be a projective birational morphism satisfying the properties (i)–(iii) in Proposition 2.10. Let Δ_Y be the $\mathbb{R}$-divisor on Y defined by $g^*(K_X + \Delta) = K_Y + \Delta_Y$. Then $(Y, \Delta_Y^{\wedge 1})$ is dlt. It follows from Lemma 4.8 that

$$Rg_*(WI_{\mathrm{Nklt}(Y,\Delta_Y),\mathbb{Q}}) \simeq WI_{\mathrm{Nklt}(X,\Delta),\mathbb{Q}}.$$

Therefore, it holds that

$$Rh_*(WI_{\mathrm{Nklt}(Y,\Delta_Y),\mathbb{Q}}) \simeq Rf_*(WI_{\mathrm{Nklt}(X,\Delta),\mathbb{Q}}), \tag{4.10.1}$$

where $h : Y \xrightarrow{g} X \xrightarrow{f} Z$ is the composition. Thanks to (4.10.1) and Proposition 4.9, it suffices to find an effective $\mathbb{R}$-divisor Ω_Y on Y such that:

(i) $(Y, \Omega_Y^{\wedge 1})$ is dlt;
(ii) $K_Y + \Omega_Y \sim_{h,\mathbb{R}} 0$;
(iii) Ω_Y is h-big; and
(iv) $\mathrm{Supp}(\Omega_Y^{>1}) = \mathrm{Supp}(\Omega_Y^{\geqslant 1}) = \mathrm{Supp}(\Delta_Y^{\geqslant 1})$.

Since X is $\mathbb{Q}$-factorial, there exists an effective $\mathbb{R}$-divisor F on Y such that $-F$ is g-ample and $\mathrm{Supp}\, F = \mathrm{Ex}(g)$. Since $-(K_Y + \Delta_Y)$ is the pullback of an f-ample $\mathbb{R}$-divisor $-(K_X + \Delta)$ on X, it follows that $-(K_Y + \Delta_Y) - \epsilon F$ is h-ample for any sufficiently small $\epsilon > 0$.

Note that $\mathrm{Supp}\, F \subset \mathrm{Supp}(\Delta_Y^{\geqslant 1})$. Thus, we can find an effective $\mathbb{R}$-divisor B on Y such that $B \geqslant \epsilon F$, $-(K_Y + \Delta_Y) - B$ is h-ample and $\mathrm{Supp}\, B = \mathrm{Supp}(\Delta_Y^{\geqslant 1})$. Lemma 2.8 enables us to find an effective $\mathbb{R}$-divisor A on Y such that $A \sim_{h,\mathbb{R}} -(K_Y + \Delta_Y) - B$ and $(Y, \Delta_Y^{\wedge 1} + 2A)$ is dlt. In particular, $(Y, \Delta_Y^{\wedge 1} + A)$ is dlt.

Set $\Omega_Y := \Delta_Y + A + B$. Then both (ii) and (iii) hold automatically. We obtain

$$\Omega_Y^{\wedge 1} = (\Delta_Y + A + B)^{\wedge 1} = (\Delta_Y + A)^{\wedge 1} = \Delta_Y^{\wedge 1} + A, \tag{4.10.2}$$

where the second equality follows from $\mathrm{Supp}\, B = \mathrm{Supp}(\Delta_Y^{\geqslant 1})$ and the third one holds by the fact that $(Y, \Delta_Y^{\wedge 1} + 2A)$ is dlt. Thus, (i) holds.

Let us prove (iv). It is clear that $\mathrm{Supp}(\Omega_Y^{>1}) \subset \mathrm{Supp}(\Omega_Y^{\geqslant 1})$. The inverse inclusion follows from

$$\begin{aligned}\mathrm{Supp}(\Omega_Y^{\geqslant 1}) &= \mathrm{Supp}(\Delta_Y + A + B)^{\geqslant 1} = \mathrm{Supp}(\Delta_Y + B)^{\geqslant 1} \\ &= \mathrm{Supp}(\Delta_Y + B)^{>1} \subset \mathrm{Supp}(\Delta_Y + A + B)^{>1} = \mathrm{Supp}(\Omega_Y^{>1}),\end{aligned}$$

where the second equality follows from the fact that $(Y, \Delta_Y^{\wedge 1} + 2A)$ is dlt and the third one holds by $\mathrm{Supp}\, B = \mathrm{Supp}(\Delta_Y^{\geqslant 1})$. Then we obtain the remaining equality as follows:

$$\mathrm{Supp}\, \Omega_Y^{\geqslant 1} = \mathrm{Supp}(\Omega_Y^{\wedge 1})^{=1} = \mathrm{Supp}(\Delta_Y^{\wedge 1} + A)^{=1} = \mathrm{Supp}\, \Delta_Y^{\geqslant 1},$$

where the second equality follows from (4.10.2) and the third one holds by the fact that $(Y, \Delta_Y^{\wedge 1} + 2A)$ is dlt. Thus, (iv) holds. This completes the proof of Step 1. □

Step 2. The assertion of Theorem 4.10 holds without any additional assumptions.

Proof of Step 2. Lemma 4.8 enables us to replace (X, Δ) by the log pair (Y, Δ_Y) appearing in Proposition 2.10 (cf. (4.10.1)). Hence, we may assume that X is $\mathbb{Q}$-factorial.

Since $-(K_X+\Delta)$ is f-nef and f-big, there exists an effective $\mathbb{R}$-divisor E such that $-(K_X+\Delta) - \epsilon E$ is f-ample for any real number ϵ satisfying $0 < \epsilon < 1$. The equation

$$\mathrm{Nklt}(X, \Delta) = \mathrm{Nklt}(X, \Delta + \epsilon E)$$

holds for sufficiently small $\epsilon > 0$. Therefore, replacing Δ by $\Delta + \epsilon E$, we may assume that $-(K_X + \Delta)$ is f-ample. Hence, Step 2 follows from Step 1. □

Step 2 completes the proof of Theorem 4.10. □

Theorem 4.11. *Let k be a perfect field of characteristic $p > 5$. Let (X, Δ) be a three-dimensional log pair over k and let $f : X \to Z$ be a projective k-morphism to a quasi-projective k-scheme Z. Let Z' be a closed subscheme of Z, set $X' := X \times_Z Z'$ and let $f' : X' \to Z'$ be the induced morphism. Assume that $-(K_X + \Delta)$ is f-nef and f-big. Then the following hold:*

(i) $R^i f_*(WI_{X' \cup \mathrm{Nklt}(X,\Delta),\mathbb{Q}}) = 0$ *for* $i > 0$;
(ii) $R^i f'_*(WI_{j^{-1}(\mathrm{Nklt}(X,\Delta)),\mathbb{Q}}) = 0$ *for* $i > 0$, *where* $j : X' \to X$ *denotes the induced closed immersion.*

Proof. Taking the Stein factorisation of f, we may assume that $f_*\mathcal{O}_X = \mathcal{O}_Z$.

Let us prove (i). Since the problem is local on Z, the problem is reduced to the case when Z is affine. Hence, we can write

$$Z = \mathrm{Spec}\, A, \quad Z' = \mathrm{Spec}(A/I), \quad I = (a_1, \ldots, a_r)$$

for some $a_1, \ldots, a_r \in A$. We show the assertion (i) by induction on r.

We now treat the case when $r = 1$. If $Z' = Z$, then there is nothing to show. If $Z' = \emptyset$, then the assertion follows from Theorem 4.10. Thus, we may assume that X' is a non-zero effective Cartier divisor on X. Since $-(K_X + \Delta + X')$ is f-nef and f-big, Theorem 4.10 implies that

$$R^i f_*(WI_{X' \cup \mathrm{Nklt}(X,\Delta),\mathbb{Q}}) = R^i f_*(WI_{\mathrm{Nklt}(X,\Delta+X'),\mathbb{Q}}) = 0$$

for $i > 0$. Thus, the assertion (i) holds if $r = 1$.

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Assume that $r \geqslant 2$ and that the assertion (i) holds for the case when I is generated by fewer than r elements. We set

$$Z'' := \mathrm{Spec}(A/(f_1, \dots, f_{r-1})), \quad Z_r := \mathrm{Spec}(A/(f_r)),$$
$$X'' := X \times_Z Z'', \quad X_r := X \times_Z Z_r.$$

For $N := \mathrm{Nklt}(X, \Delta)$, we have the exact sequence (Lemma 2.18)

$$0 \to WI_{(X'' \cup N) \cup (X_r \cup N), \mathbb{Q}} \to WI_{X'' \cup N, \mathbb{Q}} \oplus WI_{X_r \cup N, \mathbb{Q}} \to WI_{X' \cup N, \mathbb{Q}} \to 0.$$

Then the assertion (i) follows by induction.

Let us prove (ii). Thanks to the exact sequence (Lemma 2.18)

$$0 \to WI_{\mathrm{Nklt}(X,\Delta) \cup X', \mathbb{Q}} \to WI_{\mathrm{Nklt}(X,\Delta), \mathbb{Q}} \oplus WI_{X', \mathbb{Q}} \to WI_{\mathrm{Nklt}(X,\Delta) \cap X', \mathbb{Q}} \to 0,$$

(i) implies that the natural map

$$\varphi^i : R^i f_*(WI_{X', \mathbb{Q}}) \to R^i f_*(WI_{\mathrm{Nklt}(X,\Delta) \cap X', \mathbb{Q}})$$

is bijective for $i > 0$. Thanks to the following commutative diagram with exact horizontal sequence

$$\begin{array}{ccccccccc}
0 & \longrightarrow & WI_{X',\mathbb{Q}} & \longrightarrow & W\mathcal{O}_{X,\mathbb{Q}} & \longrightarrow & W\mathcal{O}_{X',\mathbb{Q}} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi} & & \| & & \downarrow & & \\
0 & \longrightarrow & WI_{\mathrm{Nklt}(X,\Delta)\cap X',\mathbb{Q}} & \longrightarrow & W\mathcal{O}_{X,\mathbb{Q}} & \longrightarrow & W\mathcal{O}_{\mathrm{Nklt}(X,\Delta)\cap X',\mathbb{Q}} & \longrightarrow & 0
\end{array}$$

the snake lemma induces the exact sequence

$$0 \to WI_{X', \mathbb{Q}} \xrightarrow{\varphi} WI_{\mathrm{Nklt}(X,\Delta) \cap X', \mathbb{Q}} \to j_* WI_{j^{-1}(\mathrm{Nklt}(X,\Delta)), \mathbb{Q}} \to 0.$$

Since the map $\varphi^i : R^i f_*(WI_{X', \mathbb{Q}}) \to R^i f_*(WI_{\mathrm{Nklt}(X,\Delta) \cap X', \mathbb{Q}})$ induced by φ is bijective for $i > 0$, we get

$$Rf_*(j_* WI_{j^{-1}(\mathrm{Nklt}(X,\Delta)), \mathbb{Q}}) \simeq f_* j_* WI_{j^{-1}(\mathrm{Nklt}(X,\Delta)), \mathbb{Q}}. \tag{4.11.1}$$

If i denotes the induced closed immersion $Z' \to Z$, then it holds that

$$\begin{aligned}
f_* j_* WI_{j^{-1}(\mathrm{Nklt}(X,\Delta)), \mathbb{Q}} &\simeq Rf_*(j_* WI_{j^{-1}(\mathrm{Nklt}(X,\Delta)), \mathbb{Q}}) \\
&\simeq Rf_* Rj_* WI_{j^{-1}(\mathrm{Nklt}(X,\Delta)), \mathbb{Q}} \\
&\simeq Ri_* Rf'_* WI_{j^{-1}(\mathrm{Nklt}(X,\Delta)), \mathbb{Q}} \\
&\simeq i_* Rf'_* WI_{j^{-1}(\mathrm{Nklt}(X,\Delta)), \mathbb{Q}},
\end{aligned}$$

where the first isomorphism follows from (4.11.1) and the second and last ones hold because j_* and i_* are exact functors. This completes the proof of (ii). □

THEOREM 4.12. *Let k be a perfect field of characteristic $p > 5$. Let (X, Δ) be a three-dimensional log pair over k and let $f : X \to Z$ be a projective k-morphism to a quasi-projective k-scheme Z such that f has connected fibres. Assume that $-(K_X + \Delta)$ is f-nef and f-big. Then the induced morphism $\mathrm{Nklt}(X, \Delta) \to Z$ has connected fibres.*

Proof. Thanks to Theorem 4.10, the homomorphism

$$W\mathcal{O}_{Z,\mathbb{Q}} = f_* W\mathcal{O}_{X,\mathbb{Q}} \to f_*(W\mathcal{O}_{\mathrm{Nklt}(X,\Delta),\mathbb{Q}})$$

is surjective. Since this homomorphism factors through $W\mathcal{O}_{f(\mathrm{Nklt}(X,\Delta)),\mathbb{Q}}$, also the induced homomorphism

$$\theta : W\mathcal{O}_{f(\mathrm{Nklt}(X,\Delta)),\mathbb{Q}} \to f_*(W\mathcal{O}_{\mathrm{Nklt}(X,\Delta),\mathbb{Q}})$$

is surjective. Since this is automatically injective, θ is bijective. Therefore, the morphism $\mathrm{Nklt}(X, \Delta) \to f(\mathrm{Nklt}(X, \Delta))$ has connected fibres (cf. Lemma 2.17), as desired. □

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5. Application to rational points on varieties over finite fields

As an application of our vanishing theorem of Nadel type (Theorems 4.10 and 4.11), we deduce some consequences for rational points on varieties over finite fields (Theorems 5.1 and 5.3).

Theorem 5.1. *Let (X,Δ) be a three-dimensional log pair over a finite field k of characteristic $p>5$. Let $f: X\to Y$ be a projective k-morphism to a quasi-projective k-scheme Y such that $f_*\mathcal{O}_X=\mathcal{O}_Y$. Assume that $-(K_X+\Delta)$ is f-nef and f-big. Then the congruence*

$$\#X(k)-\#V(k)\equiv \#Y(k)-\#f(V)(k) \mod \#k$$

holds, where $V:=\mathrm{Nklt}(X,\Delta)$.

Proof. If $Y(k)=\emptyset$, then there is nothing to show. Thus, the problem is reduced to the case when $Y(k)\neq\emptyset$. Fix a k-rational point $y\in Y$. Since the problem is local on Y, we may assume that $Y(k)=\{y\}$. If $\mathrm{Nklt}(X,\Delta)\cap f^{-1}(y)=\emptyset$, then the assertion follows from [GNT19, Theorem 5.4]. Hence, we may assume that $y\in f(\mathrm{Nklt}(X,\Delta))=f(V)$, i.e. $V_y\neq\emptyset$. In particular, it holds that

$$Y(k)=f(V)(k)=\{y\}.$$

Since

$$X(k)=X_y(k),\quad V(k)=V_y(k),$$

it suffices to prove that

$$\#X_y(k)\equiv\#V_y(k)\mod \#k.$$

Consider the exact sequence

$$0\to WI_{V_y,\mathbb{Q}}\to W\mathcal{O}_{X_y,\mathbb{Q}}\to W\mathcal{O}_{V_y,\mathbb{Q}}\to 0.$$

Thanks to Theorem 4.11(ii), it holds that

$$H^i(X_y,WI_{V_y,\mathbb{Q}})=0$$

for $i>0$. Furthermore, we get $H^0(X_y,WI_{V_y,\mathbb{Q}})=0$ by $V_y\neq\emptyset$ and the fact that X_y is connected. Hence, the natural map

$$H^i(X_y,W\mathcal{O}_{X_y,\mathbb{Q}})\to H^i(V_y,W\mathcal{O}_{V_y,\mathbb{Q}})$$

is bijective for $i\geqslant 0$. Therefore, the assertion holds by [BBE07, Proposition 6.9(i)]. □

Corollary 5.2. *Let (X,Δ) be a three-dimensional geometrically connected projective log pair over a finite field k of characteristic $p>5$. Assume that $-(K_X+\Delta)$ is nef and big and that (X,Δ) is not klt. Then the congruence*

$$\#X(k)\equiv\#V(k)\mod \#k$$

holds, where $V:=\mathrm{Nklt}(X,\Delta)$.

Proof. Applying Theorem 5.1 for $Y:=\operatorname{Spec} k$, the assertion holds. □

Theorem 5.3. *Let k be a perfect field of characteristic $p>5$. Let X be a projective normal variety over k with $\dim X\leqslant 3$. Let D be a non-zero effective $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisor on X. Assume that there exists an effective $\mathbb{R}$-divisor Δ such that:*

(a) (X, Δ) *is klt;*
(b) $-(K_X + \Delta)$ *is nef and big; and*
(c) $-(K_X + \Delta + D)$ *is nef and big.*

Then the following hold.

(i) *The equation*

$$H^i(D, W\mathcal{O}_{D,\mathbb{Q}}) = 0$$

holds for $i > 0$ *and the induced map*

$$H^0(X, W\mathcal{O}_{X,\mathbb{Q}}) \to H^0(D, W\mathcal{O}_{D,\mathbb{Q}})$$

is bijective.

(ii) *If* k *is a finite field and* X *is geometrically connected over* k*, then the congruence*

$$\#D(k) \equiv 1 \mod \#k$$

holds.

Proof. Let us prove (i). We have the exact sequence

$$0 \to WI_{D,\mathbb{Q}} \to W\mathcal{O}_{X,\mathbb{Q}} \to W\mathcal{O}_{D,\mathbb{Q}} \to 0.$$

It follows from (a), (b) and Theorem 3.1(ii) that the equation

$$H^i(X, W\mathcal{O}_{X,\mathbb{Q}}) = 0$$

holds for $i > 0$. Note that $\mathrm{Nklt}(X, \Delta + D) = \mathrm{Supp}\, D$. Hence, by (c) and Theorem 4.10, we get

$$H^i(X, WI_{D,\mathbb{Q}}) = 0$$

for $i > 0$. Since $D \neq 0$, we obtain $H^0(X, WI_{D,\mathbb{Q}}) = 0$. This completes the proof of (i).

Let us show (ii). Thanks to (i), we may apply [BBE07, Proposition 6.9(i)] and obtain the congruence $\#X(k) \equiv \#D(k) \mod \#k$. On the other hand, we have another congruence $\#X(k) \equiv 1 \mod \#k$, which is guaranteed by [GNT19, Theorem 5.4]. To summarise, we get $\#D(k) \equiv 1 \mod \#k$, as desired. □

Remark 5.4. As applications of a vanishing theorem of Nadel type (Theorem 4.10), we obtain two results: the Kollár–Shokurov connectedness theorem (Theorem 4.12) and the existence of rational points (Theorem 5.1). For certain special cases, these two consequences are also related as follows.

Let k be a perfect field of characteristic $p > 5$ and let (X, Δ) be a projective log pair over k with $\dim X \leqslant 3$ such that $-(K_X + \Delta)$ is f-nef and f-big. Assume that X is geometrically connected over k, $\mathrm{Nklt}(X, \Delta) \neq \emptyset$ and $\dim \mathrm{Nklt}(X, \Delta) = 0$. Then $\mathrm{Nklt}(X, \Delta)$ is geometrically connected over k by the Kollár–Shokurov connectedness theorem (Theorem 4.12), which implies that $\mathrm{Nklt}(X, \Delta)$ consists of a single k-rational point.

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Yusuke Nakamura nakamura@ms.u-tokyo.ac.jp

Graduate School of Mathematical Sciences, The University of Tokyo,
3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan

Hiromu Tanaka tanaka@ms.u-tokyo.ac.jp

Graduate School of Mathematical Sciences, The University of Tokyo,
3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan

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