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Quasiconformality and hyperbolic skew

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Abstract

We prove that if $f : \mathbb{B}^n \to \mathbb{B}^n$, for $n \geq 2$, is a homeomorphism with bounded skew over all equilateral hyperbolic triangles, then f is in fact quasiconformal. Conversely, we show that if $f : \mathbb{B}^n \to \mathbb{B}^n$ is quasiconformal then f is η-quasisymmetric in the hyperbolic metric, where η depends only on n and K. We obtain the same result for hyperbolic n-manifolds. Analogous results in $\mathbb{R}^n$, and metric spaces that behave like $\mathbb{R}^n$, are known, but as far as we are aware, these are the first such results in the hyperbolic setting, which is the natural metric to use on $\mathbb{B}^n$.

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1. *Introduction*

1·1. *Quasiconformal and quasisymmetric maps*

There are various equivalent definitions of quasiconformal mappings in the plane: the analytic definition via Sobolev spaces, the geometric definition involving extremal length of

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curve families and moduli of quadrilaterals, and the metric definition using linear dilatation. We refer to, for example, [5, 11] for a fuller discussion on the various characterisations of planar quasiconformal mappings.

A more recent way to define quasiconformal mappings locally was given by Hubbard [11] using a skew condition on triangles. Given a topological triangle T in $\mathbb{C}$ with vertices v_1, v_2, v_3, its skew is defined to be

$$\operatorname{skew}(T) = \frac{\max_{i\neq j} |v_i - v_j|}{\min_{i\neq j} |v_i - v_j|}. \tag{1·1}$$

Hubbard showed that if in a neighbourhood U of a point z_0 there is a constant σ so that the image of every triangle in U with skew at most $\sqrt{7/3}$ has skew at most σ, then the map is quasiconformal in U. The question of whether the constant $\sqrt{7/3}$ can be reduced to 1 was also asked in [11]. After partial progress in [2], this question was positively answered in [1], and so quasiconformal mappings may be characterised locally as mappings that distort the skew of equilateral triangles by a bounded amount.

The skew condition is closely related to the three point condition called quasisymmetry. A map f is called quasisymmetric if there is a bijective increasing homeomorphism $\eta : (0, \infty) \to (0, \infty)$ so that for every distinct triple of points u, v, w, we have

$$\left| \frac{f(u) - f(v)}{f(u) - f(w)} \right| \leq \eta \left(\left| \frac{u - v}{u - w} \right| \right).$$

In particular, a global quasiconformal map $f : \mathbb{C} \to \mathbb{C}$ is known to be quasisymmetric, which in turn implies the skew condition above with $\sigma = \eta(1)$. On the other hand, even a conformal map that is not global may not be quasisymmetric. In [11, p.135], it was shown that a conformal map from the unit disk to a slit disk is not quasisymmetric and fails the skew condition. Moreover, the family of conformal self-maps of the unit disk is not uniformly quasisymmetric, that is, there is no function η that works simultaneously for all functions in the family. To see this, one can verify that if

$$A_r(z) = \frac{z + r}{1 + rz}, \qquad r \in (0, 1),$$

and $u = 0$, $v = -r$, $w = r$ then

$$\left| \frac{u - v}{u - w} \right| = 1$$

but

$$\left| \frac{A_r(u) - A_r(v)}{A_r(u) - A_r(w)} \right| = \frac{1 + r^2}{1 - r^2},$$

which diverges as $r \to 1$.

In this paper we show that there is a global characterisation of quasiconformal mappings in hyperbolic space in dimension at least two in terms of a skew condition on equilateral triangles in the hyperbolic metric. This has the immediate advantage of making the family of conformal self-maps of the unit disk uniformly quasisymmetric with $\eta(t) = t$. The slit disk example mentioned above will then no longer be an issue since we will be using the hyperbolic metric on the slit disk, instead of the Euclidean metric.

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1·2. *Statement of results*

We will start by stating our results in the unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$, for $n \geq 2$, equipped with the hyperbolic metric ρ. Given a topological triangle $T \subset \mathbb{B}^n$ with vertices v_1, v_2, v_3, its hyperbolic skew is

$$\operatorname{skew}_\rho(T) = \frac{L(T)}{\ell(T)},$$

where

$$L(T) = \max_{i \neq j} \rho(v_i, v_j), \qquad \ell(T) = \min_{i \neq j} \rho(v_i, v_j).$$

An equilateral hyperbolic triangle T has $\operatorname{skew}_\rho(T) = 1$.

Definition 1·1. Let $n \geq 2$ and $\sigma \geq 1$. Then the family $\mathcal{F}_\sigma$ consists of homeomorphisms $f : \mathbb{B}^n \to \mathbb{B}^n$ so that $\operatorname{skew}_\rho(f(T)) \leq \sigma$ for every equilateral hyperbolic triangle $T \subset \mathbb{B}^n$.

THEOREM 1·2. *Let $n \geq 2$ and suppose that $f \in \mathcal{F}_\sigma$. Then f is quasiconformal.*

It turns out that while equilateral hyperbolic triangles of small side length are close to equilateral Euclidean triangles, it is not straightforward to immediately apply the results of [1] to this case. The point here is that equilateral Euclidean triangles are not equilateral hyperbolic triangles and so our hypothesis that $f \in \mathcal{F}_\sigma$ says nothing a priori about the boundedness of the skew of the images of equilateral Euclidean triangles. The methods employed in the proof of Theorem 1·2 are analogous to those in [1], but modifications to the hyperbolic setting are necessary and, in fact, we are able to substantially weaken some of the geometric requirements.

For the converse, we will prove the following.

THEOREM 1·3. *Let $n \geq 2$ and suppose that $f : \mathbb{B}^n \to \mathbb{B}^n$ is K-quasiconformal. Then f is η-quasisymmetric in the hyperbolic metric with η depending only on n and K.*

We will see that we can in fact take $\eta(t) = C \max\{t^K, t^{1/K}\}$, which means that f is power-quasisymmetric in the hyperbolic metric. This term was introduced by Trotsenko and Väisälä [15].

This result is likely known by experts in the field, but we were unable to find a reference and so we include a proof here. It is well known that this result is true for quasiconformal mappings in $\mathbb{R}^n$, $n \geq 2$, and there has been a substantial amount of research into generalising this to other metric spaces that are, in a sense, analogous to Euclidean spaces. Heinonen and Koskela [10, corollary 4·8 and theorem 4·9] proved that if X and Y are Ahlfors Q-regular metric spaces, X is a Loewner space, Y is locally linearly connected and $f : X \to Y$ is a quasiconformal map (in the metric sense) which maps bounded sets to bounded sets, then f is quasisymmetric with η depending only on the quasiconformality constant of f and the data associated to the spaces X and Y.

We refer to [10] for the various definitions in the above statement, except to point out that a metric space X is Ahlfors Q-regular means that there exists a constant $C \geq 1$ so that for all balls $B(x,r)\subset X$

$$\frac{r^Q}{C}\le \mathcal{H}_Q(B(x,r))\le Cr^Q,$$

where $\mathcal{H}_Q$ denotes the Q-Hausdorff measure in the underlying metric space. This means that in a Q-regular metric space, the size of balls of radius r is comparable to r^Q. However, in hyperbolic space this is not true: the size of balls grow exponentially with the radius and consequently the arguments of [10] do not apply in the context of interest to this paper. It would be interesting to see to what extent the results here can be generalised to quasiconformal mappings in spaces analogous to the ball equipped with the hyperbolic metric, for example domains in $\mathbb{R}^n$ equipped with the quasi-hyperbolic metric.

If M^n is a hyperbolic n-manifold, for $n\ge 2$, then by definition there is a covering map $\pi_M:\mathbb{B}^n\to M^n$ and an associated group of covering transformations G_M acting properly discontinuously on $\mathbb{B}^n$, so that M^n can be realized as $\mathbb{B}^n/G_M$. See [13, p.348] applied to the unit ball model of hyperbolic space. Then the hyperbolic distance ρ_M can be defined via the hyperbolic distance ρ on $\mathbb{B}^n$ and the formula

$$\rho_M(p,q)=\inf_{\pi_M(x)=p,\pi_M(y)=q}\rho(x,y).$$

If x is considered fixed with $\pi_M(x)=p$, then by the discreteness of G_M we also have

$$\rho_M(p,q)=\min_{\pi_M(y)=q}\rho(x,y).$$

We then obtain the following corollaries to Theorem 1·3.

COROLLARY 1·4. *Let $n\ge 2$ and let M^n, N^n be hyperbolic n-manifolds carrying hyperbolic distance functions ρ_M,ρ_N respectively. Then a homeomorphism $f:M\to N$ is K-quasiconformal if and only if it is η-quasisymmetric with respect to ρ_M and ρ_N, where η depends only on K and n.*

COROLLARY 1·5. *Let $n\ge 2$ and let M^n and N^n be hyperbolic n-manifolds. Then a homeomorphism $f:M^n\to N^n$ is quasiconformal if and only if there is a constant $\sigma\ge 1$ such that for all equilateral hyperbolic triangles T in M^n, we have $\mathrm{skew}_{\rho_N}(f(T))\le\sigma$. Similarly to before, we define*

$$\mathrm{skew}_{\rho_N}(f(T))=\frac{max_{i\ne j}\rho_N(v_i,v_j)}{min_{i\ne j}\rho_N(v_i,v_j)},$$

where v_1, v_2 and v_3 are the vertices of the topological triangle $f(T)$.

COROLLARY 1·6. *Let $n\ge 2$ and let M^n and N^n be hyperbolic n-manifolds. Then the family $\mathcal{F}_K$ of K-quasiconformal maps from M^n onto N^n is a uniformly quasisymmetric family with respect to the hyperbolic distances on M^n and N^n.*

Note that the $n=2$ case in the corollaries above applies to hyperbolic Riemann surfaces which, via the Uniformisation Theorem, are almost all Riemann surfaces.

The paper is organised as follows. In Section 2, we recall some facts about quasiconformal mappings and hyperbolic geometry. In Section 3, we prove Theorem 1·2. In Section 4, we prove Theorem 1·3 and its corollaries.

2. Preliminaries

2·1. *Hyperbolic geometry*

Let $n \geq 2$ and let $\mathbb{B}^n$ be the unit ball in $\mathbb{R}^n$. We equip $\mathbb{B}^n$ with the hyperbolic density

$$\lambda(x)\,|dx| = \frac{2|dx|}{1-|x|^2}. \tag{2·1}$$

The hyperbolic metric on $\mathbb{B}^n$ is defined by

$$\rho(u, v) = \inf \int_\gamma \lambda(x)\,|dx|,$$

where the infimum is taken over all paths in $\mathbb{B}^n$ joining u and v. The infimum is achieved for circular arcs which, if extended to $\partial\mathbb{B}^n$, cut through $\partial\mathbb{B}^n$ perpendicularly. We will denote by $B_\rho(x_0, r)$ the open hyperbolic ball of radius $r > 0$ centred at $x_0 \in \mathbb{B}^n$. Balls in other metric spaces will use similar notation.

2·2. *In dimension two*

We refer to [3] for a reference to the theory of hyperbolic geometry in dimension two. The formula for the hyperbolic metric on the unit disk $\mathbb{D}$ is given by

$$\rho(z, w) = \log \frac{1 + \left|\frac{z-w}{1-\overline{w}z}\right|}{1 - \left|\frac{z-w}{1-\overline{w}z}\right|}, \qquad z, w \in \mathbb{D}.$$

Isometries of the hyperbolic metric are given precisely by Möbius transformations which preserve the unit disk.

The hyperbolic metric can be defined on any simply connected proper sub-domain U of $\mathbb{C}$ via a Riemann map $\varphi : U \to \mathbb{D}$. We then define the hyperbolic density on U by

$$\lambda_U(x) = \lambda_{\mathbb{D}}(\varphi(z))|\varphi'(z)|,$$

where $\lambda_{\mathbb{D}}$ is defined in formula (2·1), and the hyperbolic metric on U by integrating λ_U. The hyperbolic metric can be defined on any plane domain and, more generally, any Riemann surface that is not covered by the sphere or plane via the Uniformisation Theorem.

An equilateral hyperbolic triangle T has three vertices v_1, v_2 and v_3 and three edges made by geodesic segments of equal length joining the vertices. The side length r of T determines the interior angles. Applying a Möbius map to send one of the vertices to 0 and another to $x > 0$, the remaining vertex must be sent to $xe^{i\alpha}$ for some α. Since

$$r = \rho(0, x) = \rho(x, xe^{i\alpha}),$$

we can compute that

$$\alpha = \cos^{-1}\left(\frac{1+x^2}{2}\right). \tag{2·2}$$

We can express α in terms of r by using the relationships

$$r = \log\frac{1+x}{1-x}, \qquad x = \frac{e^r - 1}{e^r + 1}$$

to see that

$$\alpha = \cos^{-1}\left(\frac{1+\tanh^2(r/2)}{2}\right). \qquad (2\cdot3)$$

As $r \to 0$, we observe that $\alpha \to \pi/3$ and so small equilateral hyperbolic triangles are close to equilateral Euclidean triangles. Evaluating when $r = 1$, we have the following lemma.

LEMMA 2·1. *If an equilateral hyperbolic triangle has side length* $0 < r \leq 1$*, then the internal angles satisfy* $2\pi/7 < \theta < \pi/3$.

The centroid of an equilateral hyperbolic triangle T can be found by applying a Möbius map A to send the vertices to t, $t\omega$ and $t\omega^2$, where $t > 0$ and $\omega = e^{2\pi i/3}$. Then 0 is the centroid of the resulting triangle, and is the common intersection point of the geodesic segments joining a vertex to the midpoint of the opposite side. Applying A^{-1}, we see that $A^{-1}(0)$ is the centroid of T.

We call a collection of equilateral hyperbolic triangles $T_1, \ldots, T_m$ of the same side length r in $\mathbb{D}$ a *chain* if T_j and T_{j+1} have a common side for $j = 1, \ldots, m-1$. We allow the triangles in the chain to overlap.

2·3. *Quasihyperbolic metric*

A metric that is related to the hyperbolic metric, but can be defined on any proper subdomain U of $\mathbb{R}^n$, is the quasihyperbolic metric given by density

$$\delta_U(x)|dx| = \frac{|dx|}{d(x, \partial U)},$$

where $d(x, \partial U)$ denotes the Euclidean distance from x to the boundary of U. The quasihyperbolic metric is denoted q_U and obtained by integrating the density δ_U.

The hyperbolic and quasi-hyperbolic metrics are bi-Lipschitz equivalent on simply connected proper subdomains U of $\mathbb{C}$. In fact, it follows from the Koebe 1/4-Theorem that

$$\frac{\delta_U(z)}{2} \leq \lambda_U(z) \leq 2\delta_U(z),$$

for all $z \in U$. This is not true in general, considering for example the punctured disk. In dimension three and greater, we can only define the hyperbolic metric on balls and half-spaces. This is a consequence of the generalised Liouville's Theorem, see for example [**14**, theorem I.2.5], which says that the only 1-quasiregular mappings in $\mathbb{R}^n$ with $n \geq 3$ are (restrictions of) Möbius transformations. Consequently, the quasihyperbolic metric plays the role of the hyperbolic metric in function theory in higher dimensions.

2·4. *Quasiconformal mappings*

As remarked at the outset of this paper, there are various equivalent definitions of quasiconformal mappings in $\mathbb{R}^n$, $n \geq 2$. We will give the analytic definition and the metric definition.

Definition 2·2 (Analytic definition). A *quasiconformal* mapping in a domain $U \subset \mathbb{R}^n$ for $n \geq 2$ is a homeomorphism in the Sobolev space $W^1_{n,loc}(U)$ where there is a uniform bound

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on the distortion, that is, there exists $K \geq 1$ such that

$$|f'(x)|^n \leq K J_f(x)$$

almost everywhere in U. The minimum such K for which this inequality holds is called the *outer dilatation* and denoted by $K_O(f)$. As a consequence of this, there is also $K' \geq 1$ such that

$$J_f(x) \leq K' \inf_{|h|=1} |f'(x)h|^n$$

holds almost everywhere in U. The minimum such K' for which this inequality holds is called the *inner dilatation* and denoted by $K_I(f)$. If $K(f) = \max\{K_O(f), K_I(f)\}$, then $K(f)$ is the *maximal dilatation* of f. A K-quasiconformal mapping is a quasiconformal mapping for which $K(f) \leq K$.

Definition 2·3 (Metric definition). Let $n \geq 2$ and let $U \subset \mathbb{R}^n$ be a domain. For each $x \in U$ let $H(x) = \limsup_{r\to 0} H(x, r)$, where

$$H(x, r) = \frac{\max_{|x-y|=r} |f(x) - f(y)|}{\min_{|x-y|=r} |f(x) - f(y)|}.$$

Then $f : U \to \mathbb{R}^n$ is quasiconformal if and only if there exists a constant H such that $H(x) \leq H$ for all $x \in U$.

The Analytic and Metric definitions are equivalent, see [7, corollary 4]. In this paper, we will be interested in a hyperbolic version of linear distortion. We therefore define for $x \in \mathbb{B}^n$ and $r > 0$

$$H_\rho(x, r) = \frac{L_\rho(x, r)}{\ell_\rho(x, r)},$$

where

$$L_\rho(x, r) = \max_{\rho(x,y)=r} \rho(f(x), f(y)), \quad \ell_\rho(x, r) = \min_{\rho(x,y)=r} \rho(f(x), f(y)).$$

The following result on the distortion of the hyperbolic metric was proved by Gehring and Osgood [8] for the quasihyperbolic metric with a constant depending on n and K, then improved to a dimension independent version by Vuorinen [18] (see also [17, corollary 12·20]). For our purposes with the hyperbolic metric, we just note that the hyperbolic and quasihyperbolic metrics are bi-Lipschitz equivalent on the unit ball.

THEOREM 2·4. *Let $n \geq 2$ and let $\mathbb{B}^n$ be the unit ball in $\mathbb{R}^n$ equipped with the hyperbolic metric ρ. Then if $f : \mathbb{B}^n \to \mathbb{B}^n$ is a K-quasiconformal mapping,*

$$\rho(f(x), f(y)) \leq C_1 \max\{\rho(x, y)^{1/K}, \rho(x, y)\}$$

and

$$\rho(f(x), f(y)) \geq C_2 \min\{\rho(x, y)^K, \rho(x, y)\},$$

where C_1, C_2 are constants that depend only on K.

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We will also need the following result which characterises quasiconformal mappings as local quasisymmetric mappings in a quantitative way. This result is due to Väisälä [16, theorem 2.4] and is slightly reformulated for our purposes (see also [9, theorem 11.14]).

THEOREM 2·5. *Let $n \geq 2$, suppose that $U \subset \mathbb{R}^n$ is open and suppose that $f : U \to \mathbb{R}^n$ is K-quasiconformal. Suppose also that $x_0 \in U$, $0 < \lambda < 1$ and $r > 0$ so that $B(x_0, r) \subset U$. Then f restricted to $B(x_0, \lambda r)$ is ξ-quasisymmetric, where ξ depends only on n, K and λ.*

This result is informally called the egg yolk principle: the smaller ball is the yolk, the larger ball is the egg and, however wildly f behaves near the edge of the egg, it is relatively well-behaved on the yolk.

3. *Hyperbolic equilateral triangles*

In this section, we will prove Theorem 1·2. In dimension three and higher, the proof is easier and so we will deal with this case first and then move to the dimension two case. Throughout, if $n \geq 2$, denote by $\mathbb{B}^n$ the unit ball in $\mathbb{R}^n$, by ρ the hyperbolic metric on $\mathbb{B}^n$ and by $\mathcal{F}_\sigma$, the family of homeomorphisms $f : \mathbb{B}^n \to \mathbb{B}^n$ satisfying the skew condition with constant $\sigma \geq 1$, recalling Definition 1·1.

Proof of Theorem 1·2 with $n \geq 3$. Fix $n \geq 3$. Choose r_0 small enough so that an equilateral hyperbolic triangle in $\mathbb{B}^n$ with side length $r \leq r_0$ has interior angles at least $\pi/3 - \delta$ for some small fixed δ. For such an r, denote by S_r the boundary of $B_\rho(0, r)$ in $\mathbb{B}^n$.

Suppose $f \in \mathcal{F}_\sigma$ and for now assume that f fixes 0. Since S_r is compact, $L_\rho(0, r)$ and $\ell_\rho(0, r)$ are achieved on S_r at, say, x_1 and x_0 respectively.

Consider all equilateral hyperbolic triangles which have two vertices at 0 and x_1. The locus of all possible locations for the third vertex is an $(n-2)$-sphere Σ_1 contained in S_r. Similarly, Σ_0 is the locus of all possible locations for the third vertex of an equilateral hyperbolic triangle with vertices at 0 and x_0.

If Σ_0 and Σ_1 intersect, then we can choose x_2 to be an intersection point. Otherwise we choose $x_2 \in \Sigma_1$ to be a closest point to x_0. We then define Σ_2 analogously for x_2 and check whether Σ_2 intersects Σ_0. Continuing in this fashion, we build a chain of at most four triangles where the initial triangle has vertices $0, x_1, x_2$ and the final triangle has vertices including 0 and x_0. The reason we can do this with at most four triangles is that the interior angles of each triangle are at least $\pi/3 - \delta$.

Finally, since $f \in \mathcal{F}_\sigma$, we obtain

$$L_\rho(0, r) \leq \sigma^4 \ell_\rho(0, r),$$

for all $r \leq r_0$. Hence f is quasiconformal at 0. For any other point $x \in \mathbb{B}^n$, we can apply Möbius maps A_1, A_2 which send x and $f(x)$ to 0 respectively and then apply the above argument to $A_2 \circ f \circ A_1^{-1}$.

We next turn to the dimension two case. We first need some preliminary results on hyperbolic geometry.

LEMMA 3·1. *There exists $\delta > 0$ so that any equilateral hyperbolic triangle T in $\mathbb{D}$ of side length $r \leq 1$ has the property that $B_\rho(c, 2\delta r) \subset T$, where c is the centroid of T.*

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Proof. Given any equilateral hyperbolic triangle T of side length r, we may apply a Möbius transformation A so that the vertices of $A(T)$ lie at the points $t, \omega t, \omega^2 t$, where $t > 0$ and $\omega = e^{2\pi i/3}$. By a direct computation, the quantities r and t are related via

$$r = \rho(t, t\omega) = \log \frac{1 + \frac{t\sqrt{3}}{\sqrt{1+t^2+t^4}}}{1 - \frac{t\sqrt{3}}{\sqrt{1+t^2+t^4}}}.$$

We see that $r = 2\sqrt{3}t + o(t)$ as $t \to 0$.

By the formula for the midpoint of a hyperbolic geodesic segment, see [4, proposition 3·2], the hyperbolic midpoint of $t\omega$ and $t\omega^2$ occurs on the negative real axis at

$$\frac{\sqrt{1+t^2+t^4} - 1 - t^2}{t}.$$

This implies that any Euclidean ball of radius less than $R(t) = (1 + t^2 - \sqrt{1+t^2+t^4})/t$ centred at 0 is contained in T. Therefore any hyperbolic ball of radius less then $\widetilde{R}(t) := \log((1 + R(t))/(1 - R(t)))$ centred at 0 is contained in T.

Now, R is an increasing function of t with $R(t) = t/2 + o(t)$ as $t \to 0$ (as one would expect since small equilateral hyperbolic triangles are close to small Euclidean triangles) and $\lim_{t\to 1} R(t) = 2 - \sqrt{3}$. Hence $\widetilde{R}$ is also increasing with $\widetilde{R}(t) = t + o(t)$ as $t \to 0$. Consequently, if the side length r of T is at most 1, then we can find $\delta > 0$ so that $B_\rho(0, 2\delta r)$ is contained in T.

Given an equilateral triangle of side length $r \le 1$, we will denote by $B_\delta(T)$ the ball $B_\rho(c, \delta r)$, where δ is from Lemma 3·1. Then if $p \in B_\delta(T)$, we have $B_\rho(p, \delta r) \subset B_\rho(c, 2\delta r) \subset T$.

If $E \subset \mathbb{D}$ is closed and $z \in \mathbb{D} \setminus E$, the hyperbolic distance between z and E is

$$\rho(z, E) = \min\{\rho(z, w) : w \in E\}.$$

LEMMA 3·2. *Let T be an equilateral hyperbolic triangle in $\mathbb{D}$ with side length $r \le 1$ and let $p \in \mathbb{D}$. Then there exists a chain of equilateral hyperbolic triangles $T_1, \ldots, T_m$ with side length r, $T_1 = T$, $p \in T_m$ and moreover $m \le M$, where $M = \max\{7, 700\rho(p, T)/r\}$.*

Proof. Without loss of generality, we may apply a Möbius map so that T has vertices $t, t\omega, t\omega^2$, where $t > 0$ and $\omega = e^{2\pi i/3}$ and T has centroid 0. Further, we may assume that $-\pi/3 \le \arg p \le \pi/3$, otherwise apply a rotation permuting the vertices of T.

Since $r \le 1$, Lemma 2·1 implies that the internal angles α of T are at least $2\pi/7$. Consequently, if we form a chain of triangles by rotating T in the clockwise direction through angle α about t, then by the time we add in the seventh triangle, we will intersect T.

Let U be an open $r/100$ neighbourhood of T in the hyperbolic metric and let

$$U' = \{z : z \in U \text{ and } \arg z \in [-\pi/3, \pi/3]\}.$$

Then the collection $\mathcal{C}$ of seven triangles obtained by forming the chain around the point $t \in T$ covers U'. If p lies in T or this chain, then we are done. Otherwise, consider a geodesic segment realizing the distance $\rho(p, T)$. This segment must cross U' and consequently there is a triangle $T_1 \in \mathcal{C}$ satisfying

$$\rho(p, T_1) \le \rho(p, T) - \frac{r}{100}.$$

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Repeating this process, we are able to construct a chain of triangles as required. Each step requires at most seven triangles and so the maximum number required is $700\rho(p, T)/r$.

LEMMA 3·3. *Let $0 < t \leq 1$ and let T be the hyperbolic triangle with vertices v_1, v_2, v_3 so that $v_1 = 0$, $\arg(v_2) = e^{i\pi/3}$, $\arg(v_3) = e^{-i\pi/3}$ and $\rho(v_1, v_2) = \rho(v_1, v_3) = t$. Then given $\epsilon > 0$, there exists $\xi > 0$ so that if $\rho(v_i, w_i) < t\xi$ for $i = 1, 2, 3$ and if ϕ denotes the angle $\angle w_2 w_1 w_3$ of the hyperbolic triangle T' with vertices w_1, w_2, w_3, then $|2\pi/3 - \phi| < \epsilon$.*

Proof. By the hyperbolic Law of Cosines,

$$\cos\phi = \frac{\cosh(\rho(w_1, w_2))\cosh(\rho(w_1, w_3)) - \cosh(\rho(w_2, w_3))}{\sinh(\rho(w_1, w_2))\sinh(\rho(w_1, w_3))}. \tag{3·1}$$

Clearly by construction the angle $\angle v_2 v_1 v_3$ is $2\pi/3$, and so replacing the w_i by the v_i in this formula, we obtain $\cos(2\pi/3) = -1/2$. By the hypotheses and the triangle inequality,

$$(1 - 2\xi)t < \rho(w_1, w_i) < (1 + 2\xi)t$$

for $i = 2, 3$. Writing $h(t) = \rho(v_2, v_3)$, by the triangle inequality,

$$h(t) - 2\xi t < \rho(w_2, w_3) < h(t) + 2\xi t.$$

We therefore see

$$\frac{\cosh^2((1-2\xi)t) - \cosh(h(t) + 2\xi t)}{\sinh^2((1+2\xi)t)} < \cos\phi < \frac{\cosh^2((1+2\xi)t) - \cosh(h(t) - 2\xi t)}{\sinh^2((1-2\xi)t)}.$$

By the continuity of the functions involved here and since the limit as $\xi \to 0$ of both left- and right-hand sides is $-1/2 = \cos(2\pi/3)$, the claim follows.

Our final preliminary result we need is to construct a certain self-map of the unit disk which fixes the origin, acts as a rotation on each circle centered at the origin and generates equilateral hyperbolic triangles. We recall that a homeomorphism $f : \mathbb{D} \to \mathbb{D}$ is called *locally quasiconformal* if and only if for each compact siubset $E \subset \mathbb{D}$, $f|_E$ is quasiconformal. See for example [12]. In particular, $|\mu_f(z)|$ is allowed to converge to 1 as $|z| \to 1$.

LEMMA 3·4. *The map $R_0 : \mathbb{D} \to \mathbb{D}$ defined in polar coordinates by*

$$R_0(te^{i\theta}) = t\exp\left[i\left(\theta + \cos^{-1}\left(\frac{1+t^2}{2}\right)\right)\right]$$

is locally quasiconformal, fixes 0 and, moreover, for any $w \in \mathbb{D} \setminus \{0\}$, the hyperbolic triangle with vertices 0, w and $R_0(w)$ is equilateral.

Proof. It is clear that R_0 acts as a rotation on each circle centered at the origin. Moreover, the claim on equilateral hyperbolic triangles follows from (2·2). The formula for the complex dilatation of R_0 in terms of polar coordinates is

$$\mu_{R_0} = e^{2i\theta}\left(\frac{(R_0)_t + \frac{i}{t}(R_0)_\theta}{(R_0)_t - \frac{i}{t}(R_0)_\theta}\right).$$

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Via elementary computations we have

$$(R_0)_\theta = it \exp\left[i\left(\theta + \cos^{-1}\left(\frac{1+t^2}{2}\right)\right)\right]$$

and

$$(R_0)_t = \left(1 - \frac{2it^2}{\sqrt{(3+t^2)(1-t^2)}}\right)\exp\left[i\left(\theta + \cos^{-1}\left(\frac{1+t^2}{2}\right)\right)\right].$$

Hence we obtain

$$|\mu_{R_0}(te^{i\theta})| = \frac{t^2}{|\sqrt{(3+t^2)(1-t^2)} - it^2|} = \frac{t^2}{\sqrt{3-2t^2}}.$$

Hence R_0 is quasiconformal on each compact subset of $\mathbb{D}$, but $|\mu_{R_0}(z)| \to 1$ as $|z| \to 1$.

Definition 3·5. For $w \in \mathbb{D}$, let $A_w(z) = (z-w)/(1-\overline{w}z)$. Then define $R_w : \mathbb{D} \to \mathbb{D}$ by $R_w = A_w^{-1} \circ R_0 \circ A_w$.

The key property is that for any $z \neq w$, the hyperbolic triangle with vertices w, z, $R_w(z)$ is equilateral.

With these results in hand, we can prove the remaining case of Theorem 1·2.

Proof of Theorem 1·2 *when* $n = 2$. Let $\sigma \geq 1$ and suppose that $f \in \mathcal{F}_\sigma$. Let T be an equilateral hyperbolic triangle of side length $r \leq 1$. Then by Lemma 3·1, we know that $B_\delta(T) \subset T$ for a constant $\delta > 0$ independent of r. The definition of $B_\delta(T)$ is given directly after the proof of Lemma 3·1.

We will prove the theorem in a number of steps. Given p close to the centroid of T, we find a chain of small equilateral triangles connecting a side of T to p. Then we find a particular small equilateral triangle close to p, and show that this construction implies that the image $f(T)$ contains a disk of a definite size, relative to the side length of $f(T)$, centred at $f(p)$. Finally, we show how this implies that f satisfies the metric definition of quasiconformality.

Step 1: *constructing a chain of small triangles.* Let $p \in B_\delta(T)$ and let $n \in \mathbb{N}$. We will specify how large n must be later. Select the side of T which realises $L(f(T))$, the maximum distance between two vertices of the topological triangle $f(T)$, and subdivide this side into r/n segments of equal length. Let v, w be the endpoints of the segment whose image has the largest length and T_1 be the equilateral triangle in T which has one side with vertices v, w. Therefore

$$L(f(T)) \leq n\rho(f(v), f(w)).$$

Apply Lemma 3·2 to find a chain of triangles $T_1, \ldots, T_m$ of side length r/n with $p \in T_m$. Since T has side length length less than or equal to 1 which implies $\rho(p, T_1) \leq 1$, we can achieve this with $m \leq M = 700n$ triangles. Since $f \in \mathcal{F}_\sigma$, we find by induction that if v', w' is any other side in the chain,

$$\rho(f(v), f(w)) \leq \sigma^M \rho(f(v'), f(w'))$$

and hence

$$L(f(T)) \leq n\sigma^M \rho(f(v'), f(w')).$$

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Choose v', w' to be any two vertices of T_m that are different from p (typically p will not be a vertex of T_m). Then for one of v', w', denoted by q, we are guaranteed by the triangle inequality to have

$$L(f(T)) \leq 2n\sigma^M \rho(f(p), f(q)),$$

and

$$\rho(p, q) \leq r/n.$$

Step 2: *constructing a small equilateral triangle.* Denote by μ the distance from $f(p)$ to $\partial f(T)$. We can realise μ as the length of a hyperbolic geodesic segment joining $f(p)$ to $\partial f(T)$. Let γ be the pre-image of this geodesic segment and further denote by γ_1 the component of $\gamma \cap B_\rho(p, \delta r)$ that contains p. Next, denote by γ_2 the curve $R_p(\gamma_1) \cup R_p^{-1}(\gamma_1)$. We observe that since R_p is locally quasiconformal, so is R_p^{-1} and hence γ_2 really is a curve. For $t \in \gamma_2$, we can find $s \in \gamma_1$ that arises as its pre-image under either R_p or R_p^{-1}. Then

$$\rho(f(t), f(p)) \leq \sigma\rho(f(s), f(p)) \leq \sigma\mu, \tag{3·2}$$

since the triangle with vertices s, t and p is equilateral by Lemma 3·4.

Next, we need to ensure that γ_2 is well-behaved near its endpoints. To that end, we will slightly enlarge the curve, while maintaining an inequality similar to (3·2). Given $\epsilon < 1$, take the corresponding ξ from Lemma 3·3. Denote by a, b the endpoints of γ_2 on $\partial B_\rho(p, \delta r)$ and let B_a, B_b be the disks $B_\rho(a, \delta r\xi)$ and $B_\rho(b, \delta r\xi)$ respectively. Write γ_{2a} and γ_{2b} for the components of $\gamma_2 \cap B_a$ and $\gamma_2 \cap B_b$ that have endpoints at a and b respectively.

We focus on extending γ_{2a} in B_a and will perform an analogous construction for γ_{2b} in B_b. Denote by a' the endpoint of γ_{2a} on ∂B_a. Use the hyperbolic geodesic through a tangent to $B_\rho(p, \delta r)$ at a to divide B_a into two parts and then use hyperbolic rotations through angle $\pi/3$ about a to divide B_a into six sectors, each of which has angle $\pi/3$ seen from a. Let S_a denote the middle sector that has no intersection with $B_\rho(p, \delta r)$. By Lemma 2·1 and Lemma 3·4, R_a acts on ∂B_a by rotating through an angle between $2\pi/7$ and $\pi/3$. Hence there exists $n \in \mathbb{Z}$ with $|n| \leq 3$ so that $R_a^n(a')$ lies in the sector S_a. Let the image of γ_{2a} under R_a^n be denoted by γ_{3a}

Let $t \in \gamma_{3a}$ and $t_0 \in \gamma_{2a}$ such that $R_a^n(t_0) = t$. If t_k denotes the image of t_0 under R_a^k, then a, t_{k-1} and t_k form an equilateral triangle and so

$$\rho(f(t_{k-1}), f(t_k)) \leq \sigma\rho(f(t_{k-1}), f(a)),$$

since $f \in \mathcal{F}_\sigma$. By the triangle inequality and (3·2) we have

$$\rho(f(a), f(t_0)) \leq \rho(f(a), f(p)) + \rho(f(p), f(t_0)) \leq 2\sigma\mu.$$

Then we conclude that since $|n| \leq 3$,

$$\begin{aligned} \rho(f(t), f(p)) &\leq \rho(f(a), f(p)) + \rho(f(a), f(t)) \\ &\leq \rho(f(a), f(p)) + \sigma^3\rho(f(a), f(t_0)) \\ &\leq \sigma\mu + 2\sigma^4\mu \\ &= \sigma\mu(1 + 2\sigma^3). \end{aligned} \tag{3·3}$$

If $\delta' \leq \xi\delta$ is chosen appropriately so that the endpoints of the geodesic segments forming S_a intersect ∂B_a on $\partial B_\rho(p, (\delta + \delta')r)$, then γ_{3a} must also intersect $\partial B_\rho(p, (\delta + \delta')r)$.

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We make the same construction using the point b instead of the point a and then define γ_3 to be the connected component of $(\gamma_2 \cup \gamma_{3a} \cup \gamma_{3b}) \cap B_\rho(p, (\delta+\delta')r)$ that includes p. By the construction above, we have

$$\rho(f(t), f(p)) \leq \sigma\mu(1+2\sigma^3)$$

for all $t \in \gamma_3$.

We will use this curve γ_3 to find the required equilateral triangle. Recall the point q from step 1, and let B_q be the smallest disk centred at q which contains $B_\rho(p, \delta r)$. We choose n large enough in step 1 so that $1/n < \delta\xi$ and $B_q \subset B_\rho(p, (\delta+\delta')r)$. Let γ_4 be the connected component of $\gamma_3 \cap B_q$ with endpoints $A \in B_a \cap \partial B_q$ and $B \in B_b \cap \partial B_q$.

By our construction, the angle made by the geodesics joining A to q and B to q make an angle in $(\pi/3, \pi)$. To see this, take a Möbius map M which moves p to 0. Then since $\rho(p,q) < \xi\delta r$, $A \in B_a$, $B \in B_b$ and $\epsilon < 1 < \pi/3$ (recall ϵ was selected in the second paragraph of step 2) applying Lemma 3·3 to $M(q)$, $M(A)$ and $M(B)$ gives the claim.

Since we may assume that $(\delta+\delta')r < 1$, by Lemma 2·1 R_q acts on ∂B_q by a rotation with angle strictly between $2\pi/7$ and $\pi/3$. It follows that the images $R_q(A)$ and $R_q(B)$ will separate A and B on ∂B_q. Consequently $R_q(\gamma_4)$ must intersect γ_4.

We then take t_1 to be an intersection point, t_2 to be its pre-image under R_q and obtain, via Lemma 3·4, an equilateral triangle with vertices q, t_1 and t_2. It is possible that these vertices coincide if γ_4 passes through q and then we obtain the trivial triangle. Note by (3·3) we have

$$\rho(f(t_i), f(p)) \leq \sigma\mu(1+2\sigma^3)$$

for $i \in \{1, 2\}$.

Step 3: *a disk of a definite size in the image.* In steps 1 and 2, given $p \in B_\delta(T)$, we found an equilateral triangle with vertices q, t_1, t_2 and constants D_1, D_2 so that

$$\rho(f(t_j), f(p)) \leq D_1\mu, \quad \rho(f(p), f(q)) \geq D_2 L(f(T)), \tag{3·4}$$

recalling that μ is the distance from $f(p)$ to $\partial f(T)$. If $t_1 = t_2 = q$, then by (3·4)

$$D_2 L(f(T)) \leq \rho(f(p), f(q)) \leq D_1\mu,$$

which implies there is a disk in $f(T)$ centred at $f(p)$ of radius at least $D_2 L(f(T))/D_1$. Otherwise, we have by the triangle inequality and the assumption that $f \in \mathcal{F}_\sigma$ that

$$\begin{aligned}\rho(f(p), f(q)) - \rho(f(t_1), f(p)) &\leq \rho(f(t_1), f(q)) \\ &\leq \sigma\rho(f(t_1), f(t_2)) \\ &\leq \sigma(\rho(f(t_1), f(p)) + \rho(f(p), f(t_2))).\end{aligned}$$

Now using (3·4), we obtain

$$D_2 L(f(T)) - D_1\mu \leq 2\sigma D_1\mu,$$

and so

$$\mu \geq \frac{D_2 L(f(T))}{(2\sigma+1)D_1}.$$

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Again we conclude that there is a disk of size β centred at $f(p)$ in $f(T)$, where β depends only on σ and $L(f(T))$ (note that the side length of T is $r \leq 1$ and β does not depend on r once we have fixed this upper bound).

Step 4: *showing f is quasiconformal.* We first assume that f fixes 0. Let $r \leq 1$. By precomposing f with a rotation, we may assume that $L_\rho(0, r)$ is taken at z_0 on the positive real axis, where $\rho(0, z_0) = r$. Let T_1 be the hyperbolic equilateral triangle with vertices 0, z_0 and $z_0 e^{i\alpha}$ and centroid c_0. By Step 3, $f(T_1)$ contains a disk centred at c_0 with radius at least $\beta L(f(T_1))$.

There exists a hyperbolic isometry which maps z_0 and c_0 to c_0 and 0 respectively and T_1 onto an equilateral hyperbolic triangle T_2. Moreover, $0 \in B_\delta(T_2)$ because 0 is the centroid of T_1 . Since one vertex of T_2 is contained in T_1 and the other two are outside, it follows that $L(f(T_2)) \geq \beta L(f(T_1))$. Since $0 \in B_\delta(T_2)$, we can apply step 3 again to see that $f(T_2)$ contains the disk $B_\rho(f(0), \beta L(f(T_2)))$. In conclusion,

$$\ell_\rho(0, r) \geq \beta L(f(T_2)) \geq \beta^2 L(f(T_1)) \geq \beta^2 L_\rho(0, r).$$

Since this is true for all $r \leq 1$, we see that f is quasiconformal at 0 with linear distortion bounded above by $1/\beta^2$.

If f does not fix 0, then consider any $z \in \mathbb{D}$ with image $f(z)$. Find Möbius maps A_1, A_2 which map z and $f(z)$ to 0 respectively and apply the above argument to $A_2 \circ f \circ A_1^{-1}$ to see that f is quasiconformal at z. Since z was arbitrary and the bound on the linear distortion is independent of z, the proof is complete.

4. *Quasiconformal implies quasisymmetric*

In this section, we will prove Theorem 1·3. The main idea in proving quasiconformal implies quasisymmetric in the hyperbolic ball is to split the proof into two cases. On large scales, quasiconformal maps are bi-Lipschitz by Theorem 2·4, whereas on small scales quasiconformal maps are quasisymmetric by Theorem 2·5. We just need to be a little careful in combining these two results.

Throughout this section, we fix $n \geq 2$ and equip the unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$ with the hyperbolic metric ρ.

LEMMA 4·1. *Suppose that $f : \mathbb{B}^n \to \mathbb{B}^n$ is K-quasiconformal, f fixes* 0 *and $t > 0$. Then there exists a constant η depending only on t, n and K so that*

$$\frac{L_\rho(0, tr)}{\ell_\rho(0, r)} \leq \eta$$

for all $r > 0$.

Proof. Throughout the proof, we will denote $L_\rho(0, tr)$ and $\ell_\rho(0, r)$ by $L_\rho(tr)$ and $\ell_\rho(r)$ respectively. We will denote by x a point with $\rho(0, x) = tr$ and $\rho(0, f(x)) = L_\rho(tr)$ and by y a point with $\rho(0, y) = r$ and $\rho(0, f(y)) = \ell_\rho(r)$.

Observe that if $f : \mathbb{B}^n \to \mathbb{B}^n$ is K-quasiconformal and fixes 0, then the image of the ball centred at 0 of hyperbolic radius 1 is contained in the ball centred at 0 of hyperbolic radius C_1 by Theorem 2·4. We may assume that $C_1 \geq 1$. Then if $x, y \in B_\rho(0, 1)$ it follows that $f(x), f(y) \in B_\rho(0, C_1)$. Since the Euclidean and hyperbolic metrics are bi-Lipschitz equivalent on compact subsets of $\mathbb{B}^n$, there exists a constant C_3 depending only on n and K

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so that the Euclidean and hyperbolic metrics are C_3-bi-Lipschitz equivalent on $B_\rho(0, C_1)$. Moreover, we can apply Theorem 2·5 to $B(0, \widetilde{C}_1) := B_\rho(0, C_1)$ contained in $\mathbb{B}^n$, that is with $\lambda = \widetilde{C}_1$. Thus we may conclude f is ξ-quasisymmetric on $B(0, \widetilde{C}_1)$, where ξ depends only on n and K, since $\widetilde{C}_1$ depends only on C_1 which depends only on n and K.

Putting all this together, if $x, y \in B_\rho(0, 1)$, we have

$$\frac{L_\rho(tr)}{\ell_\rho(r)} = \frac{\rho(0, f(x))}{\rho(0, f(y))} \le C_3^2 \frac{|f(x)|}{|f(y)|} \le C_3^2 \xi\left(\frac{|x|}{|y|}\right) \le C_3^2 \xi(C_3^2 t).$$

We now deal with the cases where at least one of x, y are not in $B_\rho(0, 1)$. First, suppose $t \ge 1$, so $|x| \ge |y|$, and $\rho(0, x) = tr \ge 1$. Then $r \ge 1/t$ and so

$$r^K = r^{K-1} r \ge \frac{r}{t^{K-1}}.$$

Consequently,

$$\min\{r^K, r\} \ge \min\left\{\frac{r}{t^{K-1}}, r\right\} = \frac{r}{t^{K-1}}.$$

By Theorem 2·4 and since $tr \ge 1$, it follows that

$$\frac{L_\rho(tr)}{\ell_\rho(r)} = \frac{\rho(0, f(x))}{\rho(0, f(y))} \le \frac{C_1 tr}{C_2 r t^{1-K}} = \frac{C_1 t^K}{C_2}.$$

Second, suppose $t \le 1$, so $|y| \ge |x|$, and $\rho(0, y) = r \ge 1$ since we have assumed at least one of x, y are not in $B_\rho(0, 1)$. Then $tr \ge t$ and so

$$(tr)^{1/K} = (tr)(tr)^{1/K-1} \le (tr)t^{1/K-1} = t^{1/K} r.$$

Consequently,

$$\max\{tr, (tr)^{1/K}\} \le t^{1/K} r.$$

By Theorem 2·4 and since $r \ge 1$, it follows that

$$\frac{L_\rho(tr)}{\ell_\rho(r)} = \frac{\rho(0, f(x))}{\rho(0, f(y))} \le \frac{C_1 t^{1/K} r}{C_2 r} = \frac{C_1 t^{1/K}}{C_2}.$$

Combining the above estimates, we see that for any $r > 0$,

$$\frac{L_\rho(rt)}{\ell_\rho(r)} \le \eta := \max\left\{C_3^2 \xi(C_3^2 t), \frac{C_1 t^K}{C_2}, \frac{C_1 t^{1/K}}{C_2}\right\},$$

and recall that ξ, C_1, C_2, C_3 depend only on n and K.

We may now prove our main result of the section.

Proof of Theorem 1·3. Suppose that $x, y, z \in \mathbb{B}^n$ with $\rho(x, y) = t\rho(x, z)$ for some $t > 0$. Choose Möbius mappings P, Q from $\mathbb{B}^n$ onto itself which map x to 0 and $f(x)$ to 0 respectively. Denote by $\widetilde{f}$ the map $Q \circ f \circ P^{-1}$. Since Möbius mappings are hyperbolic isometries, we have by applying Lemma 4·1 to $\widetilde{f}$ that there exists a homeomorphism

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$\eta : (0, \infty) \to (0, \infty)$ such that

$$\begin{aligned}\frac{\rho(f(x), f(y))}{\rho(f(x), f(z))} &= \frac{\rho(0, Q(f(y)))}{\rho(0, Q(f(z)))} \\ &= \frac{\rho(0, \widetilde{f}(P(y)))}{\rho(0, \widetilde{f}(P(z)))} \\ &\leq \eta\left(\frac{\rho(0, P(y))}{\rho(0, P(z))}\right) \\ &= \eta\left(\frac{\rho(x, y)}{\rho(x, z)}\right).\end{aligned}$$

This shows that f is quasisymmetric with respect to the hyperbolic metric, with quasisymmetry provided by the homeomorphism η.

In the proof of Lemma 4·1 we used Theorem 2·5 on scales with hyperbolic distance at most 1. If instead we had used [**6**, theorem 1·1] on small enough scales for it to be applicable, and modified the proof so the cases where x or y are not in $B_\rho(0, 1)$ become the cases where Theorem 1.1 does not apply, we could directly see that we can take η to be $\eta(t) = C \max\{t^K, t^{1/K}\}$, where C is a constant depending only on n and K. The proof of Theorem 1·3 then implies that a quasiconformal map $f : \mathbb{B}^n \to \mathbb{B}^n$ is power quasisymmetric. We finally prove consequences of Theorem 1·3.

Proof of Corollary 1·4. Let $n \geq 2$ and let M^n, N^n be hyperbolic n-manifolds with hyperbolic distance functions ρ_M, ρ_N respectively. If $f : M^n \to N^n$ is η-quasisymmetric, then it follows from the Metric definition of quasiconformality, see Definition 2·3, that f is quasiconformal since quasiconformality is a local condition.

On the other hand, suppose that $f : M^n \to N^n$ is K-quasiconformal. Writing π_M, π_N for covering maps from the universal cover $\mathbb{B}^n$ onto M^n, N^n respectively, we can lift f to a K-quasiconformal map $\widetilde{f} : \mathbb{B}^n \to \mathbb{B}^n$ satisfying $f \circ \pi_M = \pi_N \circ \widetilde{f}$.

Let p, q, r be three points in M^n and choose $u, v, w \in \mathbb{B}^n$ with $\pi_M(u) = p$, $\pi_M(v) = q$, $\pi_M(w) = r$ and, moreover, $\rho_M(p, q) = \rho(u, v)$ and $\rho_M(p, r) = \rho(u, w)$. By Theorem 1·3, there exists $\widetilde{\eta}$ depending only on K and n so that

$$\frac{\rho(\widetilde{f}(u), \widetilde{f}(v))}{\rho(\widetilde{f}(u), \widetilde{f}(w))} \leq \widetilde{\eta}\left(\frac{\rho(u, v)}{\rho(u, w)}\right) = \widetilde{\eta}\left(\frac{\rho_M(p, q)}{\rho_M(p, r)}\right). \tag{4·1}$$

Now, $\pi_N(\widetilde{f}(u)) = f(p)$, $\pi_N(\widetilde{f}(v)) = f(q)$ and $\pi_N(\widetilde{f}(w)) = f(r)$ but we cannot assume that, for example, $\rho_N(f(p), f(q))$ is realized by $\rho(\widetilde{f}(u), \widetilde{f}(v))$. However, we do have

$$\rho_N(f(p), f(q)) \leq \rho(\widetilde{f}(u), \widetilde{f}(v)). \tag{4·2}$$

If G_M is the covering group for the covering map $\pi_M : \mathbb{B}^n \to M^n$, then consider the orbit of w under G_M, that is, let $\Lambda = \{g(w) : g \in G_M\}$. Then for any $w' \in \Lambda \setminus \{w\}$, we have

$$\frac{\rho(\widetilde{f}(u), \widetilde{f}(w))}{\rho(\widetilde{f}(u), \widetilde{f}(w'))} \leq \widetilde{\eta}\left(\frac{\rho(u, w)}{\rho(u, w')}\right) \leq \widetilde{\eta}(1),$$

since $\rho(u, w) = \rho_M(p, r)$ and $\widetilde{\eta}$ is increasing. Since $f \circ \pi_M = \pi_N \circ \widetilde{f}$, it follows that $\rho_N(f(p), f(r))$ is realised by the infimum of $\rho(\widetilde{f}(u), \widetilde{f}(w'))$ as w' ranges over Λ. We

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therefore have

$$\rho_N(f(p), f(r)) \geq \frac{\rho(\widetilde{f}(u), \widetilde{f}(w))}{\widetilde{\eta}(1)}. \tag{4·3}$$

By combining (4·1), (4·2) and (4·3), we conclude that

$$\frac{\rho_N(f(p), f(q))}{\rho_N(f(p), f(r))} \leq \frac{\widetilde{\eta}(1)\rho(\widetilde{f}(u), \widetilde{f}(v))}{\rho(\widetilde{f}(u), \widetilde{f}(w))} \leq \widetilde{\eta}(1)\widetilde{\eta}\left(\frac{\rho_M(p, q)}{\rho_M(p, r)}\right).$$

The result now follows by taking the quasisymmetry function η to be $\widetilde{\eta}(1)\widetilde{\eta}(t)$.

Proof of Corollary 1·5. If $f : M^n \to N^n$ is quasiconformal, then by Corollary 1·4 f is η-quasisymmetric. It follows that f satisfies the skew condition with constant $\eta(1)$.

Conversely, if $f : M^n \to N^n$ satisfies the skew condition with constant σ, then while we cannot necessarily guarantee the lift $\widetilde{f}$ of f to $\mathbb{B}^n$ does, it does on small enough scales which will be enough to conclude quasiconformality.

More precisely, if $p \in M^n$, find $u \in \mathbb{B}^n$ and $\delta > 0$ so that the covering map π_M is an isometry from $B_{\mathbb{B}^n}(u, \delta)$ onto $B_{M^n}(p, \delta)$. Then every equilateral triangle in $B_{M^n}(p, \delta)$ lifts to an equilateral triangle in $B_{\mathbb{B}^n}(u, \delta)$. The proof of Theorem 1·2 then implies that $\widetilde{f}$ is quasiconformal in $B_{\mathbb{B}^n}(u, \delta)$ with distortion bounded above by a constant depending only on σ. Hence f is quasiconformal in a neighbourhood of p with the same distortion bound. Repeating this argument over all points in M^n proves the claim.

Proof of Corollary 1·6. This is immediate from Corollary 1·4, since η only depends on K and n.

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