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SYMMETRIES OF THE SELF-DUAL YANG-MILLS EQUATIONS

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It has been conjectured by R. S. Ward that the self-dual Yang-Mills Equations (SDYMEs) form a "master system" in the sense that most known integrable ordinary and partial differential equations are obtainable as reductions. We systematically construct the group of symmetries of the SDYMEs on $\mathbf{R}^4$ with semisimple gauge group of finite dimension and show that this yields only the well known gauge and conformal symmetries.

1. INTRODUCTION

The self-dual Yang-Mills equations are of central importance to the theory of integrable equations and have found a remarkable number of applications in physics and mathematics. These equations arise in the context of gauge theory [**12**], in classical general relativity [**14, 5**] and can be used as a powerful tool in the analysis of 4-manifolds [**2**].

In 1985 Ward [**13**] conjectured that

> ... many (and perhaps all?) of the ordinary or partial differential equations that are regarded as being integrable or solvable may be obtained from the self-duality equations (or its generalisations) by reduction.

Naturally one would like to explore reductions of the self-dual Yang-Mills equations as a means of discovering interesting integrable systems. One can systematically produce a large class of reductions of a system such as the self-dual Yang-Mills equations by exploiting the system's symmetries.

The Lie-point symmetries of a system of equations are transformations of the independent and dependent variables which do not contain derivatives and which leave the form of the system invariant. In the present note we employ the method of prolongation [**6**] to find the identity component of the group of Lie-point symmetries of the SDYM system over $\mathbf{R}^4$ with semi-simple gauge group of finite dimension. It will be seen that, in essence, this group contains only the well known gauge and conformal transformations [**1**], a result already known for $SU(2)$ [**3**].

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Quite early in the history of the subject, it was realised that the SDYMEs have a natural geometric interpretation in terms of the curvature of a principal fibre bundle. Let G be a Lie group with Lie algebra LG and let $\{x^\mu\}_{\mu=0,\dots,3}$ be coordinates on $\mathbf{R}^4$. Given $A_\mu(\mathrm{x}) \in LG$, $\mu = 0, \dots, 3$ we introduce the covariant derivatives

$$\nabla_\mu = \partial_\mu + A_\mu, \tag{1}$$

and their commutators

$$F_{\mu\nu} = [\nabla_\mu, \nabla_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{2}$$

The self-dual Yang-Mills equations (on $\mathbf{R}^4$) are simply

$$F^n_{01} = F^n_{23},\ F^n_{02} = F^n_{31},\ F^n_{03} = F^n_{12}. \tag{3}$$

In [4], La finds those Lie-point symmetries of (3) which do not depend explicitly on the independent variables $\{x^\mu\}$. In the present article we explicitly produce the conformal symmetries of the SDYMEs and show that spontaneous symmetry breaking cannot occur in this system, even at the PDE level. In other words, the only non-gauge symmetries are those of the underlying manifold $\mathbf{R}^4$.

A very compact way of writing the self-dual Yang-Mills equations was introduced by Pohlmeyer [11]. In terms of the null coordinates

$$\alpha = x^0 + ix^3,\ \overline{\alpha} = x^0 - ix^3,\ \beta = x^1 + ix^2,\ \overline{\beta} = x^1 - ix^2, \tag{4}$$

the SDYMEs take the form

$$\partial_{\overline{\alpha}}(J^{-1}\partial_\alpha J) + \partial_{\overline{\beta}}(J^{-1}\partial_\beta J) = 0, \tag{5}$$

where $J \in G$.

In [8], Papachristou and Harrison calculate the Lie-point symmetries of Pohlmeyer's form of the self-dual Yang-Mills equations (5). However, as Ward [13] has noted, (5) does not possess the same symmetries as (3). In particular, it no longer exhibits the SO(4) invariance which is present in the original system.

2. The Symmetries of the SDYM Equations

In this section we shall apply the prolongation method outlined in Olver [6] to find a Lie group of symmetries of the self-dual Yang-Mills equations. This procedure will only exhibit the component of the full symmetry group which is connected to the identity.

Let $\{X_m\}_{m=1,\dots,d}$ be a basis for the d-dimensional Lie algebra LG. (In general, Roman indices will take the values $1, \dots, d$ while Greek indices will range over $0, \dots, 3$.)

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With respect to this basis we write $A_\mu = A_\mu^m X_m$ and introduce structure constants $\{f_{ab}^c\}$ such that $[X_a, X_b] = f_{ab}^c X_c$. (Throughout this paper we shall be employing the Einstein summation convention.) Thus

$$F_{\alpha\beta} = F_{\alpha\beta}^c X_c, \quad \text{where } F_{\alpha\beta}^c := \partial_\alpha A_\beta^c - \partial_\beta A_\alpha^c + f_{ab}^c A_\alpha^a A_\beta^b.$$

Let $\mathbf{v}$ be an arbitrary generator for the Lie algebra of our symmetry group. In the present paper we shall only be concerned with the so called Lie-point symmetries for which $\mathbf{v}$ has the form

$$\mathbf{v} = \xi^\nu(\mathbf{x}, \mathbf{A})\frac{\partial}{\partial x^\nu} + \phi_\mu^m(\mathbf{x}, \mathbf{A})\frac{\partial}{\partial A_\mu^m}. \tag{6}$$

In the more general case of Lie-Bäcklund symmetries, ξ and ϕ may also depend on various derivatives of $\mathbf{A}$ [6]. Some interesting non-local and Lie-Bäcklund symmetries of the SDYMEs have recently been reported in Papachristou and Harrison [9], Papachristou [10] and La [4].

The symmetry transformation corresponding to $\mathbf{v}$ is given by

$$(\mathbf{x}, \mathbf{A}) \mapsto \exp(\varepsilon \mathbf{v})(\mathbf{x}, \mathbf{A}).$$

Now $\mathbf{v}$ generates a transformation which acts on the space $X \times U$ of the independent and dependent variables only, however, the self-dual Yang-Mills equations also contain first order derivatives of the connection components. Thus we need to *prolong* $\mathbf{v}$ to a vector field which acts on $X \times U^{(1)}$, the space of independent and dependent variables together with their first derivatives. Hence we introduce the *prolongation* (see [6]) of $\mathbf{v}$ given by

$$\mathrm{pr}^{(1)}(\mathbf{v}) := \xi^\nu \frac{\partial}{\partial x^\nu} + \phi_\mu^m \frac{\partial}{\partial A_\mu^m} + \eta_{\mu\nu}^m \frac{\partial}{\partial A_{\mu,\nu}^m}, \quad A_{\mu,\nu}^m := \frac{\partial}{\partial x_\nu} A_\nu^m,$$

where

$$\begin{aligned}\eta_{\mu\nu}^m &= D_\nu\left\{\phi_\mu^m - \xi^\lambda A_{\mu,\lambda}^m\right\} + \xi^\lambda A_{\mu,\nu\lambda}^m \quad (D_\sigma \text{ is the total derivative with respect to } x^\sigma)\\ &= \phi_{\mu,\nu}^m + A_{\rho,m}^n \phi_\mu^m|_n^\rho - \xi_{,\nu}^\lambda A_{\mu,\lambda}^m - \xi^\lambda|_n^\rho A_{\rho,\nu}^n A_{\mu,\lambda}^m.\end{aligned}$$

(We shall be using the notation $Q|_n^\nu \equiv (\partial Q)/(\partial A_\nu^n)$, $Q|_n^{\nu,\sigma} \equiv (\partial Q)/(\partial A_{\nu,\sigma}^n)$ et cetera.) Consider

$$\begin{aligned}\left(\mathrm{pr}^{(1)}(\mathbf{v})\left[F_{\alpha\beta}^c\right]\right)\Big|_m^{\nu,\nu} &= \delta_\alpha^\nu\left\{\phi_\beta^c|_m^\nu - \xi^\lambda|_m^\nu A_{\beta,\lambda}^c + \delta_m^c\left(\xi_{,\beta}^\nu + \xi^\nu|_n^\lambda A_{\lambda,\beta}^n\right)\right\}\\ &\quad - \delta_\beta^\nu\left\{\phi_\alpha^c|_m^\nu - \xi^\lambda|_m^\nu A_{\alpha,\lambda}^c + \delta_m^c\left(\xi_{,\alpha}^\nu + \xi^\nu|_n^\lambda A_{\lambda,\alpha}^n\right)\right\} \quad (\text{no sum on } \nu),\end{aligned} \tag{7}$$

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where δ^n_m is the Kronecker delta. From

$$\left(\mathbf{pr}^{(1)}(\mathbf{v})\,[F^c_{01} - F^c_{23}]\right)\Big|^{\nu,\nu}_m = 0 \qquad \text{(no sum on } \nu\text{)},$$

we find that $\xi\rho|^\tau_\sigma = 0$ for all ρ, τ, σ. Hence $\xi^\mu = \xi^\mu(\mathbf{x})$.

Also, on noting that the three self-dual Yang-Mills equations (3) are related to each other by cyclic permutations of the indices $(1,2,3)$, we discover that

$$\phi^m_\mu|^\nu_n = -\delta^m_n \xi^\nu_{,\mu} \qquad \mu \neq \nu. \tag{8}$$

So now the prolongation of $\mathbf{v}$ acting on the first Yang-Mills equation has the simpler form

$$\begin{aligned}
0 &= \mathbf{pr}^{(1)}(\mathbf{v})\,[F^m_{01} - F^m_{23}] \\
&= \phi^m_{1,0} - \phi^m_{0,1} - \phi^m_{3,2} + \phi^m_{2,3} + A^n_{\lambda,0}\phi^m_1|^\lambda_n - A^n_{\lambda,2}\phi^\gamma_3|^\lambda_n - A^n_{\lambda,1}\phi^m_0|^\lambda_n + A^n_{\lambda,3}\phi^m_2|^\lambda_n \\
&\quad - \xi^\lambda_{,0}A^m_{1,\lambda} + \xi^\lambda_{,2}A^m_{3,\lambda} + \xi^\lambda_{,1}A^m_{0,\lambda} - \xi^\lambda_{,3}A^m_{2,\lambda} \\
&\quad + f^m_{rs}\,\{A^r_0\phi^s_1 - A^r_2\phi^s_3 + \phi^r_0 A^s_1 - \phi^r_2 A^s_3\}
\end{aligned} \tag{9}$$

Once we have used (3) to eliminate terms of the form $A^\alpha_{i,0}$, $i = 1,2,3$ in (9), the coefficients of the remaining functions A^ν_μ, $A^\nu_{0,0}$ and $A^\nu_{\mu,i}$ must be zero. Hence we have

(10)

$$\begin{aligned}
0 = {} & \phi^m_{1,0} - \phi^m_{0,1} - \phi^m_{3,2} + \phi^m_{2,3} + A^n_{0,1}\left\{\phi^m_1|^1_n - \phi^m_0|^0_n + \delta^m_n\left(\xi^1_{,1} - \xi^0_{,0}\right)\right\} \\
& + A^n_{0,2}\left\{\phi^m_1|^2_n - \phi^m_3|^0_n + \delta^m_n\left(\xi^2_{,1} - \xi^0_{,3}\right)\right\} + A^n_{0,3}\left\{\phi^m_1|^3_n + \phi^m_2|^0_n + \delta^m_n\left(\xi^0_{,2} + \xi^3_{,1}\right)\right\} \\
& + A^n_{1,2}\left\{-\phi^m_1|^3_n - \phi^m_3|^1_n - \delta^m_n\left(\xi^2_{,0} + \xi^0_{,2}\right)\right\} + A^n_{1,3}\left\{\phi^m_1|^2_1 + \phi^m_2|^1_n - \delta^m_n\left(\xi^3_{,0} + \xi^0_{,3}\right)\right\} \\
& + A^n_{2,1}\left\{\phi^m_1|^3_1 - \phi^m_0|^2_n + \delta^m_n\left(\xi^0_{,2} - \xi^1_{,3}\right)\right\} + A^n_{2,3}\left\{-\phi^m_1|^1_n + \phi^m_2|^2_n + \delta^m_n\left(\xi^0_{,0} - \xi^3_{,3}\right)\right\} \\
& + A^n_{3,1}\left\{-\phi^m_1|^2_n - \phi^m_0|^3_n + \delta^m_n\left(\xi^1_{,2} + \xi^0_{,3}\right)\right\} + A^n_{3,2}\left\{\phi^m_1|^1_n - \phi^m_3|^3_n + \delta^m_n\left(\xi^2_{,2} - \xi^0_{,0}\right)\right\} \\
& + f^n_{rs}\big\{\left(\phi^m_1|^1_n - \delta^m_n\xi^0_{,0}\right)\left(A^r_2A^s_3 - A^r_0A^s_1\right) \\
& + \left(\phi^m_1|^2_n - \delta^m_n\xi^0_{,3}\right)\left(A^r_3A^s_1 - A^r_0A^s_2\right) + \left(\phi^m_1|^3_n + \delta^m_n\xi^0_{,2}\right)\left(A^r_1A^s_2 - A^r_0A^s_3\right) \\
& + \delta^m_n\left(A^r_0\phi^s_1 - A^r_2\phi^s_3 + \phi^r_0A^s_1 - \phi^r_2A^s_3\right)\big\}
\end{aligned}$$

Equating the coefficients of the first derivatives of the connection components to zero and cyclically permuting indices (1,2,3), yields

$$\xi^\mu_{,\nu} = -\xi^\nu_{,\mu} \qquad \mu \neq \nu, \qquad\qquad \xi^0_{,0} = \xi^1_{,1} = \xi^2_{,2} = \xi^3_{,3} =: \chi(\mathbf{x})$$

and
$$\phi^m_0|^0_n = \phi^m_1|^1_n = \phi^m_2|^2_n = \phi^m_3|^3_n = \theta^m_n(\mathbf{x}) - \delta^m_n\chi(\mathbf{x})$$

for some θ^m_n. Hence

$$\phi^m_\mu = \theta^m_n A^n_\mu - \xi^\lambda_{,\mu}A^m_\lambda + \psi^m_\mu(\mathbf{x}) \quad \text{for some } \psi^m_\mu. \tag{10}$$

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Substituting equation (11) into (10) and equating the coefficients of the A terms to zero yields

$$\theta^c_{a,\alpha} = f^c_{ab}\psi^b_\alpha.$$

From the compatibility conditions for these equations (that is, $\theta^c_{a,\alpha\beta} = \theta^c_{a,\beta\alpha}$)and using the assumption that G is semisimple we conclude that there is a collection of functions $\{h^m(\mathrm{x})\}$ such that

$$\psi^m_\mu = h^m_{,\mu}. \tag{12}$$

So

$$\theta^c_a = f^c_{ab}h^b + \mathrm{k}^c_a \tag{13}$$

for constants $\{\mathrm{k}^\nu_\mu\}$.

Equating the coefficients of the A^2-type terms to zero (and using the Jacobi identity) tells us that

$$f^c_{ad}\mathrm{k}^d_b + f^c_{db}\mathrm{k}^d_a + f^d_{ba}\mathrm{k}^c_d = 0. \tag{14}$$

Multiplying (14) by f^b_{ec}, summing over b, c and using the Jacobi identity, we obtain

$$\mathrm{g}_{ed}\mathrm{k}^d_a = f^b_{dc}f^c_{ea}\mathrm{k}^d_b, \tag{15}$$

where

$$\mathrm{g}_{mn} := f^r_{ms}f^s_{nr}$$

are the components of the Killing form (or "Cartan metric", see [7]).

Since G is semisimple, $[\mathrm{g}_{mn}]$ is non-singular. If we let $[\mathrm{g}^{mn}] = [\mathrm{g}_{mn}]^{-1}$, then we have

$$\mathrm{k}^b_a = \mathrm{g}^{be}f^l_{dc}f^c_{ea}\mathrm{k}^d_l = f^b_{ae}K^e \tag{16}$$

where

$$K^e := \mathrm{g}^{ce}f^l_{dc}\mathrm{k}^d_l$$

and we have used the fact that

$$f_{abc} := \mathrm{g}_{cd}f^d_{ab} = f^r_{cs}f^s_{dr}f^d_{ab} = -f^r_{cs}\left(f^s_{db}f^d_{ra} + f^s_{da}f^d_{br}\right)$$

is totally antisymmetric in all indices. For k^b_a of the form (16), (14) is satisfied as a result of the Jacobi identity. Hence, without loss of generality, we can take the k^n_ms to be zero in (13) by absorbing them into the h^as.

Hence

$$\phi^m_\mu = f^m_{ln}h^nA^l_\mu - \xi^\lambda_{,\mu}A^m_\lambda + h^m_{,\mu} \tag{17}$$

$$\xi^\nu_{,\mu} = -\xi^\mu_{,\nu}, \quad \mu \neq \nu; \quad \xi^0_{,0} = \xi^1_{,1} = \xi^2_{,2} = \xi^3_{,3}. \tag{18}$$

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Using the compatibility equations for (17) we find that

$$\xi^\mu = d^\mu + bx^\mu + a_\mu (x^\mu)^2 + \sum_{\nu \neq \mu} \left\{ 2a_\nu x^\nu x^\mu - a_\mu (x^\nu)^2 + c^\mu_\nu x^\nu \right\}, \tag{19}$$

(no sum on μ), where a^α, b, $c^\alpha_\beta = -c^\beta_\alpha$ and d^α are constants.

From (17), (19) and (6) we can construct a basis for the Lie Algebra of the generators of all the Lie-point symmetries of the self-dual Yang-Mills equations.

Summary of generators and corresponding transformations

Gauge Transformations.

$$\left\{ f^m_{rs} h^s A^r_\mu + h^m_{,\mu} \right\} \frac{\partial}{\partial A^m_\mu}, \qquad h^\sigma \in C^{(1)}(\mathbf{R}^4), \tag{20}$$

corresponding to

$$\mathbf{A} \mapsto H^{-1} dH + H^{-1} \mathbf{A} H \qquad \text{where } H = \exp(\varepsilon h^\sigma X_\sigma).$$

Translations.

$$\frac{\partial}{\partial x^\mu}, \qquad x^\mu \mapsto x^\mu + \varepsilon. \tag{21}$$

Scalings.

$$x^\mu \frac{\partial}{\partial x^\mu} + A^m_\mu \frac{\partial}{\partial A^m_\mu}, \qquad \mathbf{x} \mapsto \lambda \mathbf{x}, \quad \mathbf{A} \mapsto \lambda^{-1} \mathbf{A}. \tag{22}$$

Rotations.

(23)

$$x^\nu \frac{\partial}{\partial x^\mu} - x^\mu \frac{\partial}{\partial x^\nu} + A^l_\nu \frac{\partial}{\partial A^l_\mu} - A^l_\mu \frac{\partial}{\partial A^l_\nu}, \quad \begin{pmatrix} x^\mu & A_\mu \\ x^\nu & A_\nu \end{pmatrix} \mapsto \begin{pmatrix} \cos\varepsilon & \sin\varepsilon \\ -\sin\varepsilon & \cos\varepsilon \end{pmatrix} \begin{pmatrix} x^\mu & A_\mu \\ x^\nu & A_\nu \end{pmatrix}.$$

"Inversions".

Writing $y_0 = x^\mu$, $\{y_1, y_2, y_3\} = \{x^\nu\}_{\nu \neq \mu}$, the four remaining generators may be expressed as

$$\begin{aligned} \frac{1}{2}\left\{y_0^2 - y_1^2 - y_2^2 - y_3^2\right\} \frac{\partial}{\partial y_0} + y_0 y_1 \frac{\partial}{\partial y^1} + y_0 y_2 \frac{\partial}{\partial y^2} + y_0 y_3 \frac{\partial}{\partial y^3} \\ - \left\{ y_0 A^l_0 + y_1 A^l_1 + y_2 A^l_2 + y_3 A^l_3 \right\} \frac{\partial}{\partial A^l_0} + \sum_{\iota=1}^{3} \left\{ y_\iota A^l_0 - y_0 A^l_\iota \right\} \frac{\partial}{\partial A^l_\iota}. \end{aligned} \tag{24}$$

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If we set $\|\mathbf{y}\|^2 = y_0^2 + y_1^2 + y_2^2 + y_3^2$ and $\mathbf{y}.\mathbf{A} = y_0 A_0 + y_1 A_1 + y_2 A_2 + y_3 A_3$, then the corresponding transformations take the form

$$
\begin{aligned}
y_0 &\mapsto \frac{y_0 - \varepsilon\|\mathbf{y}\|^2/2}{1 - \varepsilon y_0 + \varepsilon^2\|\mathbf{y}\|^2/4}, \\
y_\iota &\mapsto \frac{y_\iota}{1 - \varepsilon y_0 + \varepsilon^2\|\mathbf{y}\|^2/4}, \\
A_0 &\mapsto A_0 - \varepsilon \mathbf{y}.\mathbf{A} + \frac{1}{4}\varepsilon^2\left\{2y_0\ \mathbf{y}.\mathbf{A} - \|\mathbf{y}\|^2 A_0\right\}, \\
A_\iota &\mapsto A_\iota + \varepsilon\left(y_\iota A_0 - x_0 A_\iota\right) + \frac{1}{4}\varepsilon^2\left\{\|\mathbf{y}\|^2 A_\iota - 2y_\iota\ \mathbf{y}.\mathbf{A}\right\}.
\end{aligned}
\tag{25}
$$

The only conformal symmetries which we have yet to find are the proper inversions, which have the form

$$I(y_0, y_1, y_2, y_3) = \frac{1}{\|\mathbf{y}\|^2}(y_0, -y_1, -y_2, -y_3).$$

These are discrete symmetries however and will not be produced explicitly by the prolongation method employed above because this procedure only yields continuous groups of transformations near the identity. If, on the other hand, we consider an inversion followed by a transformation near the identity, followed by another inversion, then the composition of these transformations will be a transformation near the identity. In particular we note that the action of (25) on the space X of independent variables is simply $I \circ T \circ I$, where T is the translation $y_0 \mapsto y_0 - \varepsilon/2$. Hence we have recovered only the well known gauge and conformal symmetries of the SDYMEs.

It should be noted that if G is not semisimple then in general there will be further symmetries which are not mentioned above. For example, if G is Abelian then the SDYMEs are linear and we must add the symmetry transformations

$$A_\mu^m \mapsto A_\mu^m + \varepsilon A_\mu^n, \quad \text{and} \quad A_\mu \mapsto A_\mu + \varepsilon\psi_\mu,$$

where $\{\psi_\mu\}_{\mu=0,\dots,3}$ is any other solution of the SDYMEs. If we take $G = E_2$, the Euclidean group in the plane whose Lie algebra is given by

$$[X_1, X_2] = X_3, \quad [X_3, X_1] = X_2, \quad [X_2, X_3] = 0,$$

which is neither semisimple nor Abelian, then the 'near linearity' of the SDYMEs is apparent through the symmetry

$$A_\mu^2 \mapsto \lambda A_\mu^2, \quad A_\mu^3 \mapsto \lambda A_\mu^3, \quad \text{for all } \mu = 0, \dots, 3.$$

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