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A New Axiomatics for Masures

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Abstract. Masures are generalizations of Bruhat–Tits buildings. They were introduced by Gaussent and Rousseau to study Kac–Moody groups over ultrametric fields that generalize reductive groups. Rousseau gave an axiomatic definition of these spaces. We propose an equivalent axiomatic definition, which is shorter, more practical, and closer to the axiom of Bruhat–Tits buildings. Our main tool to prove the equivalence of the axioms is the study of the convexity properties in masures.

1 Introduction

An important tool for studying a split reductive group G over a non-archimedean local field is its Bruhat–Tits building [BT72, BT84]. Kac–Moody groups are interesting infinite-dimensional (if not reductive) generalizations of reductive groups. In order to study them over fields endowed with a discrete valuation, Gaussent and Rousseau introduced masures (also known as hovels) [GR08] that are analogs of Bruhat–Tits buildings. Charignon and Rousseau generalized this construction [Cha10, Rou17, Rou16]: Charignon treated the almost split case and Rousseau suppressed restrictions on the base field and on the group. Rousseau also defined an axiomatics of masures [Rou11]. Recently, Freyn, Hartnick, Horn, and Köhl made an analog construction in the archimedean case [FHHK17]: with each split real Kac–Moody group, they associate a space on which the group acts, generalizing the notion of riemannian symmetric space.

Masures enable obtaining results on the arithmetic of (almost) split Kac–Moody groups over non-archimedean local fields. Let us survey them briefly. Let G be such a group and $\mathcal{I}$ its masure. Gaussent and Rousseau used $\mathcal{I}$ to prove a link between Littlemann's path model and some kind of Mirković–Vilonen cycle model of G [GR08]. Gaussent and Rousseau also associated a spherical Hecke algebra ${}^s\mathcal{H}$ with G and they obtained a Satake isomorphism in this setting [GR14]. These results generalized works of Braverman and Kazhdan obtained when G is supposedly affine [BK11]. Bardy-Panse, Gaussent, and Rousseau defined the Iwahori–Hecke algebra ${}^I\mathcal{H}$ of G [BPGR16]. Braverman, Kazhdan, and Patnaik had already done this construction when G is affine [BKP16]. In [Héb17], we obtained finiteness results on G enabling us to give a meaning to one side of the Gindikin–Karpelevich formula obtained by Braverman, Garland, Kazhdan, and Patnaik in the affine case [BGKP14]. Together with Abdellatif, we defined a completion of ${}^I\mathcal{H}$ and generalized the construction of the Iwahori–Hecke algebra of G: we associated Hecke algebras with subgroups of G

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more general than the Iwahori subgroup, the analogue of the parahoric subgroups [AH17]. Bardy-Panse, Gaussent, and Rousseau proved Macdonald's formula for G: they gave an explicit formula for the image of some basis of ${}^s\mathcal{H}$ by the Satake isomorphism [BPGR17]. Their formula generalizes a well-known formula of Macdonald [Mac71] for reductive groups, which had already been extended to affine Kac–Moody groups [BKP16].

Despite these results, some very basic questions are still open in the theory of masures. In this paper we are interested in questions of enclosure maps and of convexity in masures. Let us be more precise. The masure is an object similar to the Bruhat–Tits building. This is a union of subsets called apartments. An apartment is a finite-dimensional affine space equipped with a set of hyperplanes called walls. The group G acts by permuting these apartments that are, therefore, all isomorphic to one of them called the standard apartment $\mathbb{A}$.

To define the masure $\mathcal{I}$ associated with G, Gaussent and Rousseau (following Bruhat and Tits) first defined $\mathbb{A}$. Let us describe it briefly. Suppose that the field of definition is local. Let $Q^\vee$ be the co-root lattice of G and let Φ be its set of real roots. One can consider $Q^\vee$ as a lattice of some affine space $\mathbb{A}$ and Φ as a set of linear forms on $\mathbb{A}$. Let Y be a lattice of $\mathbb{A}$ containing $Q^\vee$ (one can consider $Y = Q^\vee$ in a first approximation). Then the set $\mathbb{M}$ of walls of $\mathbb{A}$ is the set of hyperplanes containing an element of Y and whose direction is $\ker(\alpha)$ for some $\alpha \in \Phi$. The half-spaces delimited by walls are called half-apartments. Suppose that G is reductive. Then Φ is finite and $\mathcal{I}$ is a building. A well-known property of buildings is that if A is an apartment of $\mathcal{I}$, then $A \cap \mathbb{A}$ is a finite intersection of half-apartments and there exists an isomorphism from A to $\mathbb{A}$ fixing $A \cap \mathbb{A}$ [BT72, §2.5.7 and Proposition 2.5.8]. Studying this question for masures seems natural for two reasons: first, masures generalize Bruhat–Tits buildings and have properties similar to them and second, because three of the five axioms of the axiomatic definition of Rousseau are weak forms of this property.

We study this question in the affine case and in the indefinite case. Let us begin with the affine case, where we prove that this property is true.

Theorem 1.1 *Let $\mathcal{I}$ be a masure associated with an affine Kac–Moody group. Let A be an apartment. Then $\mathbb{A} \cap A$ is a finite intersection of half-apartments of $\mathbb{A}$ and there exists an isomorphism from $\mathbb{A}$ to A fixing $\mathbb{A} \cap A$.*

We define a new axiomatics of masures and prove that it is equivalent to the one given by Rousseau (we recall it in Section 2.2.2), using Theorem 1.1. Our axioms are simpler and are closer to the usual geometric axioms of Euclidean buildings. To emphasize this analogy, we first recall one of their definitions in the case where the valuation is discrete (see [Bro89, §IV] or [Rou04, §6]; our definition is slightly modified but equivalent).

Definition 1.2 A *Euclidean building* is a set $\mathcal{I}$ equipped with a set $\mathcal{A}$ of subsets called *apartments* satisfying the following axioms.

(I0) Each apartment is a Euclidean apartment.

(I1) For any two faces F and F', there exists an apartment containing F and F'.

(I2) If A and A' are apartments, then $A \cap A'$ is a finite intersection of half-apartments and there exists an isomorphism $\phi: A \to A'$ fixing $A \cap A'$.

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In the statement of the next theorem, we use the notion of chimney. They are some kind of thickened sector faces. The word *splayed* will be explained later. We prove the following theorem.

Theorem 1.3 *Suppose G is an affine Kac–Moody group. Let $\mathbb{A}$ be the apartment associated with the root system of G. Let $(\mathcal{I}, \mathcal{A})$ be a couple such that $\mathcal{I}$ is a set and $\mathcal{A}$ is a set of subsets of $\mathcal{I}$ called apartments. Then $(\mathcal{I}, \mathcal{A})$ is a masure of type $\mathbb{A}$ in the sense of* [Rou11] *if and only if it satisfies the following axioms.*

(MA af i) *Each apartment is an apartment of type $\mathbb{A}$.*

(MA af ii) *If A and A' are two apartments, then $A \cap A'$ is a finite intersection of half-apartments and there exists an isomorphism $\phi : A \to A'$ fixing $A \cap A'$.*

(MA af iii) *If $\mathfrak{R}$ is the germ of a splayed chimney and F is a face or a germ of a chimney, then there exists an apartment containing $\mathfrak{R}$ and F.*

We now turn to the general (not necessarily affine) case. Similarly to buildings, we can still define a fundamental chamber C_f^v in the standard apartment $\mathbb{A}$. This enables one to define the Tits cone $\mathcal{T} = \bigcup_{w \in W^v} w.\overline{C_f^v}$, where W^v is the Weyl group of G. An important difference between buildings and masures is that when G is reductive, $\mathcal{T} = \mathbb{A}$ and when G is not reductive, $\mathcal{T} \neq \mathbb{A}$ is only a convex cone. This defines a preorder on $\mathbb{A}$ by saying that $x, y \in \mathbb{A}$ satisfy $x \leq y$ if $y \in x + \mathcal{T}$. This preorder extends to a preorder on $\mathcal{I}$, the Tits preorder, by using isomorphisms of apartments. Convexity properties in $\mathcal{I}$ were known only on preordered pairs of points. If A, A' are apartments and contain two points x, y such that $x \leq y$, then $A \cap A'$ contains the segment in A between x and y and there exists an isomorphism from A to A' fixing this segment [Rou11, Proposition 5.4].

A ray (half-line) of $\mathcal{I}$ is said to be generic if its direction meets the interior $\mathring{\mathcal{T}}$ of $\mathcal{T}$. A chimney is splayed if it contains a generic ray. The main result of this paper is the following theorem.

Theorem 1.4 *Let A be an apartment such that $\mathbb{A} \cap A$ contains a generic ray of $\mathbb{A}$. Then $\mathbb{A} \cap A$ is a finite intersection of half-apartments of $\mathbb{A}$ and there exists an isomorphism from $\mathbb{A}$ to A fixing $\mathbb{A} \cap A$.*

Using this theorem, we prove that the axiomatic definition of Rousseau is equivalent to a simpler one.

Theorem 1.5 *Let $\mathbb{A}$ be the apartment associated with the root system of G. Let $(\mathcal{I}, \mathcal{A})$ be a couple such that $\mathcal{I}$ is a set and $\mathcal{A}$ is a set of subsets of $\mathcal{I}$ called apartments. Then $(\mathcal{I}, \mathcal{A})$ is a masure of type $\mathbb{A}$ in the sense of* [Rou11] *if and only if it satisfies the following axioms.*

(MA i) *Each apartment is an apartment of type $\mathbb{A}$.*

(MA ii) *If two apartments A and A' are such that $A \cap A'$ contains a generic ray, then $A \cap A'$ is a finite intersection of half-apartments and there exists an isomorphism $\phi: A \to A'$ fixing $A \cap A'$.*

(MA iii) *If $\mathfrak{R}$ is the germ of a splayed chimney and F is a face or a germ of a chimney, then there exists an apartment containing $\mathfrak{R}$ and F.*

The axiom (MA iii) (very close to the axiom (MA3) of Rousseau) corresponds to the existence parts of Iwasawa, Bruhat and Birkhoff, decompositions in G, resp. for F a face and $\mathfrak{R}$ a sector-germ, F and $\mathfrak{R}$ two sector-germs of the same sign, and F and $\mathfrak{R}$ two opposite sector-germs. The axiom (MA ii), which implies the axiom (MA4) of Rousseau, corresponds to the uniqueness part of these decompositions.

The fact that if $x, y \in \mathcal{I}$ are such that $x \leq y$, the segment between x and y does not depend on the apartment containing $\{x, y\}$ was an axiom of masures (axiom (MAO)). A step of our proof of Theorem 1.5 is to show that (MAO) is actually a consequence of the other axioms of masures (see Proposition 5.3).

To define faces and chimneys, Rousseau used enclosure maps (see Section 2.1.5 for a precise definition). When G is a reductive group over a local field, the enclosure of a set P of $\mathbb{A}$ is the intersection of the half-apartments of $\mathbb{A}$ containing P. When G is not reductive, $\mathbb{M}$ can be dense in $\mathbb{A}$. Consequently, Gaussent and Rousseau defined the enclosure of a subset to be a filter and no more necessarily a set (which is already the case for buildings when the valuation of the base field is not discrete). Moreover, there are several natural choices of enclosure maps: one can use all the roots (real and imaginary) or only the real roots, one can allow arbitrary intersections of half-apartments or only finite intersections of half-apartments, etc. This led to many definitions and notations in [Rou17]. The theorem above proves that all these choices of enclosure maps lead to the same definition of masure; therefore the “good” enclosure map is the biggest one, which involves only real roots and finite intersections.

Actually we do not limit our study to masures associated with Kac–Moody groups: for us a masure is a set satisfying the axioms of [Rou11] and whose apartments are associated with a root generating system (and thus to a Kac–Moody matrix). We do not assume that there exists a group acting strongly transitively on it. Neither do we any discreteness hypothesis for the standard apartment: if M is a wall, the set of walls parallel to it is not necessarily discrete; this enables to handle masures associated with split Kac–Moody groups over any ultrametric field.

The paper is organized as follows. In Section 2, we describe the general framework and recall the definition of masures.

In Section 3 we study the intersection of two apartments A and B, without assuming that $A \cap B$ contains a generic ray. We prove that $A \cap B$ can be written as a union of enclosed subsets and that $A \cap B$ is enclosed when it is convex. If $P \subset A \cap B$, we give a sufficient condition of existence of an isomorphism from A to B fixing P.

In Section 4, we study the intersection of two apartments sharing a generic ray and prove Theorem 1.4, which is stated as Theorem 4.22. The reader only interested in masures associated with affine Kac–Moody groups can skip this Section and replace Theorem 4.22 by Lemma 5.20, which is far easier to prove.

In Section 5, we deduce new axiomatics of masures: we show Theorem 1.5 and Theorem 1.3, which correspond to Theorem 5.1 and Theorem 5.18.

2 General Framework, Masure

In this section, we define our framework and recall the definition of masures. Then we recall some notions on masures. References for this section are [Rou11, §1, §2] and [GR14, §1].

2.1 Standard Apartment

2.1.1 Root Generating System

Let A be a *Kac–Moody matrix* (also known as generalized Cartan matrix), *i.e.*, a square matrix $A = (a_{i,j})_{i,j\in I}$ with integer coefficients, indexed by a finite set I, and satisfying

- $a_{i,i} = 2$, for all $i \in I$,
- $a_{i,j} \leq 0$, for all $(i,j) \in I^2 | i \neq j$,
- $a_{i,j} = 0 \Leftrightarrow a_{j,i} = 0$, for all $(i,j) \in I^2$.

A *root generating system* of type A is a 5-tuple

$$\mathcal{S} = (A, X, Y, (\alpha_i)_{i\in I}, (\alpha_i^\vee)_{i\in I})$$

made of a Kac–Moody matrix A indexed by I, of two dual free $\mathbb{Z}$-modules X (of *characters*) and Y (of *co-characters*) of finite rank $\mathrm{rk}(X)$, a family $(\alpha_i)_{i\in I}$ (of *simple roots*) in X and a family $(\alpha_i^\vee)_{i\in I}$ (of *simple co-roots*) in Y. They must satisfy the following compatibility condition: $a_{i,j} = \alpha_j(\alpha_i^\vee)$ for all $i, j \in I$. We also suppose that the family $(\alpha_i)_{i\in I}$ is free in X and that the family $(\alpha_i^\vee)_{i\in I}$ is free in Y.

Let $\mathbb{A} = Y \otimes \mathbb{R}$. Every element of X induces a linear form on $\mathbb{A}$. We will consider X as a subset of the dual $\mathbb{A}^*$ of $\mathbb{A}$: the α_i's, $i \in I$, are viewed as linear forms on $\mathbb{A}$. For $i \in I$, we define an involution r_i of $\mathbb{A}$ by $r_i(v) = v - \alpha_i(v)\alpha_i^\vee$, for all $v \in \mathbb{A}$. Its space of fixed points is $\ker \alpha_i$. The subgroup of $\mathrm{GL}(\mathbb{A})$ generated by the α_i for $i \in I$ is denoted by W^v and is called the *Weyl group* of $\mathcal{S}$. The system $(W^v, \{r_i \mid i \in I\})$ is a Coxeter system. For $w \in W^v$, we denote by $\ell(w)$ the length of w with respect to $\{r_i \mid i \in I\}$.

One defines an action of the group W^v on $\mathbb{A}^*$ in the following way: if $x \in \mathbb{A}$, $w \in W^v$, and $\alpha \in \mathbb{A}^*$, then $(w.\alpha)(x) = \alpha(w^{-1}.x)$. Let $\Phi = \{w.\alpha_i \mid (w,i) \in W^v \times I\}$; Φ is the set of *real roots*. Then $\Phi \subset Q$, where $Q = \bigoplus_{i\in I} \mathbb{Z}\alpha_i$. Let $Q^+ = \bigoplus_{i\in I} \mathbb{N}\alpha_i$, $\Phi^+ = Q^+ \cap \Phi$, and $\Phi^- = (-Q^+) \cap \Phi$. Then $\Phi = \Phi^+ \sqcup \Phi^-$. Let Δ be the set of all roots as defined in [Kac94] and $\Delta_{\mathrm{im}} = \Delta \backslash \Phi$. Then $(\mathbb{A}, W^v, (\alpha_i)_{i\in I}, (\alpha_i^\vee)_{i\in I}, \Delta_{\mathrm{im}})$ is a vectorial datum as in [Rou11, §1].

2.1.2 Vectorial Faces and Tits Cone

Define $C_f^v = \{v \in \mathbb{A} \mid \alpha_i(v) > 0, \forall i \in I\}$. We call it the *fundamental chamber*. For $J \subset I$, one sets

$$F^v(J) = \{v \in \mathbb{A} \mid \alpha_i(v) = 0\ \forall i \in J, \alpha_i(v) > 0\ \forall i \in J\backslash I\}.$$

Then the closure $\overline{C_f^v}$ of C_f^v is the union of the $F^v(J)$ for $J \subset I$. The *positive*, resp., *negative*, *vectorial faces* are the sets $w.F^v(J)$, resp., $-w.F^v(J)$, for $w \in W^v$ and $J \subset I$. A *vectorial face* is either a positive vectorial face or a negative vectorial face. We call a *positive chamber*, resp., *negative chamber*, a cone of the form $w.C_f^v$ for some $w \in W^v$, resp., $-w.C_f^v$. For all $x \in C_f^v$ and for all $w \in W^v$, $w.x = x$ implies that $w = 1$. In particular the action of w on the positive chambers is simply transitive. The *Tits cone* $\mathcal{T}$ is defined by $\mathcal{T} = \bigcup_{w\in W^v} w.\overline{C_f^v}$. We also consider the negative cone $-\mathcal{T}$. We define a W^v invariant preorder $\leq$, resp., $\mathring{\leq}$, on $\mathbb{A}$, the *Tits preorder*, resp., the *Tits open preorder*, by $\forall (x,y) \in \mathbb{A}^2$, $x \leq y \Leftrightarrow y - x \in \mathcal{T}$, resp., $x \mathring{\leq} y \Leftrightarrow y - x \in \mathring{\mathcal{T}} \cup \{0\}$.

2.1.3 Weyl Group of $\mathbb{A}$

We now define the Weyl group W of $\mathbb{A}$. If X is an affine subspace of $\mathbb{A}$, one denotes by $\vec{X}$ its direction. One equips $\mathbb{A}$ with a family $\mathbb{M}$ of affine hyperplanes called *real walls* such that we have the following.

(1) For all $M \in \mathbb{M}$, there exists $\alpha_M \in \Phi$ such that $\vec{M} = \ker(\alpha_M)$.
(2) For all $\alpha \in \Phi$, there exists an infinite number of hyperplanes $M \in \mathbb{M}$ such that $\alpha = \alpha_M$.
(3) If $M \in \mathbb{M}$, we denote by r_M the reflexion of hyperplane M whose associated linear map is r_{α_M}. We assume that the group W generated by the r_M for $M \in \mathbb{M}$ stabilizes $\mathbb{M}$.

The group W is the Weyl group of $\mathbb{A}$. A point x is said to be *special* if every real wall is parallel to a real wall containing x. We suppose that 0 is special and thus $W \supset W^v$.

If $\alpha \in \mathbb{A}^*$ and $k \in \mathbb{R}$, one sets $M(\alpha, k) = \{v \in \mathbb{A} \mid \alpha(v) + k = 0\}$. Then for all $M \in \mathbb{M}$, there exists $\alpha \in \Phi$ and $k_M \in \mathbb{R}$ such that $M = M(\alpha, k_M)$. If $\alpha \in \Phi$, one sets $\Lambda_\alpha = \{k_M \mid M \in \mathcal{M} \text{ and } \vec{M} = \ker(\alpha)\}$. Then $\Lambda_{w.\alpha} = \Lambda_\alpha$ for all $w \in W^v$ and $\alpha \in \Phi$.

If $\alpha \in \Phi$, one denotes by $\widetilde{\Lambda}_\alpha$ the subgroup of $\mathbb{R}$ generated by Λ_α. By (3), $\Lambda_\alpha = \Lambda_\alpha + 2\widetilde{\Lambda}_\alpha$ for all $\alpha \in \Phi$. In particular, $\Lambda_\alpha = -\Lambda_\alpha$ and when Λ_α is discrete, $\widetilde{\Lambda}_\alpha = \Lambda_\alpha$ is isomorphic to $\mathbb{Z}$.

One sets $Q^\vee = \bigoplus_{\alpha\in\Phi} \widetilde{\Lambda}_\alpha \alpha^\vee$. This is a subgroup of $\mathbb{A}$ stable under the action of W^v. Then one has $W = W^v \ltimes Q^\vee$.

For a first reading, the reader can consider the situation where the walls are the $\phi^{-1}(\{k\})$ for $\phi \in \Phi$ and $k \in \mathbb{Z}$. We then have $\Lambda_\alpha = \mathbb{Z}$ for all $\alpha \in \Phi$, and $Q^\vee = \bigoplus_{i\in I} \mathbb{Z}\alpha_i^\vee$.

2.1.4 Filters

Definition 2.1 A *filter* in a set E is a nonempty set F of nonempty subsets of E such that, for all subsets S, S' of E, if $S, S' \in F$, then $S \cap S' \in F$ and, if $S' \subset S$, with $S' \in F$, then $S \in F$.

If F is a filter in a set E, and E' is a subset of E, one says that F contains E' if every element of F contains E'. If E' is nonempty, the set $F_{E'}$ of subsets of E containing E' is a filter. By abuse of language, we will sometimes say that E' is a filter by identifying $F_{E'}$ and E'. If F is a filter in E, its closure $\overline{F}$, resp., its convex envelope, is the filter of subsets of E containing the closure, resp., the convex envelope, of some element of F. A filter F is said to be contained in another filter F': $F \subset F'$ (resp., in a subset Z in E: $F \subset Z$) if and only if any set in F' (resp., if Z) is in F.

If $x \in \mathbb{A}$ and Ω is a subset of $\mathbb{A}$ containing x in its closure, then the *germ* of Ω in x is the filter $\text{germ}_x(\Omega)$ of subsets of $\mathbb{A}$ containing a neighborhood of x in Ω.

A *sector* in $\mathbb{A}$ is a set of the form $\mathfrak{s} = x + C^v$ with $C^v = \pm w.C^v_f$ for some $x \in \mathbb{A}$ and $w \in W^v$. A point u such that $\mathfrak{s} = u + C^v$ is called a *base point* of $\mathfrak{s}$ and C^v is its *direction*. The intersection of two sectors of the same direction is a sector of the same direction.

The *sector-germ* of a sector $\mathfrak{s} = x + C^v$ is the filter $\mathfrak{S}$ of subsets of $\mathbb{A}$ containing an $\mathbb{A}$-translate of $\mathfrak{s}$. It only depends on the direction C^v. We denote by $+\infty$, resp., $-\infty$, the sector-germ of C^v_f, resp., of $-C^v_f$.

A ray δ with base point x and containing $y \neq x$ (or the interval $]x, y] = [x, y]\backslash\{x\}$ or $[x, y]$ or the line containing x and y) is called *preordered* if $x \leq y$ or $y \leq x$ and *generic* if $y - x \in \pm\mathring{\mathcal{T}}$, the interior of $\pm\mathcal{T}$.

2.1.5 Enclosure Maps

Let $\Delta = \Phi \cup \Delta^+_{\text{im}} \cup \Delta^-_{\text{im}}$ be the set of all roots. For $\alpha \in \Delta$, and $k \in \mathbb{R} \cup \{+\infty\}$, let $D(\alpha, k) = \{v \in \mathbb{A} \mid \alpha(v) + k \geq 0\}$ (and $D(\alpha, +\infty) = \mathbb{A}$ for all $\alpha \in \Delta$) and $D^\circ(\alpha, k) = \{v \in \mathbb{A} \mid \alpha(v) + k > 0\}$ (for $\alpha \in \Delta$ and $k \in \mathbb{R} \cup \{+\infty\}$). If $\alpha \in \Delta_{\text{im}}$, one sets $\Lambda_\alpha = \mathbb{R}$. Let $[\Phi, \Delta]$ be the set of sets $\mathcal{P}$ satisfying $\Phi \subset \mathcal{P} \subset \Delta$.

If X is a set, one denotes by $\mathscr{P}(X)$ the set of subsets of X. Let $\mathcal{L}$ be the set of families $(\Lambda'_\alpha) \in \mathscr{P}(\mathbb{R})^\Delta$ such that for all $\alpha \in \Delta$, $\Lambda_\alpha \subset \Lambda'_\alpha$ and $\Lambda'_\alpha = -\Lambda'_{-\alpha}$.

Let $\mathscr{F}(\mathbb{A})$ be the set of filters of $\mathbb{A}$. If $\mathcal{P} \in [\Phi, \Delta]$ and $\Lambda' \in \mathcal{L}$, one defines the map $\text{cl}^{\mathcal{P}}_{\Lambda'} \colon \mathscr{F}(\mathbb{A}) \to \mathscr{F}(\mathbb{A})$ as follows. If $U \in \mathscr{F}(\mathbb{A})$,

$$\text{cl}^{\mathcal{P}}_{\Lambda'}(U) = \Big\{ V \in U \mid \exists (k_\alpha) \in \prod_{\alpha \in \mathcal{P}} (\Lambda'_\alpha \cup \{+\infty\}) \mid V \supset \bigcap_{\alpha \in \mathcal{P}} D(\alpha, k_\alpha) \supset U \Big\}.$$

If $\Lambda' \in \mathcal{L}$, let $\text{cl}^{\#}_{\Lambda'} \colon \mathscr{F}(\mathbb{A}) \to \mathscr{F}(\mathbb{A})$ defined as follows. If $U \subset \mathbb{A}$,

$$\text{cl}^{\#}_{\Lambda'}(U) = \Big\{ V \in U \mid \exists n \in \mathbb{N}, (\beta_i) \in \Phi^n, (k_i) \in \prod_{i=1}^n \Lambda'_{\beta_i} \mid V \supset \bigcap_{i=1}^n D(\beta_i, k_i) \supset U \Big\}.$$

Let $\mathcal{CL}^\infty = \{\text{cl}^{\mathcal{P}}_{\Lambda'} \mid \mathcal{P} \in [\Phi, \Delta] \text{ and } \Lambda' \in \mathcal{L}\}$. An element of $\mathcal{CL}^\infty$ is called an *infinite enclosure map*. Let $\mathcal{CL}^\# = \{\text{cl}^\#_{\Lambda'} | \Lambda' \in \mathcal{L}\}$. An element of $\mathcal{CL}^\#$ is called a *finite* enclosure map. Although $\mathcal{CL}^\infty$ and $\mathcal{CL}^\#$ might not be disjoint (for example, if $\mathbb{A}$ is associated with a reductive group over a local field), we define the set of *enclosure maps* $\mathcal{CL} = \mathcal{CL}^\infty \sqcup \mathcal{CL}^\#$. In Section 2.2.1 the definition of the set of faces associated with an enclosure map cl depends on whether cl is finite.

If $\text{cl} \in \mathcal{CL}$, $\text{cl} = \text{cl}^{\mathcal{P}}_{\Lambda'}$ with $\mathcal{P} \in [\Phi, \Delta] \cup \{\#\}$ and $\Lambda' \in \mathcal{L}$, then, for all $\alpha \in \Delta$, $\Lambda'_\alpha = \{k \in \mathbb{R} \mid \text{cl}(D(\alpha, k)) = D(\alpha, k)\}$. Therefore $\text{cl}^\# := \text{cl}^\#_{\Lambda'}$ is well defined. We do not use exactly the same notation as Rousseau [Rou17] where $\text{cl}^\#$ means $\text{cl}^\#_\Lambda$.

If $\Lambda' \in \mathcal{L}$, one sets $\mathcal{CL}_{\Lambda'} = \{\text{cl}^{\mathcal{P}}_{\Lambda'} \mid \mathcal{P} \in [\Phi, \Delta]\} \sqcup \{\text{cl}^\#_{\Lambda'}\}$.

In order to simplify, the reader can consider the situation where $\Lambda_\alpha = \Lambda'_\alpha = \mathbb{Z}$, for all $\alpha \in \Phi$, $\mathcal{P} = \Delta$, and $\text{cl} = \text{cl}^\Delta_\Lambda$; see [GR14, BPGR16, Héb17].

An *apartment* is a root-generating system equipped with a Weyl group W, *i.e.*, with a set $\mathbb{M}$ of real walls, Section 2.1.3, and a family $\Lambda' \in \mathcal{L}$. Let $\mathbb{A} = (\mathcal{S}, W, \Lambda')$ be an apartment. A set of the form $M(\alpha, k)$, with $\alpha \in \Phi$ and $k \in \Lambda'_\alpha$ is called a *wall* of $\mathbb{A}$ and a set of the form $D(\alpha, k)$, with $\alpha \in \Phi$ and $k \in \Lambda'_\alpha$ is called a *half-apartment* of $\mathbb{A}$. A subset X of $\mathbb{A}$ is said to be enclosed if there exist $k \in \mathbb{N}$, $\beta_1, \dots, \beta_k \in \Phi$, and $(\lambda_1, \dots, \lambda_k) \in \prod_{i=1}^k \Lambda'_{\beta_i}$ such that $X = \bigcap_{i=1}^k D(\beta_i, \lambda_i)$, *i.e.*, $X = \text{cl}^\#_{\Lambda'}(X)$. As we shall see, if $\Lambda' \in \mathcal{L}$ is fixed, the definition of masure does not depend on the choice of an enclosure map in $\mathcal{CL}_{\Lambda'}$ and thus it will be more convenient to choose $\text{cl}^\#_{\Lambda'}$; see Theorems 5.1 and 5.2.

Remark 2.2 Here and in the following, we can replace Δ^+_{im} by any W^v-stable subset of $\bigoplus_{i\in I}\mathbb{R}_+\alpha_i$ such that $\Delta^+_{\text{im}}\cap\bigcup_{\alpha\in\Phi}\mathbb{R}\alpha$ is empty. We then set $\Delta^-_{\text{im}}=-\Delta^+_{\text{im}}$. This is useful for including the case of almost-split Kac–Moody groups [Rou17, §6.11.3].

2.2 Masure

In this section, we define masures. They were introduced in [GR08] for symmetrizable split Kac–Moody groups over ultrametric fields whose residue field contains $\mathbb{C}$, axiomatized in [Rou11], then developed and generalized to almost-split Kac–Moody groups over ultrametric fields in [Rou16, Rou17].

2.2.1 Definitions of Faces, Chimneys, and Related Notions

Let $\mathbb{A}=(\mathcal{S},W,\Lambda')$ be an apartment. We choose an enclosure map $\text{cl}\in\mathcal{CL}_{\Lambda'}$.

A *local-face* is associated with a point x and a vectorial face F^v in $\mathbb{A}$; it is the filter $F^\ell(x,F^v)=\text{germ}_x(x+F^v)$ intersection of $x+F^v$ and the filter of neighborhoods of x in $\mathbb{A}$. A *face* F in $\mathbb{A}$ is a filter associated with a point $x\in\mathbb{A}$ and a vectorial face $F^v\subset\mathbb{A}$. More precisely, if cl is infinite, resp., finite, $\text{cl}=\text{cl}^{\mathcal{P}}_{\Lambda'}$ with $\mathcal{P}\in[\Phi,\Delta]$, resp., $\text{cl}=\text{cl}^{\#}_{\Lambda'}$, $F(x,F^v)$ is the filter made of the subsets containing an intersection, resp., a finite intersection, of half-spaces $D(\alpha,\lambda_\alpha)$ or $D^\circ(\alpha,\lambda_\alpha)$, with $\lambda_\alpha\in\Lambda'_\alpha\cup\{+\infty\}$ for all $\alpha\in\mathcal{P}$ (at most one $\lambda_\alpha\in\Lambda_\alpha$ for each $\alpha\in\mathcal{P}$), resp., Φ.

There is an order on the faces: if $F\subset\overline{F'}$, one says that F is a face of F' or F' contains F. The dimension of a face F is the smallest dimension of an affine space generated by some $S\in F$. Such an affine space is unique and is called its *support*. A face is said to be *spherical* if the direction of its support meets the open Tits cone $\mathring{\mathcal{T}}$ or its opposite $-\mathring{\mathcal{T}}$; then its pointwise stabilizer W_F in W^v is finite.

A *chamber* (or alcove) is a face of the form $F(x,C^v)$, where $x\in\mathbb{A}$ and C^v is a vectorial chamber of $\mathbb{A}$.

A *panel* is a face of the form $F(x,F^v)$, where $x\in\mathbb{A}$ and F^v is a vectorial face of $\mathbb{A}$ spanning a wall.

A *chimney* in $\mathbb{A}$ is associated with a face $F=F(x,F^v_0)$ and with a vectorial face F^v; it is the filter $\mathfrak{r}(F,F^v)=\text{cl}(F+F^v)$. The face F is the basis of the chimney and the vectorial face F^v is its direction. A chimney is *splayed* if F^v is spherical, and is *solid* if its support (as a filter, *i.e.*, the smallest affine subspace of $\mathbb{A}$ containing $\mathfrak{r}$) has a finite pointwise stabilizer in W^v. A splayed chimney is therefore solid.

A *shortening* of a chimney $\mathfrak{r}(F,F^v)$, with $F=F(x,F^v_0)$ is a chimney of the form $\mathfrak{r}(F(x+\xi,F^v_0),F^v)$ for some $\xi\in\overline{F^v}$. The *germ* of a chimney $\mathfrak{r}$ is the filter of subsets of $\mathbb{A}$ containing a shortening of $\mathfrak{r}$ (this definition of shortening is slightly different from the one of [Rou11, §1.12], but follows [Rou17, §3.6] and we obtain the same germs with these two definitions).

2.2.2 Masure

An *apartment of type* $\mathbb{A}$ is a set A with a nonempty set $\text{Isom}(\mathbb{A},A)$ of bijections (called *Weyl-isomorphisms*) such that if $f_0\in\text{Isom}(\mathbb{A},A)$, then $f\in\text{Isom}(\mathbb{A},A)$ if and only if there exists $w\in W$ satisfying $f=f_0\circ w$. We will say *isomorphism* instead of Weyl-isomorphism in the sequel. An isomorphism between two apartments $\phi\colon A\to A'$ is

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a bijection such that $f \in \mathrm{Isom}(\mathbb{A}, A)$ if and only if $\phi \circ f \in \mathrm{Isom}(\mathbb{A}, A')$. We extend all the notions that are preserved by W to each apartment. Thus sectors, enclosures, faces and chimneys are well defined in any apartment of type $\mathbb{A}$.

Definition 2.3 A masure of type $(\mathbb{A}, \mathrm{cl})$ is a set $\mathcal{I}$ endowed with a covering $\mathcal{A}$ of subsets called *apartments* such that we have the following.

(MA1) Any $A \in \mathcal{A}$ admits a structure of apartment of type $\mathbb{A}$.

(MA2, cl) is a point, a germof a preordered interval, a generic ray, or a solid chimney in an apartment A and if A' is another apartment containing F, then $A \cap A'$ contains the enclosure $\mathrm{cl}_A(F)$ of F and there exists an isomorphism from A onto A' fixing $\mathrm{cl}_A(F)$.

(MA3, cl) If $\mathfrak{R}$ is the germ of a splayed chimney and if F is a face or a germ of a solid chimney, then there exists an apartment containing $\mathfrak{R}$ and F.

(MA4, cl) If two apartments A, A' contain $\mathfrak{R}$ and F as in (MA3), then there exists an isomorphism from A to A' fixing $\mathrm{cl}_A(\mathfrak{R} \cup F)$.

(MAO) If x, y are two points contained in two apartments A and A', and if $x \le_A y$ then the two segments $[x, y]_A$ and $[x, y]_{A'}$ are equal.

In this definition, one says that an apartment contains a germ of a filter if it contains at least one element of this germ. One says that a map fixes a germ if it fixes at least one element of this germ.

The main example of masure is the masure associated with an almost-split Kac–Moody group over an ultrametric field [Rou17].

2.2.3 Example: A Masure Associated With a Split Kac–Moody Group Over an Ultrametric Field

Let A be a Kac–Moody matrix and $\mathcal{S}$ be a root-generating system of type A. We consider the group functor $\mathbf{G}$ associated with the root generating system $\mathcal{S}$ [Tit87] and [Rém02, Chapter 8]. This functor is a functor from the category of rings to the category of groups satisfying axioms (KMG 1)–(KMG 9) of [Tit87]. When R is a field, $\mathbf{G}(R)$ is uniquely determined by these axioms [Tit87, Theorem 1′]. This functor contains a toric functor $\mathbf{T}$, from the category of rings to the category of commutative groups (denoted $\mathcal{T}$ [Rém02]) and two functors $\mathbf{U}^+$ and $\mathbf{U}^-$ from the category of rings to the category of groups.

Let $\mathcal{K}$ be a field equipped with a non-trivial valuation $\omega: \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$, $\mathcal{O}$ its ring of integers, and $G = \mathbf{G}(\mathcal{K})$ (and $U^+ = \mathbf{U}^+(\mathcal{K})$, etc.). For all $\epsilon \in \{-, +\}$ and all $\alpha \in \Phi^\epsilon$, we have an isomorphism x_α from $\mathcal{K}$ to a group U_α. For all $k \in \mathbb{R}$, one defines a subgroup $U_{\alpha,k} := x_\alpha(\{u \in \mathcal{K} \mid \omega(u) \ge k\})$. Let $\mathcal{I}$ be the masure associated with G [Rou16]. Then for all $\alpha \in \Phi$, $\Lambda_\alpha = \Lambda'_\alpha = \omega(\mathcal{K}) \backslash \{+\infty\}$ and $\mathrm{cl} = \mathrm{cl}^\Delta_\Lambda$. If, moreover, $\mathcal{K}$ is local, one has (up to renormalization, see [GR14, Lemma 1.3]) $\Lambda_\alpha = \mathbb{Z}$ for all $\alpha \in \Phi$. Moreover, we have the following.

- The fixer of $\mathbb{A}$ in G is $H = \mathbf{T}(\mathcal{O})$ [GR08, Remark 3.2].
- The fixer of $\{0\}$ in G is $K_s = \mathbf{G}(\mathcal{O})$ [GR08, Example 3.14].
- For all $\alpha \in \Phi$ and $k \in \mathbb{Z}$, the fixer of $D(\alpha, k)$ in G is $H.U_{\alpha,k}$ [GR08, §4.2 7].
- For all $\epsilon \in \{-, +\}$, $H.U^\epsilon$ is the fixer of $\epsilon\infty$ (by [GR08, §4.2 4]).

If, moreover, $\mathcal{K}$ is local with residue cardinal q, each panel is contained in $1+q$ chambers.

The group G is reductive if and only if W^v is finite. In this case, $\mathcal{I}$ is the usual Bruhat–Tits building of G and one has $\mathcal{T} = \mathbb{A}$.

2.3 Preliminary Notions on Masures

In this subsection we recall notions on masures introduced in [GR08, Rou11, Héb17, Héb16].

2.3.1 Tits Preorder and Tits Open Preorder on $\mathcal{I}$

As the Tits preorder $\leq$ and the Tits open preorder $\mathring{\leq}$ on $\mathbb{A}$ are invariant under the action of W^v, one can equip each apartment A with preorders $\leq_A$ and $\mathring{\leq}_A$. Let A be an apartment of $\mathcal{I}$ and $x, y \in A$ such that $x \leq_A y$, resp., $x \mathring{\leq}_A y$. Then by [Rou11, Proposition 5.4], if B is an apartment containing x and y, then $x \leq_B y$, resp., $x \mathring{\leq}_B y$. This defines a relation $\leq$, resp., $\mathring{\leq}$, on $\mathcal{I}$. By [Rou11, Théorème 5.9], this defines a preorder $\leq$, resp., $\mathring{\leq}$, on $\mathcal{I}$. It is invariant by isomorphisms of apartments: if A, B are apartments, $\phi: A \to B$ is an isomorphism of apartments and $x, y \in A$ are such that $x \leq y$. resp., $x \mathring{\leq} y$, then $\phi(x) \leq \phi(y)$, resp., $\phi(x) \mathring{\leq} \phi(y)$. We call it the *Tits preorder on* $\mathcal{I}$, resp., the *Tits open preorder on* $\mathcal{I}$.

2.3.2 Retractions Centered at Sector-germs

Let $\mathfrak{s}$ be a sector-germ of $\mathcal{I}$ and A be an apartment containing it. Let $x \in \mathcal{I}$. By (MA3), there exists an apartment A_x of $\mathcal{I}$ containing x and $\mathfrak{s}$. By (MA4), there exists an isomorphism of apartments $\phi: A_x \to A$ fixing $\mathfrak{s}$. By [Rou11, §2.6], $\phi(x)$ does not depend on the choices we made and thus we can set $\rho_{A,\mathfrak{s}}(x) = \phi(x)$.

The map $\rho_{A,\mathfrak{s}}$ is a retraction from $\mathcal{I}$ onto A. It only depends on $\mathfrak{s}$ and A and we call it the *retraction onto A centered at* $\mathfrak{s}$.

If A and B are two apartments, and $\phi: A \to B$ is an isomorphism of apartments fixing some set X, one writes $\phi: A \xrightarrow{X} B$. If A and B share a sector-germ $\mathfrak{q}$, one denotes by $A \xrightarrow{A \cap B} B$ or by $A \xrightarrow{\mathfrak{q}} B$ the unique isomorphism of apartments from A to B fixing $\mathfrak{q}$ and also $A \cap B$. We denote by $\mathcal{I} \xrightarrow{\mathfrak{q}} A$ the retraction onto A fixing $\mathfrak{q}$. One denotes by $\rho_{+\infty}$ the retraction $\mathcal{I} \xrightarrow{+\infty} \mathbb{A}$ and by $\rho_{-\infty}$ the retraction $\mathcal{I} \xrightarrow{-\infty} \mathbb{A}$.

2.3.3 Parallelism in $\mathcal{I}$ and Building at Infinity

Let us explain briefly the notion of parallelism in $\mathcal{I}$. This was done more completely in [Rou11, §3].

Let us begin with rays. Let δ and δ' be two generic rays in $\mathcal{I}$. By (MA3) and [Rou11, §2.2 3] there exists an apartment A containing sub-rays of δ and δ' and we say that δ and δ' are *parallel* if these sub-rays are parallel in A. Parallelism is an equivalence relation and its equivalence classes are called *directions*. Let S be a sector of $\mathcal{I}$ and A an apartment containing S. One fixes the origin of A in a base point of S. Let $\nu \in S$ and $\delta = \mathbb{R}_+\nu$. Then δ is a generic ray in $\mathcal{I}$. By [Héb17, Lemma 3.2], for all $x \in \mathcal{I}$, there exists

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a unique ray $x+\delta$ of direction δ and base point x. To obtain this ray, one can choose an apartment A_x containing x and a sub-ray δ' of δ, which is possible by (MA3) and [Rou11, §2.2 3], and then we take the translate of δ' in A_x having x as a base point.

A *sector-face* f of $\mathbb{A}$, is a set of the form $x+F^v$ for some vectorial face F^v and some $x \in \mathbb{A}$. The germ $\mathfrak{F} = \text{germ}_\infty(f)$ of this sector-face is the filter containing the elements of the form $q+f$, for some $q \in \overline{F^v}$. The sector-face f is said to be spherical if $F^v \cap \mathring{\mathcal{T}}$ is nonempty. A *sector-panel* is a sector-face contained in a wall and spanning this one as an affine space. A sector-panel is spherical [Rou11, §1]. We extend these notions to $\mathcal{I}$ thanks to the isomorphisms of apartments. Let us make a summary of the notion of parallelism for sector-faces. This is also more complete in [Rou11, §3.3.4].

If f and f' are two spherical sector-faces, there exists an apartment B containing their germs $\mathfrak{F}$ and $\mathfrak{F}'$. One says that f and f' are parallel if there exists a vectorial face F^v of B such that $\mathfrak{F} = \text{germ}_\infty(x+F^v)$ and $\mathfrak{F}' = \text{germ}_\infty(y+F^v)$ for some $x, y \in B$. Parallelism is an equivalence relation. The parallelism class of a sector-face germ $\mathfrak{F}$ is denoted $\mathfrak{F}^\infty$. We denote by $\mathcal{I}^\infty$ the set of directions of spherical faces of $\mathcal{I}$.

For all $x \in \mathcal{I}$ and all $\mathfrak{F}^\infty \in \mathcal{I}^\infty$, there exists a unique sector-face $x+\mathfrak{F}^\infty$ of direction $\mathfrak{F}^\infty$ and with base point x [Rou11, Proposition 4.7.1]. The existence can be obtained in the same way as for rays.

2.3.4 Distance Between Apartments

Here we recall the notion of distance between apartments introduced in [Héb16]. It will often enable us to make inductions and to restrict our study to apartments sharing a sector. Let $\mathfrak{q}$ and $\mathfrak{q}'$ be two sector germs of $\mathcal{I}$ of the same sign ϵ. By (MA4), there exists an apartment B containing $\mathfrak{q}$ and $\mathfrak{q}'$. In B, there exists a minimal gallery between $\mathfrak{q}$ and $\mathfrak{q}'$, and the length of this gallery is called the distance between $\mathfrak{q}$ and $\mathfrak{q}'$. This does not depend on the choice of B. If A' is an apartment of $\mathcal{I}$, the distance $d(A',\mathfrak{q})$ between A' and $\mathfrak{q}$ is the minimal possible distance between a sector-germ of A' of sign ϵ and $\mathfrak{q}$. If A and A' are apartments of $\mathcal{I}$ and $\epsilon \in \{-1,1\}$, the distance of sign ϵ between A and A' is the minimal possible distance between a sector-germ of sign ϵ of A and a sector-germ of sign ϵ of A'. We denote it $d_\epsilon(A,A')$ or $d(A,A')$ if the sign is fixed.

Let $\epsilon \in \{-,+\}$. Then d_ϵ is not a distance on the apartments of $\mathcal{I}$ because, if A is an apartment, all apartment A' containing a sector of A of sign ϵ (and there are many of them by (MA3)) satisfies $d_\epsilon(A,A')=0$.

2.4 Notation

Let X be a finite-dimensional affine space. Let $C \subset X$ be a convex set and A' its support. The *relative interior*, resp., *relative frontier*, of C, denoted $\text{Int}_r(C)$, resp., $\text{Fr}_r(C)$, is the interior, resp., frontier, of C seen as a subset of A'. A set is said to be *relatively open* if it is open in its support.

If X is an affine space and $U \subset X$, one denotes by $\text{conv}(X)$ the convex hull of X. If $x, y \in \mathbb{A}$, we denote by $[x,y]$ the segment of $\mathbb{A}$ joining x and y. If A is an apartment and $x, y \in A$, we denote by $[x,y]_A$ the segment of A joining x and y.

If X is a topological space and $a \in X$, one denotes by $\mathcal{V}_X(a)$ the set of open neighborhoods of a.

If X is a subset of $\mathbb{A}$, one denotes by $\mathring{X}$ or by $\mathrm{Int}(X)$ (depending on the readability) its interior. One denotes by $\mathrm{Fr}(X)$ the boundary (or frontier) of X: $\mathrm{Fr}(X) = \overline{X} \backslash \mathring{X}$.

If X is a topological space, $x \in X$, and Ω is a subset of X containing x in its closure, then the *germ* of Ω in x is denoted $\mathrm{germ}_x(\Omega)$.

We use the same notation as in [Rou11] for segments and segment-germs in an affine space X. For example if $X = \mathbb{R}$ and $a, b \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, $[a, b] = \{x \in \overline{\mathbb{R}} \mid a \leq x \leq b\}$, $[a, b] = \{x \in \overline{\mathbb{R}} \mid a \leq x < b\}$, $[a, b) = \mathrm{germ}_a([a, b])$, etc.

3 General Properties of the Intersection of Two Apartments

In this section, we study the intersection of two apartments, without assuming that their intersection contains a generic ray. In Section 3.1, we extend results obtained for masure on which a group acts strongly transitively to our framework. In Section 3.2, we write the intersection of two apartments as a finite union of enclosed subsets. In Section 3.3, we use the results of Section 3.2 to prove that if the intersection of two apartments is convex, then it is enclosed. In Section 3.4, we study the existence of isomorphisms fixing subsets of an intersection of two apartments.

Let us sketch the proof of Theorem 1.4. The most difficult part is to prove that if A and B are apartments sharing a generic ray, then $A \cap B$ is convex. We first reduce our study to the case where $A \cap B$ has nonempty interior. We then parametrize the frontier of A and B by a map $\mathrm{Fr}\colon U \to \mathrm{Fr}(A \cap B)$, where U is an open and convex set of A. The idea is then to prove that for almost all choices of $x, y \in U$, some map associated with $\mathrm{Fr}_{x,y}\colon t \in [0,1] \mapsto \mathrm{Fr}(tx + (1-t)y)$ is convex. An important step in this proof is the fact that $\mathrm{Fr}_{x,y}$ is piecewise affine and this relies on the decomposition of Section 3.2. The convexity of $A \cap B$ is obtained by using a density argument. We then conclude, thanks to Sections 3.3 and 3.4.

3.1 Preliminaries

In this subsection, we extend some results obtained for a masure on which a group acts strongly transitively to our framework [Héb17, Héb16].

3.1.1 Splitting of Apartments

The following lemma generalizes [Héb16, Lemma 3.2] to our frameworks.

Lemma 3.1 *Let A_1 and A_2 be two distinct apartments such that $A_1 \cap A_2$ contains a half-apartment. Then $A_1 \cap A_2$ is a half-apartment.*

Proof One identifies A_1 and $\mathbb{A}$. By the proof of [Héb16, Lemma 3.2], $D = A_1 \cap A_2$ is a half-space of the form $D(\alpha, k)$ for some $\alpha \in \Phi$ and $k \in \mathbb{R}$. (Note that our terminology is not the same as in [Héb16] in which a half-apartment is a half-space of the form $D(\beta, \ell)$ with $\beta \in \Phi$ and $\ell \in \mathbb{R}$, whereas now, we ask moreover that $\ell \in \Lambda'_\beta$). Let F, F' be opposed sector-panels of $M(\alpha, k)$. Let S be a sector of D dominating F, $\mathfrak{s}$ its germ, and $\mathfrak{F}'$ the germ of F'. By (MA4), one has $A_1 \cap A_2 \supset \mathrm{cl}(\mathfrak{F}', \mathfrak{s})$. But $\mathrm{cl}(\mathfrak{F}', \mathfrak{s}) \supset \mathrm{cl}(D) \supset D = A_1 \cap A_2$, and thus $k \in \Lambda'_\alpha$: $A_1 \cap A_2$ is a half-apartment. ∎

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As a consequence, one can use [Héb16, Lemma 3.6, Proposition 3.7] in our framework. We thus have the following proposition.

Proposition 3.2 *Let A be an apartment, $\mathfrak{q}$ a sector-germ of $\mathcal{I}$ such that $\mathfrak{q} \not\subseteq A$, and $n = d(\mathfrak{q}, A)$.*

(i) *One can write $A = D_1 \cup D_2$, where D_1 and D_2 are opposite half-apartments of A such that for all $i \in \{1, 2\}$, there exists an apartment A_i containing D_i and such that $d(A_i, \mathfrak{q}) = n - 1$.*

(ii) *There exist $k \in \mathbb{N}$, enclosed subsets $P_1, \ldots, P_k$ of A such that for all $i \in [\![1, k]\!]$, there exist an apartment A_i containing $\mathfrak{q} \cup P_i$ and an isomorphism $\phi_i \colon A \xrightarrow{P_i} A_i$.*

Remark 3.3 The choice of the Weyl group W (and thus of $Q^\vee$) imposes restrictions on the walls that can bound the intersection of two apartments. Let A be an apartment and suppose that $A \cap \mathbb{A} = D(\alpha, k)$ for some $\alpha \in \Phi$ and $k \in \Lambda'_\alpha$. Then $k \in \frac{1}{2}\alpha(Q^\vee)$. Indeed, let $D = A \cap \mathbb{A}$, D_1 be the half-apartment of $\mathbb{A}$ opposed to D and D_2 the half-apartment of A opposed to D_1. Then $B = D_1 \cup D_2$ is an apartment of $\mathcal{I}$ [Rou11, Proposition 2.9 (2)]. Let $f \colon \mathbb{A} \xrightarrow{D} A$, $g \colon A \xrightarrow{D_2} B$, and $h \colon B \xrightarrow{D_1} \mathbb{A}$; these isomorphisms exist because two apartments sharing a half-apartment in particular share a sector; see Section 2.3.2. Let $s \colon \mathbb{A} \to \mathbb{A}$ making the following diagram commute:

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{f} & A \\ \downarrow s & & \downarrow g \\ \mathbb{A} & \xrightarrow{h^{-1}} & B. \end{array}$$

The map s fixes $M(\alpha, k)$. Moreover, if $x \in \mathring{D}$, then $f(x) = x$; thus $g(f(x)) \in \mathring{D}_1$ and hence $h^{-1}(g(f(x))) \in \mathring{D}_1$. We deduce $s \neq \mathrm{Id}$. The map s is an isomorphism of apartments and thus $s \in W$. As s fixes $M(\alpha, k)$, the vectorial part $\vec{s}$ of s fixes $M(\alpha, 0)$. As $W = W^v \ltimes Q^\vee$, one has $s = t \circ \vec{s}$, where t is a translation of vector $q^\vee$ in $Q^\vee$. If $y \in M(\alpha, k)$, one has $\alpha(s(y)) = k = \alpha(q^\vee) - k$ and therefore $k \in \frac{1}{2}\alpha(Q^\vee)$. This could enable more precision in Proposition 3.2.

3.1.2 A Characterization of the Points of $\mathbb{A}$

The aim of this subsubsection is to extend [Héb17, Corollary 4.4] to our framework.

Vectorial Distance on $\mathcal{I}$ We recall the definition of the vectorial distance [GR14, §1.7]. Let $x, y \in \mathcal{I}$ be such that $x \leq y$. Then there exists an apartment A containing x, y and an isomorphism $\phi \colon A \to \mathbb{A}$. One has $\phi(y) - \phi(x) \in \mathcal{T}$ and thus there exists $w \in W^v$ such that $\lambda = w.(\phi(y) - \phi(x)) \in \overline{C^v_f}$. Then λ does not depend on the choices we made; it is called the *vectorial distance between x and y* and denoted $d^v(x, y)$. The vectorial distance is invariant under isomorphisms of apartments: if x, y are two points in an apartment A such that $x \leq y$, if B is an apartment, and if $\phi \colon A \to B$ is an isomorphism of apartments, then $d^v(x, y) = d^v(\phi(x), \phi(y))$.

3.1.3 Image of a Preordered Segment by a Retraction

Gaussent and Rousseau gave a very precise description of the image of a preordered segment by a retraction centered at a sector-germ [GR08, Theorem 6.2]. However, they supposed that a group acts strongly transitively on $\mathcal{I}$. Without this assumption, they proved a simpler property of these images. We recall it here.

Let $\lambda \in \overline{C^v_f}$. A *$\lambda$-path* π in $\mathbb{A}$ is a map $\pi: [0,1] \to \mathbb{A}$ such that there exists $n \in \mathbb{N}$ and $0 \leq t_1 < \cdots < t_n \leq 1$ such that for all $i \in [\![1, n-1]\!]$, π is affine on $[t_i, t_{i+1}]$ and $\pi'(t) \in W^v.\lambda$ for all $t \in]t_i, t_{i+1}[$.

Lemma 3.4 *Let A be an apartment of $\mathcal{I}$. Let $x, y \in A$ be such that $x \leq y$ and $\rho: \mathcal{I} \to \mathbb{A}$ be a retraction of $\mathcal{I}$ onto $\mathbb{A}$ centered at a sector-germ $\mathfrak{q}$ of $\mathbb{A}$. Let $\tau: [0,1] \to A$ defined by $\tau(t) = (1-t)x + ty$ for all $t \in [0,1]$ and $\lambda = d^v(x,y)$. Then $\rho \circ \tau$ is a λ-path between $\rho(x)$ and $\rho(y)$.*

Proof We rewrite the proof of the beginning of Section 6 of [GR08]. Let $\phi: A \to \mathbb{A}$ be an isomorphism such that $\phi(y) - \phi(x) = \lambda$, which exists by definition of d^v. By the same reasoning as in the paragraph of [GR08] before Remark 4.6, there exist $n \in \mathbb{N}$, apartments $A_1, \ldots, A_n$ of $\mathcal{I}$ containing $\mathfrak{q}$, $0 = t_1 < \cdots < t_n = 1$ such that $\tau([t_i, t_{i+1}]) \subset A_i$ for all $i \in [\![1, n-1]\!]$.

Using [Rou11, Proposition 5.4], for all $i \in [\![1, n-1]\!]$, one chooses an isomorphism $\psi_i: A \xrightarrow{\tau([t_i, t_{i+1}])} A_i$. Let $\phi_i: A_i \xrightarrow{A_i \cap \mathbb{A}} \mathbb{A}$. For all $t \in [t_i, t_{i+1}]$,

$$\rho(\tau(t)) = \phi_i \circ \psi_i(\tau(t)).$$

Moreover, $\phi_i \circ \psi_i: A \to \mathbb{A}$ and by (MA1), there exists $w_i \in W$ such that $\phi_i \circ \psi_i = w_i \circ \phi$. Therefore for all $t \in]t_i, t_{i+1}[$, one has $(\rho \circ \tau)'(t) = w_i.\lambda$, which proves that $\rho \circ \tau$ is a λ-path. ∎

The projection y_ν Let $\nu \in C^v_f$ and $\delta = \mathbb{R}^+\nu$. By paragraph "Definition of y_ν and T_ν" of [Héb17], for all $x \in \mathcal{I}$, there exists $y_\nu(x) \in \mathbb{A}$ such that $x + \overline{\delta} \cap \mathbb{A} = y_\nu(x) + \overline{\delta}$, where $x + \overline{\delta}$ is the closure of $x + \delta$ (defined in Section 2.3.3) in any apartment containing it.

The $Q^\vee_\mathbb{R}$-order in $\mathbb{A}$ One sets $Q^\vee_{\mathbb{R},+} = \sum_{\alpha\in\Phi^+} \mathbb{R}_+\alpha^\vee = \bigoplus_{i\in I} \mathbb{R}_+\alpha_i$. One has $Q^\vee_{\mathbb{R},+} \subset \bigoplus_{i\in I} \mathbb{R}_+\alpha_i^\vee$. If $x, y \in \mathbb{A}$, one denotes $x \leq_{Q^\vee} y$ if $y - x \in Q^\vee_{\mathbb{R},+}$.

The following lemma paraphrases [Kac94, Proposition 3.12 (d)] in our context.

Lemma 3.5 *Let $\lambda \in \overline{C^v_f}$ and $w \in W^v$. Then $w.\lambda \leq_{Q^\vee} \lambda$.*

If $x \in \mathbb{A}$ and $\lambda \in \overline{C^v_f}$, one defines $\pi^a_\lambda: [0,1] \to \mathbb{A}$ by $\pi^a_\lambda(t) = a + t\lambda$ for all $t \in [0,1]$.

Lemma 3.6 *Let $\lambda \in \overline{C^v_f}$ and $a \in \mathbb{A}$. Then the unique λ-path from a to $a + \lambda$ is π^a_λ.*

Proof Let π be a λ-path from a to $a+\lambda$. One chooses a subdivision $0 = t_1 < \cdots < t_n = 1$ of $[0,1]$ such that for all $i \in [\![1, n-1]\!]$, there exists $w_i \in W^v$ such that $\pi'_{|[t_i, t_{i+1}]}(t) = w_i.\lambda$. By Lemma 3.5, $w_i.\lambda \leq_{Q^\vee} \lambda$ for all $i \in [\![1, n-1]\!]$. Let $h: \bigoplus_{i\in I} \mathbb{R}\alpha_i^\vee \to \mathbb{R}$ defined by $h(\sum_{i\in I} u_i\alpha_i^\vee) = \sum_{i\in I} u_i$ for all $(u_i) \in \mathbb{R}^I$. Suppose that there exists $i \in [\![1, n-1]\!]$

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such that $w_i.\lambda \neq \lambda$. Then $h(w_i.\lambda - \lambda) < 0$ and for all $j \in [\![1, n-1]\!]$, $h(w_j.\lambda - \lambda) \leq 0$. By integrating, we get that $h(0) < 0$: a contradiction. Therefore $\pi(t) = a + t\lambda = \pi_\lambda^a(t)$ for all $t \in [0,1]$, which is our assertion. ■

Proposition 3.7 ([Héb17, Corollary 4.4]) *Let $x \in \mathcal{I}$ be such that $\rho_{+\infty}(x) = \rho_{-\infty}(x)$. Then $x \in \mathbb{A}$.*

Proof Let $x \in \mathcal{I}$ such that $\rho_{+\infty}(x) = \rho_{-\infty}(x)$. Suppose that $x \in \mathcal{I} \setminus \mathbb{A}$. One has $x \leq y_\nu(x)$ and $d^v(x, y_\nu(x)) = \lambda$, with $\lambda = y_\nu(x) - \rho_{+\infty}(x) \in \mathbb{R}_+^* \nu$ [Héb17, Lemma 3.5 (a)]. Let A be an apartment containing x and $+\infty$, which exists by (MA3). Let $\tau\colon [0,1] \to A$ be defined by $\tau(t) = (1-t)x + t y_\nu(x)$ for all $t \in [0,1]$ (this does not depend on the choice of A [Rou11, Proposition 5.4]) and $\pi = \rho_{-\infty} \circ \tau$. Then by Lemma 3.4, π is a λ-path from $\rho_{-\infty}(x) = \rho_{+\infty}(x)$ to $y_\nu(x) = \rho_{+\infty}(x) + \lambda$.

By Lemma 3.6, $\pi(t) = \rho_{+\infty}(x) + t\lambda$ for all $t \in [0,1]$, and $\tau([0,1]) \subset \mathbb{A}$ [Héb17, Lemma 3.6]. Thus $x = \tau(0) \in \mathbb{A}$; this is absurd. Therefore $x \in \mathbb{A}$, which is our assertion. ■

3.1.4 Topological Considerations on Apartments

Proposition 3.8 ([Héb16, Corollary 5.9 (ii)]) *Let $\mathfrak{q}$ be a sector-germ of $\mathcal{I}$ and A be an apartment of $\mathcal{I}$. Let $\rho\colon \mathcal{I} \overset{\mathfrak{q}}{\to} \mathbb{A}$. Then $\rho_{|A}\colon A \to \mathbb{A}$ is continuous (for the affine topologies on A and $\mathbb{A}$).*

Proof Using Proposition 3.2 (ii), one writes $A = \bigcup_{i=1}^n P_i$ where the P_i's are closed subsets of A such that for all $i \in [\![1, n]\!]$, there exists an apartment A_i containing P_i and $\mathfrak{q}$ and an isomorphism $\psi_i\colon A \overset{P_i}{\to} A_i$. For all $i \in [\![1, n]\!]$, one denotes by ϕ_i the isomorphism $A_i \overset{\mathfrak{q}}{\to} \mathbb{A}$. Then $\rho_{|P_i} = \phi_i \circ \psi_{i|P_i}$ for all $i \in [\![1, n]\!]$.

Let $(x_k) \in A^{\mathbb{N}}$ be a converging sequence and $x = \lim x_k$. Then for all $k \in \mathbb{N}$, $\rho(x_k) \in \{\phi_i \circ \psi_i(x_k) \mid i \in [\![1, n]\!]\}$ and thus $(\rho(x_n))$ is bounded. Let $(x_{\sigma(k)})$ be a subsequence of (x_k) such that $(\rho(x_{\sigma(k)}))$ converges. Maybe extracting a subsequence of $(x_{\sigma(k)})$, one can suppose that there exists $i \in [\![1, n]\!]$ such $x_{\sigma(k)} \in P_i$ for all $k \in \mathbb{N}$. One has $(\rho(x_{\sigma(k)})) = (\phi_i \circ \psi_i(x_{\sigma(k)}))$ and thus $\rho(x_{\sigma(k)}) \to \phi_i \circ \psi_i(x) = \rho(x)$ (because P_i is closed) and thus $(\rho(x_k))$ converges towards $\rho(x)$. Hence $\rho_{|A}$ is continuous. ■

Proposition 3.9 ([Héb16, Corollary 5.10]) *Let A be an apartment. Then $A \cap \mathbb{A}$ is closed.*

Proof By Proposition 3.7, $A \cap \mathbb{A} = \{x \in A \mid \rho_{+\infty}(x) = \rho_{-\infty}(x)\}$, which is closed by Proposition 3.8. ■

3.2 Decomposition of the Intersection of Two Apartments into Enclosed Subsets

The aim of this subsection is to show that $\mathbb{A} \cap A$ is a finite union of enclosed subsets of $\mathbb{A}$.

We first suppose that A and $\mathbb{A}$ share a sector. One can suppose that $+\infty \subset A \cap \mathbb{A}$. By Proposition 3.2, one has $A = \bigcup_{i=1}^{k} P_i$, for some $k \in \mathbb{N}$, where the P_i's are enclosed and $P_i, -\infty$ is contained in some apartment A_i for all $i \in [\![1, k]\!]$.

Lemma 3.10 *Let X be a finite-dimensional affine space. Let $U \subset X$ be a set such that $U \subset \overline{\mathring{U}}$ and suppose that $U = \bigcup_{i=1}^{n} U_i$, where, for all $i \in [\![1, n]\!]$, the set U_i is the intersection of U and of a finite number of half-spaces. Let $J = \{j \in [\![1, n]\!] \mid \mathring{U}_j \neq \emptyset\}$. Then $U = \bigcup_{j \in J} U_j$.*

Proof Let $j \in [\![1, n]\!]$. Then $\mathrm{Fr}(U_j) \cap \mathring{U}$ is contained in a finite number of hyperplanes. Therefore, if one chooses a Lebesgue measure on X, the set $\bigcup_{i \in [\![1,n]\!]} \mathring{U} \cap \mathrm{Fr}(U_i)$ has measure 0 and thus $\mathring{U} \backslash \bigcup_{i \in [\![1,n]\!]} \mathrm{Fr}(U_i)$ is dense in $\mathring{U}$ and thus in U. Let $x \in U$. Let $(x_k) \in (\mathring{U} \backslash \bigcup_{i \in [\![1,n]\!]} \mathrm{Fr}(U_i))^{\mathbb{N}}$ be such that (x_k) converges towards x. Extracting a sequence if necessary, one can suppose that there exists $i \in [\![1, n]\!]$ such that $x_k \in U_i$ for all $k \in \mathbb{N}$. By definition of the frontier, $x_k \in \mathring{U}_i$ for all $k \in \mathbb{N}$. As U_i is closed in U, $x \in U_i$ and the lemma follows. ∎

Lemma 3.11 *Let $i \in [\![1, k]\!]$ be such that $A \cap \mathbb{A} \cap P_i$ has nonempty interior in $\mathbb{A}$. Then $A \cap \mathbb{A} \supset P_i$.*

Proof One chooses an apartment A_i containing $P_i, -\infty$ and $\phi_i \colon A \xrightarrow{P_i} A_i$. Let $\psi_i \colon A_i \xrightarrow{A_i \cap \mathbb{A}} \mathbb{A}$ (ψ_i exists and is unique by Subsection 2.3.2). Let $x \in P_i$. By definition of $\rho_{-\infty}$, one has $\rho_{-\infty}(x) = \psi_i(x)$ and thus $\rho_{-\infty}(x) = \psi_i \circ \phi_i(x)$.

Let $f \colon A \xrightarrow{A \cap \mathbb{A}} \mathbb{A}$. One has $\rho_{+\infty}(x) = f(x)$ for all $x \in A$. By Proposition 3.7,

$$A \cap \mathbb{A} \cap P_i = \{x \in P_i \mid \rho_{+\infty}(x) = \rho_{-\infty}(x)\} = P_i \cap (f - \psi_i \circ \phi_i)^{-1}(\{0\}).$$

As $f - \psi_i \circ \phi_i$ is affine, $(f - \psi_i \circ \phi_i)^{-1}(\{0\})$ is an affine subspace of A and as it has nonempty interior, $(f - \psi_i \circ \phi_i)^{-1}(\{0\}) = A$. Therefore $P_i \subset \mathbb{A} \cap A$. ∎

We recall the definition of $x + \infty$, if $x \in \mathcal{I}$ (Section 2.3.3). Let $x \in \mathcal{I}$ and B be an apartment containing x and $+\infty$. Let S be a sector of $\mathbb{A}$, parallel to C_f^v and such that $S \subset A \cap \mathbb{A}$. Then $x + \infty$ is the sector of A based at x and parallel to S. This does not depend on the choice of A.

Lemma 3.12 *One has $A \cap \mathbb{A} = \overline{\mathrm{Int}(A \cap \mathbb{A})}$.*

Proof By Proposition 3.9, $A \cap \mathbb{A}$ is closed and thus $\overline{\mathrm{Int}(A \cap \mathbb{A})} \subset A \cap \mathbb{A}$. Let $x \in A \cap \mathbb{A}$. By (MA4), one has $x + \infty \subset A \cap \mathbb{A}$. The fact that there exists $(x_n) \in \mathrm{Int}(x + \infty)^{\mathbb{N}}$ such that $x_n \to x$ proves the lemma. ∎

Lemma 3.13 *Let $J = \{i \in [\![1, k]\!] \mid \mathrm{Int}_{\mathbb{A}}(P_i \cap A \cap \mathbb{A}) \neq \emptyset\}$. Then $A \cap \mathbb{A} = \bigcup_{j \in J} P_j$.*

Proof Let $U = A \cap \mathbb{A}$. Then by Lemma 3.12 and Lemma 3.10, $U = \bigcup_{j \in J} U \cap P_j$ and Lemma 3.11 completes the proof. ∎

We no longer suppose that A contains $+\infty$. We say that $\bigcup_{i=1}^k P_i$ is a *decomposition of* $A\cap\mathbb{A}$ *into enclosed subsets* if the following hold.

- $k\in\mathbb{N}$ and for all $i\in[[1,k]]$, P_i is enclosed.
- $A\cap\mathbb{A}=\bigcup_{i=1}^k P_i$.
- For all $i\in[[1,k]]$, there exists an isomorphism $\phi_i\colon\mathbb{A}\xrightarrow{P_i}A$.

Proposition 3.14 *Let A be an apartment. Then there exists a decomposition $\bigcup_{i=1}^k P_i$ of $A\cap\mathbb{A}$ into enclosed subsets. As a consequence, there exists a finite set $\mathcal{M}$ of walls such that* $\mathrm{Fr}(A\cap\mathbb{A})\subset\bigcup_{M\in\mathcal{M}}M$. *If, moreover, $A\cap\mathbb{A}$ is convex, one has $A\cap\mathbb{A}=\bigcup_{j\in J}P_j$, where*

$$J=\{j\in[[1,k]]\mid \mathrm{supp}(P_j)=\mathrm{supp}(A\cap\mathbb{A})\}.$$

Proof Let $n\in\mathbb{N}$ and $\mathcal{P}_n$: for all apartment B such that $d(B,\mathbb{A})\leq n$, there exists a decomposition $\bigcup_{i=1}^{\ell}Q_i$ of $\mathbb{A}\cap B$ into enclosed subsets. The property $\mathcal{P}_0$ is true by Lemma 3.13. Let $n\in\mathbb{N}$, and suppose that $\mathcal{P}_n$ is true. Suppose that there exists an apartment B such that $d(B,\mathbb{A})=n+1$. Using Proposition 3.2, one writes $B=D_1\cup D_2$, where D_1,D_2 are opposite half-apartments such that for all $i\in\{1,2\}$, D_i is contained in an apartment B_i satisfying $d(B_i,\mathbb{A})=n$. If $i\in\{1,2\}$, one writes $B_i\cap\mathbb{A}=\bigcup_{j=1}^{\ell_i}Q_j^i$, where $\ell_i\in\mathbb{N}$, the Q_j^i's are enclosed and there exists an isomorphism $\psi_j^i\colon B_i\xrightarrow{Q_j^i}\mathbb{A}$. Then $B\cap\mathbb{A}=\bigcup_{j=1}^{\ell_1}(D_1\cap Q_j^1)\cup\bigcup_{j=1}^{\ell_2}(D_2\cap Q_j^2)$. If $i\in\{1,2\}$, one denotes by f^i the isomorphism $B\xrightarrow{D_i}B_i$. Then if $j\in[[1,\ell_i]]$, the isomorphism $\psi_j^i\circ f^i$ fixes $Q_j^i\cap D_i$ and thus $\mathcal{P}_{n+1}$ is true.

Therefore, $A\cap\mathbb{A}=\bigcup_{i=1}^k P_i$, where the P_i's are enclosed. One has

$$\mathrm{Fr}(A\cap\mathbb{A})\subset\bigcup_{i=1}^k\mathrm{Fr}(P_i),$$

which is contained in a finite union of walls.

Suppose that $A\cap\mathbb{A}$ is convex. Let $X=\mathrm{supp}(A\cap\mathbb{A})$. By Lemma 3.10 applied with $U=A\cap\mathbb{A}$,

$$A\cap\mathbb{A}=\bigcup_{\substack{i\in[[1,k]],\\ \mathrm{Int}_X(P_i)\neq\emptyset}}P_i,$$

which completes the proof. ■

3.3 Encloseness of a Convex Intersection

In this subsection, we prove Proposition 3.22. If A is an apartment such that $A\cap\mathbb{A}$ is convex, then $A\cap\mathbb{A}$ is enclosed. For this we study the gauge of $A\cap\mathbb{A}$, which is a map parameterizing the frontier of $A\cap\mathbb{A}$.

Lemma 3.15 *Let A be a finite-dimensional affine space, $k\in\mathbb{N}^*$, and let $D_1,\dots,D_k$ be half-spaces of A and $M_1,\dots,M_k$ be their hyperplanes. Then their exists $J\subset[[1,k]]$ (maybe empty) such that* $\mathrm{supp}(\bigcap_{i=1}^k D_i)=\bigcap_{j\in J}M_j$.

Proof Let $d \in \mathbb{N}^*$ and $\ell \in \mathbb{N}$. Let $\mathcal{P}_{d,\ell}$: for all affine spaces X such that $\dim X \leq d$ and half-spaces $E_1, \dots, E_\ell$ of X, there exists $J \subset [\![1, \ell]\!]$ such that $\operatorname{supp}(\bigcap_{i=1}^{\ell} E_i) = \bigcap_{j \in J} H_j$, where for all $j \in J$, H_j is the hyperplane of E_j.

It is clear that for all $\ell \in \mathbb{N}$, $\mathcal{P}_{1,\ell}$ is true and that for all $d \in \mathbb{N}$, $\mathcal{P}_{d,0}$ and $\mathcal{P}_{d,1}$ are true. Let $d \in \mathbb{N}_{\geq 2}$ and $\ell \in \mathbb{N}$, and suppose that (for all $d' \leq d-1$ and $\ell' \in \mathbb{N}$, $\mathcal{P}_{d',\ell'}$ is true) and that (for all $\ell' \in [\![0, \ell]\!]$, $\mathcal{P}_{d,\ell'}$ is true).

Let X be a d-dimensional affine space, $E_1, \dots, E_{\ell+1}$ be half-spaces of X, and $H_1, \dots, H_{\ell+1}$ be their hyperplanes. Let $L = \bigcap_{j=1}^{\ell} E_j$ and $S = \operatorname{supp} L$. Then $E_{\ell+1} \cap S$ is either S or a half-space of S. In the first case, $E_{\ell+1} \supset S \supset L$, thus $\bigcap_{i=1}^{\ell+1} E_i = L$ and thus by $\mathcal{P}_{d,\ell}$, $\operatorname{supp}(\bigcap_{i=1}^{\ell+1} E_i) = \bigcap_{j \in J} H_j$ for some $J \subset [\![1, \ell]\!]$.

Suppose that $E_{\ell+1} \cap S$ is a half-space of S. Then either $\mathring{E}_{\ell+1} \cap L \neq \varnothing$ or $\mathring{E}_{\ell+1} \cap L = \varnothing$. In the first case, one chooses $x \in \mathring{E}_{\ell+1} \cap L$ and a sequence $(x_n) \in (\operatorname{Int}_r(L))^{\mathbb{N}}$ converging towards x. Then for $n \gg 0$, $x_n \in \mathring{E}_{\ell+1} \cap \operatorname{Int}_r(L)$. Consequently, $L \cap E_{\ell+1}$ has a nonempty interior in S. Thus $\operatorname{supp}(\bigcap_{i=1}^{\ell+1} E_i) = S$ and by $\mathcal{P}_{d,\ell}$, $\operatorname{supp}(\bigcap_{i=1}^{\ell+1} E_i) = \bigcap_{j \in J} H_j$ for some $J \subset [\![1, \ell]\!]$.

Suppose now that $\mathring{E}_{\ell+1} \cap L$ is empty. Then $L \cap E_{\ell+1} \subset H_{\ell+1}$, where $H_{\ell+1}$ is the hyperplane of $E_{\ell+1}$. Therefore, $\bigcap_{i=1}^{\ell+1} E_i = \bigcap_{i=1}^{\ell+1} (E_i \cap H_{\ell+1})$ and thus by $\mathcal{P}_{d-1,\ell+1}$, $\operatorname{supp}(\bigcap_{i=1}^{\ell+1} E_i) = \bigcap_{j \in J} H_j$ for some $J \subset [\![1, \ell+1]\!]$. ∎

Lemma 3.16 *Let A be an apartment such that $A \cap \mathbb{A}$ is convex. Then* $\operatorname{supp}(A \cap \mathbb{A})$ *is enclosed.*

Proof Using Proposition 3.14, one writes $A \cap \mathbb{A} = \bigcup_{i=1}^{k} P_i$, where the P_i's are enclosed and $\operatorname{supp}(P_i) = \operatorname{supp}(A \cap \mathbb{A})$ for all $i \in [\![1, k]\!]$. By Lemma 3.15, if $i \in [\![1, k]\!]$, then $\operatorname{supp}(P_i)$ is a finite intersection of walls, which proves the lemma. ∎

Gauge of a Convex Set Let A be a finite-dimensional affine space. Let C be a closed and convex subset of A with nonempty interior. One chooses $x \in \mathring{C}$ and one fixes the origin of A in x. Let $j_{C,x} \colon A \to \mathbb{R}_+ \cup \{+\infty\}$ be defined by

$$j_{C,x}(s) = \inf\{t \in \mathbb{R}_+^* \mid s \in tC\}.$$

The map $j_{C,x}$ is called the *gauge* of C based at x. In the sequel, we will fix some $x \in \mathring{C}$ and we will denote j_C instead of $j_{C,x}$. Then $j_C(A) \subset \mathbb{R}_+$ and j_C is continuous [HUL12, Theorem 1.2.5 and §1.2].

The following lemma is easy to prove.

Lemma 3.17 *Let C be a convex closed set with nonempty interior. Fix the origin of A in a point of $\mathring{C}$. Then*

$$C = \{x \in A \mid j_C(x) \leq 1\},$$
$$\mathring{C} = \{x \in A \mid j_C(x) < 1\}.$$

Lemma 3.18 *Let C be a convex closed set with nonempty interior. Fix the origin of A in $\mathring{C}$. Let $U = U_C = \{s \in A \mid j_C(s) \neq 0\}$. Let* $\operatorname{Fr} = \operatorname{Fr}_C \colon U \to \operatorname{Fr}(C)$ *defined by* $\operatorname{Fr}(s) = \frac{s}{j_C(s)}$ *for all $s \in U$. Then* Fr *is well defined, continuous, and surjective.*

Proof If $s \in U$, then $j_C(\mathrm{Fr}(s)) = \frac{j_C(s)}{j_C(s)} = 1$ and thus Fr takes its values in $\mathrm{Fr}(C)$ by Lemma 3.17. The continuity of Fr is a consequence of the one of j_C. Let $f \in \mathrm{Fr}(C)$. Then $\mathrm{Fr}(f) = f$ and thus Fr is surjective. ∎

Let A be an apartment such that $A \cap \mathbb{A}$ is convex and nonempty. Let X be the support of $A \cap \mathbb{A}$ in $\mathbb{A}$. By Lemma 3.16, if $A \cap \mathbb{A} = X$, then $A \cap \mathbb{A}$ is enclosed. One now supposes that $A \cap \mathbb{A} \neq X$. One chooses $x_0 \in \mathrm{Int}_X(A \cap \mathbb{A})$ and consider it as the origin of $\mathbb{A}$. One defines $U = U_{A\cap\mathbb{A}}$ and $\mathrm{Fr}\colon U \to \mathrm{Fr}_r(A \cap \mathbb{A})$ as in Lemma 3.18. The set U is open and nonempty. Using Proposition 3.14, one writes $A \cap \mathbb{A} = \bigcup_{i=1}^r P_i$, where $r \in \mathbb{N}$, the P_i's are enclosed, and $\mathrm{supp}(P_i) = X$ for all $i \in [\![1, r]\!]$. Let $M_1, \dots, M_k$ be distinct walls not containing X such that $\mathrm{Fr}_r(A \cap \mathbb{A}) \subset \bigcup_{i=1}^k M_i$, which exists because the P_i's are intersections of half-spaces of X and $A \cap \mathbb{A} \neq X$. Let $\mathcal{M} = \{M_i \cap X \mid i \in [\![1, k]\!]\}$. If $M \in \mathcal{M}$, one sets $U_M = \mathrm{Fr}^{-1}(M)$.

Lemma 3.19 *Let $U' = \big\{x \in U \bigm| \exists (M, V) \in \mathcal{M} \times \mathcal{V}_U(x), \mathrm{Fr}(V) \subset M\big\}$. Then U' is dense in U.*

Proof Let $M \in \mathcal{M}$. By Lemma 3.18, U_M is closed in U. Let $V' \subset U$ be nonempty and open. Then $V' = \bigcup_{M\in\mathcal{M}} U_M \cap V'$. As $\mathcal{M}$ is finite, we can apply Baire's Theorem, and there exists $M \in \mathcal{M}$ such that $V' \cap U_M$ has a nonempty interior and hence U' is dense in U. ∎

Lemma 3.20 *Let $x \in U'$ and $V \in \mathcal{V}_U(x)$ be such that $\mathrm{Fr}(V) \subset M$ for some $M \in \mathcal{M}$. The wall M is unique and does not depend on V.*

Proof Suppose that $\mathrm{Fr}(V) \subset M \cap M'$, where M, M' are hyperplanes of X. Let $\alpha, \alpha' \in \Phi$, $k, k' \in \mathbb{R}$ be such that $M = \alpha^{-1}(\{k\})$ and $M' = \alpha'^{-1}(\{k'\})$. By definition of U, for all $y \in V$, $\mathrm{Fr}(y) = \lambda(y)y$ for some $\lambda(y) \in \mathbb{R}_+^*$. Suppose that $k = 0$. Then $\alpha(y) = 0$ for all $y \in V$, which is absurd, because $\alpha \neq 0$. By the same reasoning $k' \neq 0$.

If $y \in V \backslash \big(\alpha^{-1}(\{0\}) \cup \alpha'^{-1}(\{0\})\big)$, $\mathrm{Fr}(y) = \lambda(y)y$ for some $\lambda(y) \in \mathbb{R}_+^*$ and thus $\mathrm{Fr}(y) = \frac{k}{\alpha(y)}y = \frac{k'}{\alpha'(y)}y$. As $V \backslash \big(\alpha^{-1}(\{0\}) \cup \alpha'^{-1}(\{0\})\big)$ is dense in V, $k\alpha'(y) = k'\alpha(y)$ for all $y \in V$ and thus M and M' are parallel. Therefore $M = M'$. It remains to show that M does not depend on V. Let $V_1 \in \mathcal{V}_U(x)$ be such that $\mathrm{Fr}(V_1) \subset M_1$ for some $M_1 \in \mathcal{M}$. By the uniqueness we just proved applied to $V \cap V_1$, one has $M = M_1$, which completes the proof. ∎

If $x \in U'$, one denotes by M_x the wall defined by Lemma 3.20.

Lemma 3.21 *Let $x \in U'$ and D_1, D_2 be the two half-spaces of X defined by M_x. Then either $A \cap \mathbb{A} \subset D_1$ or $A \cap \mathbb{A} \subset D_2$.*

Proof Let $V \in \mathcal{V}_U(x)$ be such that $\mathrm{Fr}(V) \subset M_x$. Let us prove that $\mathrm{Fr}(V) = \mathbb{R}_+^* V \cap M_x$. As $\mathrm{Fr}(y) \in \mathbb{R}_+^* y$ for all $y \in V$, $\mathrm{Fr}(V) \subset \mathbb{R}_+^* V \cap M_x$. Let f be a linear form on X such that $M_x = f^{-1}(\{k\})$ for some $k \in \mathbb{R}$. If $k = 0$, then $f(v) = 0$ for all $v \in V$, and thus $f = 0$: this is absurd. Hence $k \neq 0$.

Let $a \in \mathbb{R}_+^* V \cap M_x$. One has $a = \lambda \mathrm{Fr}(v)$, for some $\lambda \in \mathbb{R}_+^*$ and $v \in V$. Moreover, $f(\mathrm{Fr}(v)) = k = f(a)$ and as $k \neq 0$, $a = \mathrm{Fr}(v) \in \mathrm{Fr}(V)$. Thus $\mathrm{Fr}(V) = \mathbb{R}_+^* V \cap M_x$

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and $\mathrm{Fr}(V)$ is an open set of M_x. Suppose there exists $(x_1, x_2) \in (\mathring{D}_1 \cap A \cap \mathbb{A}) \times (\mathring{D}_2 \cap A \cap \mathbb{A})$. Then $\mathrm{conv}(x_1, x_2, \mathrm{Fr}(V)) \subset A \cap \mathbb{A}$ is an open neighborhood of $\mathrm{Fr}(V)$ in X. This is absurd, because Fr takes its values in $\mathrm{Fr}_r(A \cap \mathbb{A})$. Thus the lemma is proved. ■

If $x \in U'$, one denotes by D_x the half-space delimited by M_x and containing $A \cap \mathbb{A}$.

Proposition 3.22 *Let A be an apartment such that $A \cap \mathbb{A}$ is convex. Then $A \cap \mathbb{A}$ is enclosed.*

Proof If $u \in U'$, then $A \cap \mathbb{A} \subset D_u$ and thus $A \cap \mathbb{A} \subset \bigcap_{u \in U'} D_u$.

Let $x \in U' \cap \bigcap_{u \in U'} D_u$. One has $0 \in A \cap \mathbb{A}$ and thus $0 \in D_x$. Moreover, $\mathrm{Fr}(x) \in M_x \cap A \cap \mathbb{A}$ and thus $x \in [0, \mathrm{Fr}(x)] \subset A \cap \mathbb{A}$. Therefore,

$$U' \cap \bigcap_{x \in U'} D_x \subset A \cap \mathbb{A}.$$

Let $x \in \mathrm{Int}_X(\bigcap_{u \in U'} D_u)$. If $x \notin U$, then $x \in A \cap \mathbb{A}$. Suppose $x \in U$. Then by Lemma 3.19, there exists $(x_n) \in (U' \cap \mathrm{Int}_X(\bigcap_{u \in U'} D_u))^{\mathbb{N}}$ such that $x_n \to x$. But then for all $n \in \mathbb{N}$, $x_n \in A \cap \mathbb{A}$, and by Proposition 3.9, $x \in A \cap \mathbb{A}$. As a consequence, $A \cap \mathbb{A} \supset \mathrm{Int}_X(\bigcap_{u \in U'} D_u)$. As $A \cap \mathbb{A}$ is closed,

$$A \cap \mathbb{A} \supset \overline{\mathrm{Int}_X\Big(\bigcap_{u \in U'} D_u\Big)} = \bigcap_{u \in U'} D_u$$

because $\bigcap_{u \in U'} D_u$ is closed, convex with nonempty interior in X. Thus we have proved $A \cap \mathbb{A} = \bigcap_{u \in U'} D_u$.

Let $M'_1, \dots, M'_k$ be walls of $\mathbb{A}$ such that for all $x \in U'$, there exists $i(x) \in [\![1, k]\!]$ such that $M'_{i(x)} \cap X = M_x$. One sets $M'_x = M'_{i(x)}$ for all $x \in U'$ and one denotes by D'_x the half-apartment of $\mathbb{A}$ delimited by M'_x and containing D_x. Then $X \cap \bigcap_{x \in U'} D'_x = A \cap \mathbb{A}$. Lemma 3.16 completes the proof. ■

3.4 Existence of Isomorphisms of Apartments Fixing a Convex Set

Let A be an apartment and $P \subset \mathbb{A} \cap A$. In this section, we study the existence of isomorphisms of apartments $\mathbb{A} \overset{P}{\to} A$. We give a sufficient condition of existence of such an isomorphism in Proposition 3.26. The existence of an isomorphism $A \xrightarrow{A \cap \mathbb{A}} \mathbb{A}$ when A and $\mathbb{A}$ share a generic ray will be a particular case of this Proposition; see Theorem 4.22. In the affine case, this will be a first step to prove that for every apartment A, there exists an isomorphism $A \overset{A \cap \mathbb{A}}{\to} \mathbb{A}$.

Lemma 3.23 *Let A be an apartment of $\mathcal{I}$ and $\phi: \mathbb{A} \to A$ an isomorphism of apartments. Let $P \subset \mathbb{A} \cap A$ be a nonempty, relatively open, convex set, $Z = \mathrm{supp}(P)$, and suppose that ϕ fixes P. Then ϕ fixes $P + (\mathcal{T} \cap \vec{Z}) \cap A$, where $\mathcal{T}$ is the Tits cone.*

Proof Let $x \in P + (\mathcal{T} \cap \vec{Z}) \cap A$. Write $x = p + t$, where $p \in P$ and $t \in \mathcal{T}$. Assume $t \neq 0$. Let $L = p + \mathbb{R}t$. Then L is a preordered line in $\mathcal{I}$ and ϕ fixes $L \cap P$. Moreover, $p \leq x$ and thus, by [Rou11, Proposition 5.4], there exists an isomorphism $\psi: \mathbb{A} \xrightarrow{[p,x]} A$. In particular, $\phi^{-1} \circ \psi: \mathbb{A} \to \mathbb{A}$ fixes $L \cap P$. But then $\phi^{-1} \circ \psi_{|L}$ is an affine isomorphism fixing a nonempty open set of L; this is the identity. Therefore $\phi^{-1}\circ\psi(x) = x = \phi^{-1}(x)$, which proves the lemma. ■

Lemma 3.24 *Let A be an apartment of $\mathcal{I}$. Let $U \subset \mathbb{A} \cap A$ be a nonempty relatively open set and $X = \mathrm{supp}(U)$. Then there exists a nonempty open subset V of U (in X) such that there exists an isomorphism $\phi\colon \mathbb{A} \overset{V}{\to} A$.*

Proof Let $\bigcup_{i=1}^{k} P_i$ be a decomposition into enclosed subsets of $A \cap \mathbb{A}$. Let $i \in [\![1,k]\!]$ be such that $P_i \cap U$ has nonempty interior in X and $\phi\colon \mathbb{A} \overset{P_i}{\longrightarrow} A$. Then ϕ fixes a nonempty open set of U, which proves the lemma. ■

Lemma 3.25 *Let A be an apartment of $\mathcal{I}$ and $\phi\colon \mathbb{A} \to A$ an isomorphism. Let $F = \{z \in A \mid \phi(z) = z\}$. Then F is closed in $\mathbb{A}$.*

Proof By Proposition 3.8, $\rho_{+\infty} \circ \phi\colon \mathbb{A} \to \mathbb{A}$ and $\rho_{-\infty} \circ \phi\colon \mathbb{A} \to \mathbb{A}$ are continuous. Let $(z_n) \in F^{\mathbb{N}}$ be such that (z_n) converges in $\mathbb{A}$ and $z = \lim z_n$. For all $n \in \mathbb{N}$, one has

$$\rho_{+\infty}(\phi(z_n)) = z_n = \rho_{-\infty}(\phi(z_n)) \to \rho_{+\infty}(\phi(z)) = z = \rho_{-\infty}(\phi(z)).$$

By Proposition 3.7, $z = \phi(z)$, which proves the lemma. ■

Proposition 3.26 *Let A be an apartment of $\mathcal{I}$ and $P \subset \mathbb{A} \cap A$ a convex set. Let $X = \mathrm{supp}(P)$ and suppose that $\mathcal{T} \cap \vec{X}$ has nonempty interior in $\vec{X}$. Then there exists an isomorphism of apartments $\phi\colon \mathbb{A} \overset{P}{\to} A$.*

Proof (See Figure 1.) Let $V \subset P$ be a nonempty open set of X such that there exists an isomorphism $\phi\colon \mathbb{A} \overset{V}{\to} A$ (such a V exists by Lemma 3.24). Let us show that ϕ fixes $\mathrm{Int}_r(P)$.

Let $x \in V$. One fixes the origin of $\mathbb{A}$ in x and thus X is a vector space. Let $(e_j)_{j\in J}$ be a basis of $\mathbb{A}$ such that for some subset $J' \subset J$, $(e_j)_{j\in J'}$ is a basis of X and $(x + \mathcal{T}) \cap X \supset \bigoplus_{j\in J'} \mathbb{R}_+^* e_j$. For all $y \in X$, $y = \sum_{j\in J'} y_j e_j$ with $y_j \in \mathbb{R}$ for all $j \in J'$, one sets $|y| = \max_{j\in J'} |y_j|$. If $a \in A$ and $r > 0$, one sets $B(a,r) = \{y \in X \bigm| |y-a| < r\}$.

Suppose that ϕ does not fix $\mathrm{Int}_r(P)$. Let $y \in \mathrm{Int}_r(P)$ be such that $\phi(y) \neq y$. Let

$$s = \sup\{t \in [0,1] \mid \exists U \in \mathcal{V}_X([0,ty]) \mid \phi \text{ fixes } U\}.$$

Set $z = sy$. Then by Lemma 3.25, $\phi(z) = z$.

By definition of z, for all $r > 0$, ϕ does not fix $B(z,r)$. Let $r > 0$ be such that $B(z,5r) \subset \mathrm{Int}_r P$. Let $z_1 \in B(z,r) \cap [0,z[$ and $r_1 > 0$ be such that ϕ fixes $B(z_1,r_1)$ and $z_2' \in B(z,r)$ such that $\phi(z_2') \neq z_2'$. Let $r_2' \in\,]0,r[$ be such that for all $a \in B(z_2',r_2')$, $\phi(z) \neq z$. Let $z_2 \in B(z_2',r_2')$ be such that for some $r_2 \in\,]0,r_2'[$, $B(z_2,r_2) \subset B(z_2',r_2')$ and such that there exists an isomorphism $\psi\colon \mathbb{A} \xrightarrow{B(z_2,r_2)} A$ (such z_2 and r_2 exist by Lemma 3.24). Then $|z_1 - z_2| < 3r$.

Let us prove that $(z_1 + \mathcal{T} \cap X) \cap (z_2 + \mathcal{T} \cap X) \cap \mathrm{Int}_r(P)$ contains a nonempty open set $U \subset X$. One identifies X and $\mathbb{R}^{J'}$ thanks to the basis $(e_j)_{j\in J'}$. One has

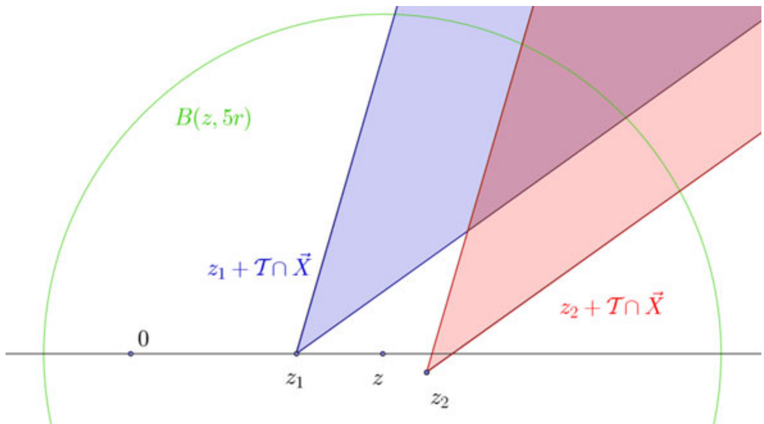

Figure 1: Proof of Proposition 3.26

$z_2 - z_1 \in \,]-3,3[^{J'}$ and thus

$$(z_1 + \mathfrak{T}) \cap (z_2 + \mathfrak{T}) = (z_1 + \mathfrak{T}) \cap (z_1 + z_2 - z_1 + \mathfrak{T}) \supset z_1 + \,]3,4[^{J'}.$$

As $P \supset B(z_1, 4r)$, the set $(z_1 + \mathfrak{T} \cap X) \cap (z_2 + \mathfrak{T} \cap X) \cap \mathrm{Int}_r(P)$ contains a nonempty open set $U \subset X$.

By Lemma 3.23, ϕ and ψ fix U. Therefore, $\phi^{-1} \circ \psi$ fixes U, and as it is an isomorphism of affine space of A, $\phi^{-1} \circ \psi$ fixes X. Therefore $\phi^{-1} \circ \psi(z_2) = \phi^{-1}(z_2) = z_2$ and thus $\phi(z_2) = z_2$; this is absurd. Hence, ϕ fixes $\mathrm{Int}_r(P)$. By Lemma 3.25, ϕ fixes $\overline{\mathrm{Int}_r(P)} = \overline{P}$ and thus ϕ fixes P, which shows the proposition. ∎

4 Intersection of Two Apartments Sharing a Generic Ray

The aim of this section is to prove Theorem 4.22. Let A and B be two apartments sharing a generic ray. Then $A \cap B$ is enclosed and there exists an isomorphism $\phi\colon A \xrightarrow{A\cap B} B$.

We first reduce our study to the case where $A \cap B$ has nonempty interior by the following lemma.

Lemma 4.1 *Suppose that for all apartments A, B such that $A \cap B$ contains a generic ray and has nonempty interior, the set $A \cap B$ is convex. Then if A_1 and A_2 are two apartments containing a generic ray, the set $A_1 \cap A_2$ is enclosed and there exists an isomorphism $\phi\colon A_1 \xrightarrow{A_1\cap A_2} A_2$.*

Proof Let us prove that $A_1 \cap A_2$ is convex. Let δ be the direction of a generic ray shared by A_1 and A_2. Let $x_1, x_2 \in A_1 \cap A_2$ and $\mathfrak{F}^\infty$ be the vectorial face direction containing δ. Let $\mathfrak{F}'^\infty$ be the vectorial face direction of A_1 opposite to $\mathfrak{F}^\infty$. Let C_1 be a chamber of A_1 containing x_1 and C_2 be a chamber of A_2 containing x_2. Set $\mathfrak{r}_1 = \mathfrak{r}(C_1, \mathfrak{F}'^\infty) \subset A_1$, $\mathfrak{r}_2 = \mathfrak{r}(C_2, \mathfrak{F}^\infty) \subset A_2$, $\mathfrak{R}_1 = \mathrm{germ}(\mathfrak{r}_1)$, and $\mathfrak{R}_2 = \mathrm{germ}(\mathfrak{r}_2)$. By (MA3) there exists an apartment A_3 containing $\mathfrak{R}_1$ and $\mathfrak{R}_2$.

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Let us prove that A_3 contains x_1 and x_2. One identifies A_1 and $\mathbb{A}$. Let $F^v = 0 + \mathfrak{F}^\infty$ and $F'^v = 0 + \mathfrak{F}'^\infty$. As $A_3 \supset \mathfrak{R}_1$, there exists $f' \in F'^v$ such that $A_3 \supset x_1 + f' + F'^v$. Moreover, $A_3 \supset \mathfrak{F}^\infty$, and thus it contains $x_1 + f' + \mathfrak{F}^\infty$. By [Rou11, Proposition 4.7.1] $x_1 + f' + \mathfrak{F}^\infty = x_1 + f' + F^v$, and thus $A_3 \ni x_1$. As $A_3 \supset \mathfrak{R}_2$, there exists $f \in F^v$ such that $A_3 \supset x_2 + f$. As $A_3 \supset \mathfrak{F}'^\infty$,

$$A_3 \supset x_2 + f + \mathfrak{F}' = x_2 + f + F'^v$$

by [Rou11, Proposition 4.7.1]. Thus $A_3 \ni x_2$.

If $i \in \{1,2\}$, each element of $\mathfrak{R}_i$ has a nonempty interior in A_i, and thus $A_i \cap A_3$ has a nonempty interior. By hypothesis, $A_1 \cap A_3$ and $A_2 \cap A_3$ are convex. By Proposition 3.26, there exist $\phi\colon A_1 \xrightarrow{A_1\cap A_3} A_3$ and $\psi\colon A_2 \xrightarrow{A_2\cap A_3} A_3$. Therefore $[x_1,x_2]_{A_1} = [x_1,x_2]_{A_3} = [x_1,x_2]_{A_2}$, and thus $A_1 \cap A_2$ is convex.

The existence of an isomorphism $A_1 \xrightarrow{A_1\cap A_2} A_2$ is a consequence of Proposition 3.26, because the direction X of $A_1 \cap A_2$ meets $\mathring{\mathcal{T}}$ and thus $\vec{X} \cap \mathcal{T}$ spans $\mathcal{T}$.

The fact that $A_1 \cap A_2$ is enclosed is a consequence of Proposition 3.22. ■

4.1 Definition of the Frontier Maps

The aim of Sections 4.1–4.5 is to prove that if A and B are two apartments containing a generic ray and such that $A \cap B$ has nonempty interior, then $A \cap B$ is convex. There is no loss of generality in assuming that $B = \mathbb{A}$ and that the direction $\mathbb{R}_+ \nu$ of δ is contained in $\pm\overline{C_f^v}$. As the roles of C_f^v and $-C_f^v$ are similar, one supposes that $\mathbb{R}_+\nu \subset \overline{C_f^v}$ and that $A \neq \mathbb{A}$. These hypotheses run until the end of Section 4.5.

In this subsection, we parametrize $\mathrm{Fr}(A \cap \mathbb{A})$ by a map and describe $A \cap \mathbb{A}$ using the values of this map.

Lemma 4.2 *Let V be a bounded subset of $\mathbb{A}$. Then there exists $a \in \mathbb{R}$ such that for all $u \in [a, +\infty[$ and $v \in V$, $v \leq u\nu$.*

Proof Let $a \in \mathbb{R}_+^*$ and $v \in V$. Then $a\nu - v = a(\nu - \frac{1}{a}v)$. As $\nu \in \mathring{\mathcal{T}}$ and V is bounded, there exists $b > 0$ such that for all $a > b$, $\nu - \frac{1}{a}v \in \mathring{\mathcal{T}}$, which proves the lemma, because $\mathring{\mathcal{T}}$ is a cone. ■

Lemma 4.3 *Let $y \in A \cap \mathbb{A}$. Then $A \cap \mathbb{A}$ contains $y + \mathbb{R}_+\nu$.*

Proof Let $x \in \mathbb{A}$ such that $A \cap \mathbb{A} \supset x + \mathbb{R}_+\nu$. The ray $x + \mathbb{R}_+\nu$ is generic and by (MA4), if $y \in \mathbb{A}$, $A \cap \mathbb{A}$ contains the convex hull of y and $x + [a, +\infty[\ \nu$, for some $a \gg 0$. In particular, it contains $y + \mathbb{R}_+\nu$. ■

Let $U = \{y \in \mathbb{A} \mid y + \mathbb{R}\nu \cap A \neq \varnothing\} = (A \cap \mathbb{A}) + \mathbb{R}\nu$.

Lemma 4.4 *The set U is convex.*

Proof Let $u, v \in U$. Let $u' \in u + \mathbb{R}_+\nu \cap A$. By Lemma 4.2 and Lemma 4.3, there exists $v' \in v + \mathbb{R}_+\nu$ such that $u' \leq v'$. By [Rou11, Proposition 5.4(2)], $[u', v'] \subset A \cap \mathbb{A}$. By definition of U, $[u', v'] + \mathbb{R}v \subset U$ and in particular $[u, v] \subset U$, which is the desired conclusion. ■

There are two possibilities: either there exists $y \in \mathbb{A}$ such that $y + \mathbb{R}v \subset A$ or for all $y \in \mathbb{A}$, $y + \mathbb{R}v \nsubseteq A$. The first case is the easiest and we treat it in the next lemma.

Lemma 4.5 *Suppose that for some $y \in \mathbb{A}$, $y - \mathbb{R}_+v \subset A \cap \mathbb{A}$. Then $A \cap \mathbb{A} = U$. In particular, $A \cap \mathbb{A}$ is convex.*

Proof By Lemma 4.3, $A \cap \mathbb{A} = (A \cap \mathbb{A}) + \mathbb{R}_+v$. By symmetry and by hypothesis on $A \cap \mathbb{A}$, one has $(A \cap \mathbb{A}) + \mathbb{R}_-v = A \cap \mathbb{A}$. Therefore $A \cap \mathbb{A} = (A \cap \mathbb{A}) + \mathbb{R}v = U$. ■

Definition of the frontier Let $u \in U$. Then by Lemma 4.3, $u + \mathbb{R}v \cap A$ is of the form $a + \mathbb{R}_+^*v$ or $a + \mathbb{R}_+v$ for some $a \in \mathbb{A}$. As $A \cap \mathbb{A}$ is closed (by Proposition 3.9), the first case cannot occur. One sets $\mathrm{Fr}_v(u) = a \in \mathbb{A} \cap A$. One fixes v until the end of Section 4.5 and one writes Fr instead of Fr_v.

Lemma 4.6 *The map* Fr *takes its values in* $\mathrm{Fr}(\mathbb{A} \cap A)$ *and* $\mathbb{A} \cap A = \bigcup_{x \in U} \mathrm{Fr}(x) + \mathbb{R}_+v$.

Proof Let $u \in U$. Then $\mathrm{Fr}(u) + \mathbb{R}_+v = (u + \mathbb{R}v) \cap A$. Thus $\mathrm{Fr}(u) \notin \mathrm{Int}(A \cap \mathbb{A})$. By Proposition 3.9, $\mathrm{Fr}(u) \in \mathrm{Fr}(A \cap \mathbb{A})$ and hence $\mathrm{Fr}(U) \subset \mathrm{Fr}(A \cap \mathbb{A})$.

Let $u \in A \cap \mathbb{A}$. One has $u \in A \cap (u + \mathbb{R}v) = \mathrm{Fr}(u) + \mathbb{R}_+v$, and we deduce that $\mathbb{A} \cap A \subset \bigcup_{x \in U} \mathrm{Fr}(x) + \mathbb{R}_+v$. The reverse inclusion is a consequence of Lemma 4.3, which finishes the proof. ■

Let us sketch the proof of the convexity of $A \cap \mathbb{A}$ (which is Lemma 4.21). If $x, y \in \mathring{U}$, one defines $\mathrm{Fr}_{x,y}\colon [0,1] \to \mathrm{Fr}(A \cap \mathbb{A})$ by $\mathrm{Fr}_{x,y}(t) = \mathrm{Fr}((1-t)x + ty)$ for all $t \in [0,1]$. For all $t \in [0,1]$, there exists a unique $f_{x,y}(t) \in \mathbb{R}$ such that $\mathrm{Fr}_{x,y}(t) = (1-t)x + ty + f_{x,y}(t)v$. We prove that for almost all $x, y \in \mathring{U}$, $f_{x,y}$ is convex. Let $x, y \in \mathring{U}$. We first prove that $f_{x,y}$ is continuous and piecewise affine. This enables us to reduce the study of the convexity of $f_{x,y}$ to the study of $f_{x,y}$ at the points where the slope changes. Let $\mathcal{M}$ be a finite set of walls such that $\mathrm{Fr}(\mathring{U}) \subset \bigcup_{M \in \mathcal{M}} M$, which exists by Proposition 3.14. Using order-convexity, we prove that if $\{x, y\}$ is such that for each point $u \in]0,1[$ at which the slope changes, $\mathrm{Fr}_{x,y}(u)$ is contained in exactly two walls of $\mathcal{M}$, then $f_{x,y}$ is convex. We then prove that there are "enough" such pairs and conclude by an argument of density.

4.2 Continuity of the Frontier

In this subsection, we prove that Fr is continuous on $\mathring{U}$, using order-convexity.

Let $\lambda\colon U \to \mathbb{R}$ such that for all $x \in U$, $\mathrm{Fr}(x) = x + \lambda(x)v$. We prove the continuity of $\mathrm{Fr}_{|\mathring{U}}$ by proving the continuity of $\lambda_{|\mathring{U}}$. For this, we begin, for $x, y \in \mathring{U}$, by dominating $\lambda([x, y])$ by a number depending on x and y (see Lemma 4.7). We use it to prove that if $n \in \mathbb{N}$ and $a_1, \dots, a_n \in \mathring{U}$, then $\lambda(\mathrm{conv}(\{a_1, \dots, a_n\}))$ is dominated, and then deduce that $\mathrm{Fr}_{|\mathring{U}}$ is continuous (which is Lemma 4.12).

Lemma 4.7 *Let $x, y \in U$, $M = \max\{\lambda(x), \lambda(y)\}$, and $k \in \mathbb{R}_+$ be such that $x + kv \geq y$. Then, for all $u \in [x, y]$, $\lambda(u) \leq k + M$.*

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Proof By Lemma 4.3, $x + Mv$ and $y + Mv$ are in A. By hypothesis, $x + kv + Mv \geq y + Mv$. Let $t \in [0,1]$ and $u = tx + (1-t)y$. By order-convexity $t(x + kv + Mv) + (1-t)(y + Mv) \in A$. Therefore $\lambda(u) \leq M + tk \leq M + k$, which is our assertion. ∎

Lemma 4.8 *Let $d \in \mathbb{N}$, X be a d-dimensional affine space and $P \subset X$. One sets* $\mathrm{conv}_0(P) = P$ *and for all* $k \in \mathbb{N}$,

$$\mathrm{conv}_{k+1}(P) = \{(1-t)p + tp' \mid t \in [0,1] \textit{ and } (p, p') \in \mathrm{conv}_k(P)^2\}.$$

Then $\mathrm{conv}_d(P) = \mathrm{conv}(P)$.

Proof By induction,

$$\mathrm{conv}_k(P) = \Big\{ \sum_{i=1}^{2^k} \lambda_i p_i \mid (\lambda_i) \in [0,1]^{2^k}, \sum_{i=1}^{2^k} \lambda_i = 1 \text{ and } (p_i) \in P^{2^k} \Big\}.$$

This is thus a consequence of Carathéodory's Theorem. ∎

Lemma 4.9 *Let P be a bounded subset of $\mathring{U}$ such that* $\sup(\lambda(P)) < +\infty$. *Then* $\sup\big(\lambda(\mathrm{conv}_1(P))\big) < +\infty$.

Proof Let $M = \sup_{x \in P} \lambda(x)$ and $k \in \mathbb{R}_+$ such that for all $x, x' \in P$, $x' + kv \geq x$, which exists by Lemma 4.2. Let $u \in \mathrm{conv}_1(P)$ and $x, x' \in P$ such that $u \in [x, x']$. By Lemma 4.7, $\lambda(u) \leq k + M$ and the lemma follows. ∎

Lemma 4.10 *Let $x \in \mathring{U}$. Then there exists $V \in \mathcal{V}_{\mathring{U}}(x)$ such that V is convex and* $\sup(\lambda(V)) < +\infty$.

Proof Let $n \in \mathbb{N}$ and $a_1, \dots, a_n \in \mathring{U}$ such that $V = \mathrm{conv}(a_1, \dots, a_n)$ contains x in its interior. Let $M \in \mathbb{R}_+$ such that for all $y, y' \in V$, one has $y + Mv \geq y'$, which is possible by Lemma 4.2. One sets $P = \{a_1, \dots, a_n\}$ and for all $k \in \mathbb{N}$, $P_k = \mathrm{conv}_k(P)$. By induction using Lemma 4.9, $\sup(\lambda(P_k)) < +\infty$ for all $k \in \mathbb{N}$ and we conclude with Lemma 4.8. ∎

Lemma 4.11 *Let $V \subset \mathring{U}$ be open, convex, bounded and such that*

$$\sup(\lambda(V)) \leq M$$

for some $M \in \mathbb{R}_+$. Let $k \in \mathbb{R}_+$ such that for all $x, x' \in V$, $x + kv \geq x'$. Let $a \in V$ and $u \in \mathbb{A}$ such that $a + u \in V$. Then for all $t \in [0,1]$, $\lambda(a + tu) \leq (1-t)\lambda(a) + t(M + k)$.

Proof By Lemma 4.3, $a + u + (M + k)v \in A$. Moreover,

$$a + u + (M + k)v \geq a + Mv, a + Mv \geq a + \lambda(a)v = \mathrm{Fr}(a),$$

and thus $a + u + (M + k)v \geq \mathrm{Fr}(a)$.

Let $t \in [0,1]$. Then by order-convexity,

$$(1-t)(a + \lambda(a)v) + t(a + u + (M + k)v) = a + tu + \big((1-t)\lambda(a) + t(M + k)\big)v \in A.$$

Therefore $\lambda(a + tu) \leq (1-t)\lambda(a) + t(M + k)$, which is our assertion. ∎

Lemma 4.12 *The map* Fr *is continuous on $\mathring{U}$.*

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Proof Let $x \in \mathring{U}$ and $V \in \mathcal{V}_{\mathring{U}}(x)$ be convex, open, bounded, and such that $\sup(\lambda(V)) \leq M$ for some $M \in \mathbb{R}_+$, which exists by Lemma 4.10. Let $k \in \mathbb{R}_+$ such that for all $v, v' \in V$, $v + kv \geq v'$. Let $|\cdot|$ be a norm on $\mathbb{A}$ and $r > 0$ such that $B(x,r) \subset V$, where $B(x,r) = \{u \in \mathbb{A} \bigm| |x-u| \leq r\}$. Let $S = \{u \in \mathbb{A} \bigm| |u-x| = r\}$. Let $N = M + k$.

Let $y \in S$ and $t \in [0,1]$. By applying Lemma 4.11 with $a = x$ and $u = y - x$, we get that $\lambda((1-t)x + ty) \leq \lambda(x) + tN$. By applying Lemma 4.11 with $a = (1-t)x + ty$ and $u = x - y$, we obtain that

$$\lambda(x) = \lambda\big((1-t)x + ty + t(x-y)\big) \leq \lambda((1-t)x + ty) + tN.$$

Therefore, for all $t \in [0,1]$ and $y \in S$,

$$\lambda(x) - tN \leq \lambda((1-t)x + ty) \leq \lambda(x) + tN.$$

Let $(x_n) \in B(x,r)^{\mathbb{N}}$ such that $x_n \to x$. Let $n \in \mathbb{N}$. One sets $t_n = \frac{|x_n - x|}{r}$. If $t_n = 0$, one chooses $y_n \in S$. It $t_n \neq 0$, one sets $y_n = x + \frac{1}{t_n}(x_n - x) \in S$. Then $x_n = t_n y_n + (1-t_n)x$ and thus $|\lambda(x_n) - \lambda(x)| \leq t_n N \to 0$. Consequently, $\lambda_{|\mathring{U}}$ is continuous, and we deduce that $\mathrm{Fr}_{|\mathring{U}}$ is continuous. ∎

4.3 Piecewise Affineness of $\mathrm{Fr}_{x,y}$

We now study the map Fr. We begin by proving that there exists a finite set $\mathcal{H}$ of hyperplanes of $\mathbb{A}$ such that Fr is affine on each connected component of $\mathring{U} \setminus \bigcup_{H \in \mathcal{H}} H$.

Let $\mathcal{M}$ be a finite set of walls such that $\mathrm{Fr}(A \cap \mathbb{A})$ is contained in $\bigcup_{M \in \mathcal{M}} M$, whose existence is provided by Proposition 3.14. Let $r = |\mathcal{M}|$. Let

$$\{\beta_1, \dots, \beta_r\} \in \Phi^r \quad \text{and} \quad (\ell_1, \dots, \ell_r) \in \prod_{i=1}^{r} \Lambda'_{\beta_i}$$

be such that $\mathcal{M} = \{M_i \mid i \in [\![1,r]\!]\}$ where $M_i = \beta_i^{-1}(\{\ell_i\})$ for all $i \in [\![1,r]\!]$.

Let $i, j \in [\![1,r]\!]$ be such that $i \neq j$. If $\beta_i(v)\beta_j(v) \neq 0$ and M_i and M_j are not parallel, one sets $H_{i,j} = \{x \in \mathbb{A} \mid \frac{\ell_i - \beta_i(x)}{\beta_i(v)} = \frac{\ell_j - \beta_j(x)}{\beta_j(v)}\}$ (this definition will appear naturally in the proof of the next lemma). Then $H_{i,j}$ is a hyperplane of $\mathbb{A}$. Indeed, otherwise $H_{i,j} = \mathbb{A}$. Hence $\frac{\beta_j(x)}{\beta_j(v)} - \frac{\beta_i(x)}{\beta_i(v)} = \frac{\ell_j}{\beta_j(v)} - \frac{\ell_i}{\beta_i(v)}$, for all $x \in \mathbb{A}$. Therefore, $\frac{\beta_j(x)}{\beta_j(v)} - \frac{\beta_i(x)}{\beta_i(v)} = 0$, for all $x \in \mathbb{A}$, and thus M_i and M_j are parallel: a contradiction. Let

$$\mathcal{H} = \{H_{i,j} \mid i \neq j, \beta_i(v)\beta_j(v) \neq 0 \text{ and } M_i \nparallel M_j\} \cup \{M_i \mid \beta_i(v) = 0\}.$$

Even if the elements of $\mathcal{H}$ can be walls of $\mathbb{A}$, we will only consider them as hyperplanes of $\mathbb{A}$. To avoid confusion between elements of $\mathcal{M}$ and elements of $\mathcal{H}$, we will try to use the letter M, resp., H, in the name of objects related to $\mathcal{M}$, resp., $\mathcal{H}$.

Lemma 4.13 *Let $M_\cap = \bigcup_{M \neq M' \in \mathcal{M}} M \cap M'$. Then $\mathrm{Fr}^{-1}(M_\cap) \subset \bigcup_{H \in \mathcal{H}} H$.*

Proof Let $x \in \mathrm{Fr}^{-1}(M_\cap)$. One has $\mathrm{Fr}(x) = x + \lambda v$, for some $\lambda \in \mathbb{R}$. There exists $i, j \in [\![1,r]\!]$ such that

- $i \neq j$,
- $\beta_i(\mathrm{Fr}(x)) = \ell_i$ and $\beta_j(\mathrm{Fr}(x)) = \ell_j$,
- M_i and M_j are not parallel.

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Therefore, if $\beta_i(v)\beta_j(v) \neq 0$, then $\lambda = \frac{\ell_i - \beta_i(x)}{\beta_i(v)} = \frac{\ell_j - \beta_j(x)}{\beta_j(v)}$ and thus $x \in H_{i,j}$. If $\beta_i(v)\beta_j(v) = 0$, then $x \in M_i \cup M_j$, which proves the lemma. ■

Lemma 4.14 *One has* $A \cap \mathbb{A} = \overline{\mathrm{Int}(A \cap \mathbb{A})}$.

Proof By Proposition 3.9, $A \cap \mathbb{A}$ is closed and thus $\overline{\mathrm{Int}(A \cap \mathbb{A})} \subset A \cap \mathbb{A}$.

Let $x \in A \cap \mathbb{A}$. Let V be an open bounded set contained in $A \cap \mathbb{A}$. By Lemma 4.2 applied to $x - V$, there exists $a > 0$ such that for all $v \in V$, one has $v + av \geq x$. One has $V + av \subset A \cap \mathbb{A}$ and by order convexity [Rou11, Proposition 5.4(2)], $\mathrm{conv}(V + av, x) \subset A \cap \mathbb{A}$. As $\mathrm{conv}(V + av, x)$ is a convex set with nonempty interior, there exists $(x_n) \in \mathrm{Int}(\mathrm{conv}(V + av, x))^{\mathbb{N}}$ such that $x_n \to x$, and the lemma follows. ■

Let $f_1, \dots, f_s$ be affine forms on $\mathbb{A}$ such that $\mathcal{H} = \{f_i^{-1}(\{0\}) \mid i \in [\![1, s]\!]\}$ for some $s \in \mathbb{N}$. Let $R = (R_i) \in \{\leq, \geq, <, >\}^s$. One sets

$$P_R = \mathring{U} \cap \{x \in \mathbb{A} \mid (f_i(x) R_i 0) \forall i \in [\![1, s]\!]\}.$$

If $R = (R_i) \in \{\leq, \geq\}^s$, one defines $R' = (R'_i) \in \{<, >\}^s$ by $R'_i = <$ if $R_i = \leq$ and $R'_i = >$ otherwise (one replaces large inequalities by strict inequalities). If $R \in \{\leq, \geq\}^s$, then $\mathrm{Int}(P_R) = P_{R'}$.

Let $X = \{R \in \{\leq, \geq\}^s \mid \mathring{P}_R \neq \emptyset\}$. By Lemma 4.14 and Lemma 3.10, $\mathring{U} = \bigcup_{R \in X} P_R$ and for all $R \in X$, $\mathring{P}_R \subset \mathbb{A} \setminus \bigcup_{H \in \mathcal{H}} H$.

Lemma 4.15 *Let $R \in X$. Then there exists $M \in \mathcal{M}$ such that* $\mathrm{Fr}(P_R) \subset M$.

Proof Let $x \in \mathring{P}_R$. Let $M \in \mathcal{M}$ be such that $\mathrm{Fr}(x) \in M$. Let us show that $\mathrm{Fr}(P_R) \subset M$. By continuity of Fr (by Lemma 4.12), it suffices to prove that $\mathrm{Fr}(\mathring{P}_R) \subset M$. By connectedness of $\mathring{P}_R$, it suffices to prove that $\mathrm{Fr}^{-1}(M) \cap \mathring{P}_R$ is open and closed. As Fr is continuous, $\mathrm{Fr}^{-1}(M) \cap \mathring{P}_R$ is closed (in $\mathring{P}_R$).

Suppose that $\mathrm{Fr}^{-1}(M) \cap \mathring{P}_R$ is not open. Then there exists $y \in \mathring{P}_R$ such that $\mathrm{Fr}(y) \in M$ and a sequence $(y_n) \in (\mathring{P}_R)^{\mathbb{N}}$ such that $y_n \to y$ and such that $\mathrm{Fr}(y_n) \notin M$ for all $n \in \mathbb{N}$. For all $n \in \mathbb{N}$, $\mathrm{Fr}(y_n) \in \bigcup_{M' \in \mathcal{M}} M'$, and thus, maybe extracting a subsequence, one can suppose that for some $M' \in \mathcal{M}$, $y_n \in M'$ for all $n \in \mathbb{N}$.

As Fr is continuous (by Lemma 4.12), $\mathrm{Fr}(y) \in M'$. Thus $\mathrm{Fr}(y) \in M \cap M'$ and by Lemma 4.13, $y \in \bigcup_{H \in \mathcal{H}} H$, which is absurd by choice of y. Therefore, $\mathrm{Fr}^{-1}(M) \cap \mathring{P}_R$ is open, which completes the proof of the lemma. ■

Lemma 4.16 *Let $R \in X$ and $M \in \mathcal{M}$ be such that* $\mathrm{Fr}(P_R) \subset M$. *Then $v \notin \vec{M}$ and there exists a (unique) affine morphism $\psi: \mathbb{A} \to M$ such that* $\mathrm{Fr}_{|P_R} = \psi_{|P_R}$. *Moreover, ψ induces an isomorphism $\overline{\psi}: \mathbb{A}/\mathbb{R}v \to M$.*

Proof If $y \in \mathring{U}$, then $\mathrm{Fr}(y) = y + k(y)v$ for some $k(y) \in \mathbb{R}$. Let $\alpha \in \Phi$ be such that $M = \alpha^{-1}(\{u\})$ for some $u \in -\Lambda'_\alpha$. For all $y \in P_R$, one has $\alpha(\mathrm{Fr}(y)) = \alpha(y) + k(y)\alpha(v) = u$ and $\alpha(v) \neq 0$ because α is not constant on P_R. Consequently, $v \notin \vec{M}$ and $\mathrm{Fr}(y) = y + \frac{u - \alpha(y)}{\alpha(v)} v$. One defines $\psi : \mathbb{A} \to M$ by $\psi(y) = y + \frac{u - \alpha(y)}{\alpha(v)} v$ for all $y \in \mathbb{A}$ and ψ has the desired properties. ■

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4.4 Local Convexity of $\mathrm{Fr}_{x,y}$

Let $M \in \mathcal{M}$ and let $\vec{M}$ be its direction. Let $\mathcal{T}_M = \mathring{\mathcal{T}} \cap \vec{M}$ and D_M be the half-apartment containing a shortening of $\mathbb{R}_+ v$ and whose wall is M.

Lemma 4.17 *Let $a \in \mathrm{Fr}(\mathring{U})$ and suppose that there exists $\mathcal{K} \in \mathcal{V}_{\mathring{U}}(a)$ such that $\mathrm{Fr}(\mathcal{K}) \subset M$ for some $M \in \mathcal{M}$. Then $\mathrm{Fr}((a \pm \mathring{\mathcal{T}}_M) \cap \mathring{U}) \subset D_M$.*

Proof Let $u \in \mathring{U} \cap (a - \mathring{\mathcal{T}}_M)$, $u \neq a$. Suppose $\mathrm{Fr}(u) \notin D_M$. Then $\mathrm{Fr}(u) = u - kv$, with $k \geq 0$. Then $\mathrm{Fr}(u) \leq u \mathring{\leq} a$ (which means that $a - u \in \mathring{\mathcal{T}}$). Therefore for some $\mathcal{K}' \in \mathcal{V}_M(a)$ such that $\mathcal{K}' \subset \mathcal{K}$, one has $\mathrm{Fr}(u) \mathring{\leq} u'$ for all $u' \in \mathcal{K}'$. As a consequence $\mathbb{A} \cap A \supset \mathrm{conv}(\mathcal{K}', \mathrm{Fr}(u))$ and thus $\mathrm{Fr}(u') \notin M$ for all $u' \in \mathcal{K}'$. This is absurd and hence $\mathrm{Fr}(u) \in D_M$.

Let $v \in \mathring{U} \cap (a + \mathring{\mathcal{T}}_M)$, $v \neq a$, and suppose that $\mathrm{Fr}(v) \notin D_M$. Then for $v' \in [\mathrm{Fr}(v), v[$ near enough from v, one has $a \leq v'$. Therefore, $[a, v'] \subset \mathbb{A} \cap A$. Thus for all $t \in]a, v[$, $\mathrm{Fr}(t) \notin D_M$, a contradiction. Therefore $\mathrm{Fr}(v) \in D_M$ and the lemma follows. ∎

The following lemma is crucial to prove the local convexity of $\mathrm{Fr}_{x,y}$ for good choices of x and y. This is here mainly so that we can use that $A \cap \mathbb{A}$ have nonempty interior.

Let $H_\cap = \bigcup_{H \neq H' \in \mathcal{H}} H \cap H'$.

Lemma 4.18 *Let $x \in \mathring{U} \cap (\bigcup_{H \in \mathcal{H}} H) \backslash H_\cap$ and $H \in \mathcal{H}$ be such that $x \in H$. Let C_1 and C_2 be the half-spaces defined by H. Then there exists $V \in \mathcal{V}_{\mathring{U}}(x)$ satisfying the following conditions.*

(i) *For $i \in \{1, 2\}$, let $V_i = V \cap \mathring{C}_i$. Then $V_i \subset \mathring{P}_{R_i}$ for some $R_i \in X$.*

(ii) *Let M be a wall containing $\mathrm{Fr}(P_{R_1})$. Then $\mathrm{Fr}(V) \subset D_M$.*

Proof (See Figure 2.) The set $\mathring{U} \backslash \bigcup_{H \in \mathcal{H} \backslash \{H\}} H$ is open in $\mathring{U}$. Let $V' \in \mathcal{V}_{\mathring{U}}(x)$ be such that $V' \cap \bigcup_{H' \in \mathcal{H} \backslash \{H\}} H' = \varnothing$ and such that V' is convex. Let $i \in \{1, 2\}$ and $V'_i = V' \cap \mathring{C}_i$. Then $V'_i \subset \mathring{U} \backslash \bigcup_{H \in \mathcal{H}} H$. Moreover, V'_i is connected. As the connected components of $\mathring{U} \backslash \bigcup_{H \in \mathcal{H}} H$ are the $\mathring{P}_R$'s for $R \in X$, we deduce that V' satisfies (i).

Let $\psi: \mathbb{A} \to M$ be the affine morphism such that $\psi_{|P_{R_1}} = \mathrm{Fr}_{|P_{R_1}}$ and $\overline{\psi}: \mathbb{A}/\mathbb{R}v \to M$ be the induced isomorphism, which exist by Lemma 4.16. Let $\pi: \mathbb{A} \to \mathbb{A}/\mathbb{R}v$ be the canonical projection. As C_1 is invariant under translation by v (by definition of the elements of $\mathcal{H}$), the set $\psi(C_1) = \overline{\psi}(\pi(C_1))$ is a half-space D of M. Let $V'' = V' \cap C_1$. Then

$$\psi(V'') = \overline{\psi}(C_1) \cap \overline{\psi}(\pi(V')) \in \mathcal{V}_D(\mathrm{Fr}(x)).$$

Let $g: \vec{M} \to \mathbb{R}$ be a linear form such that $D = g^{-1}([b, +\infty[)$, for some $b \in \mathbb{R}$. Let $\epsilon \in \{-1, 1\}$ be such that $g(u) > 0$ for some $u \in \epsilon \mathcal{T}_M$. Let $\eta > 0$. Then $\mathrm{Fr}(x + \eta u) \in x + \eta u + \mathbb{R}v$ and thus $\mathrm{Fr}(x + \eta u) = \mathrm{Fr}(x) + \eta u + kv$ for some $k \in \mathbb{R}$. If η is small enough that $x + \eta u \in V''$, then $kv = \mathrm{Fr}(x + \eta u) - (\mathrm{Fr}(x) + \eta u) \in \vec{M}$ and hence $k = 0$ (by Lemma 4.16). Let $\mathcal{K} = \psi(V'') + \mathbb{R}v$ and $a = \mathrm{Fr}(x) + \eta u$. Then $\mathcal{K} \in \mathcal{V}_{\mathring{U}}(a)$ and, for all $v \in \mathcal{K}$, $\mathrm{Fr}(v) \in M$. By Lemma 4.17,

$$\mathrm{Fr}(\mathring{U} \cap (a - \epsilon \mathcal{T}_M)) = \mathrm{Fr}(\mathring{U} \cap (a - \epsilon \mathcal{T}_M + \mathbb{R}v)) \subset D_M.$$

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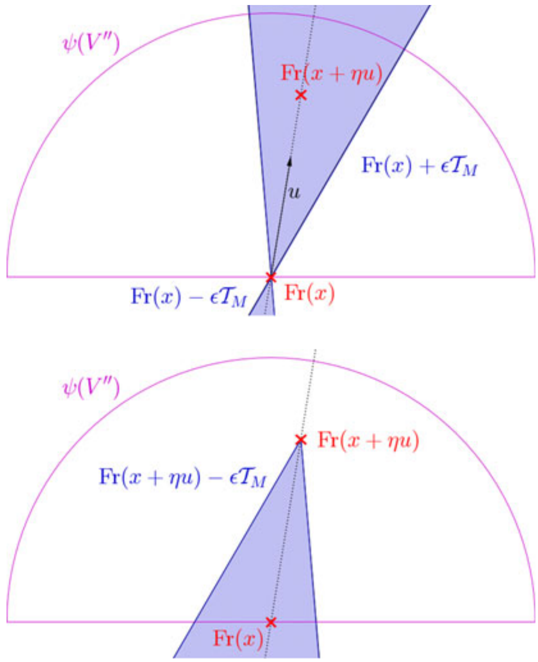

Figure 2: Proof of Lemma 4.18 when dim $H = 2$. (The illustration is made in M.)

Moreover, $a-\epsilon\mathcal{T}_M+\mathbb{R}v \in \mathcal{V}_{\mathring{U}}(x)$ and thus if one sets $V = V' \cap (a-\epsilon\mathcal{T}_M+\mathbb{R}v)$, V satisfies (i) and (ii). ∎

4.5 Convexity of $A \cap \mathbb{A}$

Let $\vec{\mathcal{H}} = \bigcup_{H\in\mathcal{H}} \vec{H}$ be the set of directions of the hyperplanes of $\mathcal{H}$.

Lemma 4.19 *Let $x, y \in \mathring{U} \cap A \cap \mathbb{A}$ be such that $y-x \notin \vec{\mathcal{H}}$ and such that the line spanned by $[x, y]$ does not meet any point of $H_\cap$. Then $[x, y] \subset \mathring{U} \cap A \cap \mathbb{A}$.*

Proof Let $\pi\colon [0,1] \to \mathbb{A}$ defined by $\pi(t) = tx+(1-t)y$, for all $t \in [0,1]$. Set $g = \mathrm{Fr}\circ\pi$. Let $f_1, \dots, f_s$ be affine forms on $\mathbb{A}$ such that

$$\mathcal{H} = \{f_i^{-1}(\{0\}) \mid i \in [\![1, s]\!]\}.$$

As $y-x \notin \vec{\mathcal{H}}$, for all $i \in [\![1, s]\!]$, the map $f_i \circ g$ is strictly monotonic and $\pi^{-1}(\bigcup_{H\in\mathcal{H}} H)$ is finite. Therefore, there exist $k \in \mathbb{N}$ and open intervals $T_1 \dots, T_k$ such that

- $[0,1] = \bigcup_{i=1}^k \overline{T_i}$,
- $T_1 < \cdots < T_k$,
- for all $i \in [\![1, k]\!]$, there exist $R_i \in X$ such that $\pi(T_i) \subset \mathring{P}_{R_i}$.

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For all $t \in [0,1]$, one has $g(t) = \pi(t) + f(t)v$ for some $f(t) \in \mathbb{R}$. By Lemma 4.16, this equation uniquely determines $f(t)$ for all $t \in [0,1]$. By Lemma 4.12, f is continuous and by Lemma 4.16, f is affine on each T_i.

Let us prove that f is convex. Let $i \in [\![1, k-1]\!]$. One writes $T_i =]a, b[$. Then for $\epsilon > 0$ small enough, one has $f(b+\epsilon) = f(b) + \epsilon c_+$ and $f(b-\epsilon) = f(b) - \epsilon c_-$. To prove the convexity of f, it suffices to prove that $c_- < c_+$. Let M be a wall containing $\mathrm{Fr}(P_{R_i})$. As $\pi(b) \in \mathring{U} \cap \bigcup_{H \in \mathcal{H}} H \backslash H_\cap$, we can apply Lemma 4.18 and there exists $V \in \mathcal{V}_{[0,1]}(b)$ such that $g(V) \subset D_M$. Let $h\colon \mathbb{A} \to \mathbb{R}$ be a linear map such that $D_M = h^{-1}([a, +\infty[)$. For $\epsilon > 0$ small enough, one has $h(g(b+\epsilon)) \geq a$ and $h(g(b-\epsilon)) = a$. For $\epsilon > 0$ small enough, one has

$$\begin{aligned} h(g(b+\epsilon)) &= h(\pi(b) + \epsilon(y-x) + (f(b) + \epsilon c_+)v) \\ &= h(g(b) + \epsilon(y - x + c_+ v)) \\ &= a + \epsilon(h(y-x) + c_+ h(v)) \geq a, \end{aligned}$$

and similarly, $h(g(b-\epsilon)) = a - \epsilon(h(y-x) + c_- h(v)) = a$.

Therefore, $h(y-x) + c_+ h(v) \geq 0$, $h(y-x) + c_- h(v) = 0$, and hence $(c_+ - c_-)h(v) \geq 0$. As D_M contains a shortening of $\mathbb{R}_+ v$, $h(v) \geq 0$ and, by Lemma 4.16, $h(v) > 0$. Consequently, $c_- \leq c_+$ and, as $i \in [\![1, k-1]\!]$ was arbitrary, f is convex.

For all $t \in [0,1]$, $f(t) \leq (1-t)f(0) + tf(1)$. Therefore,

$$\begin{aligned} (1-t)g(0) + tg(1) &= \pi(t) + ((1-t)f(0) + tf(1))v \in \pi(t) + f(t)v + \mathbb{R}_+ v \\ &= g(t) + \mathbb{R}_+ v. \end{aligned}$$

By definition of Fr, if $t \in [0,1]$, then $(1-t)g(0) + tg(1) \in A \cap \mathbb{A}$. Moreover, there exist $\lambda, \mu \geq 0$ such that $x = g(0) + \lambda v$ and $y = g(1) + \mu v$. Then

$$\pi(t) = (1-t)x + ty = (1-t)g(0) + tg(1) + ((1-t)\lambda + t\mu)v \in A \cap \mathbb{A}$$

and hence $[x, y] \subset A \cap \mathbb{A}$. ∎

Lemma 4.20 *Let $x, y \in \mathrm{Int}(\mathbb{A} \cap A)$ and $\vec{\mathcal{H}} = \bigcup_{H \in \mathcal{H}} \vec{H}$. Then there exists $(x_n), (y_n) \in \mathrm{Int}(A \cap \mathbb{A})^{\mathbb{N}}$ satisfying the following conditions.*

(i) *$x_n \to x$ and $y_n \to y$.*

(ii) *For all $n \in \mathbb{N}$, $y_n - x_n \notin \vec{\mathcal{H}}$.*

(iii) *The line spanned by $[x_n, y_n]$ does not meet any point of $H_\cap$.*

Proof Let $(x_n) \in (\mathrm{Int}(A \cap \mathbb{A}) \backslash H_\cap)^{\mathbb{N}}$ be such that $x_n \to x$. Let $|\cdot|$ be a norm on $\mathbb{A}$. Let $n \in \mathbb{N}$. Let F be the set of points $z \in \mathbb{A}$ such that the line spanned by $[x_n, z]$ meets $H_\cap$. Then F is a finite union of hyperplanes of $\mathbb{A}$, because $H_\cap$ is a finite union of spaces of dimension at most $\dim \mathbb{A} - 2$. Therefore $\mathbb{A} \backslash (F \cup x_n + \vec{\mathcal{H}})$ is dense in $\mathbb{A}$ and one can choose $y_n \in \mathbb{A} \backslash (F \cup x_n + \vec{\mathcal{H}})$ such that $|y_n - y| \leq \frac{1}{n+1}$. Then (x_n) and (y_n) satisfy the conditions of the lemma. ∎

Lemma 4.21 *The set $A \cap \mathbb{A}$ is convex.*

Proof Let $x, y \in \mathrm{Int}(A \cap \mathbb{A})$. Let $(x_n), (y_n)$ be as in Lemma 4.20. Let $t \in [0,1]$. As $\mathrm{Int}(A \cap \mathbb{A}) \subset \mathring{U}$, one has $tx_n + (1-t)y_n \in A \cap \mathbb{A}$, for all $n \in \mathbb{N}$, by Lemma 4.19. As

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$A \cap \mathbb{A}$ is closed (by Proposition 3.9), $tx + (1-t)y \in \overline{A \cap \mathbb{A}}$. Therefore $\text{Int}(A \cap \mathbb{A})$ is convex. Consequently, $A \cap \mathbb{A} = \overline{\text{Int}(A \cap \mathbb{A})}$ (by Lemma 4.14) is convex. ■

We thus have proved the following theorem.

Theorem 4.22 *Let A and B be two apartments sharing a generic ray. Then $A \cap B$ is enclosed and there exists an isomorphism $\phi\colon A \xrightarrow{A\cap B} B$.*

Proof By Lemma 4.21 and Lemma 4.1, $A \cap B$ is convex. By Proposition 3.22, $A \cap B$ is enclosed and, by Proposition 3.26, there exists an isomorphism $\phi\colon A \xrightarrow{A\cap B} B$. ■

4.6 A Partial Reciprocal

One says that a group G of automorphisms of $\mathcal{I}$ acts strongly transitively on $\mathcal{I}$ if the isomorphisms involved in (MA2) and (MA4) are induced by elements of G. For example if G is a quasi-split Kac–Moody group over an ultrametric field $\mathcal{K}$, it acts strongly transitively on the associated masure $\mathcal{I}(G,\mathcal{K})$.

We now prove a kind of weak reciprocal of Theorem 4.22 when some group G acts strongly transitively on $\mathcal{I}$ and when $\mathcal{I}$ is thick, which means that each panel is contained in at least three chambers. This implies some restrictions on Λ' by Lemma 4.24 and Remark 3.3.

Lemma 4.23 *Let P be an enclosed subset of $\mathbb{A}$ and suppose that $\mathring{P} \neq \emptyset$. One fixes the origin of $\mathbb{A}$ in some point of $\mathring{P}$. Let j_P be the gauge of P defined in Section 3.3. Let $U = \{x \in \mathbb{A} \mid j_P(x) \neq 0\}$. One defines $\text{Fr}\colon U \to P$ as in Lemma 3.18. One writes $P = \bigcap_{i=1}^k D_i$, where the D_i's are half-apartments of $\mathbb{A}$. Let $j \in [\![1,k]\!]$, M_j be the wall of D_j, and suppose that for all open subsets V of U, $\text{Fr}(V) \nsubseteq M_j$. Then $P = \bigcap_{i\in[\![1,k]\!]\setminus\{j\}} D_i$.*

Proof Suppose that $P \nsubseteq \bigcap_{i\in[\![1,k]\!]\setminus\{j\}} D_i$. Let V be a nonempty open and bounded subset contained in $\bigcap_{i\in[\![1,k]\!]\setminus\{j\}} D_i \backslash P$. Let $n \in \mathbb{N}^*$ be such that $\frac{1}{n}V \subset P$. Let $v \in V$. Then $[\frac{1}{n}v, v] \cap \text{Fr}(P) = \{\text{Fr}(v)\}$. Moreover, for all $i \in [\![1,k]\!]\backslash\{j\}$, $[\frac{1}{n}v, v] \subset \mathring{D}_i$. As $\text{Fr}(P) \subset \bigcup_{i\in[\![1,k]\!]} M_i$, we deduce that $\text{Fr}(v) \in M_j$: this is absurd and thus $P = \bigcap_{i\in[\![1,k]\!]\setminus\{j\}} D_i$. ■

Lemma 4.24 *Suppose that $\mathcal{I}$ is thick. Let D be a half-apartment of $\mathbb{A}$. Then there exists an apartment A of $\mathbb{A}$ such that $D = A \cap \mathbb{A}$.*

Proof Let F be a panel of the wall of D. As $\mathcal{I}$ is thick, there exists a chamber C dominating F and such that $C \nsubseteq \mathbb{A}$. By [Rou11, Proposition 2.9(1)], there exists an apartment A containing D and C. The set $\mathbb{A} \cap A$ is a half-apartment by Lemma 3.1 and thus $\mathbb{A} \cap A = D$, which proves the lemma. ■

Proposition 4.25 *Suppose that $\mathcal{I}$ is thick and that some group G acts strongly transitively on $\mathcal{I}$. Let P be an enclosed subset of $\mathbb{A}$ containing a generic ray and having nonempty interior. Then there exists an apartment A such that $A \cap \mathbb{A} = P$.*

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Proof One writes $P = D_1 \cap \dots \cap D_k$, where the D_i's are half-apartments of $\mathbb{A}$. One supposes that k is minimal for this writing, which means that for all $i \in [\![1,n]\!]$, $P \neq \bigcap_{j\in[\![1,k]\!]\setminus\{i\}} D_j$. For all $i \in [\![1,n]\!]$, one chooses an apartment A_i such that $\mathbb{A} \cap A_i = D_i$. Let $\phi_i \colon \mathbb{A} \xrightarrow{D_i} A_i$ and $g_i \in G$ inducing ϕ_i.

Let $g = g_1 \cdots g_k$ and $A = g.\mathbb{A}$. Then $A \cap \mathbb{A} \supset D_1 \cap \dots \cap D_k$ and g fixes $D_1 \cap \dots \cap D_k$. Let us show that $A \cap \mathbb{A} = \{x \in \mathbb{A} \mid g.x = x\}$. By Theorem 4.22, there exists $\phi \colon \mathbb{A} \xrightarrow{A\cap\mathbb{A}} A$. Moreover, $g^{-1}_{|\mathbb{A}} \circ \phi \colon \mathbb{A} \to \mathbb{A}$ fixes $D_1 \cap \dots \cap D_k$, which has nonempty interior and thus $g^{-1}_{|\mathbb{A}} \circ \phi = \mathrm{Id}_{\mathbb{A}}$, which proves that $A \cap \mathbb{A} = \{x \in \mathbb{A} \mid g.x = x\}$.

Suppose that $A \cap \mathbb{A} \supsetneq D_1 \cap \dots \cap D_k$. Let $i \in [\![1,k]\!]$ be such that there exists $a \in A \cap \mathbb{A} \setminus D_i$.

One fixes the origin of $\mathbb{A}$ in some point of $\mathring{P}$; one sets $U = \{x \in \mathbb{A} \mid j_P(x) \neq 0\}$; and one defines $\mathrm{Fr} \colon U \to \mathrm{Fr}(P)$ as in Lemma 3.18. By minimality of k and Lemma 4.23, there exists a nonempty open set V of U such that $\mathrm{Fr}(V) \subset M_i$.

By the same reasoning as in the proof of Lemma 3.21, $\mathrm{Fr}(V) \cap M_i$ is open in M_i. Consequently, there exists $v \in \mathrm{Fr}(V)$ such that $v \notin \bigcup_{j\in[\![1,k]\!]\setminus\{i\}} M_j$. Let $V' \in \mathcal{V}_U(v)$ be such that $V' \cap \bigcup_{j\in[\![1,k]\!]\setminus\{i\}} M_j = \varnothing$ and such that V' is convex. Then $V' \subset \bigcap_{j\in[\![1,k]\!]\setminus\{i\}} \mathring{D}_j$. Let $V'' = \mathrm{Fr}(V) \cap V'$. By Theorem 4.22, $[a,v] \subset A \cap \mathbb{A}$ and hence g fixes $[a,v]$. Moreover, for $u \in [a,v]$ near v, one has $u \in \bigcap_{j\in[\![1,k]\!]\setminus\{i\}} D_j$. Then $g.u = g_1 \cdots g_i.(g_{i+1} \cdots g_k.u) = g_1 \cdots g_i.u$. Moreover, $g_i.u = g_{i-1}^{-1} \cdots g_1^{-1}.u = u$. Therefore $u \in D_i$, which is absurd by choice of u. ■

Remark 4.26 In the proof above, the fact that P contains a generic ray is only used to prove that $A \cap \mathbb{A}$ is convex and that there exists an isomorphism $\phi \colon A \xrightarrow{A\cap\mathbb{A}} \mathbb{A}$. When G is an affine Kac–Moody group and $\mathcal{I}$ is its masure, we will see that these properties are true without assuming that $A \cap \mathbb{A}$ contains a generic ray. Therefore, for any enclosed subset P of $\mathbb{A}$ having nonempty interior, there exists an apartment A such that $A \cap \mathbb{A} = P$

Let $\mathbb{T}$ be a discrete homogeneous tree with valence 3 and x a vertex of $\mathbb{T}$. Then there exists no pair (A, A') of apartments such that $A \cap A' = \{x\}$. Indeed, let A be an apartment containing x and C_1, C_2 be the alcoves of A dominating x. Let A' be an apartment containing x. If A' does not contain C_1, it contains C_2 and thus $A \cap A' \neq \{x\}$. Therefore the hypothesis that "P has nonempty interior" is necessary in Proposition 4.25.

5 Axioms of Masures

5.1 Axioms of Masures in the General Case

The aim of this section is to give an axiomatics of masures other than the one of [Rou11, Rou17]. For this, we mainly use Theorem 4.22.

We fix an apartment $\mathbb{A} = (\mathcal{S}, W, \Lambda')$. A *construction* of type $\mathbb{A}$ is a set endowed with a covering of subsets called apartments and satisfying (MA1).

Let $\mathrm{cl} \in \mathcal{CL}_{\Lambda'}$. Let (MA i)=(MA1).

Let (MA ii) : if two apartments A, A' contain a generic ray, then $A \cap A'$ is enclosed and there exists an isomorphism $\phi: A \xrightarrow{A \cap A'} A'$.

Let (MA iii, cl): if $\mathfrak{R}$ is the germ of a splayed chimney and if F is a face or a germ of a chimney, then there exists an apartment containing $\mathfrak{R}$ and F.

It is easy to see that the axiom (MA ii) implies (MA4, cl) for all $\mathrm{cl} \in \mathcal{CL}_{\Lambda'}$. If $\mathrm{cl} \in \mathcal{CL}_{\Lambda'}$, then (MA iii, cl) is equivalent to (MA3, cl) because each chimney is contained in a solid chimney.

Let $\mathcal{I}$ be a construction of type $\mathbb{A}$ and $\mathrm{cl} \in \mathcal{CL}_{\Lambda'}$. One says that $\mathcal{I}$ is a *masure of type* $(1, \mathrm{cl})$ if it satisfies the axioms of [Rou11]: (MA2, cl), (MA3, cl), (MA4, cl), and (MAO). One says that $\mathcal{I}$ is a *masure of type* $(2, \mathrm{cl})$ if it satisfies (MA ii) and (MA iii, cl).

The aim of the next two subsections is to prove the following theorem.

Theorem 5.1 *Let $\mathcal{I}$ be a construction of type $\mathbb{A}$ and $\mathrm{cl} \in \mathcal{CL}_{\Lambda'}$. Then $\mathcal{I}$ is a masure of type* $(1, \mathrm{cl})$ *if and only if $\mathcal{I}$ is a masure of type* $(1, \mathrm{cl}^\#)$*, if and only if $\mathcal{I}$ is a masure of type* $(2, \mathrm{cl})$*, if and only if $\mathcal{I}$ is a masure of type* $(2, \mathrm{cl}^\#)$.

Let us introduce some other axioms and definitions. An *extended chimney* of $\mathbb{A}$ is associated with a local face $F^l = F^\ell(x, F_0^v)$ (its *basis*) and a vectorial face (its *direction*) F^v; this is the filter $\mathfrak{r}_e(F^\ell, F^v) = F^\ell + F^v$. Similarly to classical chimneys, we define shortenings and germs of extended chimneys. We use the same vocabulary for extended chimneys as for classical: splayed, solid, full, etc. We use the isomorphisms of apartments to extend these notions in constructions. Actually each classical chimney is of the form $\mathrm{cl}(\mathfrak{r}_e)$ for some extended chimney $\mathfrak{r}_e$.

Let $\mathrm{cl} \in \mathcal{CL}_{\Lambda'}$. Let (MA2′, cl): if F is a point, a germ of a preordered interval, or a splayed chimney in an apartment A, and if A' is another apartment containing F, then $A \cap A'$ contains the enclosure $\mathrm{cl}_A(F)$ of F and there exists an isomorphism from A onto A' fixing $\mathrm{cl}_A(F)$.

Let (MA2″, cl): if F is a solid chimney in an apartment A and if A' is another apartment containing F, then $A \cap A'$ contains the enclosure $\mathrm{cl}_A(F)$ of F and there exists an isomorphism from A onto A' fixing $\mathrm{cl}_A(F)$.

The axiom (MA2, cl) is a consequence of (MA2′, cl), (MA2″, cl), and (MA ii).

Let (MA iii′): if $\mathfrak{R}$ is the germ of a splayed extended chimney and if F is a local face or a germ of an extended chimney, then there exists an apartment containing $\mathfrak{R}$ and F.

Let $\mathcal{I}$ be a construction. Then $\mathcal{I}$ is said to be a *masure of type* 3 if it satisfies (MA ii) and (MA iii′).

In order to prove Theorem 5.1, we will, in fact, prove the following stronger theorem.

Theorem 5.2 *Let $\mathrm{cl} \in \mathcal{CL}_{\Lambda'}$ and $\mathcal{I}$ be a construction of type $\mathbb{A}$. Then $\mathcal{I}$ is a masure of type* $(1, \mathrm{cl})$ *if and only $\mathcal{I}$ is a masure of type* $(2, \mathrm{cl})$ *if and only if $\mathcal{I}$ is a masure of type* 3.

The proof of this theorem will be divided into two steps. In the first step, we prove that (MAO) is a consequence of variants of (MA1), (MA2), (MA3), and (MA4) (see Proposition 5.3 for a precise statement). This uses paths but not Theorem 4.22. In the second step, we prove the equivalence of the three definitions. One implication relies on Theorem 4.22.

5.1.1 Dependency of (MAO)

The aim of this subsection is to prove the following proposition.

Proposition 5.3 *Let $\mathcal{I}$ be a construction of type $\mathbb{A}$ satisfying* (MA2′), (MA iii′) *and* (MA4). *Then $\mathcal{I}$ satisfies* (MAO).

We now fix a construction $\mathcal{I}$ of type $\mathbb{A}$ satisfying (MA2′), (MA iii′), and (MA4). To prove Proposition 5.3, the key step is to prove that if B is an apartment and if $x, y \in \mathbb{A} \cap B$ are such that $x \leq_{\mathbb{A}} y$, then the image by $\rho_{-\infty}$ of the segment of B joining x to y is a $(y-x)^{++}$-path, where if $u \in \mathcal{T}$, u^{++} is the unique element of $W^v.u \cap \overline{C^v_f}$.

Let $a, b \in \mathbb{A}$. An (a,b)*-path of* $\mathbb{A}$ is a continuous piecewise linear map $[0,1] \to \mathbb{A}$ such that for all $t \in [0,1[$, $\pi'(t)^+ \in W^v.(b-a)$. When $a \leq b$, the (a,b)-paths are the $(b-a)^{++}$-paths defined in Section 3.1.2.

Let A be an apartment and $\pi\colon [0,1] \to A$ a map. Let $a, b \in A$. One says that π is an (a,b)*-path* of A if there exists $\Upsilon\colon A \to \mathbb{A}$ such that $\Upsilon \circ \pi$ is a $(\Upsilon(a), \Upsilon(b))$-path of $\mathbb{A}$.

Lemma 5.4 *Let A be an apartment and $a, b \in A$. Let $\pi\colon [0,1] \to A$ be an (a,b)-path in A and $f\colon A \to B$ an isomorphism of apartments. Then $f \circ \pi$ is an $(f(a), f(b))$-path.*

Proof Let $\Upsilon\colon A \to \mathbb{A}$ be an isomorphism such that $\Upsilon \circ \pi$ is a $(\Upsilon(a), \Upsilon(b))$-path in $\mathbb{A}$. Then $\Upsilon' = \Upsilon \circ f^{-1}\colon B \to \mathbb{A}$ is an isomorphism, $\Upsilon' \circ f \circ \pi$ is a $\big(\Upsilon'(f(a)), \Upsilon'(f(b))\big)$-path in $\mathbb{A}$, and we get the lemma. ∎

The following lemma slightly improves [Rou11, Proposition 2.7(1)]. We recall that if A is an affine space and $x, y \in A$, $[x,y)$ means the germ $\mathrm{germ}_x([x,y])$, $(x,y]$ means $\mathrm{germ}_y([x,y])$, etc.; see Section 2.4.

Lemma 5.5 *Let $\mathfrak{R}$ be the germ of a splayed extended chimney, A an apartment of $\mathcal{I}$, and $x^-, x^+ \in A$ such that $x^- \leq_A x^+$. Then there exists a subdivision $z_1 = x^-, \dots, z_n = x^+$ of $[x^-, x^+]_A$ such that for all $i \in [\![1, n-1]\!]$ there exists an apartment A_i containing $[z_i, z_{i+1}]_A \cup \mathfrak{R}$ such that there exists an isomorphism $\phi_i\colon A \xrightarrow{[z_i, z_{i+1}]_{A_i}} A_i$.*

Proof Let $u \in [x^-, x^+]$. By (MA iii′), applied to $(x^-, u]$ and $[u, x+)$, there exist apartments A^-_u and A^+_u containing $\mathfrak{R} \cup (x^-, u]$ and $\mathfrak{R} \cup [u, x^+)$ and by (MA2′), there exist isomorphisms

$$\phi^+_u\colon A \xrightarrow{(x^-,u]} A^-_u \quad \text{and} \quad \phi^-_u\colon A \xrightarrow{[u,x^+)} A^+_u.$$

For all $u \in [x^-, x^+]$ and $\epsilon \in \{-,+\}$, one chooses a convex set $V^\epsilon_u \in [u, x^\epsilon)$ such that $V^\epsilon_u \subset A \cap A^\epsilon_u$ and V^ϵ_u is fixed by ϕ^ϵ_u. If $u \in [x^-, x^+]$, one sets $V_u = \mathrm{Int}_{[x^-,x^+]_A}(V^+_u \cup V^-_u)$. By compactness of $[x^-, x^+]$, there exists a finite set K and a map $\epsilon\colon K \to \{-,+\}$ such that $[x^-, x^+] = \bigcup_{k \in K} V^{\epsilon(k)}_k$ and the lemma follows. ∎

Let $\mathfrak{q}$ be a sector-germ. Then $\mathfrak{q}$ is an extended chimney. Let A be an apartment containing $\mathfrak{q}$. The axioms (MA2′), (MA iii′), and (MA4) enable one to define a retraction $\rho: \mathcal{I} \overset{\mathfrak{q}}{\to} \mathbb{A}$ as in [Rou11, §2.6].

Lemma 5.6 *Let A and B be two apartments, $\mathfrak{q}$ a sector-germ of B, and $\rho: \mathcal{I} \overset{\mathfrak{q}}{\to} B$. Let $x, y \in A$ be such that $x \leq_A y$. Let $\tau: [0,1] \to A$ mapping each $t \in [0,1]$ on $(1-t)x +_A ty$ and $f: A \to B$ be an isomorphism. Then $\rho \circ \tau$ is a $(f(x), f(y))$-path of B.*

Proof By Lemma 5.5, there exist $k \in \mathbb{N}$ and $t_1 = 0 < \cdots < t_k = 1$ such that, for all $i \in [\![1, k-1]\!]$, there exists an apartment A_i containing $\tau([t_i, t_{i+1}]) \cup \mathfrak{q}$ such that there exists an isomorphism $\phi_i: A \xrightarrow{\tau([t_i, t_{i+1}])} A_i$.

If $i \in [\![1, k-1]\!]$, one denotes by ψ_i the isomorphism $A_i \overset{\mathfrak{q}}{\to} B$. Then for $t \in [t_i, t_{i+1}]$, one has $\rho(\tau(t)) = \psi_i \circ \phi_i(\tau(t))$. Let $\Upsilon: B \to \mathbb{A}$ be an isomorphism. By (MA1), for all $i \in [\![1, k]\!]$, there exists $w_i \in W$ such that $\Upsilon \circ \psi_i \circ \phi_i = w_i \circ \Upsilon \circ f$.

Let $i \in [\![1, k-1]\!]$ and $t \in [t_i, t_{i+1}]$. Then

$$\Upsilon \circ \rho \circ \tau(t) = \Upsilon \circ \psi_i \circ \phi_i \circ \tau(t) = (1-t)w_i \circ \Upsilon \circ f(x) + tw_i \circ \Upsilon \circ f(y).$$

Therefore, $\rho \circ \tau$ is a $(f(x), f(y))$-path in B. ∎

Lemma 5.7 *Let $\lambda \in \overline{C^v_f}$ and $\pi: [0,1] \to \mathbb{A}$ be a λ-path. Then*

$$\pi(1) - \pi(0) \leq_{Q^\vee} \lambda.$$

Proof By definition, there exists $k \in \mathbb{N}$, $(t_i) \in [0,1]^k$, and $(w_i) \in (W^v)^k$ such that $\sum_{i=1}^k t_i = 1$ and $\pi(1) - \pi(0) = \sum_{i=1}^k t_i . w_i . \lambda$. Therefore $\pi(1) - \pi(0) - \lambda = \sum_{i=1}^k t_i (w_i . \lambda - \lambda)$ and thus $\pi(1) - \pi(0) - \lambda \leq_{Q^\vee} 0$ by Lemma 3.5. ∎

Lemma 5.8 *Let $x, y \in \mathbb{A}$ be such that $x \leq_{\mathbb{A}} y$, and let B be an apartment containing x, y. Let $\tau_B: [0,1] \to B$ be defined by $\tau_B(t) = (1-t)x +_B ty$. Let $\mathfrak{s}$ be a sector-germ of $\mathbb{A}$ and $\rho_{\mathfrak{s}}: \mathcal{I} \overset{\mathfrak{s}}{\to} \mathbb{A}$. Then $x \leq_B y$ and $\pi_{\mathbb{A}} := \rho_{\mathfrak{s}} \circ \tau_B$ is an (x, y)-path of $\mathbb{A}$.*

Proof Possibly changing the choice of $\overline{C^v_f}$, one can suppose that $y - x \in \overline{C^v_f}$. Let $\mathfrak{q}$ be a sector-germ of B, $\rho_B: \mathcal{I} \overset{\mathfrak{q}}{\to} B$, and $\tau_{\mathbb{A}}: [0,1] \to \mathbb{A}$ be defined by $\tau_{\mathbb{A}}(t) = (1-t)x + ty$. Let $\phi: \mathbb{A} \to B$. By Lemma 5.6, $\pi_B := \rho_B \circ \tau_{\mathbb{A}}$ is a $(\phi(x), \phi(y))$-path of B from x to y. Therefore $x \leq_B y$. Let $\psi = \phi^{-1}: B \to \mathbb{A}$. Composing ϕ by some $w \in W^v$ if necessary, one can suppose that $\psi(y) - \psi(x) \in \overline{C^v_f}$.

By Lemma 5.6, $\pi_{\mathbb{A}}$ is a $(\psi(x), \psi(y))$-path of $\mathbb{A}$. By Lemma 5.7, we deduce that $y - x \leq_{Q^\vee} \psi(y) - \psi(x)$.

By Lemma 5.4, $\psi \circ \pi_B$ is an (x, y)-path of $\mathbb{A}$ from $\psi(x)$ to $\psi(y)$. By Lemma 5.7, we deduce that $\psi(y) - \psi(x) \leq_{Q^\vee} y - x$. Therefore $x - y = \psi(x) - \psi(y)$ and $\pi_{\mathbb{A}}$ is an (x, y)-path of $\mathbb{A}$. ∎

If $x, y \in \mathcal{I}$, one says that $x \leq y$ if there exists an apartment A containing x, y and such that $x \leq_A y$. By Lemma 5.8, this does not depend on the choice of A: if $x \leq y$, then for all apartments B containing x, y one has $x \leq_B y$. However, one does not know yet that $\leq$ is a preorder: the proof of [Rou11, Théorème 5.9] uses (MAO).

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Lemma 5.9 ([Héb17, Lemma 3.6]) *Let* $\tau: [0,1] \to \mathcal{I}$ *be a segment such that* $\tau(0) \leq \tau(1)$ *such that* $\tau(1) \in \mathbb{A}$ *and such that there exists* $v \in \overline{C_f^v}$ *such that* $(\rho_{-\infty} \circ \tau)' = v$. *Then* $\tau([0,1]) \subset \mathbb{A}$ *and thus* $\rho_{-\infty} \circ \tau = \tau$.

Proof Let A be an apartment such that τ is a segment of A. Then τ is increasing for $\leq_A$ and thus τ is increasing for $\leq$. Let $x, y \in A$ be such that $\tau(t) = (1-t)x + ty$ for all $t \in [0,1]$. Let us first prove that τ is increasing for $\leq$. It suffices to prove that $x \leq y$. By (MA iii′), there exists $u \in]0,1[$ such that there exists an apartment A containing $\tau([0,u])$ and $-\infty$. Let $\phi: A \xrightarrow{-\infty} \mathbb{A}$. One has $\phi(\tau(u)) = \rho_{-\infty}(\tau(u)) = \rho_{-\infty}(\tau(0)) + uv = \phi(\tau(0)) + uv$. Thus $\phi(\tau(u)) \geq \phi(\tau(0))$ and hence $\tau(u) \geq \tau(0)$. As τ is a segment of A, it suffices to prove that there exists $u > 0$ such that $\tau(u) \geq \tau(0)$. Therefore, τ is increasing for $\leq$.

Suppose that $\tau([0,1]) \nsubseteq \mathbb{A}$. Let $u = \sup\{t \in [0,1] \mid \tau(t) \notin \mathbb{A}\}$. Let us prove that $\tau(u) \in \mathbb{A}$. If $u = 1$, this is our hypothesis. Suppose $u < 1$. Then by (MA2′) applied to $]\tau(u), \tau(1))$, $\mathbb{A}$ contains $\mathrm{cl}_A(]\tau(u), \tau(1)))$ and thus $\mathbb{A}$ contains $\tau(u)$.

By (MA iii′), there exists an apartment B containing $\tau((0,u]) \cup -\infty$ and by (MA4), there exists an isomorphism $\phi: B \xrightarrow{\tau(u) - \overline{C_f^v}} \mathbb{A}$. For all $t \in [0,u]$ close enough to u, one has $\phi(\tau(t)) = \rho_{-\infty}(\tau(t))$. By hypothesis, for all $t \in [0,u]$, $\rho_{-\infty}(\tau(t)) \in \tau(u) - \overline{C_f^v}$. Therefore, for t close enough to u, $\phi(\tau(t)) = \tau(t) \in \mathbb{A}$; this is absurd by choice of u, and thus $\tau([0,1]) \subset \mathbb{A}$. ∎

We can now prove Proposition 5.3: $\mathcal{I}$ satisfies (MAO).

Proof Let $x, y \in \mathbb{A}$ be such that $x \leq_{\mathbb{A}} y$ and let B be an apartment containing $\{x, y\}$. We suppose that $y - x \in \overline{C_f^v}$. Let $\pi_{\mathbb{A}}: [0,1] \to \mathbb{A}$ mapping each $t \in [0,1]$ on $\rho_{-\infty}((1-t)x +_B ty)$. By Lemma 5.8, $\pi_{\mathbb{A}}$ is an (x,y)-path from x to y. By Lemma 3.6, $\pi_{\mathbb{A}}(t) = x + t(y-x)$ for all $t \in [0,1]$. Then by Lemma 5.9, $\pi_{\mathbb{A}}(t) = (1-t)x +_B ty$ for all $t \in [0,1]$. In particular $[x,y] = [x,y]_B$ and thus $\mathcal{I}$ satisfies (MAO). ∎

5.1.2 Equivalence of the Axioms

As each chimney or face contains an extended chimney or a local face of the same type, if $\mathrm{cl} \in \mathcal{CL}_{\Lambda'}$, (MA iii, cl) implies (MA iii′). Therefore a masure of type $(2, \mathrm{cl})$ is also a masure of type 3.

If A is an apartment and F is a filter of A, then $\mathrm{cl}_A(F) \subset \mathrm{cl}_A^{\#}(F)$. Therefore, for all $\mathrm{cl} \in \mathcal{CL}_{\Lambda'}$, (MA2′, $\mathrm{cl}^{\#}$) implies (MA2′, cl) and (MA iii, $\mathrm{cl}^{\#}$) implies (MA iii, cl).

Lemma 5.10 *Let* $\mathrm{cl} \in \mathcal{CL}_{\Lambda'}$ *and* $\mathcal{I}$ *be a masure of type* $(1, \mathrm{cl})$. *Then* $\mathcal{I}$ *is a masure of type* $(2, \mathrm{cl})$.

Proof By Theorem 4.22, $\mathcal{I}$ satisfies (MA ii). By [Rou11, Conséquence 2.2(3)], $\mathcal{I}$ satisfies (MA iii, cl). ∎

By abuse of notation, if $\mathcal{I}$ is a masure of any type and if $\mathfrak{q}, \mathfrak{q}'$ are adjacent sectors of $\mathcal{I}$, we denote by $\mathfrak{q} \cap \mathfrak{q}'$ the maximal face of $\overline{\mathfrak{q}} \cap \overline{\mathfrak{q}'}$. By [Rou11, §3], this has a meaning for masures of type 1 and by (MA ii) for masures of type 2 and 3.

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Lemma 5.11 *Let* $\mathcal{I}$ *be a masure of type* 3*. Let* A *be an apartment. Let* $\mathcal{X}$ *be a filter of* A *such that for all sector-germs* $\mathfrak{s}$ *of* $\mathcal{I}$*, there exists an apartment containing* $\mathcal{X}$ *and* $\mathfrak{s}$*. Then if* B *is an apartment containing* $\mathcal{X}$*,* B *contains* $\mathrm{cl}^{\#}(\mathcal{X})$ *and there exists an isomorphism* $\phi\colon A \xrightarrow{\mathrm{cl}^{\#}(\mathcal{X})} B$*.*

Proof Let $\mathfrak{q}$ and $\mathfrak{q}'$ be sector-germs of A and B of the same sign. By (MA iii′), there exists an apartment C containing $\mathfrak{q}$ and $\mathfrak{q}'$. Let $\mathfrak{q}_1 = \mathfrak{q}, \dots, \mathfrak{q}_n = \mathfrak{q}'$ be a gallery of sector-germs from $\mathfrak{q}$ to $\mathfrak{q}'$ in C. One sets $A_1 = A$ and $A_{n+1} = B$. By hypothesis, for all $i \in [\![2, n]\!]$, there exists an apartment A_i containing $\mathfrak{q}_i$ and $\mathcal{X}$. For all $i \in [\![1, n-1]\!]$, $\mathfrak{q}_i \cap \mathfrak{q}_{i+1}$ is a splayed chimney and $A_i \cap A_{i+1} \supset \mathfrak{q}_i \cap \mathfrak{q}_{i+1}$. Therefore $A_i \cap A_{i+1}$ is enclosed and there exists $\phi_i\colon A_i \xrightarrow{A_i \cap A_{i+1}} A_{i+1}$. The set $A_n \cap A_{n+1}$ is also enclosed and there exists $\phi_n\colon A_n \xrightarrow{A_n \cap A_{n+1}} A_{n+1}$.

If $i \in [\![1, n+1]\!]$, one sets $\psi_i = \phi_{i-1} \circ \cdots \circ \phi_1$. Then ψ_i fixes $A_1 \cap \cdots \cap A_i$.

Let $i \in [\![1, n]\!]$ and suppose that $A_1 \cap \cdots \cap A_i$ is enclosed in A. The isomorphism ψ_i fixes $A_1 \cap \cdots \cap A_i$ and thus we deduce that $A_1 \cap \cdots \cap A_i = \psi_i(A_1 \cap \cdots \cap A_i)$ is enclosed in A_i. Moreover, $A_i \cap A_{i+1}$ is enclosed in A_i and thus $A_1 \cap \cdots \cap A_{i+1}$ is enclosed in A_i. Consequently $A_1 \cap \cdots \cap A_{i+1} = \psi_i^{-1}(A_1 \cap \cdots \cap A_{i+1})$ is enclosed in A. Let $X = A_1 \cap \cdots \cap A_{n+1}$. By induction, X is enclosed in A and $\phi := \psi_n$ fixes X. As $X \supset \mathcal{X}$, we deduce that $X \in \mathrm{cl}^{\#}(\mathcal{X})$ and we get the lemma. ∎

Lemma 5.12 *Let* $\mathcal{I}$ *be a masure of type* 3*. Then for all* $\mathrm{cl} \in \mathcal{CL}_{\Lambda'}$*,* $\mathcal{I}$ *satisfies* (MA iii, cl).

Proof Each face is contained in the finite enclosure of a local face and each chimney is contained in the finite enclosure of an extended chimney. Thus by Lemma 5.11, applied when $\mathcal{X}$ is a local face and a germ of a chimney, $\mathcal{I}$ satisfies (MA iii, $\mathrm{cl}^{\#}$). Consequently for all $\mathrm{cl} \in \mathcal{CL}_{\Lambda'}$, $\mathcal{I}$ satisfies (MA iii, cl), hence (MA3, cl) and the lemma is proved. ∎

Lemma 5.13 *Let* $\mathcal{I}$ *be a masure of type* 3 *and* $\mathrm{cl} \in \mathcal{CL}_{\Lambda'}$*. Then* $\mathcal{I}$ *satisfies* (MA2′, cl).

Proof If A is an apartment and F is a filter of A, then $\mathrm{cl}(F) \subset \mathrm{cl}^{\#}(F)$. Therefore it suffices to prove that $\mathcal{I}$ satisfies (MA2′, $\mathrm{cl}^{\#}$). We conclude the proof by applying Lemma 5.11 applied when $\mathcal{X}$ is a point, a germ of a preordered segment. ∎

Using Proposition 5.3, we deduce that a masure of type 2 or 3 satisfies (MAO), as (MA4) is a consequence of (MA ii).

Lemma 5.14 *Let* $\mathcal{I}$ *be a masure of type* 3*. Let* $\mathfrak{r}$ *be a chimney of* $\mathbb{A}$*,* $\mathfrak{r} = \mathfrak{r}(F^{\ell}, F^{v})$*, where* F^{ℓ}*, resp.,* F^{v}*, is a local face, resp., vectorial face, of* A*. Let* $\mathfrak{R}^{\#} = \mathrm{germ}_{\infty}(\mathrm{cl}^{\#}(F^{\ell}, F^{v}))$*. Let* A *be an apartment containing* $\mathfrak{r}$ *and* $\mathfrak{R}^{\#}$ *and such that there exists* $\phi\colon \mathbb{A} \xrightarrow{\mathfrak{R}^{\#}} A$*. Then* $\phi\colon \mathbb{A} \overset{\mathfrak{r}}{\to} A$*.*

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Proof One can suppose that $F^v \subset \overline{C_f^v}$. Let $U \in \mathfrak{R}^\#$ such that U is enclosed, $U \subset A \cap \mathbb{A}$ and such that U is fixed by ϕ. One writes $U = \bigcap_{i=1}^k D(\beta_i, k_i)$, with $\beta_1, \dots, \beta_k \in \Phi$ and $(k_1, \dots, k_r) \in \prod_{i=1}^r \Lambda'_{\beta_i}$.

Let $\xi \in F^v$ be such that $U \in \mathrm{cl}(F^\ell + F^v + \xi)$. Let $J = \{i \in [\![1, k]\!] \mid \beta_i(\xi) \neq 0\}$. For all $i \in [\![1, r]\!]$, one has $D(\beta_i, k_i) \supset n\xi$ for $n \gg 0$. Thus $\beta_i(\xi) > 0$ for all $i \in J$. One has $U - \xi = \bigcap_{i=1}^k D(\beta_i, k_i + \beta_i(\xi))$. Let $\lambda \in]1, +\infty[$ be such that for all $i \in J$, there exists $\widetilde{k_i} \in \Lambda'_{\beta_i}$ such that $k_i + \beta_i(\xi) \leq \widetilde{k_i} \leq k_i + \lambda\beta_i(\xi)$. Let $\widetilde{U} = \bigcap_{i=1}^k D(\beta_i, \widetilde{k_i})$. Then $U - \xi \subset \widetilde{U} \subset U - \lambda\xi$. Therefore, $\widetilde{U} \in \mathfrak{r}$. Let $V' \in \mathfrak{r}$ be such that $V' \subset A \cap \mathbb{A}$ and such that $V' + F^v \subset V'$. Then $V := \widetilde{U} \cap V' \in \mathfrak{r}$. Let $v \in V$ and $\delta \subset F^v$ be the ray based at 0 and containing ξ. By the proof of of [Rou11, Proposition 5.4] (which uses only (MA1), (MA2′), (MA3), (MA4) and (MAO)), there exists $g_v \colon \mathbb{A} \xrightarrow{v+\delta} A$. As $V \subset U - \lambda\xi$, there exists a shortening δ' of $v + \delta$ contained in U. Then $g_v^{-1} \circ \phi \colon \mathbb{A} \to \mathbb{A}$ fixes δ'. Consequently, $g_v^{-1} \circ \phi$ fixes the support of δ' and thus ϕ fixes v: ϕ fixes V. Therefore ϕ fixes $\mathfrak{r}$, and the lemma follows. ∎

Lemma 5.15 *Let $\mathcal{I}$ be a masure of type* 3 *and* $\mathrm{cl} \in \mathcal{CL}_{\Lambda'}$. *Then $\mathcal{I}$ satisfies* (MA2″, cl).

Proof Let $\mathfrak{r} = \mathrm{cl}(F^l, F^v)$ be a solid chimney of an apartment A and A' be an apartment containing $\mathfrak{r}$. One supposes that $A = \mathbb{A}$. Let $\mathfrak{r}^\# = \mathrm{cl}^\#(F^l, F^v)$, resp., $\mathfrak{r}_e = F^l + F^v$, and $\mathfrak{R}^\#$, resp., $\mathfrak{R}_e$, be the germ of $\mathfrak{r}^\#$, resp., $\mathfrak{r}_e$). By Lemma 5.11 applied with $\mathcal{X} = \mathfrak{R}_e$, there exists $\phi \colon A \xrightarrow{\mathfrak{R}^\#} A'$. By Lemma 5.14, ϕ fixes $\mathfrak{r}$ and thus $\mathcal{I}$ satisfies (MA2", cl). ∎

We can now prove Theorem 5.2. Let $\mathrm{cl} \in \mathcal{CL}_{\Lambda'}$. By Lemma 5.10, a masure of type $(1, \mathrm{cl})$ is also a masure of type $(2, \mathrm{cl})$ and thus it is a masure of type 3. By Lemma 5.12, Lemma 5.13, and Lemma 5.15, a masure of type 3 is a masure of type $(1, \mathrm{cl})$, which concludes the proof of the theorem.

5.2 Friendly Pairs in $\mathcal{I}$

Let $\mathbb{A} = (\mathbb{A}, W, \Lambda')$ be an apartment. Let $\mathcal{I}$ be a masure of type $\mathbb{A}$. We now use the finite enclosure $\mathrm{cl} = \mathrm{cl}^\#_{\Lambda'}$, which makes sense by Theorem 5.1. A family $(F_j)_{j \in J}$ of filters in $\mathcal{I}$ is said to be *friendly* if there exists an apartment containing $\bigcup_{j \in J} F_j$. In this section, we obtain friendliness results for pairs of faces, improving results of [Rou11, §5]. We will use it to give a very simple axiomatics of masures in the affine case. These kinds of results also have interest in their own right: the definitions of the Iwahori–Hecke algebra of [BPGR16] and of the parahoric Hecke algebras of [AH17] rely on the existence of apartments containing pairs of faces.

If $x \in \mathcal{I}$, $\epsilon \in \{-, +\}$, and A is an apartment, one denotes by $\mathcal{F}_x$, resp., $\mathcal{F}^\epsilon$, $\mathcal{F}^\epsilon(A)$, $\mathcal{C}_x, \dots$, the set of faces of $\mathcal{I}$ based at x, resp., and of sign ϵ, and contained in A, the set of chambers of $\mathcal{I}$ based at $x, \dots$. If $\mathcal{X}$ is a filter, one denotes by $\mathcal{A}(\mathcal{X})$ the set of apartments containing $\mathcal{X}$.

Lemma 5.16 *Let A be an apartment of $\mathcal{I}$, $a \in A$, and $C_1, C_2 \in \mathcal{C}_a(A)$. Let $\mathcal{D}_a$ be the set of half-apartments of A whose wall contains a. Suppose that $C_1 \neq C_2$. Then there exists $D \in \mathcal{D}_a$ such that $D \supset C_1$ and $D \not\supset C_2$.*

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Proof Let C_1^v and C_2^v be vectorial chambers of A such that $C_1 = F(a, C_1^v)$ and $C_2 = F(a, C_2^v)$. Suppose that for all $D \in \mathcal{D}_a$ such that $D \supset C_1$, one has $D \supset C_2$. Let $X \in C_1$. There exist half-apartments $D_1, \dots, D_k$ and $\Omega \in \mathcal{V}_A(a)$ such that $X \supset \bigcap_{i=1}^k D_i^\circ \supset \Omega \cap (a + C_1^v)$.

Let $J = \{j \in [\![1, k]\!] \mid D_j \notin \mathcal{D}_a\}$. For all $j \in J$, one chooses $\Omega_j \in \mathcal{V}_A(a)$ such that $D_j^\circ \supset \Omega_j$. If $j \in [\![1, k]\!] \backslash J$, $D_j \supset C_1$, thus $D_j \supset C_2$ and hence $D_j^\circ \supset C_2$. Therefore, there exists $\Omega_j \in \mathcal{V}_A(a)$ such that $D_j^\circ \supset \Omega_j \cap (x + C_2^v)$. Hence

$$X \supset \bigcap_{j=1}^k D_j^\circ \supset \Big(\bigcap_{j=1}^k \Omega_j\Big) \cap (x + C_2^v),$$

thus $X \in C_2$ and $C_1 \supset C_2$.

Let $D \in \mathcal{D}_a$ such that $D \supset C_2$. Suppose that $D \not\supset C_1$. Let D' be the half-apartment opposite D. Then $D' \supset C_1$ and therefore $D' \supset C_2$: this is absurd. Therefore for all $D \in \mathcal{D}_a$ such that $D \supset C_2$, one has $D \supset C_1$. By the same reasoning as above, we deduce that $C_2 \supset C_1$ and thus $C_1 = C_2$. This is absurd, and the lemma is proved. ∎

The following proposition improves [Rou11, Proposition 5.1]. It is the analogue of axiom (I1) of buildings (see the Introduction).

Proposition 5.17 *Let $\{x, y\}$ be a friendly pair in $\mathcal{I}$.*

(i) *Let $A \in \mathcal{A}(\{x, y\})$ and δ be a ray of A based at x and containing y (if $y \neq x$, δ is unique) and $F_x \in \mathcal{F}_x$. Then (δ, F_x) is friendly. Moreover, there exists $A' \in \mathcal{A}(\delta \cup F_x)$ such that there exists an isomorphism $\phi: A \overset{\delta}{\to} A'$.*

(ii) *Let $(F_x, F_y) \in \mathcal{F}_x \times \mathcal{F}_y$. Then (F_x, F_y) is friendly.*

Proof We begin by proving (i). Let C_x be a chamber of $\mathcal{I}$ containing F_x. Let C be a chamber of A based at x and having the same sign as C_x. By [Rou11, Proposition 5.1], there exists an apartment B containing C_x and C. Let $C_1 = C, \dots, C_n = C_x$ be a gallery in B from C to C_x. If $i \in [\![1, n]\!]$, one lets $\mathcal{P}_i$ be the statement "there exists an apartment A_i containing C_i and δ such that there exists an isomorphism $\phi: A \overset{\delta}{\to} A_i$". The property $\mathcal{P}_1$ is true by taking $A_1 = A$. Let $i \in [\![1, n-1]\!]$ be such that $\mathcal{P}_i$ is true. If $C_{i+1} = C_i$, then $\mathcal{P}_{i+1}$ is true. Suppose $C_i \neq C_{i+1}$. Let A_i be an apartment containing C_i and δ. By Lemma 5.16, there exists a half-apartment D of A whose wall contains x and such that $C_i \subset D$ and $C_{i+1} \notin D$. As C_i and C_{i+1} are adjacent, the wall M of D is the wall separating C_i and C_{i+1}. By (MA2), there exists an isomorphism $\phi: B \overset{C_i}{\longrightarrow} A_i$. Let $M' = \phi(M)$ and D_1, D_2 be the half-apartments of A_i delimited by M'. Let $j \in \{1, 2\}$ such that $D_j \supset \delta$. By [Rou11, Proposition 2.9(1)], there exists an apartment A_{i+1} containing D_j and C_{i+1}. Let $\psi_i: A \overset{\delta}{\to} A_i$ and $\psi: A_i \overset{D_j}{\longrightarrow} A_{i+1}$. Then $\psi \circ \psi_i: A \overset{\delta}{\to} A_{i+1}$. Therefore $\mathcal{P}_{i+1}$ is true. Consequently, $\mathcal{P}_n$ is true, which proves (i).

Let us prove (ii), which is very similar to (i). As a particular case of (i), there exists an apartment A' containing F_x and y. Let C_y be a chamber of $\mathcal{I}$ containing F_y. Let C be a chamber of A' based at y and of the same sign as F_y. Let $C_1 = C, \dots, C_n = C_y$ be a gallery of chambers from C to C_y (which exists by [Rou11, Proposition 5.1]). By the same reasoning as above, for all $i \in [\![1, n]\!]$, there exists an apartment containing F_x and C_i, which proves (ii). ■

5.3 Axioms of Masures in the Affine Case

In this section, we study the particular case of masures associated with the irreducible affine Kac–Moody matrix A, which means that A satisfies condition (aff) of [Kac94, Theorem 4.3].

Let $\mathcal{S}$ be a generating root system associated with an irreducible and affine Kac–Moody matrix and $\mathbb{A} = (\mathcal{S}, W, \Lambda')$ be an apartment. By [Rou11, §1.3], one has $\mathring{\mathcal{T}} = \{v \in \mathbb{A} \mid \delta(v) > 0\}$ for some imaginary root $\delta \in Q^+\backslash\{0\}$ and $\mathcal{T} = \mathring{\mathcal{T}} \cup \mathbb{A}_{in}$, where $\mathbb{A}_{in} = \bigcap_{i \in I} \ker(\alpha_i)$.

We fix an apartment $\mathbb{A}$ of affine type. Let (MA af i)=(MA1).

Let (MA af ii): let A and B be two apartments. Then $A \cap B$ is enclosed and there exists $\phi: A \xrightarrow{A \cap B} B$.

Let (MA af iii)= (MA iii).

The aim of this subsection is to prove the following theorem.

Theorem 5.18 *Let $\mathcal{I}$ be a construction of type $\mathbb{A}$ and* cl $\in \mathcal{CL}_{\Lambda'}$. *Then $\mathcal{I}$ is a masure for* cl *if and only if $\mathcal{I}$ satisfies* (MA af i), (MA af ii) *and* (MA af iii, cl) *if and only if $\mathcal{I}$ satisfies* (MA af i), (MA af ii) *and* (MA af iii, $\text{cl}^\#$).

Remark 5.19 Actually, we do not know if these axioms are true for masures associated with indefinite Kac–Moody groups. We do not know if the intersection of two apartments is always convex in a masure.

The fact that we can exchange (MA af iii, $\text{cl}^\#$) and (MA af iii, cl) for all cl $\in \mathcal{CL}_{\Lambda'}$ follows from Theorem 5.2. The fact that a construction satisfying (MA af ii) and (MA af iii, $\text{cl}^\#$) is a masure is clear and does not use the fact that $\mathbb{A}$ is associated with an affine Kac–Moody matrix. It remains to prove that a masure of type $\mathbb{A}$ satisfies (MA af ii), which is the aim of this subsection.

Lemma 5.20 *Let A and B be two apartments such that there exist $x, y \in A \cap B$ such that $x \mathring{\leq} y$ and $x \neq y$. Then $A \cap B$ is convex.*

Proof One identifies A and $\mathbb{A}$. Let $a, b \in \mathbb{A} \cap B$. If $\delta(a) \neq \delta(b)$, then $a \leq b$ or $b \leq a$ and $[a, b] \subset B$ by (MAO). Suppose $\delta(a) = \delta(b)$. As $\delta(x) \neq \delta(y)$, one can suppose that $\delta(a) \neq \delta(x)$. Then $[a, x] \subset B$. Let $(a_n) \in [a, x]^{\mathbb{N}}$ be such that $\delta(a_n) \neq \delta(a)$ for all $n \in \mathbb{N}$ and $a_n \to a$. Let $t \in [0,1]$. Then $ta_n + (1-t)b \in B$ for all $n \in \mathbb{N}$ and by Proposition 3.9, $ta + (1-t)b \in B$: $\mathbb{A} \cap B$ is convex. ■

Lemma 5.21 *Let A and A' be two apartments of $\mathcal{I}$. Then $A \cap A'$ is convex. Moreover, if $x, y \in A \cap A'$, there exists an isomorphism $\phi: A \xrightarrow{[x,y]_A} A'$.*

Proof Let $x, y \in A \cap A'$ be such that $x \neq y$. Let C_x be a chamber of A based at x and C_y be a chamber of A' based at y. Let B be an apartment containing C_x and C_y,

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which exists by Proposition 5.17. By Lemma 5.20, $A \cap B$ and $A' \cap B$ are convex and by Proposition 3.26, there exist isomorphisms $\psi\colon A \xrightarrow{A\cap B} B$ and $\psi'\colon B \xrightarrow{A'\cap B} A'$. Therefore $[x,y]_A = [x,y]_B = [x,y]_{A'}$. Moreover, $\phi = \psi' \circ \psi$ fixes $[x,y]_A$ and the lemma is proved. ∎

Theorem 5.22 *Let A and B be two apartments. Then $A \cap B$ is enclosed and there exists an isomorphism $\phi\colon A \xrightarrow{A\cap B} B$.*

Proof The fact that $A \cap B$ is enclosed is a consequence of Lemma 5.21 and Proposition 3.22. By Proposition 3.14, there exist $\ell \in \mathbb{N}$, enclosed subsets $P_1, \dots, P_\ell$ of A such that $\operatorname{supp}(A \cap B) = \operatorname{supp}(P_j)$ and isomorphisms $\phi_j\colon A \xrightarrow{P_j} B$ for all $j \in [\![1,\ell]\!]$. Let $x \in \operatorname{Int}_r(P_1)$ and $y \in A \cap B$. By Lemma 5.21, there exists $\phi_y\colon A \xrightarrow{[x,y]} B$. Then $\phi_y^{-1} \circ \phi_1$ fixes a neighborhood of x in $[x,y]$ and thus ϕ_1 fixes y, which proves the theorem. ∎

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