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New variable martingale Hardy spaces

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We investigate various variable martingale Hardy spaces corresponding to variable Lebesgue spaces $\mathcal{L}_{p(\cdot)}$ defined by rearrangement functions. In particular, we show that the dual of martingale variable Hardy space $\mathcal{H}^s_{p(\cdot)}$ with $0 < p_- \leqslant p_+ \leqslant 1$ can be described as a BMO-type space and establish martingale inequalities among these martingale Hardy spaces. Furthermore, we give an application of martingale inequalities in stochastic integral with Brownian motion.

Keywords: variable exponent; rearrangement function; martingale inequalities; atomic decomposition; duality

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1. Introduction

Variable Lebesgue spaces $L_{p(\cdot)}(\mathbb{R}^n)$ in harmonic analysis nowadays have been well studied. It is generally accepted that the dividing line between the 'early' and 'modern' periods in the study of variable Lebesgue spaces is the foundational paper of Kováčik and Rákosník [**20**] from 1991. But the origin of the variable Lebesgue spaces predates their work by 60 years, since they were first studied by Orlicz [**27**] in 1931. The most influential work is due to Zhikov [**32**, **33**], who beginning in 1986 applied the variable Lebesgue spaces to problems in the calculus of variations. The connection between variable exponent spaces and variational integrals with non-standard growth and coercivity conditions was made in [**34**]. Moreover, the substantial progress on the study of variable Lebesgue spaces is due to Diening [**5**, **6**], who proposed the so-called log-Hölder condition on variable exponents to obtain the boundedness of Hardy-Littlewood maximal operator on $L_{p(\cdot)}(\mathbb{R}^n)$. Since then, the investigation on variable Lebesgue spaces has been developed rapidly. A lot of interesting work on the theory of function spaces with variable exponents appeared, such as Nakai and Sawano [**25**] defined Hardy spaces with variable exponents on $\mathbb{R}^n$ by the grand maximal function, and investigated the Littlewood-Paley characterization and the dual spaces of Hardy spaces with variable exponents. It should be mentioned that Cruz-Uribe and Wang [**4**] independently introduced the variable Hardy space $H_{p(\cdot)}(\mathbb{R}^n)$, investigated its atomic decomposition and discussed the boundedness of operators on it with the variable exponents $p(\cdot)$ satisfying some condition slightly weaker than that used in [**25**]. The variable Lorentz-Hardy spaces $H_{p(\cdot),q}(\mathbb{R}^n)$ were considered by Yan *et al.* [**31**] and Jiao *et al.* [**15**]. We refer to [**3**, **7**, **8**, **21**, **22**, **30**] and references therein for the recent progress on Lebesgue spaces with variable exponents and some applications in PDEs.

Inspired by the considerable progress of variable Lebesgue spaces in harmonic analysis, martingale variable Lebesgue spaces have gained more and more attentions in recent years. Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_n\}_{n\geqslant 0})$ be a complete probability space and $p(\cdot)$ be a measurable function on Ω. Similar to $L_{p(\cdot)}(\mathbb{R}^n)$, we may define $L_{p(\cdot)}(\Omega)$. Aoyama [**1**] proved the Doob maximal inequality under the assumption that the variable exponent $p(\cdot)$ is $\mathcal{F}_n$-measurable for all $n \geqslant 0$. This is the first attempt to study martingale variable Lebesgue space. However, the condition imposed on $p(\cdot)$ is quite strong. Indeed, Nakai and Sadasue [**26**] gave a counterexample to show that, at least, $\mathcal{F}_0$-measurability of $p(\cdot)$ is not necessary for the boundedness of the Doob maximal operator on $L_{p(\cdot)}(\Omega)$. In order to get the Doob maximal inequality on variable Lebesgue spaces, there are two major difficulties need to be overcome. First, abstract probability space generally does not enjoy nice metric structure, and thus the log-Hölder condition is not applicable any more. Second, the arguments used in classical Lebesgue spaces are no longer efficient here and the essential reason is that the space $L_{p(\cdot)}(\Omega)$ is not a rearrangement invariant space. To better describe the Doob maximal inequalities in variable exponent setting, Jiao *et al.* [**13**] introduced a condition without metric characterization of variable exponent $p(\cdot)$ to replace the log-Hölder continuous condition. Under this new condition, they obtained the weak-type and strong-type estimates of the Doob maximal operator, and formulated the duals of martingale variable Hardy spaces. Still using the same condition, Jiao *et al.* [**12**] described the boundedness of fractional integral operator in martingale variable Hardy spaces. In the very recent paper [**16**], Jiao *et al.* gave a relatively complete investigation on martingale variable Hardy(-Lorentz) spaces. However, all the results mentioned above (namely, [**12**, **13**, **16**]) only works for atomic σ-algebras $\{\mathcal{F}_n\}_{n\geqslant 0}$. We also refer the reader to [**14**, **17**, **29**] for more results about martingales in variable exponent Lebesgue spaces.

Recently, new variable Lebesgue space $\mathcal{L}_{p(\cdot)}(\mathbb{R}^n)$ [**9**, **18**] defined by rearranging function came into view. Compared with the usual variable Lebesgue space $L_{p(\cdot)}(\mathbb{R}^n)$, the advantage $\mathcal{L}_{p(\cdot)}(\mathbb{R}^n)$ possesses is that it is a rearrangement invariant space. With the emergence of this new variable Lebesgue space, one question arises: whether we can define and investigate corresponding new martingale variable Lebesgue spaces. In this paper, we shall concentrate on this question. Our variable Lebesgue space $\mathcal{L}_{p(\cdot)}(\Omega)$ here is defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (see definition 2.10 below). Note that $p(\cdot)$ now is defined in $[0, 1]$. Under the assumption that $p(\cdot)$ satisfies the local log-Hölder condition, we obtain the strong-type and weak-type estimates for the Doob maximal operators in $\mathcal{L}_{p(\cdot)}(\Omega)$. Also, we investigate various variable martingale Hardy spaces corresponding to $\mathcal{L}_{p(\cdot)}(\Omega)$. Via atomic decompositions, we show that the dual of martingale variable Hardy space $\mathcal{H}^s_{p(\cdot)}(\Omega)$ with $0 < p_- \leqslant p_+ \leqslant 1$ can be described as a BMO-type space. Moreover, we establish some new martingale inequalities among these martingale Hardy spaces defined

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in the § 2.4. Finally, we obtain an application of martingale inequalities in stochastic integral with Brownian motion. Compared with the results due to Jiao *et al.* [**12**, **13**, **16**], we do not need to assume that $\mathcal{F}_n$ is an atomic σ-algebra any more.

Throughout this paper, $\mathbb{Z}, \mathbb{N}$ and $\mathbb{C}$ denote the integer set, the nonnegative integer set and set of complex numbers, respectively. We denote the absolute positive constant by C, which can vary from line to line, and we denote by $C_{p(\cdot)}$ the constant depending only on $p(\cdot)$. The symbol $A \lesssim B$ stands for the inequality $A \leqslant CB$ or $A \leqslant C_{p(\cdot)}B$. If we write $A \approx B$, then it stands for $A \lesssim B \lesssim A$. Moreover, for each measurable set $E \subset \Omega$ (or $\subset [0,1]$), we denote $|E|$ the measure of E.

2. Preliminaries

This section contains four subsections. Firstly, we introduce the definition and some related properties of the classical variable Lebesgue space. Secondly, we give the definition of Lebesgue space $\mathcal{L}_{p(\cdot)}(\Omega)$ and its useful properties. In § 2.4, the variable exponent martingale Hardy spaces corresponding to $\mathcal{L}_{p(\cdot)}(\Omega)$ are defined.

2.1. Variable Lebesgue spaces $L_{p(\cdot)}$

Throughout the paper, we always suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space. Let (R, μ) a measure space. For any measurable set $E \subset R$, we will often denote the measure of E simply by $|E|$ whenever no confusion can occur. Indeed, (R, μ) could be $(\Omega, \mathcal{F}, \mathbb{P})$ or $(\mathbb{R}^n, m)(n \geqslant 1)$, where m is the Lebesgue measure.

A measurable function $p(\cdot) : R \to (0, \infty)$ is called a variable exponent. For a measurable set $A \subset R$, we denote

$$p_-(A) := \operatorname*{ess\,inf}_{x \in A} p(x), \quad p_+(A) := \operatorname*{ess\,sup}_{x \in A} p(x),$$

and for convenience

$$p_- := p_-(R), \quad p_+ := p_+(R), \quad \underline{p} := \min\{1, p_-\}.$$

In sequel, we always use the following symbols

$$\mathfrak{P}(R) = \{p(\cdot) : 0 < p_- \leqslant p_+ < \infty\},$$

and

$$\mathfrak{P}_a(R) = \{p(\cdot) : a < p_- \leqslant p_+ < \infty\}, \quad a \in \mathbb{R}^+.$$

Throughout the paper, given a variable exponent $p(\cdot)$, we define the conjugate variable exponent $p'(\cdot)$ by the formula

$$\frac{1}{p'(x)} + \frac{1}{p(x)} = 1, \quad x \in R.$$

The variable Lebesgue spaces $L_{p(\cdot)}(R)$ are defined as follows.

DEFINITION 2.1. Let $p(\cdot) \in \mathfrak{P}(R)$. The variable exponent Lebesgue space $L_{p(\cdot)}(R)$ is defined as the set of all measurable functions f on R such that, for some $\lambda > 0$,

$$\rho\left(\frac{f}{\lambda}\right) = \int_R \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} \mathrm{d}\mu < \infty.$$

This becomes a quasi-Banach function space when it is equipped with the quasi-norm

$$\|f\|_{L_{p(\cdot)}} = \inf\{\lambda > 0 : \int_R \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} \mathrm{d}\mu \leqslant 1\}. \tag{2.1}$$

According to [10, theorem 1.3], for any $f \in L_{p(\cdot)}(R)$, we have $\rho(f) \leqslant 1$ if and only if $\|f\|_{L_{p(\cdot)}} \leqslant 1$. For any quasi-Banach space X, we denote by X^* the dual space of X. Next, we present some basic properties for variable Lebesgue spaces $L_{p(\cdot)}(R)$.

LEMMA 2.2 [3, theorem 2.80]. *Let $p(\cdot) \in \mathfrak{P}_1(R)$. Then $(L_{p(\cdot)}(R))^* = L_{p'(\cdot)}(R)$.*

LEMMA 2.3 [20, theorem 2.8]. *Let $p(\cdot), q(\cdot) \in \mathfrak{P}(\Omega)$. If $p(\cdot) \leqslant q(\cdot)$, then for every $f \in L_{q(\cdot)}(\Omega)$, we have*

$$\|f\|_{L_{p(\cdot)}(\Omega)} \leqslant 2\|f\|_{L_{q(\cdot)}(\Omega)}.$$

LEMMA 2.4 [3, corollary 2.28]. *Let $r(\cdot), p(\cdot), q(\cdot) \in \mathfrak{P}(R)$ satisfy*

$$\frac{1}{r(\cdot)} = \frac{1}{p(\cdot)} + \frac{1}{q(\cdot)}.$$

Then there is a positive constant $C_{s(\cdot)} > 0$ such that for all $f \in L_{p(\cdot)}(R)$ and $g \in L_{q(\cdot)}(R)$,

$$\|fg\|_{L_{r(\cdot)}} \leqslant C_{s(\cdot)}\|f\|_{L_{p(\cdot)}}\|g\|_{L_{q(\cdot)}},$$

where $s(\cdot) = \frac{r(\cdot)}{p(\cdot)}$.

The following lemma is from [31].

LEMMA 2.5 [31, remark 2.1]. *Given $p(\cdot) \in \mathfrak{P}(R)$, if $0 < p_- \leqslant p_+ \leqslant 1$, then for any positive function $f, g \in L_{p(\cdot)}(R)$, we have*

$$\|f\|_{L_{p(\cdot)}(R)} + \|g\|_{L_{p(\cdot)}(R)} \leqslant \|f + g\|_{L_{p(\cdot)}(R)}.$$

2.2. The maximal operators and log-Hölder continuous condition

We begin this subsection with the following log-Hölder continuous condition for the variable exponent $p(\cdot)$ defined on $\mathbb{R}^n$ (see [3, lemma 3.24]).

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Definition 2.6. Let $p(\cdot) \in \mathfrak{P}(\mathbb{R}^n)$. We say that $p(\cdot)$ is locally log-Hölder continuous, if there exists a positive constant C such that for all $x, y \in \mathbb{R}^n$ with $|x-y| < \frac{1}{2}$,

$$|p(x) - p(y)| \leqslant \frac{C}{-\log|x-y|}. \tag{2.2}$$

We say $p(\cdot)$ is log-Hölder continuous at infinity if there exists a positive constant C such that for all $x \in \mathbb{R}^n$, $p_\infty = \lim_{x\to\infty} p(x)$,

$$|p(x) - p_\infty| \leqslant \frac{C}{\log(e+|x|)}. \tag{2.3}$$

The conditions (2.2) and (2.3) are called log-Hölder condition.

Lemma 2.7 [**13**, lemma 5.2]. *Let $p(\cdot) \in \mathfrak{P}([0,1])$ satisfy locally log-Hölder condition. For any interval $Q \subset [0,1]$, then for $x \in Q$, we have*

$$\|\chi_Q\|_{L_{p(\cdot)}} \approx |Q|^{1/p_-(Q)} \approx |Q|^{1/p(x)} \approx |Q|^{1/p_+(Q)},$$

where denote the Lebesgue measure of Q by $|Q|$.

For locally integrable function f defined on $\mathbb{R}^n$, the Hardy-littlewood maximal operator is defined by

$$\mathcal{M}(f)(x) = \sup_{x\in Q} \frac{1}{|Q|} \int_Q |f(y)| \mathrm{d}y, \quad x \in \mathbb{R}^n,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ that contain x.

The log-Hölder condition is sufficient so that the Hardy-littlewood maximal operator is bounded on $L_{p(\cdot)}(\mathbb{R}^n)$, $p_- > 1$.

Lemma 2.8 [**3**, theorem 3.16]. *Let $R \subset \mathbb{R}$ and let $p(\cdot) \in \mathfrak{P}(R)$ satisfy log-Hölder condition. Then*

$$\|\mathcal{M}(f)\|_{L_{p(\cdot)}(R)} \leqslant C_{p(\cdot)} \|f\|_{L_{p(\cdot)}(R)}, \quad p_- > 1$$

and

$$\|\mathcal{M}(f)\|_{wL_{p(\cdot)}(R)} \leqslant C_{p(\cdot)} \|f\|_{L_{p(\cdot)}(R)}, \quad p_- \geqslant 1.$$

Remark 2.9. If $|R| < \infty$, then $p(\cdot)$ is automatically log-Hölder continuous at infinity (see [**3**, chapter 2.1]). In this case, if $p(\cdot)$ is log-Hölder continuous, $p(\cdot)$ is actually local log-Hölder continuous.

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2.3. Variable Lebesgue spaces $\mathcal{L}_{p(\cdot)}$

Assume that f is a measurable function on $(\Omega, \mathcal{F}, \mathbb{P})$. We define the distribution function $d_f : [0,\infty) \to [0,1]$ associated with f by

$$d_f(s) = |\{x \in \Omega : |f(x)| > s\}|, \ s \in [0,\infty).$$

The non-increasing rearrangement function $f^* : [0,\infty) \to (0,\infty]$ of f is defined by

$$f^*(t) = \inf\left\{s \geqslant 0 : d_f(s) \leqslant t\right\}.$$

We state several basic properties of the function f^*. For all $E \in \mathcal{F}$,

$$\chi_E^* = \chi_{(0,|E|)}, \quad \int_\Omega fg\mathrm{d}\mathbb{P} \leqslant \int_0^\infty f^*(t)g^*(t)\mathrm{d}t. \tag{2.4}$$

For more properties, we refer the reader to [**11**, proposition 1.4.5].

We also define the function f^{**} associated with f^* by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\mathrm{d}s, \quad t > 0.$$

According to [**2**, proposition 3.2, theorem 3.4], f^{**} is non-negative, decreasing and continuous on $(0,\infty)$. Further, we have

$$f^*(t) \leqslant f^{**}(t), \quad t > 0$$

and

$$(f+g)^{**}(t) \leqslant f^{**}(t) + g^{**}(t), \quad t > 0. \tag{2.5}$$

We take the following definition from [**9**, **18**].

Definition 2.10. Let $p(\cdot) \in \mathfrak{P}([0,1])$. Define the variable exponent Lebesgue space $\mathcal{L}_{p(\cdot)}(\Omega)$ as the space of all measurable functions $f(x)$ on Ω such that

$$\varrho_{p(\cdot)}(f) = \int_0^\infty f^*(t)^{p(t)}\mathrm{d}t < \infty.$$

For any $f \in \mathcal{L}_{p(\cdot)}(\Omega)$, define

$$\|f\|_{\mathcal{L}_{p(\cdot)}(\Omega)} := \|f^*\|_{L_{p(\cdot)}([0,1])}.$$

Obviously, the Lebesgue space $\mathcal{L}_{p(\cdot)}(\Omega)$ goes back to the classical Lebesgue space $L_p(\Omega)$ as variable exponent $p(\cdot)$ is equal to the constant p. With the help of next lemma, we will show that $\|\cdot\|_{\mathcal{L}_{p(\cdot)}(\Omega)}$ is a quasi-norm for $p(\cdot) \in \mathfrak{P}([0,1])$ satisfying local log-Hölder condition.

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LEMMA 2.11 [**19**, theorem E]. *Let $p(\cdot) \in \mathfrak{P}_1([0,1])$ satisfy locally log-Hölder condition. Then the following Hardy-type inequalities hold*

$$\left\| \frac{1}{t} \int_0^t f(s)\, ds \right\|_{L_{p(\cdot)}([0,1])} \leqslant C_{p(\cdot)} \|f\|_{L_{p(\cdot)}([0,1])} \tag{2.6}$$

and

$$\left\| \int_t^1 \frac{f(s)}{s}\, ds \right\|_{L_{p(\cdot)}([0,1])} \leqslant C_{p(\cdot)} \|f\|_{L_{p(\cdot)}([0,1])}, \tag{2.7}$$

REMARK 2.12. In fact, in [**19**, theorem E], $p(\cdot)$ is local log-Hölder continuous on interval $[0, \delta]$ for some small $\delta > 0$. Later, it was proved in [**7**, theorem 3.1] that (2.6) holds true with a weaker condition stated in [**7**, hypothesis].

LEMMA 2.13. *Given $p(\cdot) \in \mathfrak{P}([0,1])$ and $f \in \mathcal{L}_{p(\cdot)}(\Omega)$, we have*

$$\||f|^t\|_{\mathcal{L}_{p(\cdot)}} = \|f\|^t_{\mathcal{L}_{tp(\cdot)}}, \quad t > 0.$$

Proof. Note that $(f^*)^t = (|f|^t)^*$. The desired equality follows from [**3**, proposition 2.18]. □

PROPOSITION 2.14. *Let $p(\cdot) \in \mathfrak{P}([0,1])$ and let $f, g \in \mathcal{L}_{p(\cdot)}(\Omega)$. If $p(\cdot)$ satisfies local log-Hölder condition, then $\|\cdot\|_{\mathcal{L}_{p(\cdot)}(\Omega)}$ is a quasi-norm, moreover, we have*

$$\|f + g\|_{\mathcal{L}_{p(\cdot)}} \lesssim \|f\|_{\mathcal{L}_{p(\cdot)}} + \|g\|_{\mathcal{L}_{p(\cdot)}}.$$

Proof. Take $f, g \in \mathcal{L}_{p(\cdot)}$ and $s \in (0, \underline{p})$. According to lemma 2.13 and (2.5), we have

$$\begin{aligned}\|f + g\|_{\mathcal{L}_{p(\cdot)}} &= \||f + g|^s\|^{1/s}_{\mathcal{L}_{p(\cdot)/s}} \leqslant \|\big(|f|^s + |g|^s\big)^{**}\|^{1/s}_{L_{p(\cdot)/s}} \\ &\leqslant \|\big(|f|^s\big)^{**} + \big(|g|^s\big)^{**}\|^{1/s}_{L_{p(\cdot)/s}}.\end{aligned}$$

Since $p(\cdot)/s > 1$, $\|\cdot\|_{L_{p(\cdot)/s}}$ is a norm. It follows (2.6) that $\|f^{**}\|_{L_{p(\cdot)/s}} \lesssim \|f^*\|_{L_{p(\cdot)/s}}$. Thus, using lemma 2.13, we obtain

$$\begin{aligned}\|f + g\|_{\mathcal{L}_{p(\cdot)}} &\lesssim \|\big(|f|^s\big)^{**}\|^{1/s}_{L_{p(\cdot)/s}} + \|\big(|g|^s\big)^{**}\|^{1/s}_{L_{p(\cdot)/s}} \\ &\lesssim \|(|f|^s)^*\|^{1/s}_{L_{p(\cdot)}} + \|(|g|^s)^*\|^{1/s}_{L_{p(\cdot)}} \\ &= \|f\|_{\mathcal{L}_{p(\cdot)}} + \|g\|_{\mathcal{L}_{p(\cdot)}}.\end{aligned}$$

□

We now present several useful lemmas for the variable Lebesgue space $\mathcal{L}_{p(\cdot)}(\Omega)$.

LEMMA 2.15. *Given $p(\cdot) \in \mathfrak{P}_1([0,1])$, if $f \in \mathcal{L}_{p(\cdot)}(\Omega)$ and $g \in \mathcal{L}_{p'(\cdot)}(\Omega)$, then there is a positive constant $C_{p(\cdot)} > 0$ such that*

$$\int_\Omega |fg|\, d\mathbb{P} \leqslant C_{p(\cdot)} \|f\|_{\mathcal{L}_{p(\cdot)}} \|g\|_{\mathcal{L}_{p'(\cdot)}}.$$

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Proof. Since $\int_\Omega |fg|\mathrm{d}\mathbb{P} \leqslant \int_0^1 f^*(t)g^*(t)\mathrm{d}t$, the desired Hölder inequality follows from lemma 2.2. □

LEMMA 2.16. *Let $r(\cdot), p(\cdot), q(\cdot) \in \mathfrak{P}([0,1])$ satisfy*

$$\frac{1}{r(\cdot)} = \frac{1}{p(\cdot)} + \frac{1}{q(\cdot)}.$$

Then for all $h \in \mathcal{L}_{q(\cdot)}(\Omega)$ and $E \in \mathcal{F}$, we have

$$\|\chi_E h\|_{\mathcal{L}_{r(\cdot)}} \leqslant \|\chi_E\|_{\mathcal{L}_{p(\cdot)}} \|h\|_{\mathcal{L}_{q(\cdot)}}.$$

Proof. For any $E \in \mathcal{F}$, according to the Hölder inequality (see lemma 2.4) and

$$(\chi_E h)^*(t) \leqslant h^*\chi_{[0,|E|]}(t),\ 0 \leqslant t \leqslant \infty,$$

we have

$$\|\chi_E h\|_{\mathcal{L}_{r(\cdot)}} \leqslant \left\|h^*\chi_{[0,|E|]}\right\|_{L_{r(\cdot)}} \leqslant \left\|\chi_{[0,|E|]}\right\|_{L_{p(\cdot)}} \|h^*\|_{L_{q(\cdot)}} = \|\chi_E\|_{\mathcal{L}_{p(\cdot)}} \|h\|_{\mathcal{L}_{q(\cdot)}}. \quad \square$$

By lemma 2.3, we have the following lemma.

LEMMA 2.17. *Given $p(\cdot),\ q(\cdot) \in \mathfrak{P}([0,1])$, if $p(\cdot) \leqslant q(\cdot)$, then $\mathcal{L}_{q(\cdot)} \subset \mathcal{L}_{p(\cdot)}$.*

Using lemma 2.7, we can easily prove the next lemma.

LEMMA 2.18. *Given $p(\cdot) \in \mathfrak{P}([0,1])$, and $p(\cdot)$ is locally log-Hölder continuous, then for all $I \in \mathcal{F}$, we have*

$$\|\chi_I\|_{\mathcal{L}_{p(\cdot)}} \approx |I|^{(1/p_-([0,|I|]))} \approx |I|^{(1/p(x))} \approx |I|^{(1/p_+([0,|I|]))}, \quad x \in (0,|I|).$$

PROPOSITION 2.19. *Let $p(\cdot) \in \mathfrak{P}([0,1])$ satisfy locally log-Hölder condition.*

(1) Then for all $I \in \mathcal{F}$, we have

$$\|\chi_I\|_1 \approx \|\chi_I\|_{\mathcal{L}_{p(\cdot)}} \|\chi_I\|_{\mathcal{L}_{q(\cdot)}},$$

where

$$1 = \frac{1}{p(\cdot)} + \frac{1}{q(\cdot)}.$$

(2) Let $q(\cdot) \in \mathfrak{P}([0,1])$, and $q(\cdot)$ is locally log-Hölder continuous in $[0,1]$. Then for all $I \in \mathcal{F}$, we have

$$\|\chi_I\|_{\mathcal{L}_{r(\cdot)}} \approx \|\chi_I\|_{\mathcal{L}_{p(\cdot)}} \|\chi_I\|_{\mathcal{L}_{q(\cdot)}},$$

where

$$\frac{1}{r(\cdot)} = \frac{1}{p(\cdot)} + \frac{1}{q(\cdot)}.$$

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Proof. (1) By [3, proposition 2.3], $q(\cdot) = (p(\cdot)/p(\cdot) - 1)$ satisfies locally log-Hölder condition. Then the desired result follows from lemma 2.18.

(2) For any $I \in \mathcal{F}$, it follows from $(\chi_I)^* = \chi_{(0,|I|)}$ and lemma 2.18 that for any $x \in (0, |I|)$, we have

$$\|\chi_I\|_{\mathcal{L}_{r(\cdot)}} = \left\|\chi_{[0,|I|]}\right\|_{L_{r(\cdot)}} \approx |I|^{(1/r(x))} = |I|^{(1/p(x))+(1/q(x))} \approx \|\chi_I\|_{\mathcal{L}_{p(\cdot)}} \|\chi_I\|_{\mathcal{L}_{q(\cdot)}}.$$

□

2.4. Variable martingale Hardy spaces

Now we introduce some standard notation from martingale theory. We refer to [23] and [28] for the classical martingale space theory. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with subalgebras $(\mathcal{F}_n)_{n\geqslant 0}$ and $\mathcal{F} = \sigma(\cup_{n\geqslant 0}\mathcal{F}_n)$. Recall that the conditional expectation operator relative to $\mathcal{F}_n$ is denoted by $\mathbb{E}_{\mathcal{F}_n}$(simply by $\mathbb{E}_n$), that is, $\mathbb{E}(f|\mathcal{F}_n) = \mathbb{E}_n(f)$. We also call $(\mathcal{F}_n)_{n\geqslant 0}$ a stochastic basis with convention $\mathcal{F}_{-1} = \mathcal{F}_0$. A sequence of measurable functions $f = (f_n)_{n\geqslant 0} \subset L_1(\Omega)$ is called a martingale with respect to $(\mathcal{F}_n)_{n\geqslant 0}$ if $\mathbb{E}_n(f_{n+1}) = f_n$ for every $n \geqslant 0$. For a martingale $f = (f_n)_{n\geqslant 0}$,

$$d_n f = f_n - f_{n-1}, n \geqslant 0,$$

denote the martingale difference (with convention $d_0 f = 0$). In addition, if $f_n \in \mathcal{L}_{p(\cdot)}$, f is called an $\mathcal{L}_{p(\cdot)}$-martingale with respect to $(\mathcal{F}_n)$. In this case, we set

$$\|f\|_{\mathcal{L}_{p(\cdot)}} = \sup_{n\geqslant 0} \|f_n\|_{\mathcal{L}_{p(\cdot)}}.$$

If $\|f\|_{\mathcal{L}_{p(\cdot)}} < \infty$, f is called a bounded $\mathcal{L}_{p(\cdot)}$-martingale and denoted $f \in \mathcal{L}_{p(\cdot)}$. For a martingale relative to $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_n)_{n\geqslant 0})$, define the maximal operator, the square function and the conditional square function of f, respectively, as follows ($f_{-1} = 0$)

$$M_m(f) = \sup_{n\leqslant m} |f_n|, \ Mf = \sup_{n\geqslant 1} |f_n|;$$

$$S_m(f) = \left(\sum_{n=0}^{m} |\mathrm{d}f_n|^2\right)^{1/2}, \ S(f) = \left(\sum_{n=0}^{\infty} |\mathrm{d}f_n|^2\right)^{1/2};$$

$$s_m(f) = \left(\sum_{n=0}^{m} \mathbb{E}_{\mathcal{F}_{n-1}} |\mathrm{d}f_n|^2\right)^{1/2}, \ s(f) = \left(\sum_{n=0}^{\infty} \mathbb{E}_{\mathcal{F}_{n-1}} |\mathrm{d}f_n|^2\right)^{1/2}.$$

Denote by Λ the collection of all sequences $(\lambda_n)_{n\geqslant 0}$ of non-decreasing, non-negative and adapted functions with $\lambda_\infty = \lim_{n\to\infty} \lambda_n$. Let $p(\cdot) \in \mathfrak{P}([0, 1])$. The variable exponent martingale Hardy spaces associated with variable exponent Lebesgue space $\mathcal{L}_{p(\cdot)}$ are defined as follows:

$$\mathcal{H}^M_{p(\cdot)} = \{f = (f_n)_{n\geqslant 0} : \|f\|_{\mathcal{H}^M_{p(\cdot)}} = \|M(f)\|_{\mathcal{L}_{p(\cdot)}} < \infty\};$$

$$\mathcal{H}^S_{p(\cdot)} = \{f = (f_n)_{n\geqslant 0} : \|f\|_{\mathcal{H}^S_{p(\cdot)}} = \|S(f)\|_{\mathcal{L}_{p(\cdot)}} < \infty\};$$

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$$\mathcal{H}^s_{p(\cdot)} = \{f = (f_n)_{n\geqslant 0} : \|f\|_{\mathcal{H}^s_{p(\cdot)}} = \|s(f)\|_{\mathcal{L}_{p(\cdot)}} < \infty\};$$

$$\mathcal{Q}_{p(\cdot)} = \{f = (f_n)_{n\geqslant 0} : \exists(\lambda_n)_{n\geqslant 0} \in \Lambda,\ \text{s.t. } S_n(f) \leqslant \lambda_{n-1},\ \lambda_\infty \in \mathcal{L}_{p(\cdot)}\},$$

$$\|f\|_{\mathcal{Q}_{p(\cdot)}} = \inf_{(\lambda_n)\in\Lambda} \|\lambda_\infty\|_{\mathcal{L}_{p(\cdot)}},$$

where the infimum is taken over all $(\lambda_n)_{n\geqslant 0} \in \Lambda$ such that $S_n(f) \leqslant \lambda_{n-1}$.

$$\mathcal{P}_{p(\cdot)} = \{f = (f_n)_{n\geqslant 0} :\ \exists(\lambda_n)_{n\geqslant 0} \in \Lambda, \text{s.t. } |f_n| \leqslant \lambda_{n-1},\ \lambda_\infty \in \mathcal{L}_{p(\cdot)}\},$$

$$\|f\|_{\mathcal{P}_{p(\cdot)}} = \inf_{(\lambda_n)\in\Lambda} \|\lambda_\infty\|_{\mathcal{L}_{p(\cdot)}},$$

where the infimum is taken over all $(\lambda_n)_{n\geqslant 0} \in \Lambda$ such that $|f_n| \leqslant \lambda_{n-1}$.

Remark 2.20. If $p(\cdot) = p$ is a constant, then the above definitions of variable Hardy spaces go back to the classical definitions (see [**28**]).

We are going to end this subsection with the Doob's maximal inequalities. To this end, we introduce weak variable exponent Lebesgue space $w\mathcal{L}_{p(\cdot)}(\Omega)$.

Definition 2.21. Let $p(\cdot) \in \mathfrak{P}([0,1])$. The weak variable exponent Lebesgue space $w\mathcal{L}_{p(\cdot)}(\Omega)$ is defined as follows:

$$w\mathcal{L}_{p(\cdot)}(\Omega) = \{f \in L(\Omega) : \|f\|_{w\mathcal{L}_{p(\cdot)}} < \infty\},$$

with the quasi-norm

$$\|f\|_{w\mathcal{L}_{p(\cdot)}} = \sup_{t>0} t\|\chi_{\{|f|>t\}}\|_{\mathcal{L}_{p(\cdot)}}.$$

We have

$$\|f\|_{w\mathcal{L}_{p(\cdot)}} = \|f^*\|_{wL_{p(\cdot)}} := \sup_{t>0} t\|\chi_{\{f^*>t\}}\|_{L_{p(\cdot)}}. \tag{2.8}$$

In fact, by [**11**, proposition 1.4.5(3)], we have

$$\{s \geqslant 0 :\ f^*(s) > t\} = [0, d_f(t)).$$

Applying the above equation, we obtain

$$\begin{aligned}\|f\|_{w\mathcal{L}_{p(\cdot)}} &= \sup_{t>0} t\|\chi_{\{x\in\Omega:|f(x)|>t\}}\|_{\mathcal{L}_{p(\cdot)}} = \sup_{t>0} t\|\chi_{(0,d_f(t))}\|_{L_{p(\cdot)}} \\ &= \sup_{t>0} t\|\chi_{\{s\in[0,1]:f^*(s)>t\}}\|_{L_{p(\cdot)}} \triangleq \|f^*\|_{wL_{p(\cdot)}}.\end{aligned}$$

Theorem 2.22 Doob's inequalities. *Let $p(\cdot) \in \mathfrak{P}([0,1])$ satisfy locally log-Hölder condition and $f \in \mathcal{L}_{p(\cdot)}(\Omega)$. Then there exists a positive constant $C_{p(\cdot)}$, such that*

$$\|Mf\|_{\mathcal{L}_{p(\cdot)}} \leqslant C_{p(\cdot)}\|f\|_{\mathcal{L}_{p(\cdot)}}, \quad p_->1$$

and

$$\|Mf\|_{w\mathcal{L}_{p(\cdot)}} \leqslant C_{p(\cdot)}\|f\|_{\mathcal{L}_{p(\cdot)}}, \quad p_-\geqslant 1.$$

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Proof. Take $f = (f_n)_{n\geqslant 0} \in \mathcal{L}_{p(\cdot)}(\Omega)$. According to [**23**, theorem 3.6.3], we have

$$(Mf)^*(t) \leqslant f^{**}(t), \quad \forall t > 0. \tag{2.9}$$

Then, applying lemma 2.11, we deduce that

$$\|Mf\|_{\mathcal{L}_{p(\cdot)}} = \|(Mf)^*\|_{L_{p(\cdot)}} \leqslant \|f^{**}\|_{L_{p(\cdot)}} \leqslant C_{p(\cdot)}\|f^*\|_{L_{p(\cdot)}} = C_{p(\cdot)}\|f\|_{\mathcal{L}_{p(\cdot)}}.$$

According to (2.9) and the definition of the Hardy-littlewood maximal operator $\mathcal{M}$, we have

$$(Mf)^*(t) \leqslant \mathcal{M}(f^*)(t), \quad \forall 0 < t < 1.$$

From (2.8) and lemma 2.8, we conclude that

$$\|Mf\|_{w\mathcal{L}_{p(\cdot)}} = \|(Mf)^*\|_{wL_{p(\cdot)}} \leqslant \|\mathcal{M}(f^*)\|_{wL_{p(\cdot)}} \lesssim \|f^*\|_{L_{p(\cdot)}} = \|f\|_{\mathcal{L}_{p(\cdot)}}. \qquad \square$$

3. Atomic decompositions for variable Hardy martingale spaces

In this section, we consider the atomic decomposition for the new variable Hardy martingale spaces introduced in § 2.4.

3.1. Atomic decompositions in $\mathcal{H}^s_{p(\cdot)}(\Omega)$

We begin with the following definition and denote the set of all stopping time with respect to $\{\mathcal{F}_n\}_{n\geqslant 0}$ by Γ.

Definition 3.1. Let $p(\cdot) \in \mathfrak{P}([0,1])$. A measurable function a is called a $(1, p(\cdot), \infty)$-atom (or $(2, p(\cdot), \infty)$-atom, $(3, p(\cdot), \infty)$-atom, respectively), if there exists a stopping time $\tau \in \Gamma$ such that

(1) $a_n := \mathbb{E}_n(a) = 0$ if $n \leqslant \tau$,

(2) $\|s(a)\|_\infty$(or $\|S(a)\|_\infty, \|M(a)\|_\infty$, respectively) $\leqslant \frac{1}{\|\chi_{\{\tau<\infty\}}\|_{\mathcal{L}_{p(\cdot)}}}$.

Definition 3.2. Let $p(\cdot) \in \mathfrak{P}([0,1])$. Assume that $d = 1, 2$ or 3. The atomic Hardy space $\mathcal{H}^{at,d,\infty}_{p(\cdot)}(\Omega)$ is defined as the space of all martingales $f = (f_n)_{n\geqslant 0}$ such that

$$f_n = \sum_{k\in\mathbb{Z}} \mu_k a^k_n \qquad a.e. \quad \forall n \geqslant 0, \tag{3.1}$$

where $(a^k)_{k\in\mathbb{Z}}$ is a sequence of $(d, p(\cdot), \infty)$-atoms, associated with stopping time $\tau_k \in \Gamma$ and $a^k_n = \mathbb{E}_n(a^k)$. For $f \in \mathcal{H}^{at,d,\infty}_{p(\cdot)}$, define its quasi-norm by

$$\|f\|_{\mathcal{H}^{at,d,\infty}_{p(\cdot)}} = \inf \left\| \left[\sum_{k\in\mathbb{Z}} \left(\frac{\mu_k \chi_{\{\tau_k<\infty\}}}{\| \chi_{\{\tau_k<\infty\}} \|_{\mathcal{L}_{p(\cdot)}}} \right)^t \right]^{1/t} \right\|_{\mathcal{L}_{p(\cdot)}},$$

where $0 < t < \underline{p}$ and the infimum is taken over all the decompositions of f by the form (3.1).

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THEOREM 3.3. *Let $p(\cdot) \in \mathfrak{P}([0,1])$. Then*

$$\mathcal{H}^s_{p(\cdot)}(\Omega) = \mathcal{H}^{at,1,\infty}_{p(\cdot)}(\Omega)$$

with equivalent quasi-norms.

Proof. Assume that $f \in \mathcal{H}^s_{p(\cdot)}$. Let us consider the following stopping time, for all $k \in \mathbb{Z}$,

$$\tau_k = \inf\{n \in \mathbb{N} : s_{n+1}(f) > 2^k\}, \quad \inf \phi = \infty.$$

The sequence of these stopping times is obviously non-decreasing. For each stopping τ, denote $f_n^\tau = f_{n\wedge\tau}$, where $n \wedge \tau = \min(n, \tau)$. Hence

$$f_n = \sum_{k\in\mathbb{Z}} (f_n^{\tau_{k+1}} - f_n^{\tau_k}).$$

Let

$$\mu_k = 3 \cdot 2^k \left\|\chi_{\{\tau_k<\infty\}}\right\|_{\mathcal{L}_{p(\cdot)}} \quad \text{and} \quad a_n^k = \frac{f_n^{\tau_{k+1}} - f_n^{\tau_k}}{\mu_k}.$$

If $\mu_k = 0$, then let $a_n^k = 0$ for all $k \in \mathbb{Z},\ n \in \mathbb{N}$. Thus $(a_n^k)_{n\geqslant 0}$ is a martingale for each fixed $k \in \mathbb{Z}$. By the definition of τ_k, since $s(f^{\tau_k}) = s_{\tau_k}(f) \leqslant 2^k$, we obtain

$$s\big((a_n^k)_{n\geqslant 0}\big) \leqslant \frac{s(f^{\tau_{k+1}}) + s(f^{\tau_k})}{\mu_k} \leqslant \|\chi_{\{\tau_k<\infty\}}\|^{-1}_{\mathcal{L}_{p(\cdot)}}.$$

Thus $(a_n^k)_{n\geqslant 0}$ is a L_2-bounded martingale. Consequently there exists an element $a^k \in L_2$ such that

$$a_n^k = \mathbb{E}_n(a^k), \quad \forall n \in \mathbb{N}.$$

If $n \leqslant \tau_k$, then $a_n^k = (f_n^{\tau_{k+1}} - f_n^{\tau_k}/\mu_k) = 0$, $\operatorname{supp}(a^k) \subseteq \{\tau_k < \infty\}$ and

$$\|s(a^k)\|_\infty \leqslant \frac{1}{\|\chi_{\{\tau_k<\infty\}}\|_{\mathcal{L}_{p(\cdot)}}}.$$

We conclude that a^k is really a $(1, p(\cdot), \infty)$-atom according to the above estimate.

Denote $\vartheta_k = \{\tau_k < \infty\} = \{s(f) > 2^k\}$. Recalling that τ_k is nondecreasing for each $k \in \mathbb{Z}$, we have $\vartheta_k \supset \vartheta_{k+1}$. Then

$$\sum_{k\in\mathbb{Z}} \big(3 \cdot 2^k \chi_{\vartheta_k}(x)\big)^t$$

is the sum of the geometric sequence $\{(3 \cdot 2^k \chi_{\vartheta_k}(x))^t\}_{k\in\mathbb{Z}}$, where $0 < t < \underline{p}$. Thus,

$$\sum_{k\in\mathbb{Z}} \left(3 \cdot 2^k \chi_{\vartheta_k}(x)\right)^t \approx \left(\sum_{k\in\mathbb{Z}} 3 \cdot 2^k \chi_{\vartheta_k}(x)\right)^t \approx \left(\sum_{k\in\mathbb{Z}} 3 \cdot 2^k \chi_{\vartheta_k\setminus\vartheta_{k+1}}(x)\right)^t.$$

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Note that $\vartheta_k \backslash \vartheta_{k+1} = \{2^k < s(f) \leqslant 2^{k+1}\}$. Then

$$\begin{aligned} \|f\|_{\mathcal{H}^{at,1,\infty}_{p(\cdot)}} &\leqslant \left\| \left(\sum_{k\in\mathbb{Z}} \left(3\cdot 2^k \chi_{\{\tau_k<\infty\}} \right)^t \right)^{1/t} \right\|_{\mathcal{L}_{p(\cdot)}} \lesssim \left\| \sum_{k\in\mathbb{Z}} 3\cdot 2^k \chi_{\vartheta_k\backslash\vartheta_{k+1}} \right\|_{\mathcal{L}_{p(\cdot)}} \\ &\leqslant \left\| \sum_{k\in\mathbb{Z}} 3\cdot s(f)\chi_{\vartheta_k\backslash\vartheta_{k+1}} \right\|_{\mathcal{L}_{p(\cdot)}} \lesssim \|s(f)\|_{\mathcal{L}_{p(\cdot)}} = \|f\|_{\mathcal{H}^s_{p(\cdot)}}. \end{aligned} \tag{3.2}$$

We now prove that the sum $\sum_{k\in\mathbb{Z}} \mu_k a^k$ converges in $\mathcal{H}^s_{p(\cdot)}(\Omega)$. Since $s(f-f^{\tau_k})^2 = s(f)^2 - s(f^{\tau_k})^2$, it follows that

$$s\big(f-f^{\tau_k}\big), s(f^{\tau_k}) \leqslant s(f)^2 \quad \text{and} \quad s\big(f-f^{\tau_k}\big),\ s\big(f^{\tau_{-k}}\big) \ \to\ 0 \ \text{ a.e. as } k \ \to\ \infty.$$

Consequently, by the dominated convergence theorem in variable $L_{p(\cdot)}(\Omega)$ (see [**3**, theorem 2.62]),

$$\left\| f - \sum_{k=-M}^{N} \mu_k a^k \right\|^{\underline{p}}_{\mathcal{H}^s_{p(\cdot)}} \leqslant \|f - f^{\tau_{N+1}}\|^{\underline{p}}_{\mathcal{H}^s_{p(\cdot)}} + \|f^{\tau_{-M}}\|^{\underline{p}}_{\mathcal{H}^s_{p(\cdot)}}$$

converges to 0 a.e. as $M,\ N \ \to\ \infty$.

Conversely, taking $f \in \mathcal{H}^{at,1,\infty}_{p(\cdot)},(\Omega)$ according to the definition of $\mathcal{H}^{at,1,\infty}_{p(\cdot)}(\Omega)$, we have the decomposition

$$f = \sum_{k\in\mathbb{Z}} \mu_k a^k \qquad a.e,$$

where $(a^k)_{k\in\mathbb{Z}}$ is a sequence of $(1, p(\cdot), \infty)$-atoms, associated with stopping time $\tau_k \in \Gamma$ and $a^k_n = \mathbb{E}_n(a^k)$. Since

$$\|s(a^k)\|_\infty \leqslant \frac{1}{\big\|\chi_{\{\tau_k<\infty\}}\big\|_{\mathcal{L}_{p(\cdot)}}} \quad \text{and} \quad s(a^k) = s(a^k)\chi_{\{\tau_k<\infty\}},$$

we obtain

$$\begin{aligned} \|f\|_{\mathcal{H}^s_{p(\cdot)}} = \|s(f)\|_{\mathcal{L}_{p(\cdot)}} &\leqslant \left\| \sum_{k\in\mathbb{Z}} \mu_k s(a^k) \right\|_{\mathcal{L}_{p(\cdot)}} \\ &\leqslant \left\| \sum_{k\in\mathbb{Z}} \mu_k \frac{\chi_{\{\tau_k<\infty\}}}{\|\chi_{\{\tau_k<\infty\}}\|_{\mathcal{L}_{p(\cdot)}}} \right\|_{\mathcal{L}_{p(\cdot)}} \\ &\leqslant \left\| \left[\sum_{k\in\mathbb{Z}} \left(\frac{\mu_k \chi_{\{\tau_k<\infty\}}}{\|\chi_{\{\tau_k<\infty\}}\|_{\mathcal{L}_{p(\cdot)}}} \right)^t \right]^{1/t} \right\|_{\mathcal{L}_{p(\cdot)}}, \end{aligned} \tag{3.3}$$

which implies that $\|f\|_{\mathcal{H}^s_{p(\cdot)}} \lesssim \|f\|_{\mathcal{H}^{at,1,\infty}_{p(\cdot)}}$. The proof is complete. □

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THEOREM 3.4. *Let $p(\cdot) \in \mathfrak{P}([0,1])$. Then*

$$\mathcal{Q}_{p(\cdot)}(\Omega) = \mathcal{H}^{at,2,\infty}_{p(\cdot)}(\Omega), \quad \mathcal{P}_{p(\cdot)}(\Omega) = \mathcal{H}^{at,3,\infty}_{p(\cdot)}(\Omega)$$

with equivalent quasi-norms.

Proof. The proof is similar to the one of theorem 3.3, so we only sketch the outline. We only prove the first atomic decomposition since the later one is the same. Let $f = (f_n)_{n\geqslant 0} \in \mathcal{Q}_{p(\cdot)}(\Omega)$. The stopping times τ_k are defined by

$$\tau_k = \inf\{n \in \mathbb{N} : \ \lambda_n > 2^k\},$$

where $(\lambda_n)_{n\geqslant 0}$ is the sequence in the definition of $\mathcal{Q}_{p(\cdot)}(\Omega)$. Let a^k_n and μ_k $(k \in \mathbb{Z})$ be the same as the proof of theorem 3.3. Then we get (3.1), where $(a^k)_{k\in\mathbb{Z}}$ is a sequence of $(2, p(\cdot), \infty)$-atoms. Moreover,

$$\left\| \left[\sum_{k\in\mathbb{Z}} \left(\frac{\mu_k \chi_{\{\tau_k<\infty\}}}{\|\chi_{\{\tau_k<\infty\}}\|_{\mathcal{L}_{p(\cdot)}}} \right)^t \right]^{1/t} \right\|_{\mathcal{L}_{p(\cdot)}} \lesssim \|f\|_{\mathcal{Q}_{p(\cdot)}}$$

still holds.

To prove the converse part, for any $f \in \mathcal{H}^{at,2,\infty}_{p(\cdot)}$, let

$$\lambda_n = \sum_{k\in\mathbb{Z}} \mu_k \chi_{\{\tau_k\leqslant n\}} \|S(a^k)\|_\infty.$$

Then $(\lambda_n)_{n\geqslant 0}$ is a non-decreasing, non-negative and adapted sequence such that $S_{n+1}(f) \leqslant \lambda_n$ for any $n \geqslant 0$. According to the estimate that $(a^k)_{k\in\mathbb{Z}}$ is a sequence of $(2, p(\cdot), \infty)$-atoms, we obtain $f \in \mathcal{Q}_{p(\cdot)}(\Omega)$ and

$$\|f\|_{\mathcal{Q}_{p(\cdot)}} \leqslant \inf \left\| \left[\sum_{k\in\mathbb{Z}} \left(\frac{\mu_k \chi_{\{\tau_k<\infty\}}}{\|\chi_{\{\tau_k<\infty\}}\|_{\mathcal{L}_{p(\cdot)}}} \right)^t \right]^{1/t} \right\|_{\mathcal{L}_{p(\cdot)}},$$

where the infimum is taken over all the decompositions as in (3.1). □

3.2. Atomic decompositions in $\mathcal{H}^M_{p(\cdot)}(\Omega)$

We are ready to prove the atomic decompositions for martingale Hardy spaces $\mathcal{H}^M_{p(\cdot)}(\Omega)$ and $\mathcal{H}^S_{p(\cdot)}(\Omega)$. The stochastic basis $(\mathcal{F}_n)_{n\geqslant 0}$ is said to be regular, if for $n \geqslant 0$ and $A \in \mathcal{F}_n$, there exists $B \in \mathcal{F}_{n-1}$ such that

$$A \subset B \text{ and } |B| \leqslant C|A|, \tag{3.4}$$

where C is a positive constant independent of n.

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We need the following proposition.

PROPOSITION 3.5. *Let $p(\cdot) \in \mathfrak{P}([0,1])$ satisfy $0 < p_- \leqslant p_+ \leqslant 1$. For any $n \in \mathbb{N}$, let $0 < \nu_i < \infty$ and $A_i \in \mathcal{F}$, $i = 1, \cdots, n$. If the measurable sets sequence $\{A_i\}_{1 \leqslant i \leqslant n}$ is non-increasing, then*

$$\sum_{i=1}^{n} \nu_i \|\chi_{A_i}\|_{\mathcal{L}_{p(\cdot)}} \leqslant \left\| \sum_{i=1}^{n} \nu_i \chi_{A_i} \right\|_{\mathcal{L}_{p(\cdot)}}. \tag{3.5}$$

Proof. By lemma 2.5, we obtain

$$\sum_{i=1}^{n} \nu_i \|\chi_{A_i}\|_{\mathcal{L}_{p(\cdot)}} = \sum_{i=1}^{n} \left\|\nu_i \chi_{[0,|A_i|)}\right\|_{L_{p(\cdot)}} \leqslant \left\| \sum_{i=1}^{n} \nu_i \chi_{[0,|A_i|)} \right\|_{L_{p(\cdot)}}.$$

Then, to finish the proof, it suffices to show that

$$\left(\sum_{i=1}^{n} \nu_i \chi_{A_i} \right)^* = \sum_{i=1}^{n} \nu_i \chi_{[0,|A_i|)}. \tag{3.6}$$

Since $A_n \subseteq A_{n-1} \subseteq \cdots \subseteq A_1$, we have (with convenience $A_{n+1} = \varnothing$)

$$\sum_{i=1}^{n} \nu_i \chi_{A_i} = \sum_{i=1}^{n} \left(\sum_{j=1}^{i} v_j \right) \chi_{A_i \setminus A_{i+1}}.$$

Then basic calculation gives us (see [11, example 1.4.2])

$$\left(\sum_{i=1}^{n} \left(\sum_{j=1}^{i} v_j \right) \chi_{A_i \setminus A_{i+1}} \right)^* = \sum_{i=1}^{n} \left(\sum_{j=1}^{i} v_j \right) \chi_{[|A_{i+1}|,|A_i|)} = \sum_{i=1}^{n} \nu_i \chi_{[0,|A_i|)}.$$

The proof is complete. □

THEOREM 3.6. *Let $p(\cdot) \in \mathfrak{P}([0,1])$ satisfy locally log-Hölder condition. If $\{\mathcal{F}_n\}_{n \geqslant 0}$ is regular, then*

$$\mathcal{H}^S_{p(\cdot)}(\Omega) = \mathcal{H}^{at,2,\infty}_{p(\cdot)}(\Omega), \quad \mathcal{H}^M_{p(\cdot)}(\Omega) = \mathcal{H}^{at,3,\infty}_{p(\cdot)}(\Omega)$$

with equivalent quasi-norms.

Proof. We only give the proof for the second equality since the other one is similar. Take $f \in \mathcal{H}^M_{p(\cdot)}(\Omega)$. Consider the following stopping times with respect to $(\mathcal{F}_n)_{n \geqslant 0}$,

$$\rho_k := \inf\{n \in \mathbb{N} : \ |f_n| > 2^k\}, \quad k \in \mathbb{Z}.$$

For fixed $k \in \mathbb{Z}$, according to the regularity of $(\mathcal{F}_n)_{\{n \geqslant 0\}}$, we can choose a small enough measurable set $F^k_j \in \mathcal{F}_{j-1}$ such that $\{\rho_k = j\} \subset F^k_j$ and

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$|F_j^k| \leqslant C|\{\rho_k = j\}|$. Define a new family of stopping times by

$$\tau_k(x) := \inf\{n \in \mathbb{N} :\ x \in F_{n+1}^k\}.$$

It is obvious that τ_k is non-decreasing. For any $j \in \mathbb{N}$, according to the definition of stopping times ρ_k and τ_k, we have

$$\{\tau_k = j-1\} \in \mathcal{F}_{j-1} \ \text{ and } \ \{\rho_k = j\} \subset \{\tau_k = j-1\} \subset F_j^k.$$

Denote $A_k = \{\tau_k < \infty\}$ and $A_k' = \{\rho_k < \infty\}$. By the regularity of $(\mathcal{F}_n)_{n\geqslant 0}$ and $\{\rho_k = j\} \in \mathcal{F}_j$, we obtain

$$|A_k| = \sum_{j=1}^{\infty} |\{\tau_k = j\}| \leqslant \sum_{j=1}^{\infty} |F_{j+1}^k| \leqslant C \sum_{j=1}^{\infty} |\{\rho_k = j+1\}|) = C|A_k'|,$$

where C is the constant as in (3.4), which implies that

$$|A_k'| \leqslant |A_k| \leqslant C|A_k'|. \tag{3.7}$$

Using lemma 2.18 and (3.7), we have

$$\begin{aligned}\|\chi_{A_k}\|_{\mathcal{L}_{p(\cdot)}} &\approx |A_k|^{(1/p_-[0,|A_k|])} \leqslant (C|A_k'|)^{(1/p_-[0,|A_k|])} \\ &\leqslant C^{\frac{1}{p_-}} (|A_k'|)^{(1/p_-[0,|A_k'|])} \approx \|\chi_{A_k'}\|_{\mathcal{L}_{p(\cdot)}},\end{aligned} \tag{3.8}$$

which deduces that

$$\|\chi_{A_k}\|_{\mathcal{L}_{p(\cdot)}} \lesssim \|\chi_{A_k'}\|_{\mathcal{L}_{p(\cdot)}} = \|\chi_{\{Mf>2^k\}}\|_{\mathcal{L}_{p(\cdot)}} \leqslant 2^{-k}\, \|Mf\|_{\mathcal{L}_{p(\cdot)}} \ \to \ 0$$

as $k \ \to \ \infty$, that is to say $\lim_{k\ \to\ \infty} |\tau_k = \infty| = 1$ and $\lim_{k\ \to\ \infty} \tau_k = \infty$ a.e. Thus

$$\lim_{k\ \to\ \infty} f_n^{\tau_k} = f_n \ \ a.e.\ (n \in \mathbb{N}) \ \text{ and } \ f_n = \sum_{k\in\mathbb{Z}} f_n^{\tau_{k+1}} - f_n^{\tau_k}.$$

We still define $\mu_k = 3 \cdot 2^k\, \|\chi_{A_k}\|_{\mathcal{L}_{p(\cdot)}}$ and $a_n^k = (f_n^{\tau_{k+1}} - f_n^{\tau_k}/\mu_k)$. By a similar argument as used in the proof of theorem 3.3, we can see that a^k is a $(3, p(\cdot), \infty)$-atom associated with stopping time τ_k.

Now we show

$$\|f\|_{\mathcal{H}_{p(\cdot)}^{at,3,\infty}} \lesssim \|f\|_{\mathcal{H}_{p(\cdot)}^{M}}.$$

According to (2.5), we have

$$Z := \left\| \sum_{k\in\mathbb{Z}} \left(\frac{\mu_k \chi_{A_k}}{\|\chi_{A_k}\|_{\mathcal{L}_{p(\cdot)}}} \right)^t \right\|_{\mathcal{L}_{p(\cdot)/t}}^{1/t} \leqslant \left\| \sum_{k\in\mathbb{Z}} \left(3 \cdot 2^k\right)^t \chi_{A_k}^{**} \right\|_{L_{p(\cdot)/t}}^{1/t}.$$

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Using lemma 2.2, we may choose a positive function $g \in L_{(p(\cdot)/t)'}$ with $\|g\|_{L_{(p(\cdot)/t)'}} \leqslant 1$ such that

$$Z^t \leqslant \int_0^1 \sum_{k\in\mathbb{Z}} \left(3\cdot 2^k\right)^t \chi_{A_k}^{**} g \mathrm{d}x = \sum_{k\in\mathbb{Z}} \left(3\cdot 2^k\right)^t \int_0^1 \frac{1}{x}\int_0^x \chi_{A_k}^{*}(y)\mathrm{d}y g(x)\mathrm{d}x$$
$$= \sum_{k\in\mathbb{Z}} \left(3\cdot 2^k\right)^t \int_0^1 \int_y^1 \frac{g(x)}{x}\mathrm{d}x \chi_{[0,|A_k|]}(y)\mathrm{d}y.$$

We denote $h(y) = \int_y^1 (g(x)/x)\mathrm{d}x,\ \ 0 \leqslant y \leqslant 1$. Since $(p(\cdot)/t)' > 1$, it follows from lemma 2.8 and (2.7) that

$$\|\mathcal{M}(h)\|_{L_{(p(\cdot)/t)'}} \lesssim \|h\|_{L_{(p(\cdot)/t)'}} \lesssim \|g\|_{L_{(p(\cdot)/t)'}} \leqslant 1. \tag{3.9}$$

Applying (3.7), Hölder inequality and (3.9), we find that

$$\begin{aligned}
Z^t &\leqslant \sum_{k\in\mathbb{Z}} \left(3\cdot 2^k\right)^t \int_0^1 h(y)\chi_{[0,|A_k|]}(y)\mathrm{d}y \\
&= \sum_{k\in\mathbb{Z}} \left(3\cdot 2^k\right)^t |A_k| \left(\frac{1}{|A_k|}\int_0^1 \chi_{[0,|A_k|]} h\mathrm{d}y\right) \\
&\overset{(3.7)}{\lesssim} \sum_{k\in\mathbb{Z}} \left(3\cdot 2^k\right)^t |A_k'| \left(\frac{1}{|A_k|}\int_0^1 \chi_{[0,|A_k|]} h\mathrm{d}y\right) \\
&\leqslant \sum_{k\in\mathbb{Z}} \left(3\cdot 2^k\right)^t \int_0^1 \chi_{[0,|A_k'|)} \mathcal{M}(h)\mathrm{d}y \\
&\leqslant \left\| \sum_{k\in\mathbb{Z}} \left(3\cdot 2^k\right)^t \chi_{[0,|A_k'|)} \right\|_{L_{p(\cdot)/t}} \|\mathcal{M}(h)\|_{L_{(p(\cdot)/t)'}} \\
&\lesssim \left\| \sum_{k\in\mathbb{Z}} \left(3\cdot 2^k\right)^t \chi_{[0,|A_k'|)} \right\|_{L_{p(\cdot)/t}}.
\end{aligned}$$

Note that $A_{k+1}' \subset A_k'$. By (3.6), we obtain

$$Z^t \lesssim \left\| \sum_{k\in\mathbb{Z}} \left(3\cdot 2^k\right)^t \chi_{A_k'} \right\|_{\mathcal{L}_{p(\cdot)/t}} = \left\| \sum_{k\in\mathbb{Z}} \left(3\cdot 2^k\right)^t \chi_{\{Mf>2^k\}} \right\|_{\mathcal{L}_{p(\cdot)/t}}$$

Taking the same argument as in (3.2), we can see that $\|f\|_{\mathcal{H}^{at,3,\infty}_{p(\cdot)}} \lesssim \|f\|_{\mathcal{H}^{M}_{p(\cdot)}}$. The converse inequality

$$\|f\|_{\mathcal{H}^{M}_{p(\cdot)}} \lesssim \|f\|_{\mathcal{H}^{at,3,\infty}_{p(\cdot)}}$$

can be similarly proved as (3.3). Theorem is thereby proved. □

Combining theorem 3.4 and theorem 3.6, we have the following corollary.

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COROLLARY 3.7. *Let $p(\cdot) \in \mathfrak{P}([0,1])$ satisfy locally log-Hölder condition. If $\{\mathcal{F}_n\}_{n\geqslant 0}$ is regular, then*

$$\mathcal{H}^S_{p(\cdot)}(\Omega) = \mathcal{Q}_{p(\cdot)}(\Omega), \quad \mathcal{H}^M_{p(\cdot)}(\Omega) = \mathcal{P}_{p(\cdot)}(\Omega)$$

with equivalent quasi-norms.

4. Duality

In this section, applying atomic decomposition, we now prove a duality theorem. First let us introduce the new Lipschitz spaces with variable exponents.

DEFINITION 4.1. Let $(\alpha(\cdot)+1)$ be a variable exponent and $1 < q < \infty$. Define

$$BMO_q(\alpha(\cdot))(\Omega) = \{f \in L_q(\Omega) : \|f\|_{BMO_q(\alpha(\cdot))} < \infty\},$$

where

$$\|f\|_{BMO_q(\alpha(\cdot))} = \sup_{\tau\in\Gamma} \|\chi_{\{\tau<\infty\}}\|^{-1}_{\mathcal{L}_{1/\alpha(\cdot)+1}} \|\chi_{\{\tau<\infty\}}\|_{q/q-1} \|f - f^\tau\|_q.$$

For $q = 1$, we define $BMO_1(\alpha(\cdot))(\Omega)$ with the norm

$$\|f\|_{BMO_q(\alpha(\cdot))} = \sup_{\tau\in\Gamma} \|\chi_{\{\tau<\infty\}}\|^{-1}_{\mathcal{L}_{1/\alpha(\cdot)+1}} \|f - f^\tau\|_1.$$

REMARK 4.2. If $\alpha(\cdot) = 0$, then this definition goes back to classical martingale BMO space. If $\alpha(\cdot) = \alpha_0 > 0$ is a constant, then this definition becomes the classical martingale Lipschitz space. We refer the reader to [**28**] for details.

The following corollary is a consequence of proposition 3.5.

COROLLARY 4.3. *Let $p(\cdot) \in \mathfrak{P}([0,1])$ satisfy $0 < p_- \leqslant p_+ \leqslant 1$. If $f \in \mathcal{H}^s_{p(\cdot)}(\Omega)$, then*

$$\sum_{k\in\mathbb{Z}} \mu_k \leqslant \left\| \left[\sum_{k\in\mathbb{Z}} \left(\frac{\mu_k \chi_{\{\tau_k<\infty\}}}{\|\chi_{\{\tau_k<\infty\}}\|_{\mathcal{L}_{p(\cdot)}}} \right)^t \right]^{1/t} \right\|_{\mathcal{L}_{p(\cdot)}}, \qquad 0 < t < \underline{p},$$

where μ_k and τ_k are derived from the decomposition $f = \sum_{k\in\mathbb{Z}} \mu_k a^k$ $(\mu_k = 3 \cdot 2^k \left\|\chi_{\{\tau_k<\infty\}}\right\|_{\mathcal{L}_{p(\cdot)}})$ in theorem 3.3.

Proof. It is easy to see $\{\tau_{k+1} < \infty\} = \{s(f) > 2^{k+1}\} \subset \{\tau_k < \infty\} = \{s(f) > 2^k\}$ and $0 < \mu_k < \infty$. According to the above proposition 3.5 , we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \mu_k &= \sum_{k\in\mathbb{Z}} 3 \cdot 2^k \left\|\chi_{\{\tau_k<\infty\}}\right\|_{\mathcal{L}_{p(\cdot)}} \\ &\leqslant \left\| \sum_{k\in\mathbb{Z}} 3 \cdot 2^k \chi_{\{\tau_k<\infty\}} \right\|_{\mathcal{L}_{p(\cdot)}} \leqslant \left\| \left[\sum_{k\in\mathbb{Z}} \left(\frac{\mu_k \chi_{\{\tau_k<\infty\}}}{\|\chi_{\{\tau_k<\infty\}}\|_{\mathcal{L}_{p(\cdot)}}} \right)^t \right]^{1/t} \right\|_{\mathcal{L}_{p(\cdot)}},\end{aligned}$$

where the last '$\leqslant$' is due to $0 < t \leqslant 1$. □

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THEOREM 4.4. *Let* $p(\cdot) \in \mathfrak{P}([0,1])$ *satisfy locally log-Hölder condition with* $0 < p_- \leqslant p_+ \leqslant 1$. *Then*

$$(\mathcal{H}^s_{p(\cdot)}(\Omega))^* = BMO_2(\alpha(\cdot))(\Omega), \quad \alpha(\cdot) = \frac{1}{p(\cdot)} - 1.$$

Proof. Let $\varphi \in BMO_2(\alpha(\cdot))(\Omega) \subset L_2(\Omega)$. Define

$$l_\varphi(f) = \mathbb{E}(f\varphi), \quad \forall f \in L_2(\Omega).$$

We claim that l_φ is a bounded linear functional on $\mathcal{H}^s_{p(\cdot)}(\Omega)$. Note that $L_2(\Omega) \subset \mathcal{H}^s_{p(\cdot)}(\Omega)$. It follows from theorem 3.3 that

$$f = \sum_{k\in\mathbb{Z}} \mu_k a^k, \qquad f \in L_2(\Omega),$$

and the convergence holds also in the L_2-norm, where a^k is an $(1, p(\cdot), \infty)$-atom and $\mu_k = 3 \cdot 2^k \left\|\chi_{\{\tau_k<\infty\}}\right\|_{\mathcal{L}_{p(\cdot)}}$. Hence

$$l_\varphi(f) = \mathbb{E}(f\varphi) = \sum_{k\in\mathbb{Z}} \mu_k \mathbb{E}(a^k\varphi).$$

By the definition of the atom a^k and the orthogonality of the martingale difference sequence,

$$\mathbb{E}(a^k\varphi) = \mathbb{E}(a^k(\varphi - \varphi^{\tau_k}))$$

always holds. Using Hölder's inequality, we conclude that

$$\begin{aligned}
|l_\varphi(f)| &\leqslant \sum_{k\in\mathbb{Z}} \mu_k \Big|\int_\Omega a^k(\varphi - \varphi^{\tau_k})\mathrm{d}p\Big| \leqslant \sum_{k\in\mathbb{Z}} \mu_k \left\|a^k\right\|_2 \left\|(\varphi - \varphi^{\tau_k})\chi_{\{\tau_k<\infty\}}\right\|_2 \\
&= \sum_{k\in\mathbb{Z}} \mu_k \left\|s(a^k)\chi_{\{\tau_k<\infty\}}\right\|_2 \left\|(\varphi - \varphi^{\tau_k})\chi_{\{\tau_k<\infty\}}\right\|_2 \\
&\leqslant \sum_{k\in\mathbb{Z}} \mu_k \left\|s(a^k)\|_\infty \|\chi_{\{\tau_k<\infty\}}\right\|_2 \left\|(\varphi - \varphi^{\tau_k})\chi_{\{\tau_k<\infty\}}\right\|_2 \\
&\leqslant \sum_{k\in\mathbb{Z}} \mu_k \frac{|\{\tau_k<\infty\}|^{1/2}}{\left\|\chi_{\{\tau_k<\infty\}}\right\|_{\mathcal{L}_{p(\cdot)}}} \left\|(\varphi - \varphi^{\tau_k})\chi_{\{\tau_k<\infty\}}\right\|_2 \\
&\lesssim \sum_{k\in\mathbb{Z}} \mu_k \left\|\varphi\right\|_{BMO_2(\alpha(\cdot))},
\end{aligned}$$

where the first '$=$' and the fourth '$\leqslant$' is due to $s(a^k) = s(a^k)\chi_{\{\tau_k<\infty\}}$ and (2) of the definition 3.1 respectively. Since $p_+ \leqslant 1$, we obtain from corollary 4.3 that

$$|l_\varphi(f)| \lesssim \|f\|_{\mathcal{H}^s_{p(\cdot)}} \|\varphi\|_{BMO_2(\alpha(\cdot))}.$$

We know that $L_2(\Omega)$ is dense in $\mathcal{H}^s_{p(\cdot)}(\Omega)$, see [**13**]. Consequently, l_φ can be uniquely extended to be a linear function on $\mathcal{H}^s_{p(\cdot)}(\Omega)$, and $\|l_\varphi\| \lesssim \|\varphi\|_{BMO_2(\alpha(\cdot))}$.

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On the other hand, let l be an arbitrary bounded linear function on $\mathcal{H}^s_{p(\cdot)}(\Omega)$. We shall show that there exists $\varphi \in BMO_2(\alpha(\cdot))(\Omega)$ such that $l = l_\varphi$ and

$$\|\varphi\|_{BMO_2(\alpha(\cdot))} \lesssim \|l\|.$$

Since $0 < p_- \leqslant p^+ \leqslant 1$ and lemma 2.17, we have

$$\|f\|_{\mathcal{H}^s_{p(\cdot)}} \lesssim \|s(f)\|_{p^+} \leqslant \|s(f)\|_2 = \|f\|_2, \quad \forall f \in L_2.$$

Then L_2 is embedded continuously in $\mathcal{H}^s_{p(\cdot)}(\Omega)$. Consequently, there exists $\varphi \in L_2$ such that

$$l(f) = \mathbb{E}(f\varphi), \qquad f \in L_2.$$

Take stopping time $\tau \in \Gamma$. We set

$$g = \frac{\varphi - \varphi^\tau}{\|\varphi - \varphi^\tau\|_2 \left\|\chi_{\{\tau<\infty\}}\right\|_{\mathcal{L}_{(1/\alpha(\cdot)+1)}} \left\|\chi_{\{\tau<\infty\}}\right\|_2^{-1}},$$

then g is not necessarily a $(1, p(\cdot), \infty)$-atom, but it satisfies $g = g\chi_{\{\tau<\infty\}}$. Assume that $r > 2$. Note that

$$\frac{1}{p(x)} = \frac{1}{(1/\alpha(x) + (1/r))} + \left(\frac{1}{r'} - \frac{1}{2}\right) + \frac{1}{2}, \qquad \forall x \in \Omega,$$

where $\frac{1}{r} + \frac{1}{r'} = 1$. Observe that

$$s(\varphi - \varphi^\tau) = s((\varphi - \varphi^\tau)\chi_{\{\tau<\infty\}}) = s(\varphi - \varphi^\tau)\chi_{\{\tau<\infty\}}.$$

Since $\mathcal{F} = \sigma(\cup_{n\geqslant 0}\mathcal{F}_n)$ and $\{\tau = n\} \in \mathcal{F}_n$, we know $\{\tau < \infty\} = \bigcup_{n=1}\{\tau = n\} \in \mathcal{F}$. By lemma 2.4, proposition 2.16 and proposition 2.19, we get

$$\begin{aligned}
&\|s(\varphi - \varphi^\tau)\|_{\mathcal{L}_{p(\cdot)}} \\
&\quad \lesssim \|s(\varphi - \varphi^\tau)\chi_{\{\tau<\infty\}}\|_2 \|\chi_{\{\tau<\infty\}}\|_{(1/(1/r')-(1/2)} \|\chi_{\{\tau<\infty\}}\|_{\mathcal{L}_{1/\alpha(\cdot)+(1/r)}} \\
&\quad = \|s(\varphi - \varphi^\tau)\chi_{\{\tau<\infty\}}\|_2 \|\chi_{\{\tau<\infty\}}\|_2^{-1} \|\chi_{\{\tau<\infty\}}\|_{r'} \|\chi_{\{\tau<\infty\}}\|_{\mathcal{L}_{1/\alpha(\cdot)+(1/r)}} \\
&\quad \approx \|s(\varphi - \varphi^\tau)\chi_{\{\tau<\infty\}}\|_2 \|\chi_{\{\tau<\infty\}}\|_2^{-1} \|\chi_{\{\tau<\infty\}}\|_{\mathcal{L}_{1/\alpha(\cdot)+1}}.
\end{aligned}$$

Hence, we obtain that

$$\begin{aligned}
\|g\|_{\mathcal{H}^s_{p(\cdot)}} &= \frac{\|s(\varphi - \varphi^\tau)\|_{\mathcal{L}_{p(\cdot)}}}{\left\|(\varphi - \varphi^\tau)\chi_{\{\tau<\infty\}}\right\|_2 \left\|\chi_{\{\tau<\infty\}}\right\|_{\mathcal{L}_{(1/\alpha(\cdot)+1)}} \left\|\chi_{\{\tau<\infty\}}\right\|_2^{-1}} \\
&\lesssim \frac{\left\|s(\varphi - \varphi^\tau)\chi_{\{\tau<\infty\}}\right\|_2 \left\|\chi_{\{\tau<\infty\}}\right\|_{(1/\alpha(\cdot)+1)} \left\|\chi_{\{\tau<\infty\}}\right\|_2^{-1}}{\left\|(\varphi - \varphi^\tau)\chi_{\{\tau<\infty\}}\right\|_2 \left\|\chi_{\{\tau<\infty\}}\right\|_{\mathcal{L}_{(1/\alpha(\cdot)+1)}} \left\|\chi_{\{\tau<\infty\}}\right\|_2^{-1}} \\
&= 1.
\end{aligned}$$

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Finally, we obtain

$$\begin{aligned}\|l\| &\gtrsim l(g)=\mathbb{E}(g(\varphi-\varphi^{\tau}))\\ &=\left\|(\varphi-\varphi^{\tau})\chi_{\{\tau<\infty\}}\right\|_2\left\|\chi_{\{\tau<\infty\}}\right\|^{-1}_{\mathcal{L}_{1/\alpha(\cdot)+1}}\left\|\chi_{\{\tau<\infty\}}\right\|_2.\end{aligned}$$

Then we have

$$\|\varphi\|_{BMO_2(\alpha(\cdot))}\lesssim\|l\|,$$

and the proof is complete. □

5. Martingale inequalities and its applications

In this section, we prove a σ-sublinear operator to be bounded from the martingale Hardy spaces to $\mathcal{L}_{p(\cdot)}(\Omega)$ by the tool of atomic decompositions. Applying this result, we deal with martingale inequalities between different Hardy spaces. Furthermore, we obtain an application of martingale inequalities in stochastic integral with Brownian motion.

5.1. Martingale inequalities

We firstly give the definition of σ-sublinear operator:

Definition 5.1. An operator $T: X\to Y$ is called a σ-sublinear operator, if for any $\alpha\in\mathbb{C}$

$$\left|T\left(\sum_{k=1}^{\infty}f_k\right)\right|\leqslant\sum_{k=1}^{\infty}|T(f_k)|\quad\text{and}\quad|T(\alpha f)|=|\alpha|T(f),$$

where f and f_k $(k\geqslant 1)$ belong to X, X is a martingale space and Y is a measurable function space.

The following result is proved by applying lemma 2.8 and lemma 2.11.

Theorem 5.2. *Given $1<r<\infty$, let $p(\cdot)\in\mathfrak{P}([0,1])$ satisfy locally log-Hölder condition with $p_+<r$. If $T: H_r^s(\Omega)\to L_r(\Omega)$ is a bounded σ-sublinear operator and*

$$\{|Ta|>0\}\subset\{\tau<\infty\}\tag{5.1}$$

for any $(1,p(\cdot),\infty)$-atoms a associated with stopping time τ, then

$$\|Tf\|_{\mathcal{L}_{p(\cdot)}}\lesssim\|f\|_{\mathcal{H}^s_{p(\cdot)}}.$$

Proof. Let a martingale $f\in\mathcal{H}^s_{p(\cdot)}(\Omega)$. By theorem 3.3, we know that there exists a sequence of triples $\{\mu_k,a^k,\tau_k\}$ such that $f=\sum_{k\in\mathbb{Z}}\mu_k a^k$ and

$$\left\|\left[\sum_{k\in\mathbb{Z}}\left(\frac{\mu_k\chi_{\{\tau_k<\infty\}}}{\|\chi_{\{\tau_k<\infty\}}\|_{\mathcal{L}_{p(\cdot)}}}\right)^t\right]^{1/t}\right\|_{\mathcal{L}_{p(.)}}\lesssim\|f\|_{\mathcal{H}^s_{p(\cdot)}},$$

where $0<t<\underline{p}$, a^k is a $(1,p(\cdot),\infty)$-atom associated with stopping time τ_k and $\mu_k=3\cdot 2^k\|\chi_{\{\tau_k<\infty\}}\|_{\mathcal{L}_{p(\cdot)}}$ for any $k\in\mathbb{Z}$. By the σ-sublinearity of the operator T,

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we have

$$\|Tf\|_{\mathcal{L}_{p(\cdot)}} \leqslant \left\|\sum_{k\in\mathbb{Z}} \mu_k |T(a^k)|\right\|_{\mathcal{L}_{p(\cdot)}} \leqslant \left\|\sum_{k\in\mathbb{Z}} \left(\mu_k |T(a^k)|\right)^t\right\|^{1/t}_{\mathcal{L}_{p(\cdot)/t}} =: Z.$$

By (2.5), we obtain

$$Z^t \lesssim \left\|\left[\sum_{k\in\mathbb{Z}} \left(\mu_k |T(a^k)|\right)^t\right]^{**}\right\|_{L_{p(\cdot)/t}} \leqslant \left\|\sum_{k\in\mathbb{Z}} \mu_k^t \left[T(a^k)^t\right]^{**}\right\|_{L_{p(\cdot)/t}}.$$

According to lemma 2.2, we may choose a positive function $g \in L_{(\frac{p(\cdot)}{t})'}([0,1])$ with $\|g\|_{L_{(\frac{p(\cdot)}{t})'}} \leqslant 1$ such that

$$\begin{aligned} Z^t &\lesssim \int_0^1 \sum_{k\in\mathbb{Z}} \mu_k^t \left[T(a^k)^t\right]^{**} g \mathrm{d}x \\ &= \sum_{k\in\mathbb{Z}} \mu_k^t \int_0^1 \frac{1}{x} \int_0^x \left[T(a^k)^t\right]^*(y) \mathrm{d}y g(x) \mathrm{d}x \\ &= \sum_{k\in\mathbb{Z}} \mu_k^t \int_0^1 \int_y^1 \frac{g(x)}{x} \mathrm{d}x \left[T(a^k)^t\right]^*(y) \mathrm{d}y. \end{aligned}$$

We denote $h(y) = \int_y^1 (g(x)/x)\mathrm{d}x, 0 \leqslant y \leqslant 1$. Since $((p(\cdot)/t))' > 1$, it follows from lemma 2.8 and (2.7) that

$$\|\mathcal{M}(h)\|_{L_{(p(\cdot)/t)'}} \lesssim \|h\|_{L_{(p(\cdot)/t)'}} \lesssim \|g\|_{L_{(p(\cdot)/t)'}} \leqslant 1. \tag{5.2}$$

Denote $A_k = \{\tau_k < \infty\}$. From (5.1) and Hölder inequality, we obtain

$$\begin{aligned} Z^t &\lesssim \sum_{k\in\mathbb{Z}} \mu_k^t \int_0^1 h(y) \left[T(a^k)^t\right]^*(y) \mathrm{d}y \leqslant \sum_{k\in\mathbb{Z}} \mu_k^t \int_0^1 \left[T(a^k)^t\right]^* \chi_{[0,|A_k|]} h \mathrm{d}y \\ &\lesssim \sum_{k\in\mathbb{Z}} \mu_k^t \|\left[T(a^k)^t\right]^*\|_{\frac{r}{t}} \|\chi_{[0,|A_k|]} h\|_{(\frac{r}{t})'}. \end{aligned}$$

According to the boundedness of T and the definition of $(1, p(\cdot), \infty)$-atom a^k, we have

$$\left\|T(a^k)\right\|_r \lesssim \left\|s(a^k)\right\|_r \leqslant \frac{\|\chi_{A_k}\|_r}{\|\chi_{A_k}\|_{\mathcal{L}_{p(\cdot)}}}.$$

It follows from $p_+ < r$ that

$$\frac{1}{(r/t)'} \cdot \left(\frac{p(\cdot)}{t}\right)' > 1.$$

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Hence, by (5.2), we get

$$
\begin{aligned}
Z^t &\lesssim \sum_{k\in\mathbb{Z}} \mu_k^t \left(\frac{\|\chi_{A_k}\|_r}{\|\chi_{A_k}\|_{\mathcal{L}_{p(\cdot)}}} \right)^t \left\| \chi_{[0,|A_k|]} h \right\|_{(r/t)'} \\
&= \sum_{k\in\mathbb{Z}} 3^t 2^{tk} \left\| \chi_{[0,|A_k|)} \right\|_r^t \left(\int_0^{|A_k|} h^{(r/t)'} \mathrm{d}x \right)^{1/(r/t)'} \\
&= \sum_{k\in\mathbb{Z}} 3^t 2^{tk} \int_0^1 \chi_{[0,|A_k|)} \left(\frac{1}{|A_k|} \int_0^{|A_k|} h^{(r/t)'} \mathrm{d}x \right)^{1/(r/t)'} \mathrm{d}y \\
&\leqslant \sum_{k\in\mathbb{Z}} 3^t 2^{tk} \int_0^1 \chi_{[0,|A_k|)} \left(\mathcal{M}(h^{(r/t)'}) \right)^{1/(r/t)'} \mathrm{d}y \\
&\lesssim \left\| \sum_{k\in\mathbb{Z}} 3^t 2^{tk} \chi_{[0,|A_k|)} \right\|_{L_{p(\cdot)/t}} \left\| \left(\mathcal{M}(h^{(r/t)'}) \right)^{1/(r/t)'} \right\|_{L_{(p(\cdot)/t)'}} \\
&\lesssim \left\| \sum_{k\in\mathbb{Z}} 3^t 2^{tk} \chi_{[0,|A_k|)} \right\|_{L_{p(\cdot)/t}} .
\end{aligned}
$$

Then, by (3.6) and theorem 3.3, we have

$$
Z^t \lesssim \left\| \sum_{k\in\mathbb{Z}} \left(\mu_k \frac{\chi_{A_k}}{\|\chi_{A_k}\|_{\mathcal{L}_{p(\cdot)}}} \right)^t \right\|_{\mathcal{L}_{p(\cdot)/t}} \lesssim \|f\|_{\mathcal{H}^s_{p(\cdot)}}^t ,
$$

which implies that

$$
\|Tf\|_{\mathcal{L}_{p(\cdot)}} \lesssim \|f\|_{\mathcal{H}^s_{p(\cdot)}} .
$$

The proof is complete. □

Similarly to theorem 5.2, we obtain the following theorem by applying theorem 3.4.

THEOREM 5.3. *Given* $1 < r < \infty$*, let* $p(\cdot) \in \mathfrak{P}([0,1])$ *satisfy locally log-Hölder condition with* $0 < p_+ < r$*. If* $T : H_r^S(\Omega) \to L_r(\Omega)$ *(or* $T : H_r^M(\Omega) \to L_r(\Omega)$*) is a bounded* σ*-sublinear operator and (5.1) holds for any* $(2, p(\cdot), \infty)$*-atoms (or* $(3, p(\cdot), \infty)$*-atoms), then*

$$
\begin{aligned}
\|Tf\|_{\mathcal{L}_{p(\cdot)}} &\lesssim \|f\|_{\mathcal{Q}_{p(\cdot)}}, \quad f \in \mathcal{Q}_{p(\cdot)}(\Omega), \\
(\textit{or} \ \ \|Tf\|_{\mathcal{L}_{p(\cdot)}} &\lesssim \|f\|_{\mathcal{P}_{p(\cdot)}}, \quad f \in \mathcal{P}_{p(\cdot)}(\Omega)).
\end{aligned}
$$

Now we prove our main result of this section.

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THEOREM 5.4. *Let $p(\cdot) \in \mathfrak{P}([0,1])$ satisfy locally log-Hölder condition. Then the following inequalities hold:*

$$\|f\|_{\mathcal{H}^M_{p(\cdot)}} \lesssim \|f\|_{\mathcal{H}^s_{p(\cdot)}}, \quad \|f\|_{\mathcal{H}^S_{p(\cdot)}} \lesssim \|f\|_{\mathcal{H}^s_{p(\cdot)}}, \quad \text{if } \ 0 < p_- \leqslant p_+ < 2; \tag{5.3}$$

$$\|f\|_{\mathcal{H}^M_{p(\cdot)}} \lesssim \|f\|_{\mathcal{P}_{p(\cdot)}}, \quad \|f\|_{\mathcal{H}^S_{p(\cdot)}} \lesssim \|f\|_{\mathcal{Q}_{p(\cdot)}}; \tag{5.4}$$

$$\|f\|_{\mathcal{H}^S_{p(\cdot)}} \lesssim \|f\|_{\mathcal{P}_{p(\cdot)}}, \quad \|f\|_{\mathcal{H}^M_{p(\cdot)}} \lesssim \|f\|_{\mathcal{Q}_{p(\cdot)}}; \tag{5.5}$$

$$\|f\|_{\mathcal{H}^s_{p(\cdot)}} \lesssim \|f\|_{\mathcal{P}_{p(\cdot)}}, \quad \|f\|_{\mathcal{H}^s_{p(\cdot)}} \lesssim \|f\|_{\mathcal{Q}_{p(\cdot)}}; \tag{5.6}$$

$$\|f\|_{\mathcal{P}_{p(\cdot)}} \lesssim \|f\|_{\mathcal{Q}_{p(\cdot)}} \lesssim \|f\|_{\mathcal{P}_{p(\cdot)}}. \tag{5.7}$$

Moreover, if $\{\mathcal{F}_n\}_{n\geqslant 0}$ is regular, then

$$\mathcal{H}^S_{p(\cdot)}(\Omega) = \mathcal{Q}_{p(\cdot)}(\Omega) = \mathcal{P}_{p(\cdot)}(\Omega) = \mathcal{H}^M_{p(\cdot)}(\Omega) = \mathcal{H}^s_{p(\cdot)}(\Omega) \tag{5.8}$$

with equivalent quasi-norms.

Proof. It is clear that the operators M, S and s satisfy (5.1).

Since the maximal operator $T(f) = M(f)$ is sublinear and $\|Mf\|_2 \lesssim C\|sf\|_2$ (see [**28**, theorem 2.11]), it follows from theorem 5.2 that

$$\|f\|_{\mathcal{H}^M_{p(\cdot)}} = \|Mf\|_{\mathcal{L}_{p(\cdot)}} \lesssim \|f\|_{\mathcal{H}^s_{p(\cdot)}},$$

which is just the first inequality of (5.3). The same argument can be applied to prove the second inequality of (5.3).

(5.4) comes easily from the definition of these martingale spaces.

The inequalities (5.5) follow from the combination of the Burkholder-Gundy and the Doob maximal inequality

$$\|S(f)\|_r \approx \|M(f)\|_r \approx \|f\|_r \quad (1 < r < \infty),$$

(see [**28**, theorem 2.11]) and theorem 5.3.

Applying the inequalities ([**28**, theorem 2.11(ii)])

$$\|s(f)\|_r \lesssim \|M(f)\|_r \approx \|S(f)\|_r, \quad 2 < r < \infty,$$

and theorem 5.3, we get (5.6).

To prove (5.7), we use (5.5). Take $f = (f_n)_{n\geqslant 0} \in \mathcal{Q}_{p(\cdot)}(\Omega)$. Then there exists an optimal control $(\lambda^1_n)_{n\geqslant 0}$ such that $S_n(f) \leqslant \lambda^1_{n-1}$ with $\lambda^1_\infty \in \mathcal{L}_{p(\cdot)}(\Omega)$. Since

$$|f_n| \leqslant M_{n-1}(f) + \lambda^1_{n-1},$$

it follows from the second inequality of (5.5) and lemma 2.14 that

$$\|f\|_{\mathcal{P}_{p(\cdot)}} \leqslant C(\|f\|_{\mathcal{H}^M_{p(\cdot)}} + \|\lambda^1_\infty\|_{\mathcal{L}_{p(\cdot)}}) \lesssim \|f\|_{\mathcal{Q}_{p(\cdot)}}.$$

On the other hand, if $f = (f_n)_{n\geqslant 0} \in \mathcal{P}_{p(\cdot)}(\Omega)$, then there exists an optimal control $(\lambda^2_n)_{n\geqslant 0}$ such that $|f_n| \leqslant \lambda^2_{n-1}$ with $\lambda^2_\infty \in \mathcal{L}_{p(\cdot)}(\Omega)$. Notice that

$$S_n(f) \leqslant S_{n-1}(f) + 2\lambda^2_{n-1}.$$

Using the first inequality of (5.5), we get the rest of (5.7).

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Further, assume that $\{\mathcal{F}_n\}_{n\geqslant 0}$ is regular. Then, according to [28, p.33], we have

$$S_n(f) \leqslant R^{1/2} s_n(f) \quad \text{and} \quad \|f\|_{\mathcal{H}^S_{p(\cdot)}} \lesssim \|f\|_{\mathcal{H}^s_{p(\cdot)}}.$$

Since $s_n(f)$ is $\mathcal{F}_{n-1}$-measurable, by the definition of $\mathcal{Q}_{p(\cdot)}(\Omega)$, we have

$$\|f\|_{\mathcal{Q}_{p(\cdot)}} \lesssim \|s(f)\|_{\mathcal{L}_{p(\cdot)}} = \|f\|_{\mathcal{H}^s_{p(\cdot)}}.$$

Hence, by (5.6) we obtain

$$\mathcal{Q}_{p(\cdot)} = \mathcal{H}^s_{p(\cdot)}.$$

Combining this and corollary 3.7, we get

$$\mathcal{H}^S_{p(\cdot)} = \mathcal{Q}_{p(\cdot)} = \mathcal{P}_{p(\cdot)} = \mathcal{H}^M_{p(\cdot)} = \mathcal{H}^s_{p(\cdot)}.$$ □

5.2. Stochastic integral with Brownian motion

This subsection is an application of martingale inequalities established in last subsection. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$. Let $B = \{B_t\}_{t\geqslant 0}$ be a one-dimensional Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ adapted to the filtration.

Definition 5.5. Let $p(\cdot) \in \mathfrak{P}([0,1])$ and $0 < T < \infty$. Denote by $\mathcal{L}^T_{p(\cdot)}(\Omega)$ the space of all real-valued measurable $\{\mathcal{F}_t\}$-adapted processes $f = \{f_t\}_{0\leqslant t\leqslant T}$ such that

$$\|f\|_{\mathcal{L}^T_{p(\cdot)}} = \left\| \left(\int_0^T |f_t|^2 \mathrm{d}t \right)^{1/2} \right\|_{\mathcal{L}_{p(\cdot)}(\Omega)} < \infty.$$

The functional $\|\cdot\|_{\mathcal{L}^T_{p(\cdot)}}$ defines a quasi-norm on $\mathcal{L}^T_{p(\cdot)}(\Omega)$, and the space is complete under this quasi-norm.

Let us introduce the concept of simple processes.

Definition 5.6. A real-valued stochastic process $g = \{g_t\}_{0\leqslant t\leqslant T}$ is called a simple process if there exists a partition $0 = t_0 < t_1 < \cdots < t_n = T$ of $[0,T]$, and bounded random variables ξ_i, $0 \leqslant i \leqslant n-1$ such that ξ_i is $\mathcal{F}_{t_i}$-measurable and

$$g_t = \xi_0 \chi_{[t_0,t_1]}(t) + \sum_{i=1}^{n-1} \xi_i \chi_{(t_i,t_{i+1}]}(t). \tag{5.9}$$

Denote by $\mathcal{L}^T_0(\Omega)$ the family of all simple processes. Obviously, we have the inclusion $\mathcal{L}^T_0(\Omega) \subset \mathcal{L}^T_{p(\cdot)}(\Omega)$. Next definition is the Itô integral for simple processes.

Definition 5.7. Let $B = \{B_t\}_{t\geqslant 0}$ be a one-dimensional Brownian motion. For a simple process $g \in \mathcal{L}^T_0(\Omega)$, define

$$\int_0^T g_t \mathrm{d}B_t = \sum_{i=0}^{n-1} \xi_i (B_{t_{i+1}} - B_{t_i})$$

and call it the stochastic integral of g with respect to Brownian motion B_t or the Itô integral.

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Clearly, the stochastic integral $\int_0^T g_t dB_t$ is $\mathcal{F}_T$-measurable. We shall now show that it belongs to $\mathcal{L}_{p(\cdot)}(\Omega)$. To this end, we need the following lemma.

LEMMA 5.8. *Let $p(\cdot) \in \mathfrak{P}([0,1])$ and $f = \{f_t\}_{0 \leqslant t \leqslant T} \in \mathcal{L}^T_{p(\cdot)}(\Omega)$. Then there exists a sequence process $\{g^n\} \subseteq \mathcal{L}^T_0(\Omega)$ such that*

$$f = \lim_{n\to\infty} g^n \quad in \quad \mathcal{L}^T_{p(\cdot)}.$$

Namely,

$$\lim_{n\to\infty} \|f - g^n\|_{\mathcal{L}^T_{p(\cdot)}} = 0. \tag{5.10}$$

Proof. This proof is similar to the one of [**24**, lemma 5.6], so we do not give the detailed proof. □

PROPOSITION 5.9. *Let $p(\cdot) \in \mathfrak{P}([0,1])$ satisfy locally log-Hölder condition and $B = \{B_t\}_{t\geqslant 0}$ be a one-dimensional Brownian motion. For any $g \in \mathcal{L}^T_0(\Omega)$, if $0 < p_- \leqslant p_+ < 2$, then there exists a constant $C_{p(\cdot)}$ dependent only on variable exponent $p(\cdot)$ such that*

$$\left\| \int_0^T g_t\, dB_t \right\|_{\mathcal{L}_{p(\cdot)}} \leqslant C_{p(\cdot)} \left\| \left(\int_0^T |g_t|^2\, dt \right)^{1/2} \right\|_{\mathcal{L}_{p(\cdot)}}.$$

Proof. For $g \in \mathcal{L}^T_0(\Omega)$ with the form of (5.9), we define

$$\eta_i := \int_0^{t_i} g_t \mathrm{d}B_t = \sum_{j=0}^{i-1} g_{t_j}(B_{t_{j+1}} - B_{t_j}). \tag{5.11}$$

It is easy to check that η_i is $\mathcal{F}_{t_i}$-measurable $(0 \leqslant i \leqslant n)$ and

$$\begin{aligned}
\mathbb{E}(\eta_{i+1}|\mathcal{F}_{t_i}) &= \mathbb{E}\left(\sum_{j=0}^{i} g_{t_j}(B_{t_{j+1}} - B_{t_j})|\mathcal{F}_{t_i} \right) \\
&= \mathbb{E}\left(\sum_{j=0}^{i-1} g_{t_j}(B_{t_{j+1}} - B_{t_j})|\mathcal{F}_{t_i} \right) + \mathbb{E}\left(g_{t_i}(B_{t_{i+1}} - B_{t_i})|\mathcal{F}_{t_i} \right) \\
&= \sum_{j=0}^{i-1} g_{t_j}(B_{t_{j+1}} - B_{t_j}) + g_{t_i} \cdot 0 \\
&= \eta_i,
\end{aligned}$$

where the third '=' is because $g_{t_j}(B_{t_{j+1}} - B_{t_j})$ is $\mathcal{F}_{t_i}$-measurable when $j < i$ and the increment $B_{t_{i+1}} - B_{t_i}$ is independent of $\mathcal{F}_{t_i}$. Thus $\eta = (\eta_i)_{0\leqslant i\leqslant n}$ is a martingale.

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By (5.11), the conditional square function of η can be written as follows

$$s(\eta)=\left(\sum_{j=0}^{n-1}\mathbb{E}\left(\left|\eta_{j+1}-\eta_j\right|^2\middle|\mathcal{F}_{t_j}\right)\right)^{1/2}=\left(\sum_{j=0}^{n-1}\mathbb{E}\left(\left|g_{t_j}\right|^2\left(B_{t_{j+1}-B_{t_j}}\right)^2\middle|\mathcal{F}_{t_j}\right)\right)^{1/2}$$

$$=\left(\sum_{j=0}^{n-1}\left|g_{t_j}\right|^2\mathbb{E}\left(\left(B_{t_{j+1}-B_{t_j}}\right)^2\middle|\mathcal{F}_{t_j}\right)\right)^{1/2}=\left(\sum_{j=0}^{n-1}\left|g_{t_j}\right|^2(t_{j+1}-t_j)\right)^{1/2}.$$

On the other hand, we get

$$\left(\int_0^T|g_t|^2\mathrm{d}t\right)^{1/2}=\left(\sum_{j=0}^{n-1}\left|g_{t_j}\right|^2(t_{j+1}-t_j)\right)^{1/2}=s(\eta).$$

By theorem 5.4, we deduce

$$\left\|\int_0^T g_t\mathrm{d}B_t\right\|_{\mathcal{L}_{p(\cdot)}}=\|\eta_n\|_{\mathcal{L}_{p(\cdot)}}\leqslant\|M(\eta)\|_{\mathcal{L}_{p(\cdot)}}$$

$$\leqslant C_{p(\cdot)}\|s(\eta)\|_{\mathcal{L}_{p(\cdot)}}=C_{p(\cdot)}\left\|\left(\int_0^T|g_t|^2\mathrm{d}t\right)^{1/2}\right\|_{\mathcal{L}_{p(\cdot)}}.$$

The proof is complete. □

This proposition implies that for all $g\in\mathcal{L}^T_{p(\cdot)}(\Omega)$, when $p(\cdot)$ satisfies certain conditions, $\int_0^T g_t dB_t$ belongs to $\mathcal{L}_{p(\cdot)}(\Omega)$. Next, we will show the main theorem of this subsection that for any $f\in\mathcal{L}^T_{p(\cdot)}(\Omega)$, there exists a sequence $\{g^n\}$ of simple processes such that $\lim_{n\to\infty}\int_0^T g^n_t dB_t$ exists in $\mathcal{L}_{p(\cdot)}(\Omega)$.

Theorem 5.10. *Let $p(\cdot)\in\mathfrak{P}([0,1])$ satisfy locally log-Hölder condition and $B=\{B_t\}_{t\geqslant 0}$ be a one-dimensional Brownian motion. Let $0<p_-\leqslant p_+<2$. For any $f=\{f_t\}_{0\leqslant t\leqslant T}\in\mathcal{L}^T_{p(\cdot)}(\Omega)$, if $\{g^n\}_{n\geqslant 0}\subseteq\mathcal{L}^T_0(\Omega)$ satisfies (5.10), then*

$$X_T(\omega):=\lim_{n\to\infty}\int_0^T g^n_t(\omega)\,dB_t\quad \textit{exists in}\quad \mathcal{L}_{p(\cdot)}(\Omega).$$

Proof. We set

$$X^n_T(\omega)=\int_0^T g^n_t(\omega)\mathrm{d}B_t,\quad\forall\omega\in\Omega.$$

It follows from lemma 5.8 and proposition 5.9 that

$$\|X^n_T-X^m_T\|_{\mathcal{L}_{p(\cdot)}}=\left\|\int_0^T g^n_t-g^m_t\mathrm{d}B_t\right\|_{\mathcal{L}_{p(\cdot)}}$$

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$$\leqslant C_{p(\cdot)} \left\| \left(\int_0^T \left| g_t^n - g_t^m \right|^2 \mathrm{d}t \right)^{1/2} \right\|_{\mathcal{L}_{p(\cdot)}}$$

$$\longrightarrow 0, \qquad \text{as } n, m \to \infty,$$

which shows that $\{X_T^n\}_{n\geqslant 0}$ is a Cauchy sequence in $\mathcal{L}_{p(\cdot)}(\Omega)$. Thus there exists $X_T \in \mathcal{L}_{p(\cdot)}(\Omega)$ such that

$$X_T = \lim_{n\to\infty} X_T^n \quad \text{in} \quad \mathcal{L}_{p(\cdot)}^T.$$

The proof is complete. □

This theorem leads to the following definition.

Definition 5.11. Let $p(\cdot) \in \mathfrak{P}([0,1])$ satisfy locally log-Hölder condition and $B = \{B_t\}_{t\geqslant 0}$ be a one-dimensional Brownian motion. Let $0 < p_- \leqslant p_+ < 2$. For $f = \{f_t\}_{0\leqslant t\leqslant T} \in \mathcal{L}_{p(\cdot)}^T(\Omega)$, the Itô integral of f with respect to $\{B_t\}$ is defined by

$$\int_0^T f_t \mathrm{d}B_t = \lim_{n\to\infty} \int_0^T g_t^n \mathrm{d}B_t \quad \text{in} \quad \mathcal{L}_{p(\cdot)}(\Omega),$$

where $\{g^n\}$ is a sequence of simple processes satisfying (5.10).

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