Combinatorics, Probability and Computing (2000) 9, 489–511. Printed in the United Kingdom

Convergence Theorems for Some Layout Measures on Random Lattice and Random Geometric Graphs

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Received 15 April 1999; revised 14 January 2000

This work deals with convergence theorems and bounds on the cost of several layout measures for lattice graphs, random lattice graphs and sparse random geometric graphs. Specifically, we consider the following problems: *Minimum Linear Arrangement*, *Cutwidth*, *Sum Cut*, *Vertex Separation*, *Edge Bisection* and *Vertex Bisection*. For full square lattices, we give optimal layouts for the problems still open. For arbitrary lattice graphs, we present best possible bounds disregarding a constant factor. We apply percolation theory to the study of lattice graphs in a probabilistic setting. In particular, we deal with the subcritical regime that this class of graphs exhibits and characterize the behaviour of several layout measures in this space of probability. We extend the results on random lattice graphs to random geometric graphs, which are graphs whose nodes are spread at random in the unit square and whose edges connect pairs of points which are within a given distance. We also characterize the behaviour of several layout measures on random geometric graphs in their subcritical regime. Our main results are convergence theorems that can be viewed as an analogue of the Beardwood, Halton and Hammersley theorem for the Euclidean TSP on random points in the unit square.

1. Introduction

Several well-known optimization problems on graphs can be formulated as *graph layout problems.* A (*linear*) *layout* of a graph with n vertices is a bijection between the vertex set and the set of naturals from 1 to n. Graph layout problems, also referred to in the

† This research was partially supported by ESPRIT LTR Project no. 20244 – ALCOM-IT, CICYT Project TIC97-1475-CE, and CIRIT project 1997SGR-00366.

https://doi.org/10.1017/S0963548300004454 Published online by Cambridge University Press

literature as *linear ordering problems* or *linear arrangement problems*, seek for a layout that minimizes a cost associated with each problem. The particular layout problems that we consider include *Minimum Linear Arrangement*, *Cutwidth*, *Sum Cut* and *Vertex Separation*. We also consider the *Edge Bisection* and the *Vertex Bisection* problems, which are partitioning problems, but can also be formulated as layout problems. The Sum Cut problem is equivalent to the *Profile* problem and the Vertex Separation problem to the *Pathwidth* problem. For general graphs, all these problems are **NP**-hard. Moreover, the decisional version of Cutwidth and Vertex Separation problems are **NP**-complete even when restricted to lattice graphs and unit disk graphs [10]. All of them have a long history, owing to their practical relevance in different applications [35, 15, 1, 32, 24, 23, 2, 26, 7, 17, 19, 32, 18, 5, 22, 27, 33, 34].

Our layout problems are formally defined as follows. A *layout* φ of a graph $G = (V, E)$ is a one-to-one function $\varphi : V \to \{1, \ldots, n\}$ with $n = |V|$. Given a graph G, a layout φ of G and an integer i, let us define the sets

$$L(i, \varphi, G) = \{u \in V(G) \ : \ \varphi(u) \leqslant i\} \quad \text{and} \quad R(i, \varphi, G) = \{u \in V(G) \ : \ \varphi(u) > i\},$$

the measures

$$\begin{aligned}
\theta(i, \varphi, G) &= |\{uv \in E(G) \ : \ u \in L(i, \varphi, G) \wedge v \in R(i, \varphi, G)\}|,\\
\delta(i, \varphi, G) &= |\{u \in L(i, \varphi, G) \ : \ \exists v \in R(i, \varphi, G) : uv \in E(G)\}|,\\
\lambda(uv, \varphi, G) &= |\varphi(u) - \varphi(v)|, \qquad \text{where } uv \in E(G),
\end{aligned}$$

and the following graph layout problems:

- Minimum Linear Arrangement (MinLA). Given a graph $G = (V, E)$, find $\text{MINLA}(G) = \min_\varphi \sum_{uv \in E} \lambda(uv, \varphi, G) = \min_\varphi \sum_{i=1}^n \theta(i, \varphi, G)$.
- Cutwidth (CutWidth). Given a graph $G = (V, E)$, find $\text{MINCW}(G) = \min_\varphi \max_{i=1}^n \theta(i, \varphi, G)$.
- Vertex Separation (VertSep). Given a graph $G = (V, E)$, find $\text{MINVS}(G) = \min_\varphi \max_{i=1}^n \delta(i, \varphi, G)$.
- Sum Cut (SumCut). Given a graph $G = (V, E)$, find $\text{MINSC}(G) = \min_\varphi \sum_{i=1}^n \delta(i, \varphi, G)$.
- Edge Bisection (EdgeBis). Given a graph $G = (V, E)$, find $\text{MINEB}(G) = \min_\varphi \theta(\lfloor n/2 \rfloor, \varphi, G)$.
- Vertex Bisection (VertBis). Given a graph $G = (V, E)$, find $\text{MINVB}(G) = \min_\varphi \delta(\lfloor n/2 \rfloor, \varphi, G)$.

Graphs encoding circuits or grids are typical instances of linear arrangement problems. We consider these instances as sparse graphs that have clustering and geometric properties. In this paper, we are concerned with *lattice graphs*, *random lattice graphs* and *random geometric graphs*. The study of these classes of graphs is an alternative to the standard $\mathcal{G}_{n,p}$ model of random graphs, which does not provide an informative framework to compare heuristics for layout problems [12]. In the rest of this section, we define these classes of graphs and outline and comment on our main results.

https://doi.org/10.1017/S0963548300004454 Published online by Cambridge University Press

Lattice graphs

A graph is said to be a *lattice graph* if it is a finite node-induced subgraph of the infinite lattice, that is, its vertex set is a finite subset of $\mathbb{Z}^2$ and two vertices are connected whenever they are at distance one.

The most simple lattice graph is L_m with node set $\{0, \ldots, m-1\}^2$. The optimal layouts for the MinLA, CutWidth and EdgeBis problems on L_m are already known [25, 24, 22]: $\text{MINEB}(L_m) = \text{MINCW}(L_m) = m$ (or $m+1$ if m is odd) and $\text{MINLA}(L_m) = \frac{1}{3}(4-\sqrt{2})m^3 + o(m^3)$. In the current paper we show that the *diagonal layout* (where $\mathbf{x} = (x, y)$ precedes $\mathbf{x}' = (x', y')$ whenever $x + y < x' + y'$, and whenever $x + y = x' + y'$ and $x < x'$) is optimal for the VertSep, SumCut and VertBis problems on L_m.

Theorem 1.1. *For all m, the diagonal layout is optimal for the* VertSep, VertBis *and* SumCut *problems on* L_m. *Moreover,*

$$\begin{aligned} \text{MINVS}(L_m) &= m, \\ \text{MINVB}(L_m) &= m, \\ \text{MINSC}(L_m) &= \tfrac{2}{3}m^3 + \tfrac{1}{2}m^2 - \tfrac{7}{6}m. \end{aligned}$$

In the general case where the lattice graph is arbitrary, in this paper we provide the upper bounds stated in the following theorem.

Theorem 1.2. *For any lattice graph L with n vertices,*

$$\text{MINEB}(L) \leqslant 2\sqrt{2n} + 1, \tag{1.1}$$
$$\text{MINVB}(L) \leqslant 2\sqrt{2n} + 1, \tag{1.2}$$
$$\text{MINCW}(L) \leqslant 14\sqrt{n}, \tag{1.3}$$
$$\text{MINVS}(L) \leqslant 14\sqrt{n}, \tag{1.4}$$
$$\text{MINLA}(L) \leqslant 14n\sqrt{n}, \tag{1.5}$$
$$\text{MINSC}(L) \leqslant 14n\sqrt{n}. \tag{1.6}$$

These bounds can be achieved algorithmically. Observe that, in the case of L_m (where $n = m^2$), the above upper bounds are within a constant of their optimal costs (Theorem 1.1 and [25, 24, 22]). This shows that the bounds in Theorem 1.2 are best possible disregarding a constant factor. The only previous known result for lattice graphs was an exact polynomial time algorithm for the Edge Bisection problem on lattice graphs without holes [28].

Random lattice graphs

We study lattice graphs in a probabilistic setting using *percolation* theory. We consider a *site percolation* process, where nodes from the infinite lattice ($\mathbb{Z}^2$) are selected with some probability p (selected nodes are called 'open'). Let C_0 be the connected component where the origin belongs. A basic question in percolation theory is whether or not C_0 can be infinite. Let $\vartheta(p)$ denote the probability that $|C_0| = \infty$, and set $p_c = \inf\{p : \vartheta(p) > 0\}$,

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the *critical value* of p. It is well known that $p_c \in (0.5, 1)$ [14]. In this paper, we consider only *subcritical* limiting regimes $p \in (0, p_c)$ in which all components are almost surely finite (results for *supercritical* regimes are derived in [9, 30]). In order to deal with finite graphs, we introduce the class of *random lattice graphs* with parameters m and p, denoted by $\mathscr{L}_{m,p}$, the lattice graphs whose set of vertices is obtained through the random selection of each element from $\{0, \ldots, m-1\}^2$ chosen independently with probability p.

Regarding the Vertex Separation and the Cutwidth problem, in this paper we show that, with high probability, their optima are $\Theta(\sqrt{\log m})$.

Theorem 1.3. *Let $p \in (0, p_c)$; then there exist constants $0 < c_1 < c_2$ such that*

$$\lim_{m\to\infty} \mathbf{Pr}\left[c_1 \leqslant \frac{\text{MINVS}(\mathscr{L}_{m,p})}{\sqrt{\log m}} \leqslant \frac{\text{MINCW}(\mathscr{L}_{m,p})}{\sqrt{\log m}} \leqslant c_2\right] = 1.$$

Recall (see for example [8]), that if $(X_n)_{n\geqslant 1}$ is a sequence of random variables and X is a random variable, then X_n converges *in probability* to X ($X_n \xrightarrow{\text{Pr}} X$) if, for every $\epsilon > 0$, it is the case that $\lim_{n\to\infty} \mathbf{Pr}\left[|X_n - X| > \epsilon\right] = 0$. We have the following convergence theorem regarding the Minimum Linear Arrangement and Sum Cut problems.

Theorem 1.4. *Let $p \in (0, p_c)$; then there exist two constants $\beta_{\text{LA}}(p) > 0$ and $\beta_{\text{SC}}(p) > 0$ such that, as $m \to \infty$,*

$$\frac{\text{MINLA}(\mathscr{L}_{m,p})}{m^2} \xrightarrow{\text{Pr}} \beta_{\text{LA}}(p) \qquad \textit{and} \qquad \frac{\text{MINSC}(\mathscr{L}_{m,p})}{m^2} \xrightarrow{\text{Pr}} \beta_{\text{SC}}(p).$$

In the proofs, we characterize $\beta_{\text{LA}}(p)$ and $\beta_{\text{SC}}(p)$, but do not give their exact value.

Random geometric graphs

Instead of having nodes fixed at regular points in the plane, one can spread nodes in the plane and connect nodes that are not too far away. To formalize this concept, let r be a number such that $0 < r < 1$, let $\|\cdot\|$ be a norm on $\mathbb{R}^2$ and let V be any set of n points in the unit square ($[0, 1]^2$). A *geometric graph* $G(V; r)$ with vertex set V and radius r is the graph $G = (V, E)$, where $E = \{uv \mid u, v \in V \wedge 0 < \|u - v\| \leqslant r\}$. Let $(r_i)_{i\geqslant 1}$ be a sequence of positive numbers and let $X = (X_i)_{i\geqslant 1}$ be a sequence of independently and uniformly distributed (i.u.d.) random points in $[0, 1]^2$. For any natural n, we write $\mathscr{X}_n = \{X_1, \ldots, X_n\}$ and denote by $\mathscr{G}(\mathscr{X}_n; r_n)$ the *random geometric graph* of n nodes on $\mathscr{X}_n$ and radius r_n.

Many empirical studies have used random geometric graphs as a basis for benchmarking heuristics for layout or partitioning problems [16, 4, 21, 31]; however, their theoretical study is still in its infancy (see [11] for a survey).

Like site percolation, continuum percolation and random geometric graphs exhibit a phase transition [29]. Suppose $\lim_{n\to\infty} nr_n^2 = \lambda$; then there exists a critical parameter λ_c such that, when $\lambda < \lambda_c$, graphs $\mathscr{G}(\mathscr{X}_n; r_n)$ are likely to have at most $O(\log n)$ points in each connected component, while when $\lambda > \lambda_c$, there is likely to be a component with $\Theta(n)$ vertices. In this paper, we consider only *subcritical* limiting regimes $0 < \lambda < \lambda_c$ (results for *supercritical* regimes are derived in [10, 30]).

The behaviour of the bisection problems on subcritical random geometric graphs is

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characterized by the following result, where λ'_c is a parameter that will be subsequently defined.

Theorem 1.5. *Suppose* $\lim_{n\to\infty} nr_n^2 = \lambda \in (0, \lambda'_c)$. *Then, as* $n \to \infty$,

$$\text{MINEB}(\mathscr{G}(\mathscr{X}_n; r_n)) \xrightarrow{\text{Pr}} 0 \quad \textit{and} \quad \text{MINVB}(\mathscr{G}(\mathscr{X}_n; r_n)) \xrightarrow{\text{Pr}} 0.$$

In the case of the Cutwidth and Vertex Separation problems for subcritical random geometric graphs, this paper determines that, with high probability, their orders of magnitude are $\Theta(\log n/\log\log n)$ and $\Theta((\log n/\log\log n)^2)$ respectively.

Theorem 1.6. *Suppose* $\lim_{n\to\infty} nr_n^2 = \lambda \in (0, \lambda_c)$. *Then there exist constants* $0 < c_3 < c_4$ *and* $0 < c_5 < c_6$ *such that*

$$\lim_{n\to\infty} \mathbf{Pr}\left[c_3 \leqslant \frac{\text{MINVS}(\mathscr{G}(\mathscr{X}_n; r_n))}{\log n/\log\log n} \leqslant c_4\right] = 1 = \lim_{n\to\infty} \mathbf{Pr}\left[c_5 \leqslant \frac{\text{MINCW}(\mathscr{G}(\mathscr{X}_n; r_n))}{(\log n/\log\log n)^2} \leqslant c_6\right].$$

In the case of the Minimum Linear Arrangement and Sum Cut problems, we prove in this paper another convergence result, analogous to Theorem 1.4.

Theorem 1.7. *Suppose* $\lim_{n\to\infty} nr_n^2 = \lambda \in (0, \lambda_c)$. *Then there exist two constants* $\tilde{\beta}_{\text{LA}}(\lambda) > 0$ *and* $\tilde{\beta}_{\text{SC}}(\lambda) > 0$ *such that, as* $n \to \infty$,

$$\frac{\text{MINLA}(\mathscr{G}(\mathscr{X}_n; r_n))}{n} \xrightarrow{\text{Pr}} \tilde{\beta}_{\text{LA}}(\lambda) \qquad \textit{and} \qquad \frac{\text{MINSC}(\mathscr{G}(\mathscr{X}_n; r_n))}{n} \xrightarrow{\text{Pr}} \tilde{\beta}_{\text{SC}}(\lambda).$$

As in the discrete case, we are able to characterize $\tilde{\beta}_{\text{LA}}(\lambda)$ and $\tilde{\beta}_{\text{SC}}(\lambda)$ as the expectation of some related quantity, but do not give their exact value. This last theorem can be viewed as analogous to the Beardwood, Halton and Hammersley theorem for the TSP.

The BHH Theorem [3]
Let $X = (X_i)_{i\geqslant 1}$ be a sequence of independent and uniformly distributed points in $[0,1]^d$. Let MINTSP(N) denote the length of the optimal solution of the TSP among the first n points of X. Then there exists a constant $\beta_{\text{TSP}}(d)$ such that $\text{MINTSP}(n)/n^{(d-1)/d}$ converges to $\beta_{\text{TSP}}(d)$ almost surely as $n \to \infty$.

Indeed, the search for BHH-like results was one of the initial motivations for this research (see [11]). A key property in proving BHH-like results is geometric sub-additivity [36, Chapter 3]. It is important to stress that geometric sub-additivity *does not hold* for our layout problems, and therefore we take a completely different approach using percolation theory. In passing, we also remark that determining the exact value of $\beta_{\text{TSP}}(2)$ is still an open problem.

Organization of the paper
In Section 2, we prove our results for lattice graphs. These are subsequently used in Section 3, where we present results for random lattice graphs. In Section 4, we give

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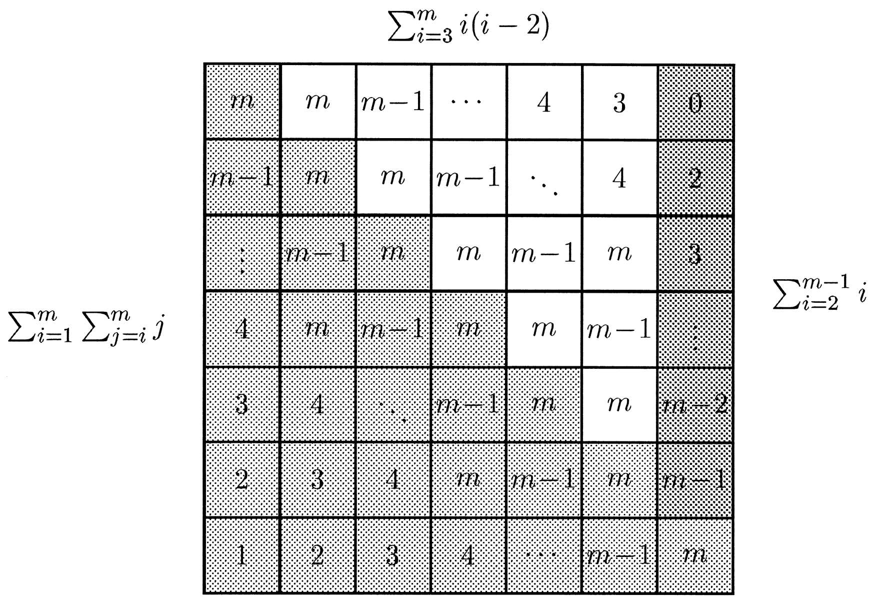

Figure 1 Values of the vertex cut in the diagonal ordering φ_D

the proofs of our results for random geometric graphs, which built on the theorems for random lattice graphs. We close the paper with a summary of our work and conclusions.

2. Lattice graphs

We begin this section by presenting a vertex isoperimetric inequality that will help us to characterize the optimal layouts for some of the problems defined in the previous section on square lattice graphs L_m. To attain this result, let φ_D denote the diagonal layout. The next lemma is a special case of Corollary 9 of Bollobás and Leader [6], who in fact prove the d-dimensional version for arbitrary d.

Lemma 2.1 (Vertex isoperimetric inequality). *For any layout φ on L_m and any $k \in \{1,\ldots,m^2\}$, it is the case that $\delta(k,\varphi,L_m) \geqslant \delta(k,\varphi_D,L_m)$.*

We are now ready to prove our first theorem.

Proof of Theorem 1.1. The previous isoperimetric inequality yields the optimality of φ_D for the costs of MINVS, MINSC and MINVB on L_m. Thus we get that $\text{MINVS}(L_m) = \text{MINVB}(L_m) = m$. To compute the sum of the cuts for φ_D, consider for each point in the lattice the value of the vertex cut produced by the diagonal ordering (see Figure 1); arranging the sum by points with the same vertex cut, we get

$$\text{MINSC}(L_m) = \sum_{i=1}^{m^2} \delta(i,\varphi_D,L_m) = \sum_{i=1}^{m}\sum_{j=i}^{m} j + \sum_{i=3}^{m} i(i-2) + \sum_{i=2}^{m-1} i = \tfrac{2}{3}m^3 + \tfrac{1}{2}m^2 - \tfrac{7}{6}m.$$

□

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We move now to general lattice graphs with n nodes.

Lemma 2.2. *For any lattice graph L with n vertices, and any $k \in \{1,\ldots,n\}$, there is a layout φ on L such that $\theta(k,\varphi,L) \leqslant 2^{3/2}\sqrt{n}+1$.*

Proof. We are looking for a subset S of L consisting of k vertices, such that there are at most $2^{3/2}\sqrt{n}+1$ edges between S and $L \setminus S$. Figure 2 illustrates this proof.

Let $\alpha > 0$ be a constant, to be chosen later. For $x \in \mathbb{Z}$ let $S_x = \{y \in \mathbb{Z} : (x,y) \in L\}$ and let $V = \{x \in \mathbb{Z} : |S_x| \geqslant \alpha\sqrt{n}\}$. For $i \in \mathbb{Z}$, let H_i denote the half-space $(-\infty, i] \times \mathbb{R}$. Set

$$i_0 = \min\{i \in \mathbb{Z} : |L \cap H_i| \geqslant k\}.$$

Consider the case $i_0 \notin V$. Then define S to be a set of the form

$$S = L \cap (H_{i_0-1} \cup (\{i_0\} \times (-\infty, j]))$$

with j chosen so that S has precisely k elements.

With this definition of S for $i_0 \notin V$, the number of horizontal edges between S and $L\backslash S$ is at most $|S_{i_0}|$, and hence is at most $\alpha\sqrt{n}$. There is at most one vertical edge between S and $L\backslash S$, so the number of edges from S to $L\backslash S$ is at most $\alpha\sqrt{n}+1$ when $i_0 \notin V$.

Now consider the other case $i_0 \in V$. Let $I = [i_1, i_2]$ be the largest integer interval which includes i_0 and is contained in V. Then $i_1 - 1 \notin V$, and $i_2 + 1 \notin V$. Also, as $|V| \leqslant \alpha^{-1}\sqrt{n}$, $i_2 - i_1 + 1 \leqslant \alpha^{-1}\sqrt{n}$. We have

$$|L \cap H_{i_1-1}| < k \leqslant |L \cap H_{i_2}|.$$

For $j \in \mathbb{Z}$ let $T_j = [i_1, i_2] \times (-\infty, j]$. Choose j_0 so that

$$|L \cap (H_{i_1-1} \cup T_{j_0-1})| < k \leqslant |L \cap (H_{i_1-1} \cup T_{j_0})|,$$

and let S be $L \cap (H_{i_1-1} \cup T_{j_0-1} \cup ([i_1, i_3] \times \{j_0\}))$, with $i_3 \in [i_1, i_2]$ chosen so that S has precisely k elements.

We estimate the number of edges between S and $L\backslash S$ for the case $i_0 \in V$. Since $i_1 - 1 \notin V$, and $i_2 + 1 \notin V$, the number of horizontal edges between S and $L\backslash S$ is at most $2\alpha\sqrt{n}+1$. Also, since $i_2 - i_1 + 1 \leqslant \alpha^{-1}\sqrt{n}$, the number of vertical edges between S and $L\backslash S$ is at most $\alpha^{-1}\sqrt{n}$. Combining these estimates we find that there are at most $(2\alpha + \alpha^{-1})\sqrt{n}+1$ edges between S and $L\backslash S$, whether or not $i_0 \in V$.

The minimum value of $2\alpha + \alpha^{-1}$ (achieved at $\alpha = 2^{-1/2}$) is $2\sqrt{2}$. Setting $\alpha = 2^{-1/2}$ in the above definition, we have the required partition. □

The previous lemma gives the key to proving our second theorem.

Proof of Theorem 1.2. Using Lemma 2.2, taking $k = \lfloor n/2 \rfloor$ and the fact that MINVB(L) $\leqslant$ MINEB(L), we immediately get equations (1.1) and (1.2).

In order to prove (1.3), first suppose we have $n = 2^k$ for an integer k. The proof is based on recursive bisection, with the cut size guaranteed by Lemma 2.2. Let $f(k)$ denote the maximum MINCW cost of all lattice graphs with 2^k vertices. Then $f(k)$ satisfies the

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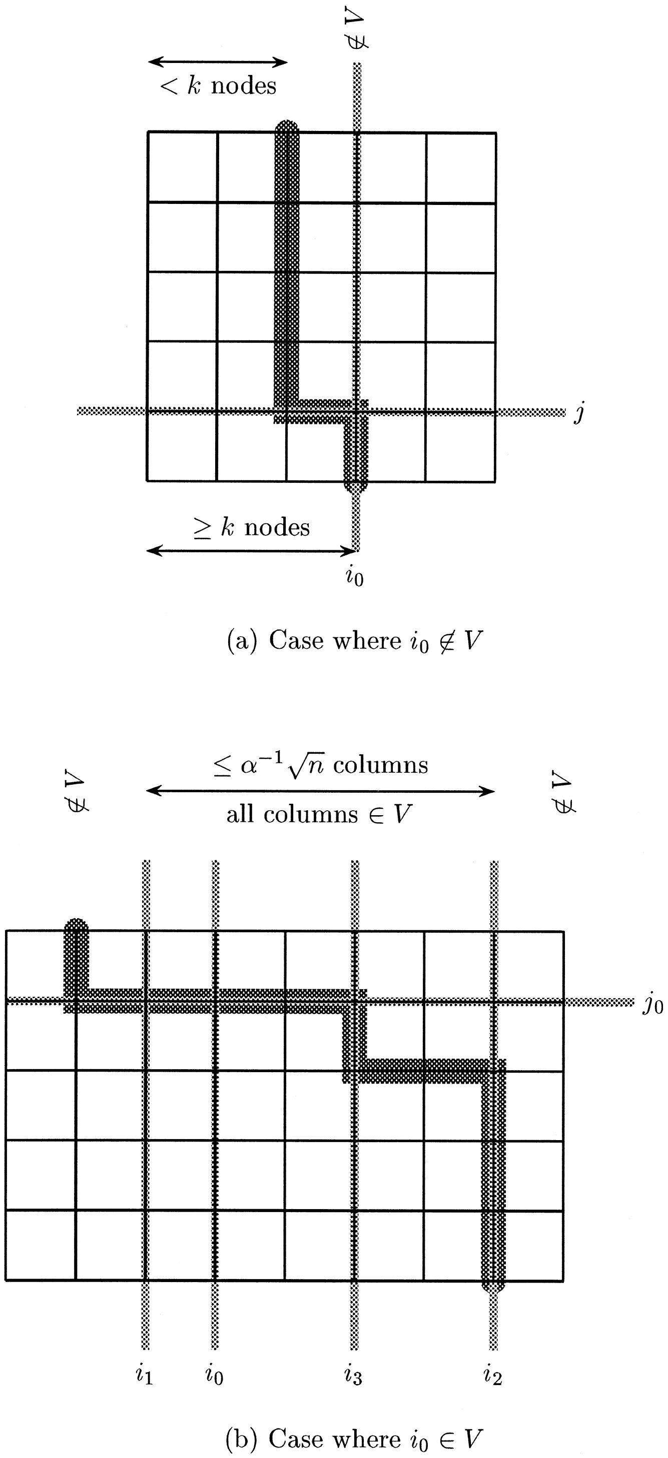

Figure 2 Illustration of the proof of Theorem 2.2.
The thick line marks the potential nodes in the cut

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following recurrence:

$$f(k) \leqslant \begin{cases} 0, & \text{if } k = 0, \\ 2^{3/2}2^{k/2} + 1 + f(k-1), & \text{otherwise.} \end{cases}$$

Then, solving the recurrence, we get

$$f(k) \leqslant \sum_{j=1}^{k} (2^{3/2}2^{j/2} + 1) = 4(2^{1/2} + 1)(2^{k/2} - 1) + k.$$

We can drop the assumption that $n = 2^k$, by taking k so that $n \leqslant 2^k < 2n$, and adding extra points until one has a set of size 2^k. By monotonicity this process does not reduce the MINCW cost, so

$$\begin{aligned} \text{MINCW}(L) &\leqslant 2^{5/2}(2^{1/2} + 1)\sqrt{n} + (\log_2(n) + 1) - 4(2^{1/2} + 1) \\ &\leqslant 13.657\sqrt{n} + \log_2(n) - 8. \end{aligned}$$

But notice that for any $x > 0$ we have $(\log_2(x) - 8)/\sqrt{x} < 0.067$; therefore the above bound for $\text{MINCW}(L)$ is at most $14\sqrt{n}$ for all n.

In order to prove equations (1.4), (1.5) and (1.6), observe that, for any graph G, it holds that $\text{MINLA}(G) \leqslant n \cdot \text{MINCW}(G)$, $\text{MINSC}(G) \leqslant n \cdot \text{MINVS}(G)$, and $\text{MINVS}(G) \leqslant \text{MINCW}(G)$. □

3. Random lattice graphs

Let us describe some basic concepts of site percolation for a lattice L with vertex set $V_m = \{0, \ldots, m-1\}^2$. Given $p \in (0, 1)$, *site percolation* with parameter p on L is obtained by taking a random set of *open* vertices of V_m with each vertex being open with probability p independently of the others. Let $\mathscr{L}_{m,p}$ be the subgraph of L obtained by taking all edges between open vertices. We say that $\mathscr{L}_{m,p}$ is a *random lattice graph*. Denote by $\mathbf{Pr}_p$ and $\mathbf{E}_p$ the probability and expectation with respect to the described process of site percolation with parameter p. By a *cluster* we mean the set of vertices in any connected component of $\mathscr{L}_{m,p}$. Let $\widetilde{C}_0$ denote the cluster in $\mathscr{L}_{m,p}$ that includes $(0, 0)$ (possibly the empty set) and let $\widetilde{C}_x$ denote the cluster in $\mathscr{L}_{m,p}$ that includes the point $x \in V_m$.

A similar site percolation process can be generated analogously on the infinite lattice with vertex set $\mathbb{Z}^2$ and edges between nearest neighbours. In the same way we can extend $\mathbf{Pr}_p$ and $\mathbf{E}_p$ to this infinite process. Let C_0 denote the cluster in $\mathscr{L}_{m,p}$ that includes $(0, 0)$ (possibly the empty set) and let C_x denote the cluster in $\mathscr{L}_{m,p}$ that includes the point x, both for site percolation on $\mathbb{Z}^2$. Notice that we can view the random lattice graph $\mathscr{L}_{m,p}$ as generated by a site percolation process on $\mathbb{Z}^2$ and taking the open vertices in V_m.

All through this section we consider random lattice graphs generated by *subcritical* limiting regimes ($p < p_c$), in which all clusters in the infinite process are almost surely finite. We begin by proving that, with high probability, the optimal costs for the CutWidth and VertSep problems on the subcritical percolation process lattice $\mathscr{L}_{m,p}$ are $\Theta(\sqrt{\log m})$.

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Proof of Theorem 1.3. Recall that, for any graph G, $\textsc{minvs}(G) \leqslant \textsc{mincw}(G)$. The $\textsc{mincw}$ cost of a disconnected graph is the maximum of the $\textsc{mincw}$ costs of its connected components. Hence, for any positive constant c_2,

$$\mathbf{Pr}\left[\textsc{mincw}(\mathscr{L}_{m,p}) \geqslant c_2\sqrt{\log m}\right] = \mathbf{Pr}\left[\cup_{x\in V_m}\left\{\textsc{mincw}(\widetilde{C}_x) \geqslant c_2\sqrt{\log m}\right\}\right].$$

By the site percolation version of equation 5.7 in [14], there exists a constant $\alpha > 0$ such that $\mathbf{Pr}\left[|C_0| \geqslant k\right] \leqslant e^{-\alpha k}$. Therefore, by equation (1.3) in Theorem 1.2,

$$\begin{aligned}\mathbf{Pr}\left[\textsc{mincw}(\mathscr{L}_{m,p}) \geqslant c_2\sqrt{\log m}\right] &\leqslant \mathbf{Pr}\left[\cup_{x\in V_m}\{|\widetilde{C}_x| \geqslant (c_2/14)^2\log m\}\right]\\ &\leqslant m^2\exp(-\alpha(c_2/14)^2\log m).\end{aligned}$$

Choosing $c_2 > 14\sqrt{2/\alpha}$ we get $\mathbf{Pr}\left[\textsc{mincw}(\mathscr{L}_{m,p}) \geqslant c_2\sqrt{\log m}\right] \to 0$.

To get a lower bound for $\textsc{minvs}(\mathscr{L}_{m,p})$, let $\delta > 0$ and let $T_1, \ldots, T_{j(m)}$ be disjoint lattice subsquares of L_m, each of side $\lfloor(\delta\log m)^{1/2}\rfloor$, where $j(m) = \lfloor m/\lfloor(\delta\log m)^{1/2}\rfloor\rfloor^2$. Set $\gamma = \log(1/p)$ so that $p = e^{-\gamma}$. Let A_j be the event that all sites in T_j are open. Then

$$\mathbf{Pr}\left[A_j\right] = \exp(-\gamma\lfloor(\delta\log m)^{1/2}\rfloor^2) \geqslant m^{-\gamma\delta}.$$

Hence, $\mathbf{Pr}\left[\cap_{i=1}^{j(m)}A_i^c\right] \leqslant (1-m^{-\gamma\delta})^{j(m)} \leqslant \exp(-m^{-\gamma\delta}j(m))$, which tends to zero provided δ is chosen so that $\gamma\delta < 2$. As $\textsc{minvs}(L_m) = m$, by Theorem 1.1, we get

$$\cup_{i=1}^{j(m)}A_j \subset \{\textsc{minvs}(\mathscr{L}_{m,p}) \geqslant (\delta\log m)^{1/2}\}.$$

Taking $c_1 = \sqrt{\delta}$ we obtain the lower bound. □

In the next lemma we prove that for subcritical site percolation with parameter p, the expected ratio between $\textsc{minla}(C_0)$ and $|C_0|$ is finite. We also give a similar result for the $\textsc{minsc}$ cost. To cover the case $C_0 = \emptyset$, we use the convention $0/0 = 0$, throughout the remainder of the paper.

Lemma 3.1. *For any $p \in (0, p_c)$,*

$$\mathbf{E}_p\left[\frac{\textsc{minla}(C_0)}{|C_0|}\right] \in (0,\infty) \quad \textit{and} \quad \mathbf{E}_p\left[\frac{\textsc{minsc}(C_0)}{|C_0|}\right] \in (0,\infty).$$

Proof. Let $R_0 = \min\{k : C_0 \subset [-k,k]^2\}$; then, by considering the lexicographic ordering of vertices one sees that $\textsc{minsc}(L_m) \leqslant m^3$ and $\textsc{minla}(L_m) \leqslant m^3$, which together with monotonicity gives us that $\textsc{minsc}(C_0) \leqslant (2R_0+1)^3$ and $\textsc{minla}(C_0) \leqslant (2R_0+1)^3$. The statement of the lemma follows from the fact that $\mathbf{Pr}_p[R_0 > k]$ decays exponentially in k (again, see [14, Chapter 5]). □

We use the previous lemma to state one of our main results on random lattice graphs, namely that the values of $\textsc{minla}$ and $\textsc{minsc}$, divided by m^2, converge in probability to a constant.

https://doi.org/10.1017/S0963548300004454 Published online by Cambridge University Press

Proof of Theorem 1.4. Consider $\mathscr{L}_{m,p}$ as being embedded in a site percolation process on the infinite lattice $\mathbb{Z}^2$. Then,

$$\begin{aligned}\frac{\textsc{minla}(\mathscr{L}_{m,p})}{m^2} &= m^{-2}\sum_{x\in V_m}\frac{\textsc{minla}(\widetilde{C}_x)}{|\widetilde{C}_x|}\\ &= m^{-2}\sum_{x\in V_m}\frac{\textsc{minla}(C_x)}{|C_x|}+m^{-2}\sum_{x\in V_m}\left(\frac{\textsc{minla}(\widetilde{C}_x)}{|\widetilde{C}_x|}-\frac{\textsc{minla}(C_x)}{|C_x|}\right). \end{aligned}\tag{3.1}$$

Using the Ergodic Theorem [13, Theorem VII.6.9] and the Kolmogorov zero-one law,

$$m^{-2}\sum_{x\in V_m}\frac{\textsc{minla}(C_x)}{|C_x|}\xrightarrow{\text{Pr}}\mathbf{E}_p\left[\frac{\textsc{minla}(C_0)}{|C_0|}\right].$$

Writing ∂V_m for the set of $x\in V_m$ with lattice neighbours in $\mathbb{Z}^2\backslash V_m$, we get

$$\begin{aligned}m^{-2}\sum_{x\in V_m}\left|\frac{\textsc{minla}(\widetilde{C}_x)}{|\widetilde{C}_x|}-\frac{\textsc{minla}(C_x)}{|C_x|}\right| &\leqslant 2m^{-2}\sum_{x\in V_m, C_x\neq\widetilde{C}_x}\frac{\textsc{minla}(C_x)}{|\widetilde{C}_x|}\\ &\leqslant 2m^{-2}\sum_{y\in\partial V_m}\textsc{minla}(C_y),\end{aligned}$$

By the proof of Lemma 3.1, $\mathbf{E}_p[\textsc{minla}(C_y)]$ is finite and does not depend on y. Hence the mean of the above expression tends to zero. The result for MINLA then follows from (3.1), taking

$$\beta_{\text{LA}}(p)=\mathbf{E}_p\left[\frac{\textsc{minla}(C_0)}{|C_0|}\right].$$

The proof for MINSC is just the same, taking

$$\beta_{\text{SC}}(p)=\mathbf{E}_p\left[\frac{\textsc{minsc}(C_0)}{|C_0|}\right].$$

□

4. Random geometric graphs

Recall that, given a set of points V in the plane and a positive number r, $\mathscr{G}(V;r)$ denotes the geometric graph with vertex set V and radius r. In the following, $(r_i)_{i\geqslant 1}$ is a sequence of positive numbers and $X=(X_i)_{i\geqslant 1}$ is a sequence of independently and uniformly distributed (i.u.d.) random points in $[0,1]^2$. Also, $\mathscr{X}_n$ is the set of the first n points of X.

For an infinite-volume analogue, let $\mathscr{P}_\lambda$ denote a homogeneous Poisson process on $\mathbb{R}^2$ of intensity λ, and set $\mathscr{P}_{\lambda,0}=\mathscr{P}_\lambda\cup\{(0,0)\}$. For large n, after appropriate scaling and centring at a randomly chosen point of $\mathscr{X}_n$, the graph $\mathscr{G}(\mathscr{X}_n;r_n)$ looks locally like $\mathscr{G}(\mathscr{P}_{\lambda,0};1)$. We consider a continuum site percolation process based on the Poisson process; let $\widetilde{\vartheta}(\lambda)$ be the probability that the added point at the origin lies in an infinite component of $\mathscr{G}(\mathscr{P}_{\lambda,0};1)$. Then define the critical intensities $\lambda_c=\inf\{\lambda>0:\widetilde{\vartheta}(\lambda)\geqslant 0\}$ and $\lambda_c'=\inf\{\lambda>0:\widetilde{\vartheta}(\lambda)\geqslant\frac{1}{2}\}$. It is well known (see [14]) that $\lambda_c\in(0,\infty)$. In this section we shall deal with random geometric graphs satisfying the condition $\lim_{n\to\infty}nr_n^2=\lambda$, for the subcritical regime $\lambda<\lambda_c$ (or for bisection problems, $\lambda<\lambda_c'$).

We first deal with the bisection problems EDGEBIS and VERTBIS.

Proof of Theorem 1.5. We need to show that, with high probability, there is a subset W of $\mathscr{X}_n$, of cardinality $\lfloor n/2 \rfloor$, with no edges between W and $\mathscr{X}_n \backslash W$. By hypothesis, $\widetilde{\vartheta}(\lambda) < \frac{1}{2}$.

For $k \in \mathbb{N}$, set $\pi_k = \mathbf{Pr}_\lambda[|C_0| = k]$, and note $\pi_k > 0$. Let $N_n(k)$ denote the number of points of $G(\mathscr{X}_n; r_n)$ lying in clusters of size k.

Let $p_k(x)$ denote the probability that, when adding a point x to a set of $n-1$ uniformly distributed points, the new point will be in a cluster of size k. Then,

$$\mathbf{E}[N_n(k)] = n \int_{[0,1]^2} p_k(x)\, dx.$$

For x not on the boundary of $[0,1]^2$, we claim that

$$p_k(x) \to \pi_k. \tag{4.1}$$

This is because the question of whether x lies in a cluster of size k is determined by the configuration of points within a distance kr_n of x, and, for any bounded set $B \subset \mathbb{R}^2$, the restriction to B of the point process $\{r_n^{-1}(X_i - x) : 1 \leqslant i \leqslant n-1\}$ converges in distribution to the restriction to B of $\mathscr{P}_\lambda$, as a consequence of the standard Poisson approximation to the binomial distribution.

So, by the dominated convergence theorem, $\mathbf{E}[N_n(k)]/n \to \pi_k$. To look at the variance, notice that, since $N_n(k)(N_n(k)-1)$ is twice the number of pairs of points both in clusters of size k, if we denote by $p_{k,k}(x,y)$ the probability that when inserting points at x and y into a set of $n-2$ uniform points they will both be in a cluster of size k, then

$$\mathbf{E}[N_n(k)(N_n(k)-1)] = n(n-1) \int_{[0,1]^2} \int_{[0,1]^2} p_{k,k}(x,y)\, dx\, dy.$$

For points x and y not on the boundary with $x \neq y$, by an argument similar to the proof of (4.1), we have that $p_{k,k}(x,y) \to (\pi_k)^2$, hence using again the dominated convergence theorem $\mathbf{E}[(N_n(k)/n)^2] \to \pi_k^2$. So $\mathbf{Var}\left[N_n(k)/n\right] \to 0$, and by Chebyshev's inequality we can conclude

$$n^{-1} N_n(k) \xrightarrow{\Pr} \pi_k. \tag{4.2}$$

As $1 - \sum_k \pi_k = \widetilde{\vartheta}(\lambda) < 1/2$, we can choose k_1 such that $\sum_{k \leqslant k_1} \pi_k > 1/2$. This inequality together with (4.2) implies that, with probability tending to 1 as n tends to infinity,

$$\sum_{k > k_1} N_n(k) < \left\lfloor \frac{n}{2} \right\rfloor,$$

and $N_n(k)$ are non-zero for $k = 1, 2, \ldots, k_1$.

We generate a subset W of $\mathscr{X}_n$ as follows. First take the union of all clusters of size greater than k_1. Then add clusters of size k_1 until there are none left. Then add clusters of size $k_1 - 1$ until there are none left. Continue in this way. At some point, having just added a set of size i, we will have a set of size $\lfloor \frac{n}{2} \rfloor - m$ with $0 \leqslant m < i$. If $m = 0$, stop. If $m > 0$, then add a cluster of size m and stop. This gives a set $W \subset \mathscr{X}_n$, of size $\lfloor \frac{n}{2} \rfloor$, with no edges connecting W to $\mathscr{X}_n \backslash W$, as desired. □

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Results analogous to those in Theorem 1.5 also hold for a percolation process in the lattice with $p < p'_c$, defined in the same way as λ'_c.

We now present our results for the CutWidth and VertSep problems. In the following, B_m denotes the box $[0, m)^2$. The following lemma establishes lower bounds for the cost of the optimal cutwidth and vertex separation of random geometric graphs in the subcritical regime.

Lemma 4.1. *Suppose $\lambda < \lambda_c$ and that there exists a constant $c > 0$ such that $m_n = c/r_n$ for all n. Then,*

$$\lim_{n\to\infty} \mathbf{Pr}\left[\text{MINVS}(\mathcal{G}(\mathcal{P}_\lambda \cap B_{m_n}; 1)) \geqslant \log n / \log\log n\right] = 1$$

and

$$\lim_{n\to\infty} \mathbf{Pr}\left[\text{MINCW}(\mathcal{G}(\mathcal{P}_\lambda \cap B_{m_n})) \geqslant \tfrac{1}{5}(\log n / \log\log n)^2\right] = 1.$$

Proof. Let us dissect B_{m_n} into boxes of size $v \times v$ where v is chosen to be as big as possible so that all pairs of points in the same box are connected by an edge in $G = \mathcal{G}(\mathcal{P}_\lambda \cap B_{m_n}; 1)$. There are $b_n = \lfloor m_n/v \rfloor^2$ boxes that completely fall in B_{m_n}. For all $1 \leqslant i \leqslant b_n$, let N_i be the number of points of the process that fall in box i. By construction, we have that N_i follows a Poisson distribution with mean $\mu = \lambda v^2$, and that N_i is independent of N_j for all $i \neq j$. Let $M_n = \max_{i=1..b_n} N_i$ be the maximal number of points in some box. Then, $\text{MINVS}(G) \geqslant M_n - 1$ and $\text{MINCW}(G) \geqslant \frac{1}{2} M_n(\frac{1}{2} M_n - 1) \geqslant \frac{1}{5} M_n$, because G contains at least a clique with M_n points.

In the following, we show that some box contains at least $f(n) = \log n / \log\log n$ points with probability tending to one as n tends to infinity. Fix a box i. Then,

$$\mathbf{Pr}\left[N_i > f(n)\right] = \sum_{k > f(n)} \frac{\mu^k e^{-\mu}}{k!} \geqslant e^{-\mu} \frac{\mu^{f(n)+1}}{(f(n)+1)!}.$$

Now consider all the boxes. We have

$$\begin{aligned} \mathbf{Pr}\left[\bigcap_{i=1..b_n} N_i \leqslant f(n)\right] &= \prod_{i=1..b_n} \mathbf{Pr}\left[N_i \leqslant f(n)\right] \\ &\leqslant \left(1 - e^{-\mu} \frac{\mu^{f(n)+1}}{(f(n)+1)!}\right)^{b_n} \leqslant \left(1 - e^{-\mu} \frac{\mu^{f(n)+1}}{(f(n)+1)!}\right)^{4c^2 n/\mu} \end{aligned}$$

and thus the above probability goes to zero as n goes to infinity. □

In order to determine upper bounds of the same order than the lower bounds, we will need two auxiliary results, which may be useful in other situations. Let us define $\text{CW}(\varphi, G) = \max_{i=1..n} \theta(i, \varphi, G)$ and $\text{VS}(\varphi, G) = \max_{i=1..n} \delta(i, \varphi, G)$ for a graph G and a layout φ of G.

Definition 1. Given a graph $G = (V, E)$ and an integer s, let us define the *s-explosion of* G as the graph $G' = (V', E')$ where $V' = \{(u, i) \mid u \in V,\ 1 \leqslant i \leqslant s\}$ and $E' = \{(u, i)(u, j) \mid u \in V,\ 1 \leqslant i < j \leqslant s\} \cup \{(u, i)(v, j) \mid uv \in E,\ 1 \leqslant i \leqslant j \leqslant s\}$.

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Lemma 4.2. *Let $G' = (V', E')$ be the s-explosion of an arbitrary graph $G = (V, E)$. Then,* $\text{MINCW}(G') \leqslant s^2(\text{MINCW}(G) + 1)$.

Proof. Let φ be a layout of G with minimal cutwidth. Let φ' be a layout of G' defined by $\varphi'((u, i)) = (\varphi(u) - 1)s + i$ for all $u \in V$ and all $1 \leqslant i \leqslant s$. In plain words, the exploded nodes are placed in φ' in the same relative order that their 'parents' are in φ.

Let i be an integer such that $1 \leqslant i \leqslant |V|$. Then we have $\theta(si, \varphi', G') = s^2\theta(i, \varphi, G)$ because each original edge from E corresponds to s^2 edges in E'. Moreover, for all $1 \leqslant j < s$, we have $\theta(si + j, \varphi', G') \leqslant \theta(i + 1, \varphi, G) \cdot s^2 + s^2$ because we have to count s^2 edges in E' for each edge in E plus the edges inside a clique of size s. As a consequence,

$$\text{MINCW}(G') \leqslant \text{CW}(G', \varphi') \leqslant s^2(\text{CW}(G, \varphi) + 1) = s^2(\text{MINCW}(G) + 1),$$

which proves the stated result. □

Definition 2. Given two integers m and l, let us define an (m, l)-*mesh* as the graph $G = (V, E)$ with vertex set $V_m = \{0, \dots, m - 1\}^2$ and edge set $E = \{uv \mid u, v \in V, \ \|u - v\|_\infty \leqslant l\}$.

Lemma 4.3. *Given an integer $l > 0$, there exists a constant $\kappa_l > 0$ such that, for any node-induced subgraph L of an (m, l)-mesh with n nodes,* $\text{MINVS}(L) \leqslant \text{MINCW}(L) \leqslant \kappa_l\sqrt{n}$.

Proof. First assume that $l = 1$. An argument similar to the proof of equation (1.3) in Theorem 1.2 shows that $\text{MINCW}(L) \leqslant \kappa_1\sqrt{n}$ for some constant $\kappa_1 > 0$. Since, for any graph, its minimal vertex separation can never be greater than its minimal cutwidth, Lemma 4.3 is proved for the case $l = 1$.

Assume now that $l > 1$. Let $m' = \lceil m/l \rceil$ and let H be a node-induced subgraph of the $(m', 1)$-mesh where a potential node u with coordinates x_u and y_u belongs to H if and only if some node v from L with coordinates x_v and y_v satisfies $\lfloor x_v/l \rfloor = x_u$ and $\lfloor y_v/l \rfloor = y_u$. As H is an $(m', 1)$-mesh and $|H| \leqslant |L| = n$, we have $\text{MINCW}(H) \leqslant \kappa_1\sqrt{n}$. Construct a graph G by s-exploding H with $s = l^2$. By Lemma 4.2, $\text{MINCW}(G) \leqslant l^4(\kappa_1\sqrt{n} + 1)$. Now observe that, by construction, L is a subgraph of G! As a consequence,

$$\text{MINVS}(L) \leqslant \text{MINCW}(L) \leqslant \text{MINCW}(G) \leqslant \kappa_l\sqrt{n}$$

where $\kappa_l = l^4(\kappa_1 + 1)$. □

We are now ready to find upper bounds for the Vertex Separation and the Cutwidth problems.

Lemma 4.4. *Suppose $\lambda < \lambda_c$ and $m_n = c/r_n$ for some constant $c > 0$ and all integer n. Then there exist constants $\gamma_1 > 0$ and $\gamma_2 > 0$ such that*

$$\lim_{n\to\infty} \mathbf{Pr}\left[\text{MINVS}(\mathscr{G}(\mathscr{P}_\lambda \cap B_{m_n}; 1)) \leqslant \gamma_1 \log n / \log\log n\right] = 1$$

and

$$\lim_{n\to\infty} \mathbf{Pr}\left[\text{MINCW}(\mathscr{G}(\mathscr{P}_\lambda \cap B_{m_n}; 1)) \leqslant \gamma_2 (\log n / \log\log n)^2\right] = 1.$$

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Proof. We only give the proof for the Vertex Separation problem, as the proof for the Cutwidth result is similar.

Let us dissect B_{m_n} in boxes of size $v \times v$, where v is a (small) constant that will be determined later. There is a total number of $b_n = \lceil m_n/v \rceil^2$ boxes, which we will consider. Let $H = H(P_\lambda)$ be the node-induced subgraph of the $(\lceil m_n/v \rceil), 1/v)$-mesh whose nodes are present if and only if their corresponding box contains at least one point in $\mathscr{P}_\lambda$. Let us denote $\mathscr{G}(\mathscr{P}_\lambda \cap B_{m_n}; 1)$ by G.

Our proof consists of three steps; let us first outline them. In Step 1, we prove that, with high probability, H does not contain 'too big' connected components. Assuming that H does not contain any big connected component, in Step 2, we upper-bound $\textsc{minvs}(H)$ by a certain function $f(n)$, and show that, if ψ is a layout such that $\textsc{vs}(\psi, H) \leqslant f(n)$, then $\textsc{minvs}(G)$ is bounded by M_n, a magnitude related to the number of points of $\mathscr{P}_\lambda$ in $f(n)$ boxes and to the layout ψ. Finally, in Step 3 we prove that, with high probability, $M_n \leqslant v \log n / \log\log n$. The combination of these steps will imply the proposed result.

Step 1. For all $i = 1, \ldots, b_n$, let N_i be the number of points of $\mathscr{P}_\lambda$ that fall in box i. By construction, if box i completely falls in B_{m_n} square, we have that N_i follows a Poisson distribution with mean $\mu = \lambda v^2$; otherwise N_i is bounded by a Poisson distribution with mean μ^2. Moreover, for all $i \neq j$, the variables N_i and N_j are independent.

We identify each box with a node of an infinite lattice with connections between boxes whose centres lie at distance not greater than $1/v$. We define on this lattice a percolation process where each node is deemed open if its corresponding box contains at least one point from $\mathscr{P}_\lambda$. Observe that our lattice is a 'general lattice' in the sense of Kesten [20, pp. 10–12] and Grimmett [14, p. 349]. Therefore, this percolation process must exhibit a phase transition at some critical probability, and a sufficiently small value of v sets us in the subcritical phase [29, Lemma 2]. As a consequence [14], there is an exponential decay in the size of a connected component C: for some constant α and all integer m,

$$\mathbf{Pr}\left[|C| > m\right] \leqslant \exp(-\alpha m).$$

Let $c_1 = 3/\alpha$. The probability that some connected component of H has size bigger than $c_1 \log n$ is

$$\mathbf{Pr}\left[\bigcup_{x \in V(H)} \{|C_x| > c_1 \log n\}\right] \leqslant b_n \cdot \exp(-\alpha c_1 \log n) \leqslant 4n^{-2}/\mu \to 0.$$

Step 2. We now suppose that no component of H has more than $c_1 \log n$ points. By Lemma 4.3, this implies that, for some constant $\kappa > 0$, $\textsc{minvs}(H) \leqslant \kappa\sqrt{\log n}$.

Set $f(n) = \kappa\sqrt{\log n}$ and let ψ be a layout of H with $\textsc{vs}(\psi, G) \leqslant f(n)$. We use the layout ψ as the basis to build a layout φ of the points in $\mathscr{P}_\lambda \cap B$, taking the ordering of points within multiply occupied boxes in an arbitrary way. As $f(n)$ boxes separate any two connected boxes of H in ψ, we have that $\textsc{vs}(G, \varphi) \leqslant M_n$, where M_n is defined as the maximum number of points in $\mathscr{P}_\lambda$ that belong to the union of any $f(n)$ consecutive boxes according to the layout ψ.

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Step 3. In the following we show that, with high probability, $f(n)$ consecutive boxes contain at most $g(n) = v \log n / \log\log n$ points.

The number of points in an open box follows a Poisson distribution with parameter μ, conditioned to be at least one. The reader can easily check that this is stochastically dominated by an unconditioned Poisson distribution with parameter μ plus one unit. Also, the number of points in $f(n)$ boxes follows the sum of $f(n)$ independent Poisson distributions, each one with parameter μ and conditioned to be at least one. This is stochastically dominated by $f(n)$ plus the sum of $f(n)$ unconditioned Poisson distributions with parameter μ, which is the same as the variable

$$S = f(n) + \mathscr{P}(f(n)\mu) = \kappa\sqrt{\log n} + \mathscr{P}\left(\mu\kappa\sqrt{\log n}\right).$$

Since each connected component of H contains at most $c_1 \log n$ boxes, and H contains at most b_n connected components, the probability that any $f(n)$ consecutive boxes contain more than $g(n)$ points is at most

$$c_1 \log n \cdot b_n \cdot \mathbf{Pr}\left[S \geqslant g(n)\right].$$

In order to prove that the previous probability goes to zero as n goes to infinity, let X be some random variable following a Poisson distribution with mean μ_n to be defined later, and let α_n be a sequence to be given later such that $\mu_n = \omega(\alpha_n)$ and that $\mu_n = \omega(1)$. Then, we claim that

$$\mathbf{Pr}\left[X \geqslant \alpha_n\right] \leqslant e^{-\mu_n}\mu_n^{\alpha_n}e^{\alpha_n}\alpha_n^{-\alpha_n}$$

and so

$$\log \mathbf{Pr}\left[X \geqslant \alpha_n\right] \leqslant -\mu_n + \alpha_n \log \mu_n + \alpha_n - \alpha_n \log \alpha_n. \tag{4.3}$$

Take X as a Poisson distribution with mean $\mu_n = \mu\kappa\sqrt{\log n}$ and take

$$\alpha_n = \gamma_1 \log n / \log\log n - \kappa\sqrt{\log n}$$

for some constant γ_1. Let $\gamma_1' = \frac{3}{4}\gamma_1$ so that

$$\gamma_1' \log n / \log\log n < \alpha_n < \gamma_1 \log n / \log\log n.$$

Substituting in the right-hand side of (4.3) and taking exponentials, we find

$$\mathbf{Pr}\left[X \geqslant \alpha_n\right] \geqslant n^{-\gamma_1/4}$$

and thus

$$c_1 \log n \cdot b_n \cdot \mathbf{Pr}\left[X \geqslant \alpha_n\right] \to 0$$

provided that γ_1 is chosen to be big enough. □

We can now finally prove Theorem 1.6.

Proof of Theorem 1.6. We first couple $\mathscr{X}_n$ to a Poisson process with a slightly lower density of points, as follows. Take $\lambda_1 \in (0, \lambda)$ and set $m_n = \lceil r_n^{-1} \rceil$. Let M_n be a Poisson variable with mean $\lambda_1 m_n^2$. Then, $\lim_{n\to\infty} \mathbf{Pr}\left[M_n > n\right] = 0$. Let us set $m_n\mathscr{X}_n = \{m_n X_i : 1 \leqslant i \leqslant n\}$, $\mathscr{P} = \{m_n X_i : 1 \leqslant i \leqslant M_n\}$. Notice that $\mathscr{P}$ is a Poisson process on B_{m_n} with intensity λ_1. If

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$M_n \leqslant n$ then $\mathscr{G}(\mathscr{P};1)$ is a subgraph of $\mathscr{G}(m_n\mathscr{X}_n;m_nr_n)$, which is isomorphic to $\mathscr{G}(\mathscr{X}_n;r_n)$. By monotonicity,

$$\lim_{n\to\infty} \mathbf{Pr}\left[\mathrm{vs}(\mathscr{G}(\mathscr{X}_n;r_n)) \geqslant \mathrm{vs}(\mathscr{G}(\mathscr{P};1))\right] = 1.$$

As a consequence, by Lemma 4.1, we have

$$\lim_{n\to\infty} \mathbf{Pr}\left[\mathrm{vs}(\mathscr{G}(\mathscr{X}_n;r_n)) \geqslant \log n/\log\log n\right] = 1. \tag{4.4}$$

We now couple $\mathscr{X}_n$ to a Poisson process with a slightly higher density of points. Take $\lambda_2 \in (\lambda, \lambda_c)$ and set $m'_n = \lfloor r_n^{-1} \rfloor$. Let M'_n be a Poisson variable with mean $\lambda_2(m'_n)^2$. Then, $\lim_{n\to\infty} \mathbf{Pr}\left[M'_n < n\right] = 0$. Let $\mathscr{P}' = \{m'_nX_i : 1 \leqslant i \leqslant M'_n\}$. Notice that $\mathscr{P}'$ is a Poisson process on $B_{m'_n}$ with intensity λ_2. If $M'_n \leqslant n$ then $\mathscr{G}(\mathscr{P};1)$ is a super-graph of $\mathscr{G}(m_n\mathscr{X}_n;m_nr_n)$, which is isomorphic to $\mathscr{G}(\mathscr{X}_n;r_n)$. By monotonicity,

$$\lim_{n\to\infty} \mathbf{Pr}\left[\mathrm{vs}(\mathscr{G}(\mathscr{X}_n;r_n)) \leqslant \mathrm{vs}(\mathscr{G}(\mathscr{P}';1))\right] = 1.$$

As a consequence, by Lemma 4.4, we have

$$\lim_{n\to\infty} \mathbf{Pr}\left[\mathrm{vs}(\mathscr{G}(\mathscr{X}_n;r_n)) \leqslant \gamma_1 \log n/\log\log n\right] = 1. \tag{4.5}$$

Combining equations (4.4) and (4.5) we obtain

$$\lim_{n\to\infty} \mathbf{Pr}\left[c_3 \leqslant \frac{\textsc{minvs}(\mathscr{G}(\mathscr{X}_n;r_n))}{\log n/\log\log n} \leqslant c_4\right] = 1$$

for the suitable constants c_3 and c_4.

The proof for the Cutwidth result is similar. □

We now turn to the convergence results for the MinLA and SumCut problems. We shall first prove that, in the subcritical case, the expected values of MINLA and MINSC on the induced graph on C_0 are finite.

Proposition 4.5. *Let $\lambda < \lambda_c$. Then*

$$\mathbf{E}_\lambda[\textsc{minla}(C_0)] \in (0,\infty) \quad \textit{and} \quad \mathbf{E}_\lambda[\textsc{minsc}(C_0)] \in (0,\infty).$$

Proof. Recall that, for any graph G with n nodes, $\textsc{minla}(G) \leqslant n^3$ and $\textsc{minsc}(G) \leqslant n^3$. Hence, to prove the statement it is enough to show that $\mathbf{E}_\lambda[|C_0|^3] < \infty$. To show this, let $B(r)$ be the ball of radius r centred at the origin and let

$$\mathscr{P}_{\lambda,0}(B(r)) = |\{x \text{ of } \mathscr{P}_{\lambda,0} \mid x \in B(r)\}|.$$

Then, for any $m > 0$, the event $\{|C_0| \geqslant m^{1/3}\}$ is contained in the union of the events $\{\mathscr{P}_{\lambda,0}(B(m^{1/(6d)}) \geqslant m^{1/3}\}$ and $\{\mathrm{diam}(C_0) \geqslant m^{1/(6d)}\}$; therefore, using Boole's inequality, we get

$$\mathbf{Pr}\left[|C_0| \geqslant m^{1/3}\right] \leqslant \mathbf{Pr}\left[\mathscr{P}_{\lambda,0}(B(m^{1/(6d)})) \geqslant m^{1/3}\right] + \mathbf{Pr}\left[\mathrm{diam}(C_0) \geqslant m^{1/(6d)}\right].$$

The first term in the right-hand side is summable in m by standard estimates of the

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Poisson distribution. The second term is summable in m by Lemma 2 in [29]. Hence

$$\sum_{m\geqslant 1} \mathbf{Pr}\left[|C_0|^3 \geqslant m\right] < \infty$$

and the statement follows. □

Next we prove a technical lemma that will be needed later.

Lemma 4.6. *The functions* $\lambda \mapsto \mathbf{E}_\lambda\left[\frac{\text{MINSC}(C_0)}{|C_0|}\right]$ *and* $\lambda \mapsto \mathbf{E}_\lambda\left[\frac{\text{MINLA}(C_0)}{|C_0|}\right]$ *are continuous in* λ *on* $(0, \lambda_c)$.

Proof. We give the proof for the MINLA case; the proof for the MINSC is similar. Define coupled versions of the Poisson process $\mathscr{P}_\lambda, \lambda > 0$, in the following standard way. Let $\mathscr{P}$ be a Poisson process on $\mathbb{R}^2 \times [0, \infty)$ of rate 1, and let $\mathscr{P}_\lambda$ consist of the projections onto the 2 coordinates of the points of $\mathscr{P} \cap (\mathbb{R}^2 \times [0, \lambda])$. Using this coupling, write $C_0(\lambda)$ for the component including the origin of $C(\mathscr{P}_\lambda \cup \{0\}; 1)$.

Suppose (λ_n) is a sequence with $\lambda_n \to \lambda \in (0, \lambda_c)$. With this coupling, with probability one it is the case that, for all sufficiently large n, the components $C_0(\lambda_n)$ and $C_0(\lambda)$ are identical. Hence, by the dominated convergence theorem,

$$\mathbf{E}_{\lambda n}[\text{MINLA}(C_0(\lambda_n))/|C_0(\lambda_n)|] \to \mathbf{E}_\lambda[\text{MINLA}(C_0(\lambda))/|C_0(\lambda)|].$$ □

We now give asymptotics for the MINLA and MINSC costs of the graphs $\mathscr{G}(\mathscr{P}_\lambda \cap B_m; 1)$.

Proposition 4.7. *Suppose* $\lambda < \lambda_c$*, and let* $\mathscr{G}_m = \mathscr{G}(\mathscr{P}_\lambda \cap B_m; 1)$*. Then, as* $m \to \infty$,

$$\frac{\text{MINLA}(\mathscr{G}_m)}{m^2} \xrightarrow{\text{Pr}} \lambda \mathbf{E}_\lambda\left[\frac{\text{MINLA}(C_0)}{|C_0|}\right] \quad \text{and} \quad \frac{\text{MINSC}(\mathscr{G}_m)}{m^2} \xrightarrow{\text{Pr}} \lambda \mathbf{E}_\lambda\left[\frac{\text{MINSC}(C_0)}{|C_0|}\right].$$

Proof. We sketch a proof for MINLA. For each point x of $\mathscr{P}_\lambda \cap B_m$, let C_x denote the component of $\mathscr{G}(\mathscr{P}_\lambda \cap B_m; 1)$ that includes the point x, and let $\widetilde{C}_x$ denote the component of $\mathscr{G}(\mathscr{P}_\lambda; 1)$ that includes the point x. By an argument similar to the proof of Theorem 1.5, it suffices to prove that

$$\mathbf{E}_\lambda\left[m^{-d} \sum_{x \in \mathscr{P}_\lambda \cap B_m} \left|\frac{\text{MINLA}(\widetilde{C}_x)}{|\widetilde{C}_x|} - \frac{\text{MINLA}(C_x)}{|C_x|}\right|\right] \to 0. \tag{4.6}$$

For $l > 0$, let $\partial_l B_m$ be the set of points $z \in B_m$ with $\|z - y\|_\infty \leqslant l$ for some $y \notin B_m$. The quantity inside the sum in (4.6) is at most $\text{MINLA}(\widetilde{C}_x) \cdot (C_x \neq \widetilde{C}_x)$, where, for any statement S, (S) stands for 1 if S is true, 0 otherwise. Hence the random variable inside the expectation in (4.6) is at most

$$\left(m^{-d} \sum_{x \in \mathscr{P}_\lambda \cap \partial_l B_m} \text{MINLA}(\widetilde{C}_x)\right) + \left(m^{-d} \sum_{x \in B_m \setminus \partial_l B_m} \text{MINLA}(\widetilde{C}_x) \cdot (\text{diam}(\widetilde{C}_x) > l)\right).$$

The expectation of the first term tends to zero, while the expectation of the second term equals $\lambda \mathbf{E}_\lambda[\text{MINLA}(C_0)(|C_0| > l)]$, which can be made arbitrarily small by the choice of l. Then (4.6) follows. □

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We can now finally prove Theorem 1.7.

Proof of Theorem 1.7. The same coupling that in the proof of Theorem 1.6 shows that

$$\mathbf{Pr}\left[\textsc{minla}(\mathscr{G}(\mathscr{P}_n;1)) \leqslant \textsc{minla}(\mathscr{G}(\mathscr{X}_n;r_n)) \leqslant \textsc{minla}(\mathscr{G}(\mathscr{P}'_n;1))\right] \to 1.$$

By Proposition 4.7,

$$\frac{\textsc{minla}(\mathscr{G}(\mathscr{P}_n;1))}{m_n^2} \xrightarrow{\text{Pr}} \lambda_1 \mathbf{E}_{\lambda_1}\left[\frac{\textsc{minla}(C_0)}{|C_0|}\right],$$

so that

$$\frac{\textsc{minla}(\mathscr{G}(\mathscr{P}_n;1))}{n} \xrightarrow{\text{Pr}} \left(\frac{\lambda_1}{\lambda}\right) \mathbf{E}_{\lambda_1}\left[\frac{\textsc{minla}(C_0)}{|C_0|}\right].$$

Similarly,

$$\frac{\textsc{minla}(\mathscr{G}(\mathscr{P}'_n;1))}{n} \xrightarrow{\text{Pr}} \left(\frac{\lambda_2}{\lambda}\right) \mathbf{E}_{\lambda_2}\left[\frac{\textsc{minla}(C_0)}{|C_0|}\right].$$

Taking $\lambda_1 \uparrow \lambda$ and $\lambda_2 \downarrow \lambda$ and using Lemma 4.6,

$$\frac{\textsc{minla}(\mathscr{G}(\mathscr{X}_n;r_n))}{n} \xrightarrow{\text{Pr}} \mathbf{E}_{\lambda}\left[\frac{\textsc{minla}(C_0)}{|C_0|}\right].$$

The proof for the convergence of MINSC is analogous. □

5. Conclusions

In this paper we have considered several layout problems for specific classes of sparse graphs: lattice graphs, random lattice graphs and random geometric graphs. We have first identified the optimal solutions for the Vertex Separation, Sum Cut, and Vertex Bisection problems for the complete square lattice, and presented several worst-case upper bounds for arbitrary lattices for all the addressed problems. We have then determined the order of magnitude of the Vertex Separation and Cutwidth of random lattice graphs generated by selecting at random nodes of an $m \times m$ complete lattice with probability $p < p_c$. For the Minimum Linear Arrangement and the Sum Cut problems, we have even been able to prove that their optimal cost divided by m^2 converges to a particular constant. Finally, we have considered random geometric graphs in the subcritical regime. We have shown that the Minimum Linear Arrangement and the Sum Cut measures divided by the number of nodes of the graph converge to another particular constant. For the Vertex Separation and Cutwidth of random geometric graphs, we have identified their order of magnitude. We also have shown that one may obtain an empty bisection of random geometric graphs and random lattice graphs (both in the subcritical case). Table 1 summarizes the obtained results.

For the sake of clarity, we have contented ourselves in this paper with demonstrating convergence in probability; however, the convergence in our theorems actually holds in the stronger sense of *complete convergence*, which implies convergence almost surely (see [37]). In all the cases, our results are given for the bidimensional space; the extension to higher dimensions remains open.

Our results show that the optimal costs for all problems are different in the supercritical

F	L_m	L	$\mathscr{L}_{m,p}$	$\mathscr{G}(\mathscr{X}_n;r_n)$
MINEB	$F = m + (\text{odd } m)$ [22]	$F \leqslant 2\sqrt{2n}+1$	$F \xrightarrow{\text{Pr}} 0$	$F \xrightarrow{\text{Pr}} 0$
MINVB	$F = m$	$F \leqslant 2\sqrt{2n}+1$	$F \xrightarrow{\text{Pr}} 0$	$F \xrightarrow{\text{Pr}} 0$
MINCW	$F = m + (\text{odd } m)$ [22]	$F \leqslant 14\sqrt{n}$	$c_1 \leqslant \frac{F}{\sqrt{\log m}} \leqslant c_2$ w.h.p.	$c_3 \leqslant \frac{F}{(\log n/\log\log n)^2} \leqslant c_4$ w.h.p.
MINVS	$F = m$	$F \leqslant 14\sqrt{n}$	$c_1 \leqslant \frac{F}{\sqrt{\log m}} \leqslant c_2$ w.h.p.	$c_5 \leqslant \frac{F}{\log n/\log\log n} \leqslant c_6$ w.h.p.
MINLA	$F = (4-\sqrt{2})m^3/3 + o(m^3)$ [24, 25]	$F \leqslant 14n\sqrt{n}$	$F/m^2 \xrightarrow{\text{Pr}} \beta_{\text{LA}}(p)$	$F/n \xrightarrow{\text{Pr}} \tilde{\beta}_{\text{LA}}(\lambda)$
MINSC	$F = 2m^3/3 + o(m^3)$	$F \leqslant 14n\sqrt{n}$	$F/m^2 \xrightarrow{\text{Pr}} \beta_{\text{SC}}(p)$	$F/n \xrightarrow{\text{Pr}} \tilde{\beta}_{\text{SC}}(\lambda)$

Table 1 Summary of results for full $m \times m$ square lattice graphs (L_m), general lattice graphs with n nodes (L), random lattice graphs in $m \times m$ lattices with probability p ($\mathscr{L}_{m,p}$), and random geometric graphs with n nodes and radius r_n where $\lim nr_n^d = \lambda$ ($\mathscr{G}(\mathscr{X}_n;r_n)$). In the first two rows we require that $p < p_c'$ and $\lambda < \lambda_c'$. For the remaining rows $p < p_c$ and $\lambda < \lambda_c$

https://doi.org/10.1017/S0963548300004454 Published online by Cambridge University Press

and subcritical phases. Let us recall the order of magnitudes in the supercritical case [30]. For random lattice graphs $\mathscr{L}_{m,p}$ with $p > p_c$,

$$
\begin{aligned}
\mathbf{Pr}\left[\text{MINLA}(\mathscr{L}_{m,p}) = \Theta(m^3)\right] &= 1,\\
\mathbf{Pr}\left[\text{MINCW}(\mathscr{L}_{m,p}) = \Theta(m)\right] &= 1,\\
\mathbf{Pr}\left[\text{MINSC}(\mathscr{L}_{m,p}) = \Theta(m^3)\right] &= 1,\\
\mathbf{Pr}\left[\text{MINVS}(\mathscr{L}_{m,p}) = \Theta(m)\right] &= 1.
\end{aligned}
$$

For random geometric graphs $\mathscr{G}(\mathscr{X}_n; r_n)$ with $\lim_{n\to\infty} nr_n = \lambda$ and $\lambda > \lambda_c$, it is the case that

$$
\begin{aligned}
\mathbf{Pr}\left[\text{MINLA}(\mathscr{L}_{m,p}) = \Theta(n^3 r_n^3)\right] &= 1,\\
\mathbf{Pr}\left[\text{MINCW}(\mathscr{L}_{m,p}) = \Theta(n^2 r_n^3)\right] &= 1,\\
\mathbf{Pr}\left[\text{MINSC}(\mathscr{L}_{m,p}) = \Theta(n^2 r_n)\right] &= 1,\\
\mathbf{Pr}\left[\text{MINVS}(\mathscr{L}_{m,p}) = \Theta(nr_n)\right] &= 1.
\end{aligned}
$$

Definitively, there also exists a transition phase for the considered layout measures on random geometric graphs and random lattice graphs.

As with many other optimization problems in the plane that exhibit convergence phenomena, an interesting problem is to find good methods for evaluating numerically the constants in Theorem 1.4 as functions of the open vertex density p and the analogous constants in Theorem 1.7 as functions of λ. Preliminary estimates for $\beta_{\text{LA}}(p)$ were given in [9]; the method used was a raw simulation of the percolation process on the lattice and computation of lower and upper bounds with several heuristics [31]. According to [36, Section 2.4], in the case of the TSP, D. S. Johnson obtained empirically that $0.70 \leqslant \beta_{\text{TSP}}(2) \leqslant 0.73$ with high confidence. Our experimental results are not so tight: the search for new empirical or analytical methods to estimate these constants is left as an open problem.

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