Existence, non-existence and blow-up behaviour of minimizers for the mass-critical fractional non-linear Schrödinger equations with periodic potentials

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We consider the minimizing problem for the energy functional with prescribed mass constraint related to the fractional non-linear Schrödinger equation with periodic potentials. Using the concentration-compactness principle, we show a complete classification for the existence and non-existence of minimizers for the problem. In the mass-critical case, under a suitable assumption of the potential, we give a detailed description of blow-up behaviour of minimizers once the mass tends to a critical value.

1. Introduction

In this paper, we consider the following time-independent fractional non-linear Schrödinger equation

$$(-\Delta)^s u + V u = |u|^\alpha u, \quad x \in \mathbb{R}^d, \tag{1.1}$$

where $d \geqslant 1$, $0 < s < 1$, $\alpha > 0$ and $V : \mathbb{R}^d \to \mathbb{R}$ is an external potential. The operator $(-\Delta)^s$ is the fractional Laplacian operator which is defined by $(-\Delta)^s u = \mathcal{F}^{-1}[|\xi|^{2s} \mathcal{F} u]$, where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transform and inverse Fourier transform respectively.

The problem (1.1) comes from the following time-dependent Schrödinger-type equation

$$i\partial_t \psi = (-\Delta)^s \psi + (V - \omega)\psi - |\psi|^\alpha \psi, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d \tag{1.2}$$

by looking for standing wave solutions $\psi(t, x) = e^{i\omega t} u(x)$ with $\omega \in \mathbb{R}$ a frequency. The equation (1.2) is the fractional non-linear Schrödinger equation which was introduced by Laskin [24, 25] as a result of extending the Feynman path integral,

from the Brownian-like to Lévy-like quantum mechanical paths. The equation (1.2) also appears in the continuum limit of discrete models with long-range lattice interactions (see e.g., [**23**]) and in the description of Boson stars (see e.g., [**16**]) as well as in water wave dynamics (see e.g., [**28**]).

The equation (1.1) involves the fractional Laplacian $(-\Delta)^s, 0 < s < 1$ which is a non-local operator. A general approach to deal with this problem due to Caffarelli–Silvestre [**5**] is to transform (1.1) into a local one via the Dirichlet–Neumann map. That is, one consider the extension $U : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}$ which satisfies

$$\begin{cases} -\text{div}(y^{1-2s}\nabla U) = 0 & \text{in } \mathbb{R}^d \times [0, \infty), \\ U(y = 0) = u & \text{on } \mathbb{R}^d. \end{cases}$$

It was shown in [**5**] that

$$(-\Delta)^s u(x) = A(d, s) \lim_{y \to 0^+} -y^{1-2s} U_y(x, y)$$

and

$$\|(-\Delta)^s u\|_{L^2}^2 = \int_{\mathbb{R}^d} |\xi|^{2s} |\mathcal{F}u(\xi)|^2 \, \mathrm{d}\xi = A(d, s) \iint_{\mathbb{R}^d \times [0, \infty)} |\nabla U(x, y)|^2 y^{1-2s} \, \mathrm{d}x \, \mathrm{d}y,$$

where $A(d, s)$ is an appropriate constant depending on d and s. This method has been applied successfully to study equations involving the fractional Laplacian, and a series of significant results have been obtained (see e.g., [**3**, **4**, **6**, **7**, **32**] and references therein).

In the sequel, we are interested in the existence, non-existence and blow-up behaviour of minimizers for the energy functional related to (1.1) under the prescribed mass constraint. More precisely, we consider the minimizing problem: for $a > 0$,

$$I(a) := \inf \left\{ E(u) : u \in \mathcal{H}(\mathbb{R}^d), \|u\|_{L^2}^2 = a \right\},$$

where

$$E(u) := \frac{1}{2} \int_{\mathbb{R}^d} |(-\Delta)^{s/2} u(x)|^2 \, \mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}^d} V(x)|u(x)|^2 \, \mathrm{d}x - \frac{1}{\alpha + 2} \int_{\mathbb{R}^d} |u(x)|^{\alpha+2} \, \mathrm{d}x$$

and the exponent α satisfies $0 < \alpha < s^*$ with

$$s^* := \begin{cases} \dfrac{4s}{d - 2s} & \text{if } d > 2s, \\ \infty & \text{if } d \leqslant 2s. \end{cases} \tag{1.3}$$

Here $\mathcal{H}(\mathbb{R}^d)$ is a subspace of $H^s(\mathbb{R}^d)$ which is defined by

$$\mathcal{H}(\mathbb{R}^d) := \left\{ u \in H^s(\mathbb{R}^d) : \int_{\mathbb{R}^d} V|u|^2 \, \mathrm{d}x < \infty \right\}.$$

Note that the space $\mathcal{H}(\mathbb{R}^d)$ is a Hilbert space with scalar product and norm

$$\langle u, v \rangle_{\mathcal{H}} := \int_{\mathbb{R}^d} u\overline{v}\,\mathrm{d}x + \int_{\mathbb{R}^d} (-\Delta)^{s/2}u\overline{(-\Delta)^{s/2}v}\,\mathrm{d}x$$
$$+ \int_{\mathbb{R}^d} Vu\overline{v}\,\mathrm{d}x, \quad \|u\|_{\mathcal{H}} = \sqrt{\langle u, u \rangle_{\mathcal{H}}}.$$

Let us recall some known results on the existence, non-existence and blow-up behaviour of minimizers for the energy functional with prescribed mass constraint. In the case $s = 1$, i.e., the classical non-linear Schrödinger equation, the problem has been studied by many mathematicians.

In the case of constant potentials, using the symmetric rearrangement argument or the concentration-compactness principle of Lions [26, 27], it holds that (see e.g., [8, Proposition 8.3.6]) for any $a > 0$, there exists at least a minimizer for $I(a)$ in the mass-subcritical case $0 < \alpha < 4/d$. In the mass-critical case $\alpha = 4/d$, one can prove that for any $a > 0$ and $a \neq \|R\|_{L^2}^2$, there is no minimizer for $I(a)$; and for $a = \|R\|_{L^2}^2$, there is a unique (up to symmetries) minimizer for $I(a)$, where R is the unique (up to symmetries) positive radial solution to

$$-\Delta R + R - |R|^{4/d}R = 0. \tag{1.4}$$

In the case of harmonic potential, i.e., $V = |x|^2$, using the compact embedding $\mathcal{H} \hookrightarrow L^q$ for any $2 \leqslant q < 2d/(d-2)$ if $d \geqslant 3$ (or $2 \leqslant q < \infty$ if $d = 1, 2$), Zhang [34] proved that for any $a > 0$, there exists at least a minimizer for $I(a)$ in the mass-subcritical case. In the mass-critical case, he proved that for $0 < a < \|R\|_{L^2}^2$, there exists at least a minimizer for $I(a)$.

In [17], Guo-Seiringer studied the minimizing problem

$$J(b) = \inf\left\{E_b(u) : u \in \mathcal{H}(\mathbb{R}^2), \|u\|_{L^2}^2 = 1\right\}, \tag{1.5}$$

where $b > 0$ is a given parameter,

$$E_b(u) = \int_{\mathbb{R}^2} |\nabla u(x)|^2\,\mathrm{d}x + \int_{\mathbb{R}^2} V(x)|u(x)|^2\,\mathrm{d}x - \frac{b}{2}\int_{\mathbb{R}^2} |u(x)|^4\,\mathrm{d}x$$

and V is a trapping potential, i.e.,

$$0 \leqslant V \in L^\infty_{\mathrm{loc}}(\mathbb{R}^2), \quad \lim_{|x| \to \infty} V(x) = \infty. \tag{1.6}$$

Note that by a simple scaling argument, it is easy to see that $I(a) = (a/2)J(a)$. Using the same argument as in [34], they proved that $J(b)$ has at least a minimizer if $0 < b < b^* := \|R\|_{L^2}^2$, whereas there is no minimizer for $J(b)$ if $b \geqslant b^*$. They also proved that if u_b is a minimizer for $J(b)$ with $0 < b < b^*$, then u_b blows up as $b \nearrow b^*$ in the sense that

$$\lim_{b \nearrow b^*} \|\nabla u_b\|_{L^2} = \infty. \tag{1.7}$$

Moreover, they gave a detailed description of the blow-up behaviour of minimizers for $J(b)$ by assuming that the trap potential V has a finite number of isolated

minima, and that in their vicinity, V behaves like a power of the distance from these points. For instance, if $V(x) = \kappa|x - x_0|^p$ for some $\kappa, p > 0$, then as $b \nearrow b^*$,

$$(b^* - b)^{1/(2+p)} u_b \left((b^* - b)^{1/(2+p)} \cdot + x_0 \right) \to \lambda_0 R_0(\lambda_0 \cdot) \text{ strongly in } L^q(\mathbb{R}^2)$$

for any $2 \leqslant q < \infty$, where

$$R_0 = \frac{R}{\|R\|_{L^2}}, \quad \lambda_0 = \left(\frac{\kappa p}{2} \int_{\mathbb{R}^2} |x|^p [R(x)]^2 \, \mathrm{d}x \right)^{1/(2+p)}. \tag{1.8}$$

In the case of bounded potentials satisfying

$$V \in C^1(\mathbb{R}^d), \quad 0 = \inf_{\mathbb{R}^d} V < \sup_{\mathbb{R}^d} V = \lim_{|x| \to \infty} V(x) < \infty, \tag{1.9}$$

by using the concentration-compactness principle of Lions, one can prove (see e.g., [**29**]) that for any $a > 0$, there exists at least a minimizer for $I(a)$ in the mass-subcritical case. In the mass-critical case, it can be proved that for any $0 < a < \|R\|_{L^2}^2$, there exists at least a minimizer for $I(a)$. Moreover, by the same argument in [**17**], there is no minimizer for $I(a)$ with $a \geqslant \|R\|_{L^2}^2$. In [**29**], Maeda studied the uniqueness, concentration and symmetry of minimizers for $I(a)$ as $a \to \infty$ in the mass-subcritical case.

In [**33**], Wang-Zhao studied the existence and non-existence of minimizers for $J(b)$ (see (1.5)) with continuous periodic potentials satisfying

$$0 = \min V < \inf \sigma(-\Delta + V),$$

where

$$\inf \sigma(-\Delta + V)$$
$$:= \inf \left\{ \int_{\mathbb{R}^2} |\nabla u(x)|^2 \, \mathrm{d}x + \int_{\mathbb{R}^2} V(x)|u(x)|^2 \, \mathrm{d}x : u \in H^1(\mathbb{R}^2), \|u\|_{L^2}^2 = 1 \right\}.$$

Using the concentration-compactness principle of Lions, they proved the existence of minimizers for $J(b)$ for $b_* < b < b^*$ with some $0 < b_* < b^*$ (see above (1.7) for the definition of b^*). On the other hand, there is no minimizer for $J(b)$ when $b \geqslant b^*$. Moreover, under the assumption

$$V^{-1}(0) = x_0 + \mathbb{Z}^2 \text{ for some } x_0 \in [0, 1]^2,$$
$$V(x) = O(|x - x_0|^p) \text{ as } x \to x_0 \text{ for some } p > 0,$$

they proved that if u_b is a minimizer for $J(b)$ with $b_* < b < b^*$, then as $b \nearrow b^*$,

$$(b^* - b)^{1/(2+p)} u_b \left((b^* - b)^{1/(2+p)} \cdot + x_0 \right) \to \lambda_0 R_0(\lambda_0 \cdot) \text{ strongly in } L^q(\mathbb{R}^2)$$

for any $2 \leqslant q < \infty$, where R_0 and λ_0 are as in (1.8).

In [**31**], Phan studied the existence and non-existence of minimizers for $J(b)$ with attractive potentials satisfying

$$0 \geqslant V \in L^q(\mathbb{R}^2) + L^r(\mathbb{R}^2), \quad \inf \sigma(-\Delta + V) < 0$$

for some $1 < q < r < \infty$. He proved that for any $0 < b < b^*$, there exists at least a minimizer for $J(b)$, and there is no minimizer for $J(b)$ if $b \geqslant b^*$. Moreover, under a suitable assumption on the external potential, he gave a detailed description of the blow-up behaviour of minimizers for $J(b)$ as $b \nearrow b^*$. The result implies in particular that if $V(x) = -\kappa|x - x_0|^{-p}$ for some $\kappa > 0$ and $0 < p < 2$, then as $b \nearrow b^*$,

$$(b^* - b)^{1/(2-p)} u_b \left((b^* - b)^{1/(2-p)} \cdot + x_0 \right) \to \lambda_0 R_0(\lambda_0 \cdot) \text{ strongly in } H^1(\mathbb{R}^2),$$

where

$$\lambda_0 = \left(\frac{\kappa p}{2} \int_{\mathbb{R}^2} |x|^{-p} [R(x)]^2 \, \mathrm{d}x \right)^{1/(2-p)}.$$

In the case of an inverse-square potential $V(x) = c|x|^{-2}$ with $c > -((d-2)/2)^2$, using the profile decomposition, Bensouilah-Dinh-Zhu [**2**] proved the existence of minimizers for $I(a)$ for any $a > 0$ in the mass-subcritical case.

The existence, non-existence and blow-up behaviour of minimizers for $J(b)$ has been extended to ring-shaped potentials in [**19**], multi-well potentials [**20**], ellipse-shaped potentials [**18**] and rotating trap potentials [**21**].

In the case $0 < s < 1$, the existence, non-existence and blow-up behaviour of minimizers for $I(a)$ has been considered in several works. In [**22**], He-Long proved the existence and non-existence of minimizers for $I(a)$ with trapping potentials (1.6) and bounded potentials satisfying (1.9). They also studied the blow-up behaviour of minimizers for $I(a)$ as a tends to a critical value in the mass-critical case $\alpha = 4s/d$.

Recently, Du-Tian-Wang-Zhang [**11**] gave a complete classification of the existence and non-existence of minimizers for $I(a)$ with trapping potentials (1.6) for $0 < \alpha < s^*$ (see (1.3) for the definition of s^*). Moreover, under a suitable assumption of the external potential, they showed a detailed analysis of the blow-up behaviour of minimizers for $I(a)$ in the mass-critical case $\alpha = 4s/d$.

Motivated by the aforementioned papers, we study the existence and non-existence of minimizers for the energy functional related to (1.1) with the prescribed mass constraint. In this paper, we focus mainly on the fractional Schrödinger equation with periodic potentials. Before stating our main results, we introduce the following notion of ground states related to the fractional Schrödinger equation.

DEFINITION 1.1 Ground states. Let $d \geqslant 1$, $0 < s < 1$ and $0 < \alpha < s^*$. A non-zero, non-negative H^s function Q_α is called a **ground state** related to

$$(-\Delta)^s u + u - |u|^\alpha u = 0 \tag{1.10}$$

if it solves (1.10) and is a minimizer for the Weinstein's functional

$$J(u) := \left[\|(-\Delta)^{s/2} u\|_{L^2}^{d\alpha/2s} \|u\|_{L^2}^{\alpha+2-d\alpha/2s} \right] \div \|u\|_{L^{\alpha+2}}^{\alpha+2}, \tag{1.11}$$

that is,

$$J(Q_\alpha) = \inf \left\{ J(u) : u \in H^s(\mathbb{R}^d) \backslash \{0\} \right\}.$$

The existence, uniqueness, symmetry, regularity and decay of the ground state related to (1.10) have been established in celebrated papers [**14**, **15**] (see theorem 2.3 more details).

From now on, we denote

$$a^* := \|Q\|_{L^2}^2, \tag{1.12}$$

where $Q := Q_{4s/d}$ is the unique (up to translations) positive radial ground state related to

$$(-\Delta)^s u + u - |u|^{4s/d}u = 0. \tag{1.13}$$

Our first result is the following existence and non-existence of minimizers for $I(a)$ in the case of no external potential $V \equiv 0$.

THEOREM 1.2 No potential. *Let $d \geqslant 1$ and $0 < s < 1$. Let $V \equiv 0$. Then it holds that*:

- [**13**] *If $0 < \alpha < 4s/d$, then for any $a > 0$, there exits at least a minimizer for $I(a)$ and $-\infty < I(a) < 0$.*

- *If $\alpha = 4s/d$, then for any $a > 0$ and $a \neq a^*$, there is no minimizer for $I(a)$; and for $a = a^*$, there is a unique (up to symmetries) minimizer for $I(a)$. Moreover, $I(a) \geqslant 0$ if $0 < a < a^*$, $I(a^*) = 0$ and $I(a) = -\infty$ if $a > a^*$.*

- *If $4s/d < \alpha < s^*$, then for any $a > 0$, there is no minimizer for $I(a)$ and $I(a) = -\infty$.*

Since adding a constant in the potential does not change the minimizing problem, the above result still holds in the case of constant potentials. In the mass-subcritical case $0 < \alpha < 4s/d$, the existence of minimizers for $I(a)$ has been studied in [**13**] by using a fractional version of the concentration-compactness principle of P. L. Lions [**26**, **27**]. We will give an alternative proof using the symmetric rearrangement argument. We refer the reader to §3 for more details.

Our next result is the following existence and non-existence of minimizers for $I(a)$ in the case of periodic potentials.

THEOREM 1.3 Periodic potential. *Let $d \geqslant 1$ and $0 < s < 1$. Let $V \in C(\mathbb{R}^d)$ be such that*

(V1) *$V(x + z) = V(x)$ for all $x \in \mathbb{R}^d$ and $z \in \mathbb{Z}^d$.*

Then it holds that:

- *If $0 < \alpha < 4s/d$, then there exists $a_* > 0$ such that for any $a > a_*$, there exists at least a minimizer for $I(a)$ and $-\infty < I(a) < (a/2)\inf \sigma((-\Delta)^s + V)$.*

- *If $\alpha = 4s/d$ and assume in addition that*

(V2) $\min V < \inf \sigma((-\Delta)^s + V)$, where $\inf \sigma((-\Delta)^s + V)$ is the infimum of the spectrum of $(-\Delta)^s + V$, i.e.,

$$\inf \sigma((-\Delta)^s + V) = \inf \left\{ \int_{\mathbb{R}^d} |(-\Delta)^{s/2} u(x)|^2 \, \mathrm{d}x \right.$$

$$\left. + \int_{\mathbb{R}^d} V(x)|u(x)|^2 \, \mathrm{d}x : \|u\|_{L^2} = 1 \right\},$$

then there exists $0 < a_* < a^*$ such that for any $a_* < a < a^*$, there exists at least a minimizer for $I(a)$; and for $a \geqslant a^*$, there is no minimizer for $I(a)$. Moreover, $(a/2) \min V < I(a) < (a/2) \inf \sigma((-\Delta)^s + V)$ if $a_* < a < a^*$, $I(a^*) = (a^*/2) \min V$ and $I(a) = -\infty$ if $a > a^*$.

- If $4s/d < \alpha < s^*$, then for any $a > 0$, there is no minimizer for $I(a)$ and $I(a) = -\infty$.

This result gives a complete classification of the existence and non-existence of minimizers for $I(a)$ with periodic potentials. The proof is based on the concentration-compactness principle in the same spirit of Lions [26, 27]. However, since we are dealing with the non-local opeartor $(-\Delta)^s$, we need a careful analysis in order to get this concentration-compactness principle. We refer the reader to §2 for more details.

Our next result is the blow-up behaviour of minimizers for $I(a)$ as $a \nearrow a^*$ in the mass-critical case $\alpha = 4s/d$. Note that since $E(|u|) \leqslant E(u)$, we only need to consider non-negative minimizers for $I(a)$.

THEOREM 1.4. Let $d \geqslant 1$, $0 < s < 1$, $\alpha = 4s/d$ and $V \in C(\mathbb{R}^d)$ satisfy (V1) and (V2). Let u_a be a non-negative minimizer for $I(a)$ with $a_* < a < a^*$ given in theorem 1.3. Then u_a blows up as $a \nearrow a^*$ in the sense that

$$\varepsilon_a^{-s} := \|(-\Delta)^{s/2} u_a\|_{L^2} \to \infty \tag{1.14}$$

as $a \nearrow a^*$. Moreover, up to a subsequence, there exist $(x_a)_{a \nearrow a^*} \subset [0,1]^d$ and $(z_a)_{a \nearrow a^*} \subset \mathbb{Z}^d$ such that $x_a \to x^0 \in [0,1]^d$ with $V(x^0) = \min V$ and

$$\varepsilon_a^{d/2} u_a(\varepsilon_a \cdot + x_a + z_a) \to Q \text{ strongly in } H^s(\mathbb{R}^d)$$

as $a \nearrow a^*$, where Q is the unique (up to translations) positive radial ground state related to (1.13).

The proof of this result is based on the compactness of optimizing sequence for the Gagliardo–Nirenberg inequality (see lemma 2.11) which is another consequence of the concentration-compactness principle.

Note that theorem 1.4 does not give any information on the blow-up rate of $\|(-\Delta)^{s/2} u_a\|_{L^2}$. Under an additional assumption on the external potential, we gave a detailed description of the blow-up behaviour of minimizers for $I(a)$. More precisely, we have the following result.

THEOREM 1.5. *Let $d \geqslant 1$ and $0 < s < 1$. Let $\alpha = 4s/d$. Let $V \in C(\mathbb{R}^d)$ satisfy* (V1) *and* (V2). *Assume in addition that V satisfies*

(V3) $\min V = 0$, $V^{-1}(0) = x_0 + \mathbb{Z}^d$ *for some $x_0 \in [0,1]^d$ and there exist $0 < p < d + 4s$ and $\kappa > 0$ such that*

$$\lim_{x \to x_0} \frac{V(x)}{|x - x_0|^p} = \kappa.$$

Let u_a be a non-negative minimizer for $I(a)$ with $a_* < a < a^*$ given in theorem 1.3. *Then up to a subsequence, there exist sequences $(x_a)_{a \nearrow a^*} \subset [0,1]^d$, $(z_a)_{a \nearrow a^*} \subset \mathbb{Z}^d$ such that $x_a \to x_0$ with*

$$\lim_{a \nearrow a^*} \frac{x_a - x_0}{\beta_a^{1/(2s+p)}} = 0$$

and

$$\beta_a^{d/2(2s+p)} u_a(\beta_a^{1/(2s+p)} \cdot + x_a + z_a) \to \lambda_0^{d/2} Q(\lambda_0 \cdot) \text{ strongly in } H^s(\mathbb{R}^d) \text{ as } a \nearrow a^*,$$

where Q is the unique (up to translations) positive radial ground state related to (1.13) *and*

$$\beta_a := 1 - \left(\frac{a}{a^*}\right)^{2s/d}, \quad \lambda_0 = \left(\frac{\kappa p}{d} \int_{\mathbb{R}^d} |x|^p [Q_0(x)]^2 \, dx\right)^{1/(2s+p)}, \quad Q_0 = \frac{Q}{\|Q\|_{L^2}}.$$
$$(1.15)$$

Theorem 1.5 gives a detailed description of the blow-up behaviour of minimizers for $I(a)$ as $a \nearrow a^*$. In particular, the minimizer blows up at speed $\beta_a^{-1/(2s+p)}$ as $a \nearrow a^*$. The proof of theorem 1.5 is based on energy estimates (see lemma 4.1). In [**11, 22**], these energy estimates were obtained by analysing the corresponding Euler–Lagrange equation related to the minimizers. In this paper, we give an alternative approach based on the compactness of optimizing sequence for the fractional Gagliardo–Nirenberg inequality. Our argument is simpler and more direct than the ones in [**11, 22**]. We finally point out that in contrast of the classical Schrödinger equation $s = 1$ in which the ground state decays exponentially at infinity, the ground state for fractional Schrödinger equation $0 < s < 1$ decays only polynomially at infinity. This is the reason why we have to impose an additional condition $p < d + 4s$ in (V3) which ensures that $|\cdot|^p [Q_0]^2$ is integrable.

The paper is organized as follows. In § 2, we give some preliminaries related to our problem including the fractional Sobolev spaces, the fractional Gagliardo–Nirenberg inequality, the radial compactness embedding and the concentration-compactness principle. In § 3, we prove the existence and non-existence of minimizers in the case of no external potential and in the case of periodic potentials. Finally, we give the proof of blow-up behaviour of minimizers in § 4.

2. Preliminaries

2.1. Fractional Sobolev spaces

For the reader's convenience, we recall the definition and some properties of the fractional Sobolev spaces. The fractional Laplacian $(-\Delta)^s$ is defined for $u \in \mathcal{S}(\mathbb{R}^d)$ by

$$(-\Delta)^s u(x) := \mathcal{F}^{-1} \left(|\xi|^{2s} \mathcal{F}u(\xi) \right)(x),$$

where $\mathcal{F}$ is the isometric Fourier transform in $L^2(\mathbb{R}^d)$

$$\mathcal{F}(u)(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} u(x)\,\mathrm{d}x.$$

An alternative equivalent definition (see [10]) is

$$(-\Delta)^s u(x) = C(d,s) \lim_{\varepsilon\searrow 0} \int_{\mathbb{R}^d \setminus B(0,\varepsilon)} \frac{u(x)-u(y)}{|x-y|^{d+2s}}\,\mathrm{d}y$$

$$= -\frac{1}{2} C(d,s) \int_{\mathbb{R}^d} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{d+2s}}\,\mathrm{d}y,$$

where

$$C(d,s) := \left(\int_{\mathbb{R}^d} \frac{1-\cos\zeta_1}{|\zeta|^{d+2s}}\,\mathrm{d}\zeta \right)^{-1}. \tag{2.1}$$

For any $s \in (0,1)$, the fractional Sobolev space $H^s(\mathbb{R}^d)$ is defined by

$$H^s(\mathbb{R}^d) := \left\{ u \in L^2(\mathbb{R}^d) : (-\Delta)^{s/2} u \in L^2(\mathbb{R}^d) \right\}.$$

The space $H^s(\mathbb{R}^d)$ is a Hilbert space endowed with the inner product

$$\langle u, v \rangle_{H^s} := \mathfrak{Re} \int_{\mathbb{R}^d} u\overline{v}\,\mathrm{d}x + \mathfrak{Re} \int_{\mathbb{R}^d} (-\Delta)^{s/2} u \overline{(-\Delta)^{s/2} v}\,\mathrm{d}x$$

and norm

$$\|u\|_{H^s}^2 = \|u\|_{L^2}^2 + \|(-\Delta)^{s/2} u\|_{L^2}^2.$$

Note that [10, Proposition 3.4]

$$\|(-\Delta)^{s/2} u\|_{L^2}^2 = \int_{\mathbb{R}^d} |\xi|^{2s} |\mathcal{F}u(\xi)|^2\,\mathrm{d}\xi = \frac{1}{2} C(d,s) \iint_{\mathbb{R}^{2d}} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y. \tag{2.2}$$

We have the following Sobolev embedding for fractional Sobolev spaces.

LEMMA 2.1 Sobolev embedding [1, 10]. *Let $s \in (0,1)$. Then the following embeddings are continuous.*

- $H^s(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$, $2 \leqslant q \leqslant 2d/(d-2s)$ *if $d > 2s$.*
- $H^s(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$, $2 \leqslant q < \infty$ *if $d = 2s$.*

- $H^s(\mathbb{R}) \hookrightarrow C_b(\mathbb{R})$ *if* $1 < 2s$, *where* $C_b(\mathbb{R})$ *is the space of continuous bounded functions on* $\mathbb{R}$.

Moreover, $H^s(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$ *is locally compact for any* $1 \leqslant q < 2^*$, *where*

$$2^* := \begin{cases} \dfrac{2d}{d-2s} & \text{if } d > 2s, \\ \infty & \text{if } d \leqslant 2s. \end{cases} \tag{2.3}$$

LEMMA 2.2 [**1**]. *Let* $s \in (0,1)$. *The space* $\mathcal{S}(\mathbb{R}^d)$ *is dense in* $H^s(\mathbb{R}^d)$. *Moreover, the multiplication by a Schwarz function is a continuous map from* $H^s(\mathbb{R}^d)$ *to itself.*

2.2. Fractional Gagliardo–Nirenberg inequality

Let us recall the following fractional Gagliardo–Nirenberg inequality

$$\|u\|_{L^{\alpha+2}}^{\alpha+2} \leqslant C_{\text{opt}} \|(-\Delta)^{s/2} u\|_{L^2}^{d\alpha/2s} \|u\|_{L^2}^{\alpha+2-d\alpha/2s}. \tag{2.4}$$

The sharp constant $C_{\text{opt}} > 0$ can be obtained by

$$\frac{1}{C_{\text{opt}}} = \inf \left\{ J(u) : u \in H^s(\mathbb{R}^d) \backslash \{0\} \right\},$$

where

$$J(u) := \left[\|(-\Delta)^{s/2} u\|_{L^2}^{d\alpha/2s} \|u\|_{L^2}^{\alpha+2-d\alpha/2s} \right] \div \|u\|_{L^{\alpha+2}}^{\alpha+2}$$

is the Weinstein functional. We have the following result due to [**11**, **14**, **15**].

THEOREM 2.3 [**11**, **14**, **15**]. *Let* $d \geqslant 1$, $0 < s < 1$ *and* $0 < \alpha < s^*$. *Then the fractional Gagliardo–Nirenberg inequality* (2.4) *is attained at a function* $Q_\alpha \in H^s(\mathbb{R}^d)$ *with the following properties*:

- $Q_\alpha(x)$ *is radially symmetric, positive and strictly decreasing in* $|x|$.

- Q_α *belongs to* $H^{2s+1}(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$ *and satisfies*

$$\frac{C_1}{1 + |x|^{d+2s}} \leqslant Q_\alpha(x) \leqslant \frac{C_2}{1 + |x|^{d+2s}}, \tag{2.5}$$

$$|\partial_j Q_\alpha(x)| \leqslant \frac{C_3}{1 + |x|^{d+2s}}, \quad j = 1, \cdots, d, \tag{2.6}$$

for all $x \in \mathbb{R}^d$, *where* C_1, C_2 *and* C_3 *are positive constants depending on* s, d, α *and* Q_α.

- Q_α *solves the elliptic equation* (1.10).

Moreover, every minimizer $\varphi \in H^s(\mathbb{R}^d)$ *for the fractional Gagliardo–Nirenberg inequality* (2.4) *is of the form* $\varphi(x) = \beta Q_\alpha(\gamma(x + y))$ *with some* $\beta \in \mathbb{C}, \beta \neq 0, \gamma > 0$ *and* $y \in \mathbb{R}^d$.

Note that (2.6) follows from (2.5) and lemma C.2 in [**15**].

REMARK 2.4. We have the following Pohozaev's identities related to (1.10)

$$\|(-\Delta)^{s/2}Q_\alpha\|_{L^2}^2 = \frac{d\alpha}{2s(\alpha+2)}\|Q_\alpha\|_{L^{\alpha+2}}^{\alpha+2} = \frac{d\alpha}{4s-(d-2s)\alpha}\|Q_\alpha\|_{L^2}^2. \qquad (2.7)$$

In particular,

$$C_{\text{opt}} = \|Q_\alpha\|_{L^{\alpha+2}}^{\alpha+2} \div \left[\|(-\Delta)^{s/2}Q_\alpha\|_{L^2}^{d\alpha/2s}\|Q_\alpha\|_{L^2}^{\alpha+2-d\alpha/2s}\right]$$

$$= \left(\frac{2s(\alpha+2)-d\alpha}{d\alpha}\right)^{d\alpha/4s}\frac{2s(\alpha+2)}{2s(\alpha+2)-d\alpha}\frac{1}{\|Q_\alpha\|_{L^2}^\alpha}.$$

In the case $\alpha = 4s/d$, we denote $Q := Q_{4s/d}$ and have that

$$\|(-\Delta)^{s/2}Q\|_{L^2}^2 = \frac{d}{d+2s}\|Q\|_{L^{4s/d+2}}^{4s/d+2} = \frac{d}{2s}\|Q\|_{L^2}^2 \qquad (2.8)$$

and

$$C_{\text{opt}} = \frac{d+2s}{d}\frac{1}{\|Q\|_{L^2}^{4s/d}}.$$

2.3. Radial compactness embedding

Denote

$$H_{\mathrm{r}}^s(\mathbb{R}^d) := \{u \in H^s(\mathbb{R}^d) : u \text{ is radially symmetric}\}.$$

We have the following estimate for radially symmetric functions in $H^s(\mathbb{R}^d)$.

LEMMA 2.5 [9]. *Let $d \geqslant 2$ and $\frac{1}{2} < s < d/2$. Then every radially symmetric function $u \in H^s(\mathbb{R}^d)$ is almost everywhere equal to a function U, continuous for $x \neq 0$ such that*

$$|U(x)| \leqslant C|x|^{s-d/2}\|(-\Delta)^{s/2}u\|_{L^2},$$

for all $x \neq 0$, where $C > 0$ depends only on d and s.

Using this radial estimate, we obtain the following radial compactness embedding for fractional Sobolev spaces.

LEMMA 2.6. *Let $d \geqslant 2$ and $\frac{1}{2} < s < d/2$. The injection $H_{\mathrm{r}}^s(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$ is compact for any $2 < q < 2d/(d-2s)$.*

If we assume that $u \in H_{\mathrm{r}}^s(\mathbb{R}^d)$ is radially decreasing, i.e., $u(x) \leqslant u(y)$ if $|x| \geqslant |y|$, then we have the following compact embedding.

LEMMA 2.7. *Let $d \geqslant 1$ and $s \in (0,1)$. The injection $H_{\mathrm{rd}}^s(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$ is compact for any $2 < q < 2^*$, where*

$$H_{\mathrm{rd}}^s(\mathbb{R}^d) := \left\{u \in H_{\mathrm{r}}^s(\mathbb{R}^d) : u \text{ is radially decreasing}\right\}.$$

Proof. It is enough to show that if $(u_n)_{n \geqslant 1} \subset H^s_{\mathrm{rd}}$ is such that $u_n \rightharpoonup 0$ weakly in $H^s(\mathbb{R}^d)$, then $u_n \to 0$ strongly in $L^q(\mathbb{R}^d)$. We can assume that $u_n \to 0$ strongly in $L^q_{\mathrm{loc}}(\mathbb{R}^d)$ and $u_n \to 0$ almost everywhere. It follows that

$$\|u_n\|^q_{L^q} = \|u_n\|^q_{L^q(B)} + \|u_n\|^q_{L^q(B^c)} = \|u_n\|^q_{L^q(B^c)} + o_n(1),$$

where B is the unit ball in $\mathbb{R}^d$ and B^c is its complement in $\mathbb{R}^d$. Here $o_n(1)$ means that $o_n(1) \to 0$ as $n \to \infty$. It remains to estimate the term on B^c. To do this, we use the fact that u is radially symmetric and radially decreasing to have that

$$\|u\|^2_{L^2} = \int |u(x)|^2 \, \mathrm{d}x \geqslant |\mathbb{S}^{d-1}| \int_0^r |u(\tau)|^2 \tau^{d-1} \, \mathrm{d}\tau \geqslant |\mathbb{S}^{d-1}| |u(r)|^2 \frac{r^d}{d}.$$

It follows that

$$|u(x)| \leqslant \left(\frac{d}{|\mathbb{S}^{d-1}|} \right)^{1/2} |x|^{-d/2} \|u\|_{L^2}$$

for all $x \neq 0$. Using this inequality and the fact $(u_n)_{n \geqslant 1}$ is bounded in $H^s(\mathbb{R}^d)$, we infer that there exists $C(d) > 0$ independent of n such that

$$|u_n(x)|^q \leqslant C(d)|x|^{-dq/2}.$$

Note that the last term is integrable on B^c since $q > 2$. It follows from the dominated convergence theorem that $\|u_n\|^q_{L^q(B^c)} \to 0$ as $n \to \infty$. The proof is complete. $\qquad \square$

We next recall the symmetric rearrangement which is useful in variational calculus. The symmetric rearrangement of a measurable function $u : \mathbb{R}^d \to \mathbb{C}$ vanishing at infinity is defined by

$$u^*(x) = \int_0^\infty \chi_{\{|u|>t\}^*}(x) \, \mathrm{d}t,$$

where $\{|u| > t\}^*$ is a ball centred at the origin whose volume equals to the volume of $\{x \in \mathbb{R}^d : |u(x)| > t\}$ and χ_B is the characteristic function of B. We see that u^* is non-negative, radially symmetric and radially decreasing. We recall some basic properties of the symmetric rearrangement:

- L^q-norm preserving: $\|u\|_{L^q} = \|u^*\|_{L^q}$ for any $1 \leqslant q \leqslant \infty$.

- Order-preserving: if $u(x) \leqslant v(x)$ for all $x \in \mathbb{R}^d$, then $u^*(x) \leqslant v^*(x)$ for all $x \in \mathbb{R}^d$.

- Hardy–Littlewood inequality: if u, v are non-negative measurable functions that vanish at infinity, then

$$\int_{\mathbb{R}^d} u(x)v(x) \, \mathrm{d}x \leqslant \int_{\mathbb{R}^d} u^*(x)v^*(x) \, \mathrm{d}x,$$

in the sense that the left-hand side is finite whenever the right-hand side is finite.

- Decreasing L^q-distance: $\|u - v\|_{L^q} \geqslant \|u^* - v^*\|_{L^q}$ for any $1 \leqslant q \leqslant \infty$.

- Polya–Szegö inequality: $\|\nabla u^*\|_{L^q} \leqslant \|\nabla u\|_{L^q}$ for any $1 \leqslant q \leqslant \infty$. In particular,

$$\|\nabla |u|^*\|_{L^q} \leqslant \|\nabla |u|\|_{L^q} \leqslant \|\nabla u\|_{L^q}, \quad 1 \leqslant q \leqslant \infty.$$

  The last inequality follows from the diamagnetic inequality.

- Fractional Polya–Szegö inequality [30]: for $0 < s < 1$, $\|(-\Delta)^{s/2} u^*\|_{L^2} \leqslant \|(-\Delta)^{s/2} u\|_{L^2}$. The equality holds for radially decreasing function. In particular,

$$\|(-\Delta)^{s/2} |u|^*\|_{L^2} \leqslant \|(-\Delta)^{s/2} |u|\|_{L^2} \leqslant \|(-\Delta)^{s/2} u\|_{L^2}.$$

  Here the last inequality follows from (2.2) and the fact $||u(x)| - |u(y)|| \leqslant |u(x) - u(y)|$.

2.4. Concentration-compactness principle

LEMMA 2.8 Concentration-compactness lemma. *Let $d \geqslant 1$ and $s \in (0,1)$. Let $(u_n)_{n \geqslant 1}$ be a bounded sequence in $H^s(\mathbb{R}^d)$ satisfying*

$$\lim_{n \to \infty} \|u_n\|_{L^2}^2 = a \tag{2.9}$$

for some fixed constant $a > 0$. Then there exists a subsequence $(u_{n_k})_{k \geqslant 1}$ satisfying one of the following three possibilities.

- **Vanishing:**

$$\lim_{k \to \infty} \sup_{y \in \mathbb{R}^d} \int_{B(y,R)} |u_{n_k}(x)|^2 \, \mathrm{d}x = 0$$

  for all $R > 0$.

- **Compactness:** *There exists a sequence $(y_k)_{k \geqslant 1} \subset \mathbb{R}^d$ such that for all $\varepsilon > 0$, there exist $R(\varepsilon) > 0$ and $k(\varepsilon) \geqslant 1$ such that for all $k \geqslant k(\varepsilon)$,*

$$\int_{B(y_k, R(\varepsilon))} |u_{n_k}(x)|^2 \, \mathrm{d}x \geqslant a - \varepsilon.$$

- **Dichotomy:** *There exist $\mu \in (0, a)$ and sequences $(u_k^1)_{k \geqslant 1}, (u_k^2)_{k \geqslant 1}$ bounded in $H^s(\mathbb{R}^d)$ such that*

$$\begin{cases} \|u_{n_k} - u_k^1 - u_k^2\|_{L^q} \to 0 \text{ as } k \to \infty \text{ for all } 2 \leqslant q < 2^*, \\ \|u_k^1\|_{L^2}^2 \to \mu, \quad \|u_k^2\|_{L^2}^2 \to a - \mu \text{ as } k \to \infty, \\ \mathrm{dist}(\mathrm{supp}(u_k^1), \mathrm{supp}(u_k^2)) \to \infty \text{ as } k \to \infty, \\ \liminf_{k \to \infty} \|(-\Delta)^{s/2} u_{n_k}\|_{L^2}^2 - \|(-\Delta)^{s/2} u_k^1\|_{L^2}^2 - \|(-\Delta)^{s/2} u_k^2\|_{L^2}^2 \geqslant 0. \end{cases} \tag{2.10}$$

Proof. The proof of this result follows by the same argument as in [26, 27]. For the reader's convenience, we give some details. Define the so-called Lévy concentration

functions $M_n : [0, \infty) \to [0, a]$ by

$$M_n(R) := \sup_{y \in \mathbb{R}^d} \int_{B(y,R)} |u_n(x)|^2 \, \mathrm{d}x.$$

Since $\|u_n\|_{L^2}^2 = a$ for all $n \geqslant 1$, we see that $(M_n)_{n \geqslant 1}$ is a uniformly bounded sequence of non-decreasing, non-negative functions on $[0, \infty)$. By Helly's collection theorem, there exists a subsequence $(M_{n_k})_{k \geqslant 1}$ and a non-negative, non-decreasing function $M : [0, \infty) \to [0, a]$ such that $\lim_{k \to \infty} M_{n_k}(R) = M(R)$ for any $R \geqslant 0$. Set

$$\mu := \lim_{R \to \infty} M(R) = \lim_{R \to \infty} \lim_{k \to \infty} M_{n_k}(R).$$

Obviously, $\mu \in [0, a]$. We will consider three possibilities: $\mu = 0$, $\mu = a$ and $\mu \in (0, a)$.

If $\mu = 0$, then since M is a non-negative, non-decreasing function it follows that $M(R) = 0$ for all $R \geqslant 0$, equivalently, $\lim_{k \to \infty} M_{n_k}(R) = 0$ for all $R \geqslant 0$.

If $\mu = a$, we proceed as follows. We first find $R_0 > 0$ large enough such that $M(R_0) > a/2$. For any $k \geqslant 1$, there exists $y_k \in \mathbb{R}^d$ such that

$$M_{n_k}(R_0) \leqslant \int_{B(y_k, R_0)} |u_{n_k}(x)|^2 \, \mathrm{d}x + \frac{1}{k}.$$

Now, for $0 < \varepsilon < a/2$, we fix $\tau(\varepsilon) > 0$ such that $M(\tau(\varepsilon)) > a - \varepsilon > a/2$ and let $x_k \in \mathbb{R}^d$ satisfying

$$M_{n_k}(\tau(\varepsilon)) \leqslant \int_{B(x_k, \tau(\varepsilon))} |u_{n_k}(x)|^2 \, \mathrm{d}x + \frac{1}{k}.$$

We thus have that

$$\int_{B(y_k, R_0)} |u_{n_k}(x)|^2 \, \mathrm{d}x + \int_{B(x_k, R(\varepsilon))} |u_{n_k}(x)|^2 \, \mathrm{d}x \geqslant M_{n_k}(R_0) + M_{n_k}(\tau(\varepsilon)) + o_k(1)$$

$$\geqslant M(R_0) + M(\tau(\varepsilon)) + o_k(1),$$

where $o_k(1)$ means that $o_k(1) \to 0$ as $k \to \infty$. Since $M(R_0) + M(\tau(\varepsilon)) > a$, we infer that there exists $k(\varepsilon)$ large enough such that the right-hand side is strictly larger than a for all $k \geqslant k(\varepsilon)$. This implies that for $k \geqslant k(\varepsilon)$,

$$B(y_k, R_0) \cap B(x_k, \tau(\varepsilon)) \neq \emptyset,$$

that is, $B(x_k, \tau(\varepsilon)) \subset B(y_k, R_0(\varepsilon))$ with $R_0(\varepsilon) = R_0 + 2\tau(\varepsilon)$. Hence, by enlarging $k(\varepsilon)$ if necessary,

$$\int_{B(y_k, R_0(\varepsilon))} |u_{n_k}(x)|^2 \, \mathrm{d}x \geqslant \int_{B(x_k, \tau(\varepsilon))} |u_{n_k}(x)|^2 \, \mathrm{d}x \geqslant a - 2\varepsilon \qquad (2.11)$$

for all $k \geqslant k(\varepsilon)$ which shows the compactness with $R(\varepsilon) = R_0(\varepsilon)$.

We now consider the case $\mu \in (0, a)$. Take $\varphi, \psi \in C_0^\infty(\mathbb{R}^d)$ satisfying $0 \leqslant \varphi, \psi \leqslant 1$ and

$$\varphi(x) = \begin{cases} 1 & \text{if } |x| \leqslant 1, \\ 0 & \text{if } |x| \geqslant 2, \end{cases} \quad \text{and} \quad \psi(x) = \begin{cases} 0 & \text{if } |x| \leqslant 1, \\ 1 & \text{if } |x| \geqslant 2. \end{cases}$$

Denote for $R > 0$ the functions $\varphi_R(x) = \varphi(x/R)$ and $\psi_R(x) = \psi(x/R)$. We first note that for $v \in H^s(\mathbb{R}^d)$,

$$\begin{aligned}
&2[C(d,s)]^{-1}\|(-\Delta)^{s/2}(\varphi_R v)\|_{L^2}^2 \\
&= \iint \frac{|\varphi_R(x)v(x) - \varphi_R(y)v(y)|^2}{|x-y|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y \\
&= \iint \frac{[\varphi_R(x)]^2|v(x) - v(y)|^2}{|x-y|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y + \iint \frac{|\varphi_R(x) - \varphi_R(y)|^2|v(y)|^2}{|x-y|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + 2\mathfrak{Re}\iint \frac{\varphi_R(x)(v(x) - v(y))(\varphi_R(x) - \varphi_R(y))\overline{v}(y)}{|x-y|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y \\
&= \iint \frac{[\varphi_R(x)]^2|v(x) - v(y)|^2}{|x-y|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y + \mathrm{I} + \mathrm{II},
\end{aligned}$$

where $C(d,s)$ is given in (2.1). By Hölder's inequality,

$$\begin{aligned}
|\mathrm{II}| &\leqslant 2\left(\iint \frac{[\varphi_R(x)]^2|v(x) - v(y)|^2}{|x-y|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2} \\
&\quad \times \left(\iint \frac{|\varphi_R(x) - \varphi_R(y)|^2|v(y)|^2}{|x-y|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2} \\
&\leqslant 2\|(-\Delta)^{s/2}v\|_{L^2}\left(\iint \frac{|\varphi_R(x) - \varphi_R(y)|^2|v(y)|^2}{|x-y|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2},
\end{aligned}$$

where we have used that $0 \leqslant \varphi_R \leqslant 1$. We thus only consider I. Using the fact $|\nabla\varphi_R| \lesssim R^{-1}$, we have that

$$\begin{aligned}
\mathrm{I} &= \iint_{|x-y|\leqslant R} \frac{|\varphi_R(x) - \varphi_R(y)|^2|v(y)|^2}{|x-y|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \iint_{|x-y|\geqslant R} \frac{|\varphi_R(x) - \varphi_R(y)|^2|v(y)|^2}{|x-y|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y \\
&\lesssim R^{-2}\iint_{|x-y|\leqslant R} \frac{|v(y)|^2}{|x-y|^{d+2s-2}}\,\mathrm{d}x\,\mathrm{d}y + \iint_{|x-y|\geqslant R} \frac{|v(y)|^2}{|x-y|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y \\
&= R^{-2}\int |v(y)|^2\,\mathrm{d}y \int_{|\zeta|\leqslant R} |\zeta|^{-(d+2s-2)}\,\mathrm{d}\zeta + \int |v(y)|^2\,\mathrm{d}x \int_{|\zeta|\geqslant R} |\zeta|^{-(d+2s)}\,\mathrm{d}\zeta \\
&= CR^{-2s}\|v\|_{L^2}^2.
\end{aligned}$$

We thus prove that

$$\|(-\Delta)^{s/2}(\varphi_R v)\|_{L^2}^2 - \frac{1}{2}C(d,s)\iint \frac{[\varphi_R(x)]^2|v(x)-v(y)|^2}{|x-y|^{d+2s}}\,dx\,dy \leqslant CR^{-s}\|v\|_{H^s}^2.$$
(2.12)

In particular, since $(u_n)_{n\geqslant 1}$ is a bounded sequence in $H^s(\mathbb{R}^d)$, we see that

$$\|(-\Delta)^{s/2}(\varphi_R u_n)\|_{L^2}^2 - \frac{1}{2}C(d,s)\iint \frac{[\varphi_R(x)]^2|u_n(x)-u_n(y)|^2}{|x-y|^{d+2s}}\,dx\,dy \leqslant CM^2R^{-s},$$
(2.13)

where $M := \sup_{n\geqslant 1}\|u_n\|_{H^s}$. A similar estimate holds for ψ_R in place of φ_R.

Now let $\varepsilon > 0$. We choose $R_0 > 0$ large enough such that $M(R) \geqslant \mu - \varepsilon/4$ for all $R \geqslant R_0$. We next choose $R_1 > 0$ large enough so that $CM^2R_1^{-s} \leqslant \varepsilon$. Of course, we may assume $R_1 \geqslant R_0$ and thus $M(R_1) \geqslant \mu - \varepsilon/4$. Since $\lim_{k\to\infty}M_{n_k}(R_1) = M(R_1)$, there exists $k_1 \geqslant 1$ large enough such that for $k \geqslant k_1$,

$$\mu - \frac{\varepsilon}{2} < M_{n_k}(R_1) \leqslant M_{n_k}(2R_1) < \mu + \varepsilon.$$

On the other hand, for each $k \geqslant 1$, there exists $y_k \in \mathbb{R}^d$ such that

$$M_{n_k}(R_1) \leqslant \int_{B(y_k,R_1)} |u_{n_k}(x)|^2\,dx + \frac{1}{k}.$$

By enlarging k_1 if necessary, we have for $k \geqslant k_1$,

$$\mu - \varepsilon < \int_{B(y_k,R_1)} |u_{n_k}(x)|^2\,dx \leqslant M_{n_k}(R_1) \leqslant M_{n_k}(2R_1) < \mu + \varepsilon.$$

We now set $u_k^1(x) = \varphi_{R_1}(x-y_k)u_{n_k}$. It follows that for $k \geqslant k_1$,

$$\|u_k^1\|_{L^2}^2 - \mu \geqslant \int_{B(y_k,R_1)} |u_{n_k}(x)|^2\,dx - \mu > -\varepsilon,$$

$$\|u_k^1\|_{L^2}^2 - \mu \leqslant \int_{B(y_k,2R_1)} |u_{n_k}(x)|^2\,dx - \mu \leqslant M_{n_k}(2R_1) - \mu < \varepsilon,$$

which imply that $\lim_{k\to\infty}\|u_k^1\|_{L^2}^2 = \mu$. Moreover, by the choice of R_1 and (2.13), we have for $k \geqslant k_1$,

$$\|(-\Delta)^{s/2}u_k^1\|_{L^2}^2 - \frac{1}{2}C(d,s)\iint \frac{[\varphi_{R_1}(x-y_k)]^2|u_{n_k}(x)-u_{n_k}(y)|^2}{|x-y|^{d+2s}}\,dx\,dy$$
$$\leqslant CM^2R_1^{-s} \leqslant \varepsilon.$$
(2.14)

We next set $u_k^2(x) = \psi_{R_k}(x - y_k)u_{n_k}$, where $R_k \to \infty$ as $k \to \infty$ is such that $M_{n_k}(2R_k) \leqslant \mu + \varepsilon$. Observe that

$$
\begin{aligned}
\|u_{n_k} - u_k^1 - u_k^2\|_{L^2}^2 &= \int \left(1 - \varphi_{R_1}^2(x - y_k) - \psi_{R_k}^2(x - y_k)\right) |u_{n_k}(x)|^2 \, \mathrm{d}x \\
&\leqslant \int_{R_1 \leqslant |x - y_k| \leqslant 2R_k} |u_{n_k}(x)|^2 \, \mathrm{d}x \\
&= \int_{B(y_k, 2R_k)} |u_{n_k}(x)|^2 \, \mathrm{d}x - \int_{B(y_k, R_1)} |u_{n_k}(x)|^2 \, \mathrm{d}x \\
&\leqslant M_{n_k}(2R_k) - (\mu - \varepsilon) \leqslant 2\varepsilon.
\end{aligned}
$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$
\|u_{n_k} - u_k^1 - u_k^2\|_{L^2}^2 \to 0 \tag{2.15}
$$

as $k \to \infty$, hence $\lim_{k \to \infty} \|u_k^2\|_{L^2}^2 = a - \mu$ and also for k large enough

$$
\begin{aligned}
\|(-\Delta)^{s/2} u_k^2\|_{L^2}^2 &- \frac{1}{2} C(d, s) \iint \frac{[\psi_{R_k}(x - y_k)]^2 |u_{n_k}(x) - u_{n_k}(y)|^2}{|x - y|^{d + 2s}} \, \mathrm{d}x \, \mathrm{d}y \\
&\leqslant CM^2 R_k^{-s} \leqslant \varepsilon. \tag{2.16}
\end{aligned}
$$

By using the fractional Gagliardo–Nirenberg inequality, we have for $2 \leqslant q < 2^*$,

$$
\begin{aligned}
\|u_{n_k} - u_k^1 - u_k^2\|_{L^q} &\leqslant C\|(-\Delta)^{s/2}(u_{n_k} - u_k^1 - u_k^2)\|_{L^2}^{d/2s - d/sq} \\
&\times \|u_{n_k} - u_k^1 - u_k^2\|_{L^2}^{1 - d/2s + d/sq}.
\end{aligned}
$$

Since the multiplication by a Schwarz function is a continuous map from $H^s(\mathbb{R}^d)$ to itself, we see that $(u_k^1)_{k \geqslant 1}$ and $(u_k^2)_{k \geqslant 1}$ are bounded in $H^s(\mathbb{R}^d)$. It follows from (2.15) that

$$
\|u_{n_k} - u_k^1 - u_k^2\|_{L^q} \to 0
$$

as $k \to \infty$ for any $2 \leqslant q < 2^*$. Moreover, by construction

$$
\mathrm{dist}\left(\mathrm{supp}(u_k^1), \mathrm{supp}(u_k^2)\right) \geqslant 2R_k - R_1 \to \infty
$$

as $k \to \infty$. Finally, the last conclusion follows from (2.14), (2.16) and the fact

$$
1 - [\varphi_{R_1}(x - y_k)]^2 - [\psi_{R_k}(x - y_k)]^2 \geqslant 0.
$$

The proof is complete. $\square$

REMARK 2.9. If instead of (2.9), we assume $\|u_n\|_{L^2}^2 = a$ for all $n \geqslant 1$, then the compactness holds for all $k \geqslant 1$, i.e., there exists a sequence $(y_k)_{k \geqslant 1} \subset \mathbb{R}^d$ such

that for all $\varepsilon > 0$, there exists $R(\varepsilon) > 0$ such that for all $k \geqslant 1$,

$$\int_{B(y_k, R(\varepsilon))} |u_{n_k}(x)|^2 \, \mathrm{d}x \geqslant a - \varepsilon.$$

In fact, since $\|u_{n_k}\|_{L^2}^2 = a$ for all $k \geqslant 1$, we see that for each $1 \leqslant k \leqslant k(\varepsilon)$, there exists $R_k(\varepsilon) > 0$ such that

$$\int_{B(y_k, R_k(\varepsilon))} |u_{n_k}(x)|^2 \, \mathrm{d}x \geqslant a - \varepsilon,$$

where $k(\varepsilon)$ is given in (2.11). Taking $R(\varepsilon) := \max\{R_0(\varepsilon), R_1(\varepsilon), \cdots, R_{k(\varepsilon)}(\varepsilon)\}$ with $R_0(\varepsilon)$ as in (2.11), we obtain

$$\int_{B(y_k, R(\varepsilon))} |u_{n_k}(x)|^2 \, \mathrm{d}x \geqslant a - \varepsilon$$

for all $k \geqslant 1$.

REMARK 2.10.

- If the vanishing occurs, then $(u_{n_k})_{k \geqslant 1}$ converges strongly to 0 in $L^q(\mathbb{R}^d)$ for any $2 < q < 2^*$. Indeed, it follows from the following fact (see e.g., [**12**]): if $(f_n)_{n \geqslant 1}$ is a bounded sequence in $H^s(\mathbb{R}^d)$ and satisfies

$$\lim_{n \to \infty} \sup_{y \in \mathbb{R}^d} \int_{B(y, R)} |f_n(x)|^2 \, \mathrm{d}x = 0$$

for some $R > 0$, then $f_n \to 0$ in $L^q(\mathbb{R}^d)$ for any $2 < q < 2^*$.

- If the compactness occurs, then we infer that up to a subsequence, $u_{n_k}(\cdot + y_k)$ converges strongly to u in $L^q(\mathbb{R}^d)$ for any $2 \leqslant q < 2^*$. Indeed, there exists a sequence $(y_k)_{k \geqslant 1} \subset \mathbb{R}^d$ such that for each $l \geqslant 1$, there exist $R_l > 0$ and $k_l \geqslant 1$ such that for all $k \geqslant k_l$,

$$\int_{B(y_k, R_l)} |u_{n_k}(x)|^2 \, \mathrm{d}x \geqslant a - \frac{1}{l} \quad \text{or} \quad \int_{B(0, R_l)} |\tilde{u}_{n_k}(x)|^2 \, \mathrm{d}x \geqslant a - \frac{1}{l}, \quad (2.17)$$

where $\tilde{u}_{n_k}(x) := u_{n_k}(x + y_k)$. Since the translated sequence $(\tilde{u}_{n_k})_{k \geqslant 1}$ is bounded in $H^s(\mathbb{R}^d)$, so up to subsequence, $\tilde{u}_{n_k} \rightharpoonup u$ weakly in $H^s(\mathbb{R}^d)$. By the lower semi-continuity of weak convergence, $\|u\|_{L^2}^2 \leqslant a$. For each $l \geqslant 1$, the embedding $H^s(B(0, R_l)) \hookrightarrow L^2(B(0, R_l))$ is compact, so up to a subsequence, we have $\tilde{u}_{n_k} \to u$ strongly in $L^2(B(0, R_l))$. By a standard diagonalization

argument, one may assume that there exists a subsequence still denoted by $(\tilde{u}_{n_k})_{k \geqslant 1}$ satisfies

$$\tilde{u}_{n_k} \to u \text{ strongly in } L^2(B(0, R_l))$$

for every $l \geqslant 1$. Taking $k \to \infty$ in (2.17), we obtain

$$a - \frac{1}{l} \leqslant \lim_{k \to \infty} \int_{B(0, R_l)} |\tilde{u}_{n_k}(x)|^2 \, \mathrm{d}x = \|u\|^2_{L^2(B(0, R_l))} \leqslant \|u\|^2_{L^2}.$$

Since $l \geqslant 1$ is arbitrary, it follows that $\|u\|^2_{L^2} = a$. Thus $\tilde{u}_{n_k} \to u$ strongly in $L^2(\mathbb{R}^d)$. By interpolating between $L^2(\mathbb{R}^d)$ and $L^{2^*}(\mathbb{R}^d)$ and using Sobolev embedding together with the boundedness of $(u_{n_k})_{k \geqslant 1}$ in $H^s(\mathbb{R}^d)$, we prove that $\tilde{u}_{n_k} \to u$ strongly in $L^q(\mathbb{R}^d)$ for any $2 \leqslant q < 2^*$.

A direct application of the concentration-compactness lemma is the following compactness of minimizing sequence for the fractional Gagliardo–Nirenberg inequality.

LEMMA 2.11. *Let $(u_n)_{n \geqslant 1}$ be a non-negative bounded sequence in $H^s(\mathbb{R}^d)$ satisfying*

$$\lim_{n \to \infty} \|u_n\|^2_{L^2} = a^*, \quad \lim_{n \to \infty} \|(-\Delta)^{s/2} u_n\|^2_{L^2} - \frac{d}{d+2s} \|u_n\|^{4s/d+2}_{L^{4s/d+2}} = 0 \qquad (2.18)$$

and

$$\liminf_{n \to \infty} \|u_n\|^{4s/d+2}_{L^{4s/d+2}} > 0, \qquad (2.19)$$

where a^* is defined in (1.12). Then there exist a subsequence $(u_{n_k})_{k \geqslant 1}$ and a sequence $(y_k)_{k \geqslant 1} \subset \mathbb{R}^d$ such that

$$u_{n_k}(\cdot + y_k) \to \gamma^{d/2} Q(\gamma \cdot) \text{ strongly in } H^s(\mathbb{R}^d)$$

for some $\gamma > 0$ as $k \to \infty$.

Proof. By lemma 2.8, there exists a subsequence $(u_{n_k})_{k \geqslant 1}$ satisfying one of the following three possibilities: vanishing, dichotomy and compactness.

No vanishing. Suppose that the vanishing occurs. We have from remark 2.10 that $u_{n_k} \to 0$ strongly in L^q for any $2 < q < 2^*$. In particular, $\|u_{n_k}\|^{4s/d+2}_{L^{4s/d+2}} \to 0$ which contradicts to (2.19).

No dichotomy. If the dichotomy occurs, then there exist $\mu \in (0, a^*)$ with $a^* := \|Q\|^2_{L^2}$ and sequences $(u^1_k)_{k \geqslant 1}, (u^2_k)_{k \geqslant 1}$ bounded in $H^s(\mathbb{R}^d)$ such that (2.10) holds.

We infer that

$$\|u_{n_k}\|_{L^{4s/d+2}}^{4s/d+2} = \|u_k^1\|_{L^{4s/d+2}}^{4s/d+2} + \|u_k^2\|_{L^{4s/d+2}}^{4s/d+2} + o_k(1). \tag{2.20}$$

By the fractional Gagliardo–Nirenberg inequality,

$$\|(-\Delta)^{s/2}u_k^1\|_{L^2}^2 \geqslant \frac{d}{d+2s} \left(\frac{a^*}{\|u_k^1\|_{L^2}^2}\right)^{2s/d} \|u_k^1\|_{L^{4s/d+2}}^{4s/d+2}$$

$$= \frac{d}{d+2s} \left(\frac{a^*}{\mu}\right)^{2s/d} \|u_k^1\|_{L^{4s/d+2}}^{4s/d+2} + o_k(1).$$

Similarly,

$$\|(-\Delta)^{s/2}u_k^2\|_{L^2}^2 \geqslant \frac{d}{d+2s} \left(\frac{a^*}{a^*-\mu}\right)^{2s/d} \|u_k^2\|_{L^{4s/d+2}}^{4s/d+2} + o_k(1).$$

It follows from (2.10) that

$$\|(-\Delta)^{s/2}u_{n_k}\|_{L^2}^2$$
$$\geqslant \|(-\Delta)^{s/2}u_k^1\|_{L^2}^2 + \|(-\Delta)^{s/2}u_k^2\|_{L^2}^2 + o_k(1)$$
$$\geqslant \frac{d}{d+2s} \left[\left(\frac{a^*}{\mu}\right)^{2s/d} \|u_k^1\|_{L^{4s/d+2}}^{4s/d+2} + \left(\frac{a^*}{a^*-\mu}\right)^{2s/d} \|u_k^2\|_{L^{4s/d+2}}^{4s/d+2}\right] + o_k(1)$$
$$\geqslant \frac{d}{d+2s} \min\left\{\left(\frac{a^*}{\mu}\right)^{2s/d}, \left(\frac{a^*}{a^*-\mu}\right)^{2s/d}\right\} \left(\|u_k^1\|_{L^{4s/d+2}}^{4s/d+2} + \|u_k^2\|_{L^{4s/d+2}}^{4s/d+2}\right) + o_k(1)$$
$$= \frac{d}{d+2s} \min\left\{\left(\frac{a^*}{\mu}\right)^{2s/d}, \left(\frac{a^*}{a^*-\mu}\right)^{2s/d}\right\} \|u_{n_k}\|_{L^{4s/d+2}}^{4s/d+2} + o_k(1).$$

Since $\mu \in (0, a^*)$, it follows from (2.19) that

$$\liminf_{k\to\infty} \|(-\Delta)^{s/2}u_{n_k}\|_{L^2}^2 - \frac{d}{d+2s}\|u_{n_k}\|_{L^{4s/d+2}}^{4s/d+2} > 0$$

which contradicts to (2.18).

Compactness. Therefore, the compactness occurs. By remark 2.10, there exist a subsequence still denoted by $(u_{n_k})_{k\geqslant 1}$ and $(\tilde{y}_k)_{k\geqslant 1} \subset \mathbb{R}^d$ such that $(u_{n_k}(\cdot + \tilde{y}_k))_{k\geqslant 1}$ converges weakly in $H^s(\mathbb{R}^d)$ and strongly in L^q for any $2 \leqslant q < 2^*$ to some function u. It follows that

$$\|u\|_{L^2}^2 = \lim_{k\to\infty} \|u_{n_k}(\cdot + \tilde{y}_k)\|_{L^2}^2 = \lim_{k\to\infty} \|u_{n_k}\|_{L^2}^2 = a^*$$

and by the lower semicontinuity of weak convergence,

$$\|(-\Delta)^{s/2}u\|_{L^2}^2 - \frac{d}{d+2s}\|u\|_{L^{4s/d+2}}^{4s/d+2}$$

$$\leqslant \liminf_{k\to\infty} \|(-\Delta)^{s/2}[u_{n_k}(\cdot + \tilde{y}_k)]\|_{L^2}^2 - \frac{d}{d+2s}\|u_{n_k}(\cdot + \tilde{y}_k)\|_{L^{4s/d+2}}^{4s/d+2}$$

$$= \lim_{k\to\infty} \|(-\Delta)^{s/2}u_{n_k}\|_{L^2}^2 - \frac{d}{d+2s}\|u_{n_k}\|_{L^{4s/d+2}}^{4s/d+2} = 0.$$

By the fractional Gagliardo–Nirenberg inequality, we obtain

$$\|(-\Delta)^{s/2}u\|_{L^2}^2 - \frac{d}{d+2s}\|u\|_{L^{4s/d+2}}^{4s/d+2} = 0$$

which implies $\|(-\Delta)^{s/2}u\|_{L^2} = \lim_{k\to\infty} \|(-\Delta)^{s/2}[u_{n_k}(\cdot + \tilde{y}_k)]\|_{L^2}$. In particular,

$$u_{n_k}(\cdot + \tilde{y}_k) \to u \text{ strongly in } H^s(\mathbb{R}^d)$$

as $k \to \infty$. We also have that u is an optimizer for the fractional Gagliardo–Nirenberg inequality. By theorem 2.3, $u(x) = \beta Q(\gamma(x + y))$ for some $\beta, \gamma > 0$ and $y \in \mathbb{R}^d$. Since $\|u\|_{L^2}^2 = a^* = \|Q\|_{L^2}^2$, we infer that $\beta = \gamma^{d/2}$, hence $u(x) = \gamma^{d/2}Q(\gamma(x + y))$ for some $\gamma > 0$ and $y \in \mathbb{R}^d$. We thus obtain

$$u_{n_k}(\cdot + y_k) \to \gamma^{d/2}Q(\gamma\cdot) \text{ strongly in } H^s(\mathbb{R}^d)$$

as $k \to \infty$, where $y_k := \tilde{y}_k - y$. The proof is complete. $\qquad\square$

3. Existence and non-existence of minimizers

3.1. No potential

In this subsection, we show the existence and non-existence of minimizers for $I(a)$ in the case of no external potential.

Proof of theorem 1.2. **The case $0 < \alpha < 4s/d$.** In this case, the existence of minimizers for $I(a)$ was established in [**13**] via the concentration-compactness principle. We now give an alternative simple proof using the radial compact embedding.

Step 1: We first show that the minimizing problem $I(a)$ is well-defined and there exists $C > 0$ such that $I(a) \leqslant -C < 0$. To see this, we take $u \in H^s(\mathbb{R}^d)$ be such that $\|u\|_{L^2}^2 = a$. By the fractional Gagliardo–Nirenberg inequality, we have

$$\|u\|_{L^{\alpha+2}}^{\alpha+2} \lesssim \|(-\Delta)^{s/2}u\|_{L^2}^{d\alpha/2s}\|u\|_{L^2}^{(4s-(d-2s)\alpha)/2}$$

which by Young's inequality and the fact $0 < d\alpha/2s < 2$ imply for any $\varepsilon > 0$,

$$\frac{1}{\alpha+2}\|u\|_{L^{\alpha+2}}^{\alpha+2} \leqslant \frac{\varepsilon}{2}\|(-\Delta)^{s/2}u\|_{L^2}^2 + C(\varepsilon, \alpha, a). \tag{3.1}$$

This shows that for any $\varepsilon > 0$, there exists $C(\varepsilon, \alpha, a) > 0$ such that

$$E(u) \geqslant \frac{1}{2}(1 - \varepsilon)\|(-\Delta)^{s/2}u\|_{L^2}^2 - C(\varepsilon, \alpha, a). \tag{3.2}$$

By choosing $0 < \varepsilon < 1$, we see that $E(u) \geqslant -C(\varepsilon, \alpha, a)$. The minimizing problem $I(a)$ is thus well-defined. Next, we define

$$u_\lambda(x) = \lambda^{d/2} u(\lambda x), \quad \lambda > 0. \tag{3.3}$$

It follows that $\|u_\lambda\|_{L^2}^2 = \|u\|_{L^2}^2 = a$ and

$$E(u_\lambda) = \frac{\lambda^{2s}}{2}\|(-\Delta)^{s/2}u\|_{L^2}^2 - \frac{\lambda^{d\alpha/2}}{\alpha+2}\|u\|_{L^{\alpha+2}}^{\alpha+2}.$$

Since $0 < d\alpha/2 < 2s$, we can find $\lambda_0 > 0$ small enough so that $E(u_{\lambda_0}) < 0$. Taking $C = -E(u_{\lambda_0}) > 0$, we get $I(a) \leqslant -C < 0$.

Step 2: We will show that there exists at least a minimizer for $I(a)$. Let $(u_n)_{n \geqslant 1}$ be a minimizing sequence for $I(a)$, i.e., $\|u_n\|_{L^2}^2 = a$ for all $n \geqslant 1$ and $E(u_n) \to I(a)$ as $n \to \infty$. We may assume u_n is radially symmetric and radially decreasing function. In fact, let u_n^* be the symmetric rearrangement of u_n. Since the symmetric rearrangement preserves $L^q(\mathbb{R}^d)$ norm for any $1 \leqslant q \leqslant \infty$ and by fractional Polya–Szegö's inequality $\|(-\Delta)^{s/2}u_n^*\|_{L^2} \leqslant \|(-\Delta)^{s/2}u_n\|_{L^2}$, we see that $E(u_n^*) \leqslant E(u_n)$ and $\|u_n^*\|_{L^2}^2 = \|u_n\|_{L^2}^2 = a$. This shows that $(u_n^*)_{n \geqslant 1}$ is also a minimizing sequence for $I(a)$. Moreover, it follows from (3.2) that $(u_n)_{n \geqslant 1}$ is bounded in $H^s(\mathbb{R}^d)$. Indeed, since $E(u_n) \to I(a)$ as $n \to \infty$, there exists $C > 0$ such that $E(u_n) \leqslant I(a) + C$ for any $n \geqslant 1$. By (3.2), we have that

$$\frac{1}{2}(1-\varepsilon)\|(-\Delta)^{s/2}u_n\|_{L^2}^2 \leqslant E(u_n) + C(\varepsilon, \alpha, a) \leqslant I(a) + C(\varepsilon, \alpha, a)$$

for any $n \geqslant 1$. This implies that $(u_n)_{n \geqslant 1}$ is bounded in $H^s(\mathbb{R}^d)$ by taking $0 < \varepsilon < 1$. We thus obtain a bounded sequence in $H_{\mathrm{rd}}^s(\mathbb{R}^d)$. By using the compact embedding $H_{\mathrm{rd}}^s(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$ for any $2 < q < 2^*$, there exist $u \in H^s(\mathbb{R}^d)$ and a subsequence $(u_{n_k})_{k \geqslant 1}$ such that $u_{n_k} \rightharpoonup u$ weakly in $H^s(\mathbb{R}^d)$ and $u_{n_k} \to u$ strongly in $L^q(\mathbb{R}^d)$ for any q as above. We will show that u is indeed a minimizer for $I(a)$. Since $I(a) < 0$, we have that $u \neq 0$. In fact, assume by contradiction that $u \equiv 0$, then

$$0 \leqslant \liminf_{k \to \infty} \frac{1}{2}\|(-\Delta)^{s/2}u_{n_k}\|_{L^2}^2 = \liminf_{k \to \infty} E(u_{n_k}) = I(a) < 0$$

which is a contradiction. We have from the fact $u_{n_k} \rightharpoonup u$ weakly in $H^s(\mathbb{R}^d)$ and $u_{n_k} \to u$ strongly in $L^{\alpha+2}(\mathbb{R}^d)$ that

$$E(u) \leqslant \liminf_{k \to \infty} E(u_{n_k}) = I(a).$$

Moreover, by the lower semi-continuity of weak convergence,

$$\|u\|_{L^2}^2 \leqslant \liminf_{k \to \infty} \|u_{n_k}\|_{L^2}^2 = a.$$

Now set $\lambda = \sqrt{a/\|u\|_{L^2}^2} \geqslant 1$. We have that

$$E(\lambda u) = \lambda^2 E(u) + \frac{\lambda^2(1-\lambda^\alpha)}{\alpha+2}\|u\|_{L^{\alpha+2}}^{\alpha+2}$$

or

$$E(u) = \frac{E(\lambda u)}{\lambda^2} + \frac{\lambda^\alpha - 1}{\alpha + 2} \|u\|_{L^{\alpha+2}}^{\alpha+2}.$$

Since $u \neq 0$ and $\|\lambda u\|_{L^2}^2 = a$, it follows that

$$I(a) \geqslant E(u) \geqslant \frac{E(\lambda u)}{\lambda^2} \geqslant \frac{I(a)}{\lambda^2}.$$

This implies that $\lambda \leqslant 1$ since $I(a) < 0$, hence $\lambda = 1$ and $\|u\|_{L^2}^2 = a$. We thus get $I(a) \leqslant E(u)$, hence $E(u) = I(a)$ or u is a minimizer for $I(a)$. This shows that for any $a > 0$, there exists at least a minimizer for $I(a)$ and $-\infty < I(a) < 0$.

The case $\alpha = 4s/d$. We first show the non-existence of minimizers for $I(a)$ in the case $0 < a < a^*$. Assume that there exists a minimizer u for $I(a)$ with $0 < a < a^*$. By the Gagliardo–Nirenberg inequality,

$$I(a) = E(u) \geqslant \frac{1}{2}\left(1 - \left(\frac{a}{a^*}\right)^{2s/d}\right) \|(-\Delta)^{s/2} u\|_{L^2}^2 > 0. \qquad (3.4)$$

On the other hand, we take $u \in H^s(\mathbb{R}^d)$ satisfying $\|u\|_{L^2}^2 = a$ and consider u_λ as in (3.3). We see that $\|u_\lambda\|_{L^2}^2 = \|u\|_{L^2}^2 = a$ and

$$I(a) \leqslant E(u_\lambda) = \frac{\lambda^{2s}}{2} \|(-\Delta)^{s/2} u\|_{L^2}^2 - \frac{\lambda^{2s}}{2(d+2s)} \|u\|_{L^{4s/d+2}}^{4s/d+2} \to 0$$

as $\lambda \to 0$ which contradicts (3.4).

We next prove the non-existence of minimizers for $I(a)$ when $a > a^*$. To this end, we need the following estimate due to [**11**, Lemma 3.2].

LEMMA 3.1 [**11**]. *Let $d \geqslant 1$ and $s \in (0, 1)$. Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be a smooth compactly supported function satisfying $0 \leqslant \varphi \leqslant 1$, $\varphi = 1$ on $|x| \leqslant 1$. Denote for $\tau > 0$,*

$$Q_\tau(x) := \varphi(\tau^{-1} x) Q(x), \qquad (3.5)$$

where Q is the unique (up to translations) positive radial ground state related to (1.13). Then it holds that

$$\|(-\Delta)^{s/2} Q_\tau\|_{L^2}^2 = \|(-\Delta)^{s/2} Q\|_{L^2}^2 + O(\tau^{-4s})$$

as $\tau \to \infty$.

This estimate is in fact a refined estimate of (2.12) thanks to the exact decay of Q at infinity given in theorem 2.3. Note that in [**11**, Lemma 3.2], the above estimate is stated in dimensions $d \geqslant 2$, but it still holds in 1D.

Let φ be as in lemma 3.1 and denote

$$u_\tau(x) = A_\tau \tau^{d/2} \varphi(x) Q_0(\tau x), \quad \tau > 0, \tag{3.6}$$

where $Q_0 = Q/\|Q\|_{L^2}$ and $A_\tau > 0$ is such that $\|u_\tau\|_{L^2}^2 = a$ for all $\tau > 0$. By definition,

$$a A_\tau^{-2} = \int \varphi^2(\tau^{-1} x) Q_0^2(x) \, dx.$$

Since $\|Q_0\|_{L^2} = 1$ and $Q_0(x), |\nabla Q_0(x)| = O(|x|^{-d-2s})$ for some $c > 0$ as $|x| \to \infty$, we see that for τ sufficiently large,

$$\left| \int (1 - \varphi^2(\tau^{-1} x)) Q_0^2(x) \, dx \right| \lesssim \int_{|x| \geqslant \tau} |x|^{-2d-4s} \, dx \lesssim \int_{|x| \geqslant \tau} |x|^{-2d-4s} \, dx \lesssim \tau^{-d-4s}.$$

This shows that

$$a A_\tau^{-2} = 1 + O\left(\tau^{-d-4s}\right) \quad \text{or} \quad A_\tau^2 = a + O\left(\tau^{-d-4s}\right) \tag{3.7}$$

as $\tau \to \infty$, where $A = O(\tau^{-d-4s})$ means $A \leqslant C\tau^{-d-4s}$ for some constant $C > 0$ independent of τ. Using the fact

$$(-\Delta)^{s/2} u_\tau(x) = A_\tau \frac{\tau^{d/2}}{\|Q\|_{L^2}} (-\Delta)^{s/2} [\varphi(x) Q(\tau x)] = A_\tau \tau^s \frac{\tau^{d/2}}{\|Q\|_{L^2}} [(-\Delta)^{s/2} Q_\tau](\tau x),$$

lemma 3.1 implies that

$$\begin{aligned}
\|(-\Delta)^{s/2} u_\tau\|_{L^2}^2 &= \frac{A_\tau^2 \tau^{2s}}{\|Q\|_{L^2}^2} \|(-\Delta)^{s/2} Q_\tau\|_{L^2}^2 \\
&= \frac{A_\tau^2 \tau^{2s}}{\|Q\|_{L^2}^2} \left(\|(-\Delta)^{s/2} Q\|_{L^2}^2 + O(\tau^{-4s}) \right)
\end{aligned}$$

as $\tau \to \infty$. By (3.7), we see that

$$\|(-\Delta)^{s/2} u_\tau\|_{L^2}^2 = \tau^{2s} a \|(-\Delta)^{s/2} Q_0\|_{L^2}^2 + O(\tau^{-d-6s}) \tag{3.8}$$

as $\tau \to \infty$. Similarly,

$$\|u_\tau\|_{L^{4s/d+2}}^{4s/d+2} = \tau^{2s} a_\tau^{2s/d+1} \|Q_0\|_{L^{4s/d+2}}^{4s/d+2} + O(\tau^{-d-6s}) \tag{3.9}$$

as $\tau \to \infty$. By the definition of $I(a)$,

$$\begin{aligned}
\frac{I(a)}{a} \leqslant \frac{E(u_\tau)}{a} &= \frac{\tau^{2s}}{2} \left(\|(-\Delta)^{s/2} Q_0\|_{L^2}^2 - a^{2s/d} \frac{d}{d+2s} \|Q_0\|_{L^{4s/d+2}}^{4s/d+2} \right) + O(\tau^{-d-6s}) \\
&= \tau^{2s} \frac{d}{4s} \left(1 - \left(\frac{a}{a^*} \right)^{2s/d} \right) + O(\tau^{-d-6s}) \tag{3.10}
\end{aligned}$$

as $\tau \to \infty$. Here we have used (2.8) to get

$$\frac{d}{2s} = \|(-\Delta)^{s/2} Q_0\|_{L^2}^2 = \frac{d}{d+2s} \|Q_0\|_{L^{4s/d+2}}^{4s/d+2} \|Q\|_{L^2}^{4s/d}.$$

It follows from (3.10) that for $a > a^*$,

$$I(a) \leqslant \lim_{\tau \to \infty} E(u_\tau) = -\infty.$$

This shows that $I(a) = -\infty$ for $a > a^*$, hence there is no minimizer for $I(a)$ when $a > a^*$.

We now consider the case $a = a^*$, we also have from (3.10) that $I(a^*) \leqslant 0$ which together with $I(a^*) \geqslant 0$ (by the fractional Gagliardo–Nirenberg inequality) imply $I(a) = 0$. In this case, we observe that Q is a minimizer for $I(a^*)$ since $E(Q) = 0$. If u is another minimizer for $I(a^*)$, then u is an optimizer for the fractional Gagliardo–Nirenberg inequality, hence by theorem 2.3, $u(x) = \beta Q(\gamma(x + y))$ for some $\beta \in \mathbb{C}, \beta \neq 0, \gamma > 0$ and $y \in \mathbb{R}^d$. Since $\|u\|_{L^2}^2 = a^* = \|Q\|_{L^2}^2$, we infer that $|\beta| = \gamma^{d/2}$ or $u(x) = e^{i\theta}\gamma^{d/2}Q(\gamma x + x^0)$ for some $\theta \in \mathbb{R}, \gamma > 0$ and $x^0 \in \mathbb{R}^d$.

The case $4s/d < \alpha < s^*$**.** Let $u \in H^s(\mathbb{R}^d)$ be such that $\|u\|_{L^2}^2 = a$. Let u_λ be as in (3.3). We see that $\|u_\lambda\|_{L^2}^2 = \|u\|_{L^2}^2 = a$ and

$$E(u_\lambda) = \frac{\lambda^{2s}}{2}\|(-\Delta)^{s/2}u\|_{L^2}^2 - \frac{\lambda^{d\alpha/2}}{\alpha + 2}\|u\|_{L^{\alpha+2}}^{\alpha+2}.$$

Since $d\alpha/2 > 2s$, we see that $E(u_\lambda) \to -\infty$ as $\lambda \to \infty$. This implies that $I(a) = -\infty$ and thus there is no minimizer for $I(a)$. $\qquad\qquad\square$

3.2. Periodic potential

In this subsection, we show the existence and non-existence of minimizers for $I(a)$ in the case of periodic potentials.

Proof of theorem 1.3. **The case** $0 < \alpha < 4s/d$**.** Let $u \in H^s(\mathbb{R}^d)$ be such that $\|u\|_{L^2}^2 = a$. By the fractional Gagliardo–Nirenberg inequality and Young's inequality with $0 < d\alpha/2s < 2$ (see (3.1)), we see that

$$E(u) \geqslant \frac{1}{2}(1 - \varepsilon)\|(-\Delta)^{s/2}u\|_{L^2}^2 + \frac{a}{2}\min V - C(\varepsilon, \alpha, a). \tag{3.11}$$

Taking $0 < \varepsilon < 1$, we see that $E(u) \geqslant (a/2)\min V - C(\varepsilon, \alpha, a)$ for all $u \in H^s(\mathbb{R}^d)$ with $\|u\|_{L^2}^2 = a$. This shows that $I(a) > -\infty$.

Let $x^0 \in \mathbb{R}^d$ and φ be as in lemma 3.1. Denote

$$u_\tau(x) = A_\tau \tau^{d/2}\varphi(x - x^0)Q_0^\alpha(\tau(x - x^0)), \quad \tau > 0, \tag{3.12}$$

where $Q_0^\alpha = Q_\alpha/\|Q_\alpha\|_{L^2}$ and $A_\tau > 0$ is such that $\|u_\tau\|_{L^2}^2 = a$ for all $\tau > 0$. By the same argument as in (3.7), we have that

$$A_\tau^2 = a + O(\tau^{-d-4s})$$

as $\tau \to \infty$. We also have from (3.8) and (3.9) that

$$\|(-\Delta)^{s/2}u_\tau\|_{L^2}^2 = \tau^{2s}a\|(-\Delta)^{s/2}Q_0^\alpha\|_{L^2}^2 + O(\tau^{-d-6s})$$

and

$$\|u_\tau\|_{L^{\alpha+2}}^{\alpha+2} = \tau^{d\alpha/2}a^{\alpha/2+1}\|Q_0^\alpha\|_{L^{\alpha+2}}^{\alpha+2} + O(\tau^{-d-6s})$$

as $\tau \to \infty$. On the other hand, we note that $\tau^d [Q_0^\alpha(\tau(x-x^0))]^2$ converges weakly to the Dirac delta function at x^0 when $\tau \to \infty$. Indeed, we take $\psi \in C_0^\infty(\mathbb{R}^d)$ and compute

$$\int \tau^d [Q_0^\alpha(\tau(x-x^0))]^2 \psi(x)\,\mathrm{d}x = \int \tau^d [Q_0^\alpha(\tau(x-x^0))]^2 (\psi(x) - \psi(x^0))\,\mathrm{d}x$$
$$+ \int \tau^d [Q_0^\alpha(\tau(x-x^0))]^2 \psi(x^0)\,\mathrm{d}x$$
$$= \int [Q_0^\alpha(x)]^2 \left(\psi(\tau^{-1}x + x^0) - \psi(x^0) \right)\,\mathrm{d}x + \psi(x^0)$$

where we have used that $\|Q_0^\alpha\|_{L^2}^2 = 1$. Note that the integral tends to zero as $\tau \to \infty$ due to the dominated convergence. Since $x \mapsto V(x)[\varphi(x-x^0)]^2$ is integrable, it follows that

$$\int V|u_\tau|^2\,\mathrm{d}x = A_\tau^2 \int V(x)[\varphi(x-x^0)]^2 \tau^d [Q_0^\alpha(\tau(x-x^0))]^2\,\mathrm{d}x \to aV(x^0)$$

as $\tau \to \infty$ for almost everywhere $x^0 \in \mathbb{R}^d$. By the definition of $I(a)$, we have that

$$\frac{I(a)}{a} \leqslant \frac{E(u_\tau)}{a} = \frac{\tau^{2s}}{2} \|(-\Delta)^{s/2} Q_0^\alpha\|_{L^2}^2 - \frac{\tau^{d\alpha/2}}{\alpha+2} a^{\alpha/2} \|Q_0^\alpha\|_{L^{\alpha+2}}^{\alpha+2} + \frac{1}{2}V(x^0) + o_\tau(1)$$
$$= \frac{\mathrm{d}\alpha}{4s - (d-2s)\alpha} \left(\frac{\tau^{2s}}{2} - \frac{2s\tau^{d\alpha/2}}{\mathrm{d}\alpha} \left(\frac{a}{\|Q_\alpha\|_{L^2}^2} \right)^{\alpha/2} \right)$$
$$+ \frac{1}{2}V(x^0) + o_\tau(1) \tag{3.13}$$

as $\tau \to \infty$ for almost everywhere $x^0 \in \mathbb{R}^d$. Here we have used (2.7) to get

$$\frac{\mathrm{d}\alpha}{4s - (d-2s)\alpha} = \|(-\Delta)^{s/2} Q_0^\alpha\|_{L^2}^2 = \frac{\mathrm{d}\alpha}{2s(\alpha+2)} \|Q_\alpha\|_{L^2}^\alpha \|Q_0^\alpha\|_{L^{\alpha+2}}^{\alpha+2}.$$

Taking $\tau = (a/\|Q_\alpha\|_{L^2}^2)^{\alpha/(4s-d\alpha)}$ and noting that $\tau \to \infty$ as $a \to \infty$, we infer from (3.13) that

$$\frac{I(a)}{a} \leqslant -\frac{4s - \mathrm{d}\alpha}{4s - (d-2s)\alpha} \left(\frac{a}{\|Q_\alpha\|_{L^2}^2} \right)^{2s\alpha/(4s-d\alpha)} + \frac{1}{2}V(x^0) + o_a(1) \tag{3.14}$$

as $a \to \infty$ for almost everywhere $x^0 \in \mathbb{R}^d$. Since $I(a)/a \to -\infty$ as $a \to \infty$, there exists $a_* > 0$ large enough such that

$$\frac{I(a)}{a} < \frac{1}{2} \inf \sigma((-\Delta)^s + V) \tag{3.15}$$

for any $a > a_*$. We will show the existence of minimizers for $I(a)$ with $a > a_*$. Let $(u_n)_{n \geqslant 1}$ be a minimizing sequence for $I(a)$. By (3.11), $(u_n)_{n \geqslant 1}$ is a bounded sequence in $H^s(\mathbb{R}^d)$. By the concentration-compactness principle given in lemma 2.8

(see also remark 2.9), there exists a subsequence $(u_{n_k})_{k \geqslant 1}$ satisfying one of the following three possibilities: vanishing, compactness and dichotomy.

No vanishing. If $(u_{n_k})_{k \geqslant 1}$ is vanishing, then by remark 2.10, $u_{n_k} \to 0$ strongly in $L^q(\mathbb{R}^d)$ for any $2 < q < 2^*$. Thus

$$I(a) = \lim_{k \to \infty} E(u_{n_k}) = \liminf_{k \to \infty} \left(\frac{1}{2} \|(-\Delta)^{s/2} u_{n_k}\|_{L^2}^2 + \frac{1}{2} \int V |u_{n_k}|^2 \, \mathrm{d}x \right)$$

$$\geqslant \frac{a}{2} \inf \sigma((-\Delta)^s + V)$$

which contradicts to (3.15).

No dichotomy. Assume the dichotomy occurs. Let $(u_k^1)_{k \geqslant 1}$ and $(u_k^2)_{k \geqslant 1}$ be the corresponding sequences in lemma 2.8. We first claim that

$$\liminf_{k \to \infty} E(u_{n_k}) - E(u_k^1) - E(u_k^2) \geqslant 0. \tag{3.16}$$

By lemma 2.8, we have that

$$\liminf_{k \to \infty} \|(-\Delta)^{s/2} u_{n_k}\|_{L^2}^2 - \|(-\Delta)^{s/2} u_k^1\|_{L^2}^2 - \|(-\Delta)^{s/2} u_k^2\|_{L^2}^2 \geqslant 0.$$

Since u_k^1 and u_k^2 have disjoint supports for k large and $\lim_{k \to \infty} \|u_{n_k} - u_k^1 - u_k^2\|_{L^q} = 0$ for $2 \leqslant q < 2^*$, we see that

$$\|u_{n_k}\|_{L^{\alpha+2}}^{\alpha+2} - \|u_k^1\|_{L^{\alpha+2}}^{\alpha+2} - \|u_k^2\|_{L^{\alpha+2}}^{\alpha+2} \to 0$$

as $k \to \infty$. Since V is bounded,

$$\int V(|u_{n_k}|^2 - |u_k^1|^2 - |u_k^2|^2) \, \mathrm{d}x = \int V(|u_{n_k}|^2 - |u_k^1 + u_k^2|^2) \, \mathrm{d}x \to 0$$

as $k \to \infty$. Collecting the above estimates, we prove (3.16).

We next have for any $\lambda > 0$ that

$$E(\lambda u) = \lambda^2 \left(\frac{1}{2} \|(-\Delta)^{s/2} u\|_{L^2}^2 + \frac{1}{2} \int V |u|^2 \, \mathrm{d}x \right) - \frac{\lambda^{\alpha+2}}{\alpha+2} \|u\|_{L^{\alpha+2}}^{\alpha+2}$$

$$= \lambda^{\alpha+2} E(u) + \lambda^2 (1 - \lambda^\alpha) \left(\frac{1}{2} \|(-\Delta)^{s/2} u\|_{L^2}^2 + \frac{1}{2} \int V |u|^2 \, \mathrm{d}x \right)$$

or equivalently

$$E(u) = \frac{1}{\lambda^{\alpha+2}} E(\lambda u) + \left(1 - \frac{1}{\lambda^\alpha} \right) \left(\frac{1}{2} \|(-\Delta)^{s/2} u\|_{L^2}^2 + \frac{1}{2} \int V |u|^2 \, \mathrm{d}x \right).$$

Set

$$\lambda_k^1 := \sqrt{\frac{a}{\|u_k^1\|_{L^2}^2}}, \quad \lambda_k^2 := \sqrt{\frac{a}{\|u_k^2\|_{L^2}^2}}.$$

It follows that

$$E(u_k^1) = \frac{1}{(\lambda_k^1)^{\alpha+2}} E(\lambda_k^1 u_k^1) + \left(1 - \frac{1}{(\lambda_k^1)^\alpha}\right)\left(\frac{1}{2}\|(-\Delta)^{s/2} u_k^1\|_{L^2}^2 + \frac{1}{2}\int V|u_k^1|^2 \, \mathrm{d}x\right)$$

$$\geqslant \left(\frac{\|u_k^1\|_{L^2}^2}{a}\right)^{\alpha/2+1} I(a) + \|u_k^1\|_{L^2}^2 \left(1 - \left(\frac{\|u_k^1\|_{L^2}^2}{a}\right)^{\alpha/2}\right)\frac{1}{2}\inf \sigma((-\Delta)^s + V).$$

Using the fact that $\|u_k^1\|_{L^2}^2 \to \mu$ as $k \to \infty$, we see that

$$E(u_k^1) \geqslant \left(\frac{\mu}{a}\right)^{\alpha/2+1} I(a) + \frac{\mu}{a}\left(1 - \left(\frac{\mu}{a}\right)^{\alpha/2}\right)\frac{a}{2}\inf \sigma((-\Delta)^s + V) + o_k(1)$$

as $k \to \infty$. Similarly, since $\|u_k^2\|_{L^2}^2 \to a - \mu$ as $k \to \infty$,

$$E(u_k^2) \geqslant \left(\frac{a-\mu}{a}\right)^{\alpha/2+1} I(a) + \frac{a-\mu}{a}\left(1 - \left(\frac{a-\mu}{a}\right)^{\alpha/2}\right)$$

$$\times \frac{a}{2}\inf \sigma((-\Delta)^s + V) + o_k(1)$$

as $k \to \infty$. We infer from (3.16) that

$$E(u_{n_k}) \geqslant E(u_k^1) + E(u_k^2) + o_k(1)$$

$$\geqslant \left(\left(\frac{\mu}{a}\right)^{\alpha/2+1} + \left(\frac{a-\mu}{a}\right)^{\alpha/2+1}\right) I(a)$$

$$+ \left(1 - \left(\frac{\mu}{a}\right)^{\alpha/2+1} - \left(\frac{a-\mu}{a}\right)^{\alpha/2+1}\right)\frac{a}{2}\inf \sigma((-\Delta)^s + V) + o_k(1)$$

as $k \to \infty$. Taking $k \to \infty$, we obtain

$$I(a) \geqslant \left(\left(\frac{\mu}{a}\right)^{\alpha/2+1} + \left(\frac{a-\mu}{a}\right)^{\alpha/2+1}\right) I(a)$$

$$+ \left(1 - \left(\frac{\mu}{a}\right)^{\alpha/2+1} - \left(\frac{a-\mu}{a}\right)^{\alpha/2+1}\right)\frac{a}{2}\inf \sigma((-\Delta)^s + V).$$

Since $0 < \mu < a$, we obtain

$$I(a) \geqslant \frac{a}{2}\inf \sigma((-\Delta)^s + V)$$

which contradicts to (3.15).

Compactness. There thus exists a sequence $(y_k)_{k \geqslant 1} \subset \mathbb{R}^d$ such that up to a subsequence, $(u_{n_k}(\cdot + y_k))_{k \geqslant 1}$ converges weakly in $H^s(\mathbb{R}^d)$ and strongly in $L^q(\mathbb{R}^d)$

to some u for all $2 \leqslant q < 2^*$. This implies that

$$\|u\|_{L^2}^2 = \lim_{k \to \infty} \|u_{n_k}(\cdot + y_k)\|_{L^2}^2 = a$$

and

$$\|(-\Delta)^{s/2} u\|_{L^2}^2 \leqslant \liminf_{k \to \infty} \|(-\Delta)^{s/2} u_{n_k}(\cdot + y_k)\|_{L^2}^2 = \liminf_{k \to \infty} \|(-\Delta)^{s/2} u_{n_k}\|_{L^2}^2$$

and

$$\|u\|_{L^{\alpha+2}}^{\alpha+2} = \lim_{k \to \infty} \|u_{n_k}(\cdot + y_k)\|_{L^{\alpha+2}}^{\alpha+2} = \lim_{k \to \infty} \|u_{n_k}\|_{L^{\alpha+2}}^{\alpha+2}.$$

We next have that

$$\int V(x)|u_{n_k}(x)|^2 \, \mathrm{d}x = \int V(x + y_k)|u_{n_k}(x + y_k)|^2 \, \mathrm{d}x$$

$$= \int V(x + y_k)|u(x)|^2 \, \mathrm{d}x$$

$$+ \int V(x + y_k) \left(|u_{n_k}(x + y_k)|^2 - |u(x)|^2 \right) \, \mathrm{d}x.$$

Note that

$$\left| \int V(x + y_k) \left(|u_{n_k}(x + y_k)|^2 - |u(x)|^2 \right) \right|$$

$$\leqslant \|V\|_{L^\infty} \|u_{n_k}(\cdot + y_k) - u\|_{L^2} \left(\|u_{n_k}(\cdot + y_k)\|_{L^2} + \|u\|_{L^2} \right) \to 0$$

as $k \to \infty$. This implies that

$$E(u_{n_k}) = \frac{1}{2}\|(-\Delta)^{s/2} u_{n_k}\|_{L^2}^2 + \frac{1}{2}\int V|u_{n_k}|^2 \, \mathrm{d}x - \frac{1}{\alpha+2}\|u_{n_k}\|_{L^{\alpha+2}}^{\alpha+2}$$

$$\geqslant \frac{1}{2}\|(-\Delta)^{s/2} u\|_{L^2}^2 + \frac{1}{2}\int V(x + y_k)|u(x)|^2 \, \mathrm{d}x - \frac{1}{\alpha+2}\|u\|_{L^{\alpha+2}}^{\alpha+2} + o_k(1)$$

as $k \to \infty$. Since V is periodic, we write

$$y_k = x_k + z_k, \quad x_k \in [0,1]^d, z_k \in \mathbb{Z}^d.$$

Since $(x_k)_{k \geqslant 1}$ is bounded in $\mathbb{R}^d$, up to a subsequence, $x_k \to x^0$ as $k \to \infty$. Thus

$$\int V(x + y_k)|u(x)|^2 \, \mathrm{d}x = \int V(x + x_k)|u(x)|^2 \, \mathrm{d}x \to \int V(x + x^0)|u(x)|^2 \, \mathrm{d}x$$

as $k \to \infty$ by the dominated convergence. Hence

$$E(u_{n_k}) \geqslant \frac{1}{2}\|(-\Delta)^{s/2} u\|_{L^2}^2 + \frac{1}{2}\int V(x + x^0)|u(x)|^2 \, \mathrm{d}x - \frac{1}{\alpha+2}\|u\|_{L^{\alpha+2}}^{\alpha+2} + o_k(1)$$

$$= E(u(\cdot - x^0)) + o_k(1)$$

as $k \to \infty$. Taking $k \to \infty$, we get $I(a) \geqslant E(u(\cdot - x^0))$. On the other hand, $\|u(\cdot - x^0)\|_{L^2}^2 = \|u\|_{L^2}^2 = a$ which implies that $I(a) \leqslant E(u(\cdot - x^0))$. This shows that

$u(\cdot - x^0)$ is a minimizer for $I(a)$. This shows the existence of minimizers for $I(a)$ for $a > a_*$.

The case $\alpha = 4s/d$. We first claim that

$$\lim_{a \nearrow a^*} \frac{I(a)}{a} = \frac{I(a^*)}{a^*} = \frac{1}{2} \min V. \tag{3.17}$$

Let u_τ be as in (3.12). By the same argument as in the mass-subcritical case (see also (3.10)), we have that

$$\|(-\Delta)^{s/2} u_\tau\|_{L^2}^2 = \tau^{2s} a \|(-\Delta)^{s/2} Q_0\|_{L^2}^2 + O(\tau^{-d-6s}),$$

$$\|u_\tau\|_{L^{4s/d+2}}^{4s/d+2} = \tau^{2s} a^{2s/d+1} \|Q_0\|_{L^{4s/d+2}}^{4s/d+2} + O(\tau^{-d-6s}),$$

where $Q_0 = Q/\|Q\|_{L^2}$ and

$$\int V |u_\tau|^2 \, \mathrm{d}x \to a V(x^0)$$

as $\tau \to \infty$ for almost everywhere $x^0 \in \mathbb{R}^d$. It follows that

$$\frac{I(a)}{a} \leqslant \frac{E(u_\tau)}{a} = \frac{\tau^{2s}}{2} \left(\|(-\Delta)^{s/2} Q_0\|_{L^2}^2 - a^{2s/d} \frac{d}{d+2s} \|Q_0\|_{L^{4s/d+2}}^{4s/d+2} \right)$$

$$+ \frac{1}{2} V(x^0) + o_\tau(1)$$

$$= \tau^{2s} \frac{d}{4s} \left(1 - \left(\frac{a}{a^*} \right)^{2s/d} \right) + \frac{1}{2} V(x^0) + o_\tau(1) \tag{3.18}$$

as $\tau \to \infty$ for almost everywhere $x^0 \in \mathbb{R}^d$. Taking $\tau = (1 - (a/a^*)^{2s/d})^{-1/4s}$ so that $\tau \to \infty$ as $a \nearrow a^*$, we get

$$\frac{I(a)}{a} \leqslant \frac{d}{4s} \left(1 - \left(\frac{a}{a^*} \right)^{2s/d} \right)^{1/2} + \frac{1}{2} V(x^0) + o_{a \nearrow a^*}(1) \tag{3.19}$$

as $a \nearrow a^*$ for almost everywhere $x^0 \in \mathbb{R}^d$. Letting $a \nearrow a^*$ and optimizing the right-hand side, we obtain

$$\limsup_{a \nearrow a^*} \frac{I(a)}{a} \leqslant \frac{1}{2} \min V.$$

On the other hand, by the fractional Gagliardo–Nirenberg inequality, we have for $\|u\|_{L^2}^2 = a$ that

$$E(u) \geqslant \frac{1}{2} \left(1 - \left(\frac{a}{a^*} \right)^{2s/d} \right) \|(-\Delta)^{s/2} u\|_{L^2}^2 + \frac{a}{2} \min V.$$

Assume at the moment that $I(a)/a$ is a decreasing function in a. This implies that

$$\frac{I(a)}{a} \geqslant \frac{I(a^*)}{a^*} \geqslant \frac{1}{2} \min V,$$

hence

$$\liminf_{a \nearrow a^*} \frac{I(a)}{a} \geqslant \frac{I(a^*)}{a^*} \geqslant \frac{1}{2} \min V$$

which proves the claim. To see that $a \mapsto I(a)/a$ is a decreasing function, we take $0 < a \leqslant b$. We will show that $I(a)/a \geqslant I(b)/b$. Let $u \in H^s(\mathbb{R}^d)$ be such that $\|u\|_{L^2}^2 = a$ and set $\lambda = \sqrt{b/a} \geqslant 1$. We see that $\|\lambda u\|_{L^2}^2 = b$ and

$$E(\lambda u) = \lambda^2 \left(\frac{1}{2} \|(-\Delta)^{s/2} u\|_{L^2}^2 + \frac{1}{2} \int V |u|^2 \, \mathrm{d}x \right) - \frac{\lambda^{\alpha+2}}{\alpha+2} \|u\|_{L^{\alpha+2}}^{\alpha+2}$$

$$= \lambda^2 E(u) + \frac{\lambda^2 (1 - \lambda^\alpha)}{\alpha+2} \|u\|_{L^{\alpha+2}}^{\alpha+2} \leqslant \lambda^2 E(u).$$

By the definition of $I(b)$,

$$I(b) \leqslant E(\lambda u) \leqslant \lambda^2 E(u).$$

Taking the infimum over all $u \in H^s(\mathbb{R}^d)$ with $\|u\|_{L^2}^2 = a$, we get $I(b) \leqslant (b/a) I(a)$ which shows that $I(a)/a$ is a decreasing function in a.

It also follows from (3.18) that for $a > a^*$,

$$\frac{I(a)}{a} \leqslant \lim_{\tau \to \infty} \frac{E(u_\tau)}{a} = -\infty$$

which implies that there is no minimizer for $I(a)$.

We next show that there is no minimizer for $I(a^*)$. In fact, assume by contradiction that there exists a minimizer for $I(a^*)$, says u. Then by the fractional Gagliardo–Nirenberg inequality,

$$\frac{a^*}{2} \min V = I(a^*) = E(u) = \frac{1}{2} \|(-\Delta)^{s/2} u\|_{L^2}^2$$

$$+ \frac{1}{2} \int V |u|^2 \, \mathrm{d}x - \frac{d}{2(d+2s)} \|u\|_{L^{4s/d+2}}^{4s/d+2} \geqslant \frac{a^*}{2} \min V.$$

This implies that

$$\int V |u|^2 \, \mathrm{d}x = a^* \min V \tag{3.20}$$

and

$$\frac{1}{2} \|(-\Delta)^{s/2} u\|_{L^2}^2 - \frac{d}{2(d+2s)} \|u\|_{L^{4s/d+2}}^{4s/d+2} = 0. \tag{3.21}$$

By (3.21), u is an optimizer for the fractional Gagliardo–Nirenberg inequality, hence u is equal to Q up to symmetries. In this case, (3.20) cannot occur except V is a constant, but it contradicts to the assumption $\min V < \inf \sigma((-\Delta)^s + V)$.

We next show the existence of minimizers for $I(a)$ when $a_* < a < a^*$ for some $0 < a_* < a^*$. By (3.17) and the assumption $\min V < \inf \sigma((-\Delta)^s + V)$, there exists

$0 < a_* < a^*$ such that

$$\frac{I(a)}{a} < \frac{1}{2} \inf \sigma((-\Delta)^s + V) \tag{3.22}$$

for any $a_* < a < a^*$. Using (3.22), we can repeat the same argument as in the mass-subcritical case to show the existence of minimizers for $I(a)$ when $a_* < a < a^*$.

The case $4s/d < \alpha < s^*$**.** Let $a > 0$. We take $u \in H^s(\mathbb{R}^d)$ such that $\|u\|_{L^2}^2 = a$. Let u_λ be as in (3.3). We see that $\|u_\lambda\|_{L^2}^2 = \|u\|_{L^2}^2 = a$ and

$$E(u_\lambda) = \frac{\lambda^{2s}}{2} \|(-\Delta)^{s/2} u\|_{L^2}^2 + \frac{1}{2} \int V(\lambda^{-1} x) |u(x)|^2 \, \mathrm{d}x - \frac{\lambda^{d\alpha/2}}{\alpha+2} \|u\|_{L^{\alpha+2}}^{\alpha+2}.$$

Since $\mathrm{d}\alpha/2 > 2s$, we see that the right-hand side tends to $-\infty$ as $\lambda \to \infty$. Note that the second term in the right-hand side is bounded due to the fact that V is bounded. This shows that $I(a) = -\infty$ and there is no minimizer for $I(a)$. The proof is complete. □

4. Blow-up behaviour of minimizers

In this section, we study the blow-up behaviour of minimizers for $I(a)$ in the mass-critical case given in theorems 1.4 and 1.5.

Proof of theorem 1.4. We first prove that u_a blows up as $a \nearrow a^*$ in the sense of (1.14). Assume that it is not true, then $(u_a)_{a \nearrow a^*}$ is a bounded sequence in $H^s(\mathbb{R}^d)$. Applying the concentration-compactness principle with the fact $\|u_a\|_{L^2}^2 = a \nearrow a^*$ as $a \nearrow a^*$, there exists a subsequence still denoted by $(u_a)_{a \nearrow a^*}$ satisfying one of the three possibilities: vanishing, dichotomy and compactness. Using the same argument as in the proof of theorem 1.3 together with the fact (see (3.17))

$$\frac{I(a^*)}{a^*} = \frac{1}{2} \min V < \frac{1}{2} \inf \sigma((-\Delta)^s + V),$$

we see that the vanishing and dichotomy cannot occur. Thus the compactness must occur, and there thus exist $x^0 \in [0,1]^d$ and $u \in H^s(\mathbb{R}^d)$ such that $u(\cdot - x^0)$ is a minimizer for $I(a^*)$ which is a contradiction.

Let ε_a be as in (1.14). We define

$$v_a(x) := \varepsilon_a^{d/2} u_a(\varepsilon_a x).$$

It is easy to see that

$$\|v_a\|_{L^2}^2 = \|u_a\|_{L^2}^2 = a, \quad \|(-\Delta)^{s/2} v_a\|_{L^2}^2 = \varepsilon_a^{2s} \|(-\Delta)^{s/2} u_a\|_{L^2}^2 = 1.$$

By the fractional Gagliardo–Nirenberg inequality and (3.17),

$$\begin{aligned} 0 &\leqslant \|(-\Delta)^{s/2} v_a\|_{L^2}^2 - \frac{d}{d+2s} \|v_a\|_{L^{4s/d+2}}^{4s/d+2} \\ &= \varepsilon_a^{2s} \left(\|(-\Delta)^{s/2} u_a\|_{L^2}^2 - \frac{d}{d+2s} \|u_a\|_{L^{4s/d+2}}^{4s/d+2} \right) \end{aligned}$$

$$= \varepsilon_a^{2s} \left(2E(u_a) - \int V|u_a|^2 \, dx \right)$$

$$\leqslant \varepsilon_a^{2s}(2I(a) - a \min V) \to 0$$

as $a \nearrow a^*$ which implies

$$\lim_{a \nearrow a^*} \|(-\Delta)^{s/2} v_a\|_{L^2}^2 - \frac{d}{d+2s} \|v_a\|_{L^{4s/d+2}}^{4s/d+2} = 0.$$

By lemma 2.11, there exist a subsequence still denoted by $(v_a)_{a \nearrow a^*}$ and a sequence $(y_a)_{a \nearrow a^*} \subset \mathbb{R}^d$ such that

$$v_a(\cdot + y_a) \to \gamma^{d/2} Q(\gamma \cdot) \text{ strongly in } H^s(\mathbb{R}^d)$$

for some $\gamma > 0$ as $a \nearrow a^*$. Since $\|(-\Delta)^{s/2} v_a\|_{L^2} = 1$, it follows that $\gamma = 1$. We next write

$$\varepsilon_a y_a = x_a + z_a, \quad x_a \in [0,1]^d, \ z_a \in \mathbb{Z}^d. \tag{4.1}$$

Since $(x_a)_{a \nearrow a^*}$ is bounded in $\mathbb{R}^d$, up to a subsequence, $x_a \to x^0 \in [0,1]^d$ as $a \nearrow a^*$. We will show that $V(x^0) = \min V$. Indeed, by the fractional Gagliardo–Nirenberg inequality and (3.17),

$$\min V \leqslant \frac{1}{a} \int V(x)|u_a(x)|^2 \, dx \leqslant \frac{2I(a)}{a} \to \min V$$

as $a \nearrow a^*$. By the Fatou's lemma,

$$\min V = \lim_{a \nearrow a^*} \frac{1}{a} \int V(x)|u_a(x)|^2 \, dx = \lim_{a \nearrow a^*} \frac{1}{a} \int V(\varepsilon_a x)|v_a(x)|^2 \, dx$$

$$= \lim_{a \nearrow a^*} \frac{1}{a} \int V(\varepsilon_a x + \varepsilon_a y_a)|v_a(x + y_a)|^2 \, dx$$

$$= \lim_{a \nearrow a^*} \frac{1}{a} \int V(\varepsilon_a x + x_a)|v_a(x + y_a)|^2 \, dx$$

$$\geqslant \int \lim_{a \nearrow a^*} \frac{1}{a} V(\varepsilon_a x + x_a)|v_a(x + y_a)|^2 \, dx$$

$$= V(x^0) \frac{1}{a^*} \int [Q(x)]^2 \, dx = V(x^0) \tag{4.2}$$

which implies that $V(x^0) = \min V$. In the last equality, we have used the fact that up to a subsequence, $v_a(\cdot + y_a)$ converges to Q almost everywhere. $\qquad \square$

The rest of this section is devoted to the proof of theorem 1.5. Before giving the proof, we need the following lemmas.

LEMMA 4.1. *Let $d \geqslant 1$, $0 < s < 1$ and $V \in C(\mathbb{R}^d)$ satisfy (V1), (V2) and (V3). Then there exist positive constants $C_1 < C_2$ independent of a such that as $a \nearrow a^*$,*

$$C_1 \beta_a^{p/(2s+p)} \leqslant \frac{I(a)}{a} \leqslant C_2 \beta_a^{p/(2s+p)}, \tag{4.3}$$

where β_a is given in (1.15).

Proof. Taking $x^0 \equiv x_0$ with x_0 in (V3), we have from (3.18) that

$$\frac{I(a)}{a} \leqslant \frac{d}{4s}\tau^{2s}\beta_a + o_\tau(1)$$

as $\tau \to \infty$. The upper bound follows by taking $\tau = C\beta_a^{-1/(2s+p)}$ for some constant $C > 0$. Note that the constant C_2 in (4.3) can be made as large as we want by enlarging the constant C.

To see the lower bound, we need the following claim.

CLAIM 4.2. Let $d \geqslant 1$, $0 < s < 1$ and V satisfy (V1), (V2) and (V3). Let ε_a and y_a be as in the proof of theorem 1.4 and write

$$\varepsilon_a y_a = x_a + z_a, \quad x_a \in [0,1]^d, \ z_a \in \mathbb{Z}^d.$$

Then there exists a subsequence still denoted by $(x_a)_{a \nearrow a^*}$ such that $x_a \to x_0$, where x_0 is given in (V3). Moreover, there exists $C > 0$ independent of a such that

$$\lim_{a \nearrow a^*} \varepsilon_a^{-p} \int_{\mathbb{R}^d} V(x)|u_a(x)|^2 \, \mathrm{d}x \geqslant C, \tag{4.4}$$

where p is given in (V3).

Proof of claim 4.2. Since $(x_a)_{a \nearrow a^*}$ is bounded in $\mathbb{R}^d$, up to a subsequence, $x_a \to x^0$ for some $x^0 \in [0,1]^d$ as $a \nearrow a^*$. We need to show that $x^0 \equiv x_0$, i.e., $V(x^0) = 0$ which is done by the same argument as in (4.2). It remains to prove (4.4). To this end, we use Fatou's lemma and the assumption (V3) to get

$$\lim_{a \nearrow a^*} \varepsilon_a^{-p} \int V(x)|u_a(x)|^2 \, \mathrm{d}x = \lim_{a \nearrow a^*} \varepsilon_a^{-p} \int V(\varepsilon_a x + x_a)|v_a(x + y_a)|^2 \, \mathrm{d}x$$

$$\geqslant \int \lim_{a \nearrow a^*} \varepsilon_a^{-p} V(\varepsilon_a x + x_a)|v_a(x + y_a)|^2 \, \mathrm{d}x$$

$$= \kappa \int \lim_{a \nearrow a^*} \left| x + \frac{x_a - x_0}{\varepsilon_a} \right|^p [Q(x)]^2 \, \mathrm{d}x. \tag{4.5}$$

We will show that $(x_a - x_0)/\varepsilon_a$ is uniformly bounded as $a \nearrow a^*$. Indeed, if $|(x_a - x_0)/\varepsilon_a| \to \infty$ as $a \nearrow a^*$, then for any $N > 0$ large, $|(x_a - x_0)/\varepsilon_a| \geqslant N$ for a close to a^*. It follows that

$$\lim_{a \nearrow a^*} \varepsilon_a^{-p} \int V(x)|u_a(x)|^2 \, \mathrm{d}x \geqslant \kappa \int_{|x| \leqslant N/2} \lim_{a \nearrow a^*} \left| x + \frac{x_a - x_0}{\varepsilon_a} \right|^p [Q(x)]^2 \, \mathrm{d}x$$

$$\geqslant \kappa \left(\frac{N}{2} \right)^p \int_{|x| \leqslant \frac{N}{2}} [Q(x)]^2 \, \mathrm{d}x = CN^p,$$

for some $C > 0$ independent of N. Using the fact $\|(-\Delta)^{s/2}v_a\|_{L^2} = 1$ and $\|v_a\|_{L^2}^2 = a$, we infer that as $a \nearrow a^*$,

$$
\begin{aligned}
I(a) = E(u_a) &= \frac{1}{2}\|(-\Delta)^{s/2}u_a\|_{L^2}^2 - \frac{d}{2(d+2s)}\|u_a\|_{L^{4s/d+2}}^{4s/d+2} + \frac{1}{2}\int V(x)|u_a(x)|^2 \,\mathrm{d}x \\
&= \frac{1}{2}\varepsilon_a^{-2s}\left(\|(-\Delta)^{s/2}v_a\|_{L^2}^2 - \frac{d}{d+2s}\|v_a\|_{L^{4s/d+2}}^{4s/d+2}\right) \\
&\quad + \frac{1}{2}\int V(x)|u_a(x)|^2 \,\mathrm{d}x \\
&\geqslant \frac{1}{2}\varepsilon_a^{-2s}\left(1 - \left(\frac{a}{a^*}\right)^{2s/d}\right) + C\varepsilon_a^p N^p \\
&= \frac{1}{2}\varepsilon_a^{-2s}\beta_a + C\varepsilon_a^p N^p \\
&\geqslant C(p,s)N^{2sp/(2s+p)}\beta_a^{p/(2s+p)}, \tag{4.6}
\end{aligned}
$$

where the last inequality follows by Young's inequality. This however contradicts to the upper bound in (4.3) by taking $N > 0$ sufficiently large. It follows that $|(x_a - x_0)/\varepsilon_a|$ is uniformly bounded as $a \nearrow a^*$, says $\lim_{a \nearrow a^*} |(x_a - x_0)/\varepsilon_a| \leqslant C$. We see that

$$
\lim_{a \nearrow a^*} \varepsilon_a^{-p}\int V(x)|u_a(x)|^2 \,\mathrm{d}x \geqslant \kappa \int_{|x| \geqslant 2C} \lim_{a \nearrow a^*} \left|x + \frac{x_a - x_0}{\varepsilon_a}\right|^p [Q(x)]^2 \,\mathrm{d}x \geqslant C
$$

for some $C > 0$ independent of a. Note that the constant $C > 0$ may change from line to line. $\square$

We now prove the lower bound in (4.3). By (4.4) and the same argument as (4.6), we have that for $a \nearrow a^*$,

$$
I(a) \geqslant C\beta_a^{p/(2s+p)}.
$$

The lower bound follows by taking $C_1 = C/a^*$. $\square$

LEMMA 4.3. *Let $V \in C(\mathbb{R}^d)$ satisfy (V1), (V2) and (V3). Let u_a be a minimizer for $I(a)$ with $a_* < a < a^*$. Then there exist constants C_3, C_4 independent of a such that as $a \nearrow a^*$,*

$$
\frac{\|(-\Delta)^{s/2}u_a\|_{L^2}^2}{a} \leqslant C_3\beta_a^{-2s/(2s+p)}, \quad C_4\beta_a^{-2s/(2s+p)} \leqslant \frac{\|u_a\|_{L^{4s/d+2}}^{4s/d+2}}{a^{2s/d+1}}. \tag{4.7}
$$

Proof. Using the fractional Gagliardo–Nirenberg inequality and the fact $V \geqslant 0$, we see that

$$
\begin{aligned}
I(a) = E(u_a) &\geqslant \frac{d}{2(d+2s)}\left(\left(\frac{a_*}{a}\right)^{2s/d} - 1\right)\|u_a\|_{L^{4s/d+2}}^{4s/d+2} \\
&= \frac{d}{2(d+2s)}\frac{\beta_a}{1-\beta_a}\|u_a\|_{L^{4s/d+2}}^{4s/d+2} \\
&\geqslant \frac{d}{2(d+2s)}\beta_a\|u_a\|_{L^{4s/d+2}}^{4s/d+2}.
\end{aligned}
$$

It follows from the upper bound in (4.3) and $V \geqslant 0$ that

$$\frac{\|(-\Delta)^{s/2}u_a\|_{L^2}^2}{a} \leqslant 2\frac{I(a)}{a} + \frac{d}{d+2s}\frac{\|u_a\|_{L^{4s/d+2}}^{4s/d+2}}{a}$$

$$\leqslant 2\frac{I(a)}{a\beta_a}(\beta_a + 1) \leqslant 4C_2\beta_a^{-2s/(2s+p)}$$

which proves the first estimate in (4.7).

Let $0 < b < a$ be such that $b^{2s/d} = a^{2s/d} - \rho^{2s/d}\beta_a$ for some $\rho > 0$ to be chosen later. Note that as $a \nearrow a^*$, b stays close to a. Now set $\lambda = \sqrt{b/a}$. By the choice of b, we have

$$1 - \lambda^{4s/d} = \left(\frac{\rho}{a}\right)^{2s/d}\beta_a.$$

This together with

$$E(\lambda u_a) = \lambda^2 E(u_a) + \frac{\lambda^2\left(1 - \lambda^{4s/d}\right)}{\frac{4s}{d} + 2}\|u_a\|_{L^{4s/d+2}}^{4s/d+2}$$

imply that

$$\frac{\left(\frac{\rho}{a}\right)^{2s/d}\beta_a}{\frac{4s}{d} + 2}\lambda^2\|u_a\|_{L^{4s/d+2}}^{4s/d+2} = E(\lambda u_a) - \lambda^2 E(u_a) \geqslant I(b) - \lambda^2 I(a).$$

In particular,

$$\frac{\rho^{2s/d}\beta_a}{\frac{4s}{d} + 2}\frac{\|u_a\|_{L^{4s/d+2}}^{4s/d+2}}{a^{2s/d+1}} \geqslant \frac{I(b)}{b} - \frac{I(a)}{a}$$

$$\geqslant C_1\beta_b^{p/(2s+p)} - C_2\beta_a^{p/(2s+p)}$$

$$\geqslant \beta_a^{p/(2s+p)}\left(C_1\left(\frac{\beta_b}{\beta_a}\right)^{p/(2s+p)} - C_2\right).$$

By the choice of b, we see that

$$\beta_b = 1 - \left(\frac{b}{a^*}\right)^{2s/d} = \beta_a\left(1 + \left(\frac{\rho}{a^*}\right)^{2s/d}\right).$$

This implies that

$$\frac{\rho^{2s/d}}{\frac{4s}{d} + 2}\frac{\|u_a\|_{L^{4s/d+2}}^{4s/d+2}}{a^{2s/d+1}} \geqslant \beta_a^{-2s/(2s+p)}\left[C_1\left(1 + \left(\frac{\rho}{a^*}\right)^{2s/d}\right)^{p/(2s+p)} - C_2\right].$$

By taking $\rho > 0$ sufficiently large, the right-hand side is bounded from below by $C\beta_a^{-2s/(2s+p)}$ for some constant C independent of a. This proves the second estimate in (4.7). $\qquad\square$

Proof of theorem 1.5. Set

$$w_a(x) := \beta_a^{d/2(2s+p)} u_a \left(\beta_a^{1/(2s+p)} x \right).$$

It follows that $\|w_a\|_{L^2}^2 = \|u_a\|_{L^2}^2 = a \nearrow a^*$ and by (4.7),

$$\|(-\Delta)^{s/2} w_a\|_{L^2}^2 = \beta_a^{2s/(2s+p)} \|(-\Delta)^{s/2} u_a\|_{L^2}^2 \leqslant C_3 a,$$

$$\|w_a\|_{L^{4s/d+2}}^{4s/d+2} = \beta_a^{2s/(2s+p)} \|u_a\|_{L^{4s/d+2}}^{4s/d+2} \geqslant C_4 a^{2s/d+1}$$

as $a \nearrow a^*$. Moreover, by (4.3),

$$0 \leqslant \frac{1}{a} \beta_a^{2s/(2s+p)} \left(\frac{1}{2} \|(-\Delta)^{s/2} u_a\|_{L^2}^2 - \frac{d}{2(d+2s)} \|u_a\|_{L^{4s/d+2}}^{4s/d+2} \right)$$

$$\leqslant \beta_a^{2s/(2s+p)} \frac{I(a)}{a} \to 0$$

as $a \nearrow a^*$. In particular,

$$\lim_{a \nearrow a^*} \|(-\Delta)^{s/2} w_a\|_{L^2}^2 - \frac{d}{d+2s} \|w_a\|_{L^{4s/d+2}}^{4s/d+2} = 0.$$

Applying lemma 2.11 to $(w_a)_{a \nearrow a^*}$, there exist a subsequence still denoted by $(w_a)_{a \nearrow a^*}$ and a sequence $(y_a)_{a \nearrow a^*} \subset \mathbb{R}^d$ such that

$$w_a(\cdot + y_a) \to \lambda_0^{d/2} Q(\lambda_0 \cdot) \text{ strongly in } H^s(\mathbb{R}^d) \tag{4.8}$$

for some $\lambda_0 > 0$ as $a \nearrow a^*$. We next write $\beta_a^{1/(2s+p)} y_a = x_a + z_a$ with $x_a \in [0,1]^d$ and $z_a \in \mathbb{Z}^d$. By the same argument as in claim 4.2, $x_a \to x_0$, where x_0 is as in (V3). Moreover, $\beta_a^{-1/(2s+p)}(x_a - x_0)$ is uniformly bounded as $a \nearrow a^*$. Passing to a subsequence if necessary, we assume that $\beta_a^{-1/(2s+p)}(x_a - x_0) \to x^0 \in \mathbb{R}^d$. By the same argument as in (4.5), we have that

$$\lim_{a \nearrow a^*} \beta_a^{-p/(2s+p)} \int V(x) |u_a(x)|^2 \, dx$$

$$= \lim_{a \nearrow a^*} \beta_a^{-p/(2s+p)} \int V \left(\beta_a^{1/(2s+p)} x + x_a \right) |w_a(x + y_a)|^2 \, dx$$

$$\geqslant \int \lim_{a \nearrow a^*} \beta_a^{-p/(2s+p)} V \left(\beta_a^{1/(2s+p)} x + x_a \right) |w_a(x + y_a)|^2 \, dx$$

$$= \kappa \int \lim_{a \nearrow a^*} \left| x + \beta_a^{-1/(2s+p)}(x_a - x_0) \right|^p \lambda_0^d [Q(\lambda_0 x)]^2 \, dx$$

$$= \kappa \int |x + x^0|^p \lambda_0^d [Q(\lambda_0 x)]^2 \, dx$$

$$= \kappa \lambda_0^{-p} \int |x + x^0 \lambda_0|^p [Q(x)]^2 \, dx$$

$$\geqslant \kappa \lambda_0^{-p} \int |x|^p [Q(x)]^2 \, dx, \tag{4.9}$$

where we use the fact Q is radially symmetric and decreasing to get the last inequality. Note that the equality holds if and only if $x^0 \equiv 0$. We infer from the above

estimate, (4.8) and the fractional Gagliardo–Nirenberg inequality that

$$
\beta_a^{-p/(2s+p)} \frac{E(u_a)}{a} = \frac{1}{2a} \beta_a^{-p/(2s+p)} \left(\|(-\Delta)^{s/2} u_a\|_{L^2}^2 - \frac{d}{d+2s} \|u_a\|_{L^{4s/d+2}}^{4s/d+2} \right)
$$

$$
+ \frac{1}{2a} \beta_a^{-p/(2s+p)} \int V(x) |u_a(x)|^2 \, \mathrm{d}x
$$

$$
= \frac{1}{2a} \beta_a^{-1} \left(\|(-\Delta)^{s/2} w_a\|_{L^2}^2 - \frac{d}{d+2s} \|w_a\|_{L^{4s/d+2}}^{4s/d+2} \right)
$$

$$
+ \frac{1}{2a} \beta_a^{-p/(2s+p)} \int V(x) |u_a(x)|^2 \, \mathrm{d}x
$$

$$
\geqslant \frac{1}{2a} \beta_a^{-1} \left(1 - \left(\frac{a}{a^*} \right)^{2s/d} \right) \|(-\Delta)^{s/2} w_a\|_{L^2}^2
$$

$$
+ \frac{1}{2a} \beta_a^{-p/(2s+p)} \int V(x) |u_a(x)|^2 \, \mathrm{d}x
$$

$$
= \frac{1}{2a^*} \left(\lambda_0^{2s} \|(-\Delta)^{s/2} Q\|_{L^2}^2 + \kappa \lambda_0^{-p} \int |x|^p [Q(x)]^2 \, \mathrm{d}x \right) + o_{a \nearrow a^*}(1)
$$

$$
= \frac{\lambda_0^{2s} d}{4s} + \frac{1}{2} \kappa \lambda_0^{-p} \int |x|^p [Q_0(x)]^2 \, \mathrm{d}x + o_{a \nearrow a^*}(1),
$$

where we have used that $Q_0 = Q/\|Q\|_{L^2}$, $\|(-\Delta)^{s/2} Q_0\|_{L^2}^2 = d/2s$. It follows that

$$
\liminf_{a \nearrow a^*} \beta_a^{-p/(2s+p)} \frac{I(a)}{a} \geqslant \frac{\lambda_0^{2s} d}{4s} + \frac{1}{2} \kappa \lambda_0^{-p} \int |x|^p [Q_0(x)]^2 \, \mathrm{d}x. \tag{4.10}
$$

We next show that the limit in (4.10) exists, and is equal to the right-hand side. To see this, we take

$$
u_\tau(x) = \sqrt{a} \tau^{d/2} Q_0(\tau(x - x_0)),
$$

where x_0 is given in (V3). We see that

$$
\frac{E(u_\tau)}{a} = \frac{1}{2} \left[\tau^{2s} \left(\|(-\Delta)^{s/2} Q_0\|_{L^2}^2 - \frac{d}{d+2s} a^{2s/d} \|Q_0\|_{L^{4s/d+2}}^{4s/d+2} \right) \right.
$$

$$
\left. + \int V(\tau^{-1} x + x_0) [Q_0(x)]^2 \, \mathrm{d}x \right]
$$

$$
= \frac{1}{2} \left[\tau^{2s} \left(1 - \left(\frac{a}{a^*} \right)^{2s/d} \right) \|(-\Delta)^{s/2} Q_0\|_{L^2}^2 + \int V(\tau^{-1} x + x_0) [Q_0(x)]^2 \, \mathrm{d}x \right]
$$

$$
= \tau^{2s} \beta_a \frac{d}{4s} + \frac{\kappa}{2} \tau^{-p} \int |x|^p [Q_0(x)]^2 \, \mathrm{d}x
$$

$$
+ \frac{1}{2} \int (V(\tau^{-1} x + x_0) - \kappa \tau^{-p} |x|^p) [Q_0(x)]^2 \, \mathrm{d}x.
$$

Taking $\tau = \lambda \beta_a^{-1/(2s+p)}$ for $\lambda > 0$, we get that

$$\beta_a^{-p/(2s+p)} \frac{E(u_\tau)}{a}$$

$$= \frac{\lambda^{2s} d}{4s} + \frac{\kappa}{2} \lambda^{-p} \int |x|^p [Q_0(x)]^2 \, dx$$

$$+ \frac{1}{2} \int \left[\beta_a^{-p/(2s+p)} V \left(\beta_a^{1/(2s+p)} \lambda^{-1} x + x_0 \right) - \kappa \lambda^{-p} |x|^p \right] [Q_0(x)]^2 \, dx.$$

Using (V3) and the fact $Q_0 = O(|x|^{-d-2s})$ as $|x| \to \infty$, it is easy to see that the integral term tends to zero as $a \nearrow a^*$ due to the dominated convergence theorem. It yields that for any $\lambda > 0$,

$$\beta_a^{-p/(2s+p)} \frac{I(a)}{a} \leqslant \frac{\lambda^{2s} d}{4s} + \frac{\kappa}{2} \lambda^{-p} \int |x|^p [Q_0(x)]^2 \, dx + o_{a \nearrow a^*}(1).$$

This implies that

$$\limsup_{a \nearrow a^*} \beta_a^{-p/(2s+p)} \frac{I(a)}{a} \leqslant \inf_{\lambda > 0} \left(\frac{\lambda^{2s} d}{4s} + \frac{\kappa}{2} \lambda^{-p} \int |x|^p [Q_0(x)]^2 \, dx \right).$$

This together with (4.10) show that

$$\lim_{a \nearrow a^*} \beta_a^{-p/(2s+p)} \frac{I(a)}{a} = \frac{\lambda_0^{2s} d}{4s} + \frac{\kappa}{2} \lambda_0^{-p} \int |x|^p [Q_0(x)]^2 \, dx$$

$$= \inf_{\lambda > 0} \left(\frac{\lambda^{2s} d}{4s} + \frac{\kappa}{2} \lambda^{-p} \int |x|^p [Q_0(x)]^2 \, dx \right)$$

and also $x^0 \equiv 0$ because of the equality in (4.9). Therefore, we obtain

$$\lambda_0 = \left(\frac{\kappa p}{d} \int |x|^p [Q_0(x)]^2 \, dx \right)^{1/(2s+p)}.$$

The proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Acknowledgements

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