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Hyperfiniteness of boundary actions of cubulated hyperbolic groups

JINGYIN HUANG†, MARCIN SABOK†‡ and FORTE SHINKO†

† *McGill University, Department of Mathematics and Statistics, 805 Sherbrooke Street W, Montreal, QC, H3A 0B9 Canada (e-mail: jingyin.huang@mail.mcgill.ca, marcin.sabok@mcgill.ca, forte.shinko@mail.mcgill.ca)*

‡ *Instytut Matematyczny PAN, Śniadeckich 8, 00-656 Warszawa, Poland*

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Abstract. We show that if a hyperbolic group acts geometrically on a CAT(0) cube complex, then the induced boundary action is hyperfinite. This means that for a cubulated hyperbolic group, the natural action on its Gromov boundary is hyperfinite, which generalizes an old result of Dougherty, Jackson and Kechris for the free group case.

Key words: group actions, topological dynamics, hyperbolic groups, hyperfinite equivalence relations, Gromov boundary
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1. *Introduction*

The complexity theory for countable Borel equivalence relations has been an active topic of study over the last few decades. By the classical result of Feldman and Moore [**FM77**], countable Borel equivalence relations correspond to Borel actions of countable groups, and there has been a lot of effort to understand how the structure of the actions of a group depends on the group itself.

Recall that if Z is a standard Borel space, then a *Borel equivalence relation* on Z is an equivalence relation $E \subseteq Z^2$ which is Borel in Z^2. If E and F are Borel equivalence relations on Z and Y respectively, we say that E is *Borel-reducible* to F (denoted $E \leq_B F$) if there is a Borel function $f : Z \to Y$ such that $z_1 \; E \; z_2$ if and only if $f(z_1) \; F \; f(z_2)$ for all $z_1, z_2 \in Z$ (f is then called a *reduction* from E to F). A *smooth* equivalence relation is a Borel equivalence relation which is reducible to $\mathrm{id}_{2^{\mathbb{N}}}$, the equality relation on the Cantor set. The relation E_0 is defined on the Cantor set $2^{\mathbb{N}}$ as follows: $x \; E_0 \; y$ if there exists n such that $x(m) = y(m)$ for all $m > n$. A *finite* (respectively *countable*) equivalence relation is an equivalence relation whose classes are finite (respectively countable). An equivalence relation E on X is *hyperfinite* (respectively *hypersmooth*) if there is a sequence F_n of finite (respectively smooth) equivalence relations on X such that $F_n \subseteq F_{n+1}$ and $E = \bigcup_n F_n$.

Note that if $E \leq_B F$ and F is hypersmooth, then E is also hypersmooth. The relation E_1 is defined on $(2^{\mathbb{N}})^{\mathbb{N}}$ similarly to the definition of E_0 on $2^{\mathbb{N}}$: $x\ E_1\ y$ if there exists n such that $x(m) = y(m)$ for all $m > n$. Note that a Borel equivalence relation is hypersmooth if and only if it is Borel-reducible to E_1.

Among countable equivalence relations, hyperfinite equivalence relations are exactly those which are Borel-reducible to E_0 [**DJK94**]. The classical dichotomy of Harrington, Kechris and Louveau [**HKL90**] implies that if a countable Borel equivalence relation is not smooth, then E_0 is Borel-reducible to it. Interestingly, a very recent result of Conley and Miller [**CM17**] implies that among countable Borel equivalence relations which are not hyperfinite there is no countable basis with respect to Borel-reducibility.

Hyperfinite equivalence relations have a particular structure, observed by Slaman and Steel and independently by Weiss (see [**Gao09**, Theorem 7.2.4]). An equivalence relation E on Z is hyperfinite if and only if there exists a Borel action of the group of integers $\mathbb{Z}$ on Z which induces E as its orbit equivalence relation. In recent years, there has been a lot of effort to understand which groups induce hyperfinite equivalence relations. For instance, Gao and Jackson [**GJ15**] showed that Borel actions of all abelian groups induce hyperfinite equivalence relations. It is still unknown if all Borel actions of amenable groups induce hyperfinite equivalence relations.

In this paper, we are mainly interested in actions of hyperbolic groups. Recall that a geodesic metric space X is *hyperbolic* if there exists $\delta > 0$ such that all geodesic triangles in X are *δ-thin*, i.e. each of their sides is contained in the δ-neighbourhood of the union of the other two sides. In such case we also say that X is *δ-hyperbolic*. A finitely generated group G is *hyperbolic* if its Cayley graph (with respect to an arbitrary finite generating set) is hyperbolic. An isometric action of a group G on a metric space X is *proper* if for every compact subset $K \subseteq X$, the set $\{g \in G : gK \cap K \neq \emptyset\}$ is finite. An isometric action of G on X is *cocompact* if there exists a compact subset A of X such that $GA = X$. If X is a combinatorial complex, then an isometric action of a group on X is proper if and only if the stabilizers of all vertices are finite. An action of a group is *geometric* if it is both proper and cocompact. If a group G acts geometrically on a geodesic metric space X by isometries, then G is hyperbolic if and only if X is hyperbolic, since hyperbolicity is invariant under quasi-isometries.

Given a geodesic hyperbolic space X, we denote by ∂X its *Gromov boundary* (for definition, see e.g. [**KB02**]). Any geometric action of a hyperbolic group G on a hyperbolic space X induces a natural action of G on ∂X by homeomorphisms. If X is the Cayley graph of a hyperbolic group G, then the Gromov boudary of X is called the *Gromov boundary of the group G*.

Hyperbolic groups often admit geometric actions on CAT(0) cube complexes. Recall that a *cube complex* is obtained by taking a disjoint collection of unit cubes in Euclidean spaces of various dimensions and gluing them isometrically along their faces. A geodesic metric space X is CAT(0) if for every geodesic triangle Δ in X and a comparison triangle Δ' in the Euclidean plane, with sides of the same length as the sides of Δ, the distances between points on Δ are less than or equal to the distances between the corresponding points on Δ'. This is one way of saying that a metric space has non-positive curvature. For more details on CAT(0) cube complexes, see §2.

If a hyperbolic group admits a geometric action on a CAT(0) cube complex, then we say that it is *cubulated*. Examples of cubulated hyperbolic groups include:

- fundamental groups of hyperbolic surfaces and hyperbolic closed 3-manifolds (Kahn and Markovic [**KM12**] and Bergeron and Wise [**BW12**]);
- uniform hyperbolic lattices of 'simple type' (Haglund and Wise [**HW12**]);
- hyperbolic Coxeter groups (Niblo and Reeves [**NR97**] and Caprace and Mühlherr [**CM05**]);
- $C'(1/6)$ or $C'(1/4)$-$T(4)$ metric small cancellation groups (Wise [**Wis04**]);
- certain cubical small cancellation groups (Wise [**Wis17**]);
- Gromov's random groups with density $<1/6$ (Ollivier and Wise [**OW11**]); and
- hyperbolic free-by-cyclic groups (Hagen and Wise [**HW16**] in the irreducible case and [**HW15**] in the general case).

It is worth noting that cubulations of hyperbolic groups played an important role in recent breakthroughs on the virtual Haken conjecture by Agol [**Ago13**] and Wise [**Wis17**]. The main result of this paper is the following.

THEOREM 1.1. *If a hyperbolic group G acts geometrically on a* CAT(0) *cube complex X, then the induced action on ∂X is hyperfinite.*

Note that if G acts geometrically on X and Y, then there is a G-equivariant homeomorphism of ∂X and ∂Y [**Gro87**]. Hence, the above theorem implies the following.

COROLLARY 1.2. *If G is a hyperbolic cubulated group, then its natural boundary action on ∂G is hyperfinite.*

The boundary actions of hyperbolic groups have been studied from the perspective of their complexity. Recall that if μ is a probability measure on a standard Borel space X, and E is a countable Borel equivalence relation on X, then E is said to be *μ-hyperfinite* if there exists a μ-conull set $A \subseteq X$ such that $E \cap A^2$ is hyperfinite. A Borel probability measure μ is *E-quasi-invariant* if it is quasi-invariant with respect to any group action inducing E. Kechris and Miller [**KM04**, Corollary 10.2] showed that if E is μ-hyperfinite for all E-quasi-invariant Borel measures, then E is μ-hyperfinite for all Borel measures μ. It is worth noting, however, that for boundary actions of hyperbolic groups there is usually no unique quasi-invariant measure on the boundary.

In the case of the free group, its boundary action induces the equivalence relation which is Borel bi-reducible with the so-called *tail equivalence relation* on the Cantor set: $x\ E_t\ y$ if there exists n, m for all k such that $x(n+k) = y(m+k)$. It follows from the results of Connes, Feldman and Weiss [**CFW81**, Corollary 13] and Vershik [**Ver78**] that if G is the free group, then the action of G on its Gromov boundary (which is the Cantor set) is μ-hyperfinite for every Borel quasi-invariant probability measure. Dougherty, Jackson and Kechris [**DJK94**, Corollary 8.2] showed later that the tail equivalence relation is actually hyperfinite. On the other hand, Adams [**Ada94**] showed that for every hyperbolic group G, the action on ∂G is μ-hyperfinite for all Borel quasi-invariant probability measures. We do not know how to generalize Corollary 1.2 to all hyperbolic groups.

Our proof uses an idea of Dougherty, Jackson and Kechris [**DJK94**], and the main ingredient of the proof is a result that seems to be interesting in its own right. Given a hyperbolic group G acting geometrically on a cube complex C, $\gamma \in \partial C$ and an element $x \in C$, define the *interval* $[x, \gamma)$ to be the set of all vertices of the complex which lie on a geodesic ray in the 1-skeleton of C from x to γ. We would like to emphasize that we consider here only the 1-skeleton of C, and all geodesics we consider are the *combinatorial geodesics*, i.e. those taken in the 1-skeleton.

LEMMA 1.3. *If a hyperbolic group G acts geometrically on a* CAT(0) *cube complex C and $\gamma \in \partial C$, then for every $x, y \in C$, the sets $[x, \gamma)$ and $[y, \gamma)$ differ by a finite set.*

Theorem 1.1 is obtained using Lemma 1.3 and the following result.

THEOREM 1.4. *Suppose a hyperbolic group G acts freely and cocompactly on a locally finite graph V such that for every $\gamma \in \partial V$ and every $x, y \in V$, the sets $[x, \gamma)$ and $[y, \gamma)$ differ by a finite set. Then the action of G on ∂V induces a hyperfinite equivalence relation.*

The assumption that G acts freely on V means that the action on the set of vertices of V is free.

Given a fixed finite set of generators for a hyperbolic group G, the group acts on its Cayley graph. For $g \in G$ and $\gamma \in \partial G$, the set $[g, \gamma)$ is defined as above. The following question seems natural.

Question 1.5. Suppose G is a hyperbolic group with a fixed finite generating set and $\gamma \in \partial G$. Is it true that for any two group elements $g, h \in G$, the sets $[g, \gamma)$ and $[h, \gamma)$ differ by a finite set?

However, the above question was recently answered in the negative by Touikan [**Tou18**]. Of course, it may turn out that the answer to the above question depends on the choice of the generating set. Or, more generally, one can ask the following question.

Question 1.6. Is it true that for every hyperbolic group G, there exists a locally finite graph V such that G acts geometrically (or even freely and cocompactly) on V, and V has the property that $[x, \gamma)$ and $[y, \gamma)$ have finite symmetric difference for every $\gamma \in \partial V$ and $x, y \in V$?

We should add here that the class of groups for which we can prove the positive answer to the above question is limited to groups with the Haagerup property. However, a recent preprint of Marquis [**Mar18**] provides a class of new examples of such groups, which contains property (T) groups.

2. CAT(0) *cube complexes*

Here, we give a summary of several basic properties of CAT(0) cube complexes without proof. We refer the reader to [**BH99**, Ch. II.5] and [**Sag14**] for more details.

Recall that a *cube complex* is obtained by taking a disjoint collection of unit cubes in Euclidean spaces of various dimensions and gluing them isometrically along their faces. In particular, every cube complex has a piecewise Euclidean metric.

A cube complex X is *uniformly locally finite* if there exists $D > 0$ such that each vertex is contained in at most D edges. Note that if X admits a cocompact group action, then it is automatically uniformly locally finite.

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Now, for each vertex v in a cube complex X, draw an ε-sphere S_v around v. Note that the cubes of X divide S_v into simplices (a priori, these simplices may not be embedded in S_v, since a cube may not be embedded in X). Thus, S_v has the structure of a combinatorial cell complex made of various simplices glued along the faces. This complex is called the *link* of the vertex v.

Recall that a simplicial complex K is said to be *flag* if every complete subgraph of the 1-skeleton of K is actually the 1-skeleton of a simplex in K.

Definition 2.1. A CAT(0) cube complex is a cube complex which is simply connected and such that the link of each vertex is a flag simplicial complex.

The above is a combinatorial equivalent definition of the CAT(0) property for cube complexes (for more details, see [**BH99**, Definition II.1.2]).

Let X be a CAT(0) cube complex with its piecewise Euclidean metric. A subset of $C \subseteq X$ is *convex* if for any two points $x, y \in C$, any geodesic segment connecting x and y is contained in C. A *convex subcomplex* of X is a subcomplex which is also convex.

Recall that a *mid-cube* of $C = [0, 1]^n$ is a subset of the form $f_i^{-1}(\{\frac{1}{2}\})$, where f_i is one of the coordinate functions.

Definition 2.2. A *hyperplane* h in X is a subset such that:

(1) h is connected; and
(2) for each cube $C \subseteq X$, $h \cap C$ is either empty or a mid-cube of C.

It was proved by Sageev [**Sag95**] that for each edge $e \in X$, there exists a unique hyperplane which intersects e in one point. This is called the hyperplane *dual* to the edge e. Actually, given an edge e, we can always build locally a piece of hyperplane that cuts through e. In order to extend this piece to a hyperplane, one needs to make sure that the piece does not run into itself when one extends it. It is shown in [**Sag95**] that this can never happen in a CAT(0) cube complex, and thus such extensions exist.

Let X be a CAT(0) cube complex, and let $e \subseteq X$ be an edge. Denote the hyperplane dual to e by h_e. The following facts about hyperplanes are well known [**Sag95**, **Sag14**].

(1) The hyperplane h_e is a convex subset of X and h_e with the induced cell structure from X is also a CAT(0) cube complex.
(2) $X \setminus h_e$ has exactly two connected components, which are called *halfspaces*.

Two points in X are *separated* by a hyperplane h if they are different connected components of $X \setminus h$.

We use the following metric on the 0-skeleton $X^{(0)}$ of X. Given two vertices in $X^{(0)}$, the ℓ^1-distance between them is defined to be the length of the shortest path joining them in the 1-skeleton $X^{(1)}$. By [**HW08**, Lemma 13.1], the ℓ^1-distance between any two vertices is equal to the number of hyperplanes separating them.

Given two vertices $u, v \in X$, a *combinatorial geodesic* between them is an edge path in $X^{(1)}$ joining u and v which realizes the ℓ^1-distance between u and v. Note that there may be several different combinatorial geodesics joining u and v. By [**HW08**, Lemma 13.1], an edge path $\omega \subseteq X^{(1)}$ is a combinatorial geodesic if and only if for each pair of different edges $e_1, e_2 \subseteq \omega$, the hyperplane dual to e_1 and the hyperplane dual to e_2 are different.

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An edge path ω *crosses* a hyperplane $h \subseteq X$ if there exists an edge $e \subseteq \omega$ such that h is the dual to e. So, in other words, ω is a combinatorial geodesic if and only if there does not exist a hyperplane $h \subseteq X$ such that ω crosses h more than once.

Let $Y \subseteq X$ be a convex subcomplex (with respect to the piecewise Euclidean metric). Then, by [**HW08**, Proposition 13.7], Y is also convex with respect to the ℓ^1-metric in the following sense: for any vertices $u, v \in Y^{(0)}$, every combinatorial geodesic joining u and v is contained in Y.

In the rest of this paper, we will always use the ℓ^1-metric on $X^{(0)}$ and use d to denote this metric.

Let $Y \subseteq X$ be a convex subcomplex. By [**HW08**, Lemma 13.8], for any vertex $v \in X$, there exists a unique vertex $u \in Y$ such that $d(u, v) = d(v, Y^{(0)})$. Thus, we have a nearest point projection map $\pi_Y : X^{(0)} \to Y^{(0)}$.

LEMMA 2.3. *Let $Y \subseteq X$ be a convex subcomplex and $v \in X$. Let ω be a combinatorial geodesic from v to $\pi_Y(v)$. Then each hyperplane dual to an edge in ω separates v from Y. Conversely, each hyperplane which separates v from Y is dual to an edge in ω.*

Proof. This is a special case of [**HW08**, Proposition 13.10]. □

The following is a consequence of Lemma 2.3 and the fact that the ℓ^1-distance between any two vertices is equal to the number of hyperplanes separating them [**HW08**, Lemma 13.1].

COROLLARY 2.4. *Let $Y \subseteq X$ be a convex subcomplex. For every $v \in X$, the distance $d(v, Y^{(0)})$ is the number of hyperplanes that separate v from Y.*

LEMMA 2.5. *Let $Y \subseteq X$ be a convex subcomplex and $\pi_Y : X^{(0)} \to Y^{(0)}$ be the nearest point projection. Given two adjacent vertices $u, v \in X$, write $u' = \pi_Y(u)$ and $v' = \pi_Y(v)$. Suppose h is the hyperplane separating u and v.*

(1) *If $h \cap Y = \emptyset$, then $u' = v'$.*
(2) *If $h \cap Y \neq \emptyset$, then u' and v' are adjacent vertices in Y. Moreover, the hyperplane separating u' and v' is exactly h.*

Proof. Suppose without loss of generality that $d(v, Y^{(0)}) \leq d(u, Y^{(0)})$. Let ω_v and ω_u be the combinatorial geodesics which realize the ℓ^1-distance from v to $Y^{(0)}$ and u to $Y^{(0)}$ respectively.

Suppose first that $h \cap Y = \emptyset$. Then $h \cap \omega_v = \emptyset$, otherwise we would have $d(v, Y^{(0)}) > d(u, Y^{(0)})$. Thus, h separates u from Y. Moreover, each hyperplane dual to an edge in ω_v separates u from Y. By Corollary 2.4, we have $d(v, Y^{(0)}) + 1 \leq d(u, Y^{(0)})$. On the other hand, the concatenation of the edge $\overline{uv}$ with ω_v has length $\leq d(v, Y^{(0)}) + 1$. Thus, this concatenation realizes the ℓ^1-distance from u to $Y^{(0)}$. It follows that $u' = v'$.

Now suppose $h \cap Y \neq \emptyset$. First, by Lemma 2.3 we get $\omega_v \cap h = \omega_u \cap h = \emptyset$, because otherwise h would be dual to some edge in ω_v or ω_u and thus separate u or v from Y and hence be disjoint from Y. Let ω be a geodesic joining v' and u'. Note that ω is contained in Y. The path obtained by concatenating ω_v, ω and ω_u must intersect h because v and u lie on different sides of h. Thus, h must intersect ω and thus separate v' and u'. To see

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that v' and u' are adjacent, it is enough to show that h is the only hyperplane separating u' and v'. Note, however, that if h' is a hyperplane separating u' from v', then h' must intersect the path obtained by contatenating ω_v, the edge from v to u and ω_u. By Lemma 2.3, we get $h' \cap \omega_v = h' \cap \omega_u = \emptyset$, as above. Thus, h' intersects the edge from u to v, and hence $h' = h$. □

The above lemma implies that we can naturally extend the nearest point projection map $\pi_Y : X^{(0)} \to Y^{(0)}$ to $\pi_Y : X^{(1)} \to Y^{(1)}$. The next result follows from Lemma 2.5.

COROLLARY 2.6. *Let $Y \subseteq X$ be a convex subcomplex. Let $\omega \subseteq X$ be a combinatorial geodesic. Then $\pi_Y(\omega)$ is also a combinatorial geodesic.*

Note that it is possible that $\pi_Y(\omega)$ is a single point.

3. *The geodesics lemma*

Throughout this section, X will be a uniformly locally finite Gromov-hyperbolic CAT(0) cube complex. Let ∂X be the boundary of X.

Definition 3.1. Let $x \in X$ be a vertex and let $\eta \in \partial X$. Define the interval

$$[\eta, x) = \{y \in X^{(0)} : y \text{ lies on a combinatorial geodesic from } x \text{ to } \eta\}.$$

Recall that if X is δ-hyperbolic, then for any $x \in X$ and $\eta \in \partial X$, any two combinatorial geodesic rays ω_1 and ω_2 from x to η satisfy $d(\omega_1(t), \omega_2(t)) \leq 2\delta$ for each $t \geq 0$ and $d_H(\omega_1, \omega_2) \leq 2\delta$. Here, $d_H(\omega_1, \omega_2)$ denotes the Hausdorff distance between ω_1 and ω_2.

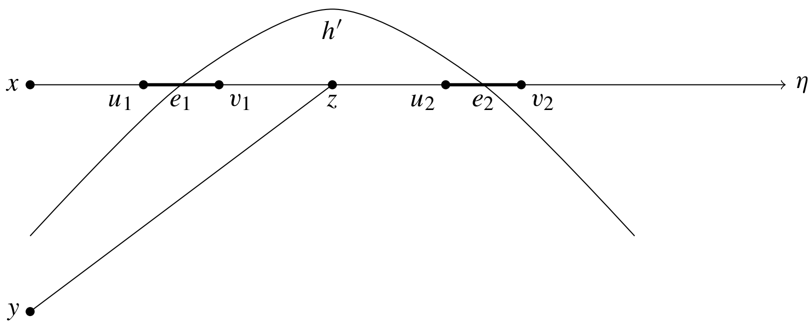
Now we will prove Lemma 1.3. Note that it suffices to prove the case where x and y are adjacent. Thus, in the rest of this section, we will assume x and y are two adjacent vertices in X.

LEMMA 3.2. *Let h be the hyperplane separating x and y and let $\overline{y\eta}$ be a combinatorial geodesic ray from y to η.*

(1) *If $\overline{y\eta}$ never crosses h, then each vertex of $\overline{y\eta}$ is contained in $[\eta, x)$.*
(2) *If $\overline{y\eta}$ crosses h, let $z \in \overline{y\eta}$ be the first vertex after $\overline{y\eta}$ crosses h and let $\overline{z\eta} \subseteq \overline{y\eta}$ be the ray after z. Pick a combinatorial geodesic segment $\overline{xz}$. Then $\overline{xz}$ and $\overline{z\eta}$ fit together to form a combinatorial geodesic ray. In particular, each vertex of $\overline{z\eta}$ is contained in $[\eta, x)$.*

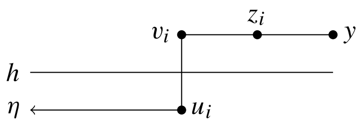
Proof. To see (1), let $\overline{xy}$ be the edge joining x and y. Then h is the hyperplane dual to $\overline{xy}$. Since $\overline{y\eta}$ never crosses h, each hyperplane which is dual to some edge of $\overline{y\eta}$ is different from h. Thus, the concatenation of $\overline{xy}$ and $\overline{y\eta}$ is a combinatorial geodesic ray, because all hyperplanes dual to its edges are distinct [**HW08**, Lemma 13.1]. Thus, each vertex of $\overline{y\eta}$ in contained in $[\eta, x)$.

Now we prove (2). Since $\overline{y\eta}$ is a combinatorial geodesic ray, it follows [**HW08**, Lemma 13.1] that $\overline{z\eta}$ does not cross h. Suppose the concatenation of $\overline{xz}$ and $\overline{z\eta}$ is not a combinatorial geodesic ray. Since $\overline{xz}$ and $\overline{z\eta}$ are already geodesic, the only possibility is that there exist edges $e_1 \subseteq \overline{xz}$ and $e_2 \subseteq \overline{z\eta}$ such that they are dual to the same hyperplane h', again by [**HW08**, Lemma 13.1]. Let u_i and v_i be endpoints of e_i, as indicated in the picture below.



Since $\overline{xz}$ is a combinatorial geodesic, it crosses h' only once [**HW08**, Lemma 13.1]. Thus, the segments $\overline{xu_1}$ and $\overline{v_1z}$ stay on different sides of h'. In particular, x and z are in different sides of h'. Since $\overline{y\eta}$ is a combinatorial geodesic ray, it crosses h' only once, and thus the segment $\overline{yz} \cup \overline{zu_2}$ is on one side of h'. In particular, y and z are on the same side of h'. Thus, we deduce that x and y are separated by h'. Since x and y are adjacent, there is only one hyperplane separating them, and thus $h' = h$. This is a contradiction since $\overline{z\eta}$ does not cross h. □

Proof of Lemma 1.3. We assume x and y are adjacent. Let h be the hyperplane separating them. We argue by contradiction and suppose there exists a sequence $\{z_i : i \in \mathbb{N}\}$ in $[\eta, y) \setminus [\eta, x)$ with $z_i \neq z_j$ for $i \neq j$. Since X is uniformly locally finite, we can assume $d(z_i, y) \to \infty$ as $i \to \infty$. Let ω_i be a combinatorial geodesic segment from y to η such that $z_i \in \omega_i$. By Lemma 3.2, each ω_i crosses h. Let $\overline{yv_i} \subseteq \omega_i$ be the segment before ω_i crosses h, and let $\overline{u_i\eta} \subseteq \omega_i$ be the segment after ω_i crosses h (see the picture below). It follows from Lemma 3.2(2) that $z_i \subseteq \overline{yv_i}$. In particular, $d(v_i, y) \to \infty$ as $i \to \infty$.



Recall that h gives rise to two combinatorial hyperplanes, one containing x, which we denote by h_x, and one containing y, which we denote by h_y. Note that $v_i \in h_y$ by construction. Since h_y is a convex subcomplex, it follows [**HW08**, Proposition 13.7] that $\overline{yv_i} \subseteq h_y$. Since X is uniformly locally finite (and hence locally compact) and $d(v_i, y) \to \infty$, up to passing to a subsequence, we can assume the sequence of segments $\{\overline{yv_i}\}_{i=1}^{\infty}$ converges to a combinatorial geodesic ray ω. Since $\overline{yv_i} \subseteq h_y$ for each i, we have $\omega \subseteq h_y$. Moreover, by δ-hyperbolicity, the Hausdorff distance between ω and any of ω_i is less than 2δ. Thus, ω is a combinatorial geodesic ray joining y and η. Since ω is contained in h_y, we get that for every i and every vertex $w \in \omega_i$, we have $d(w, h_y) \leq 2\delta$.

Let $\pi : X^{(1)} \to h_y^{(1)}$ be the nearest point projection from $X^{(1)}$ to the 1-skeleton of convex subcomplex h_y. Then $\pi(\omega_i)$ is a combinatorial geodesic by Corollary 2.6. It follows from the above remarks and the definition of π that $d_H(\omega_i, \pi(\omega_i)) \leq 2\delta$.

Thus, $\pi(\omega_i)$ is a combinatorial geodesic ray joining y and η. Since $\overline{yv_i} \subseteq h_y$, $\pi(\overline{yv_i}) = \overline{yv_i}$. Thus, $\overline{yv_i}$ is contained in $\pi(\omega_i)$. In particular, $z_i \in \pi(\omega_i)$. Since $\pi(\omega_i) \subseteq h_y$, it never crosses h, and thus Lemma 3.2(1) implies $z_i \in [\eta, x)$, which is a contradiction. □

4. *Finite Borel equivalence relations*

We will use the following standard application of the second reflection theorem [**Kec95**, Theorem 35.16]. Below, if E is an equivalence relation of Z and $A \subseteq Z$, then $E|A$ denotes $E \cap A \times A$.

Notation and standard facts from descriptive set theory can be found in the textbook [**Kec95**]. Here, we give a couple of definitions for a non-descriptive set theorist. A subset A of a Polish space X is *analytic* if there exists a Borel set $B \subseteq X \times Y$ for a Polish space Y such that A is the projection of B. In other words,

$$A = \{x \in X : \exists y \in Y \; B(x, y)\}.$$

The class of all analytic sets is denoted by $\boldsymbol{\Sigma}^1_1$. Note that if $A \subseteq X \times Y$ is analytic, then so is its projection on X. Thus, for a Borel set $B \subseteq X \times Y_1 \times \cdots Y_n$, the set $\{x \in X : \exists y_1 \in Y_1 \cdots \exists y_n \in Y_n \; B(x, y_1, \ldots, y_n)\}$ is also analytic. Complements of analytic sets are said to be *coanalytic*, and the class of coanalytic sets is denoted by $\boldsymbol{\Pi}^1_1$. Borel sets are both analytic and coanalytic, and conversely, if a set is both analytic and coanalytic, then it is Borel. Note that for a coanalytic (in particular, Borel) set $C \subseteq X \times Y$, the set $\{x \in X : \forall y \in Y \; C(x, y)\}$ is also coanalytic. The classes of analytic and coanalytic sets are both closed under countable unions and countable intersections. This implies that if $A \subseteq X \times \mathbb{N}$ is analytic, then $\{x \in X : \forall n \; (x, n) \in A\}$ is analytic, and if $C \subseteq X \times \mathbb{N}$ is coanalytic, then $\{x \in X : \exists n \; (x, n) \in C\}$ is coanalytic.

A collection Φ of subsets of a Polish space Y is called $\boldsymbol{\Pi}^1_1$ *on* $\boldsymbol{\Sigma}^1_1$ if for any Polish space X and an analytic set $A \subseteq X \times Y$, the set

$$\{x \in X : \Phi(A_x)\}$$

is coanalytic. Similarly, if Φ is a collection of pairs of subsets of X, then Φ is said to be $\boldsymbol{\Pi}^1_1$ *on* $\boldsymbol{\Sigma}^1_1$ if for every two analytic sets $A, B \subseteq X \times Y$, the set

$$\{x \in X : \Phi(A_x, B_x)\}$$

is coanalytic.

A family Φ of pairs of subsets of X is said to be *hereditary* if it is closed under taking subsets in both coordinates, and Φ is *continuous upward in the second variable* if whenever $B_n \subseteq B_{n+1}$ and $\Phi(A, B_n)$ holds for all n, then $\Phi(A, \bigcup_n B_n)$ holds as well. The second reflection theorem (stated in the dual form, see [**Kec95**, 35.16] and the discussion after its proof) says that if Φ is $\boldsymbol{\Pi}^1_1$ on $\boldsymbol{\Sigma}^1_1$, hereditary and continuous upward in the second variable, then for any analytic set $A \subseteq X$ such that $\Phi(A, A^c)$ holds, there exists a Borel set $B \subseteq X$ with $A \subseteq B$ such that $\Phi(B, B^c)$ holds.

LEMMA 4.1. *Let Z be a Polish space, $A \subseteq Z$ be analytic and E be an analytic equivalence relation on Z such that there is some $n > 1$ such that every $E|A$-class has size less than n. Then there is a Borel equivalence relation F on Z with $E|A \subseteq F$ such that every F-class has size less than n.*

Proof. Note that $G = E|A \cup \{(z, z) : z \in Z\}$ is an analytic equivalence relation on Z whose classes have size less than n. Now consider $\Phi \subseteq \mathrm{Pow}(Z^2) \times \mathrm{Pow}(Z^2)$ defined as follows:

$$
\begin{aligned}
(B, C) \in \Phi \iff & \forall x \ \neg x \ C \ x \\
& \wedge \forall (x, y) \ \neg x \ B \ y \vee \neg y \ C \ x \\
& \wedge \forall (x, y, z) \ \neg x \ B \ y \vee \neg y \ B \ z \vee \neg x \ C \ z \\
& \wedge \forall_{i=1}^{n} x_i \left(\bigvee_{i \neq j} x_i = x_j \right) \vee \left(\bigvee_{i \neq j} \neg x_i \ B \ x_j \right).
\end{aligned}
$$

Note that $\Phi(B, B^c)$ holds if and only if B is an equivalence relation on Z whose classes have size less than n, so in particular we have $\Phi(G, G^c)$. Now, Φ is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$, hereditary and continuous upward in the second variable, so, by the second reflection theorem [**Kec95**, Theorem 35.16], there is a Borel set $F \supset G$ such that $\Phi(F, F^c)$ holds, and we are done. □

5. *Proof of main theorem*

The following fact lets us reduce our problem to the case of free actions.

LEMMA 5.1. *Every cubulated hyperbolic group has a finite index subgroup acting freely and cocompactly on a* CAT(0)*-cube complex.*

Proof. If G is a hyperbolic group acting properly and cocompactly on a CAT(0) cube complex X, then by Agol's theorem [**Ago13**, Theorem 1.1] (see also Wise [**Wis17**]) there is a finite index subgroup F acting faithfully and specially on X (see Haglund and Wise [**HW08**, Definition 3.4] for the definition of special action). Now, F embeds into a right-angled Artin group which is torsion-free, so F is torsion-free. Since every stabilizer is finite by properness of the action, it must be trivial, since F is torsion-free, and thus F acts freely on X. □

Proof of Theorem 1.4. Let V be the set of vertices of the graph. Note that V as a metric space is hyperbolic, since the action of G is geometric. Below, by ∂V we denote the Gromov boundary of V. Fix $v_0 \in V$, and fix a total order on V such that $d(v_0, v) \leq d(v_0, w) \implies v \leq w$, where d denotes the graph distance on V. Fix a transversal $\tilde{V}$ of the action of G on V (the transversal is finite since the action is cocompact). For $v \in V$, we denote by $\tilde{v}$ the unique element of $\tilde{V}$ in the orbit of v. By a directed edge of V, we mean a pair $(v, v') \in V^2$ such that there is an edge from v to v'. We colour the directed edges of V as follows. We assign a distinct colour to every directed edge (v, v') with $v \in \tilde{V}$, and this extends uniquely (by freeness) to a G-invariant colouring on all directed edges. Let C be the set of colours (which is finite since V is locally finite), and let $c(v, v')$ be the colour of (v, v'). Fix any total order on C. This induces a lexicographical order on $C^{<\mathbb{N}}$ (the set of all finite sequences of elements of C).

For any combinatorial geodesic $\eta \in V^{<\mathbb{N}}$ and $m, n \in \mathbb{N}$, define

$$c(\eta, m, n) = (c(\eta_m, \eta_{m+1}), c(\eta_{m+1}, \eta_{m+2}), \ldots, c(\eta_{m+n-1}, \eta_{m+n})) \in C^{<\mathbb{N}}.$$

For every $a \in \partial V$, define $S^a \subseteq V \times C^{<\mathbb{N}}$ as follows:

$$
\begin{aligned}
S^a = \{(\eta_m, c(\eta, m, n)) \in V \times C^{<\mathbb{N}} : & \ \eta \text{ is a combinatorial geodesic} \\
& \text{from } v_0 \text{ to } a \text{ and } m, n \in \mathbb{N}\}.
\end{aligned}
$$

Let $s_n^a \in C^{<\mathbb{N}}$ be the least string of length n which appears infinitely often in S^a, i.e. such that there are infinitely many $v \in V$ for which $(v, s_n^a) \in S^a$. Note that each s_n^a is an initial segment of s_{n+1}^a. Let

$$T_n^a = \{v \in V : (v, s_n^a) \in S^a\}$$

and let $v_n^a = \min T_n^a$ (with respect to the ordering on V). Note that every vertex in T_n^a has an edge coloured by s_1^a leaving it, so every vertex of T_n^a is in the same orbit. Let

$$k_n^a = d(v_0, v_n^a),$$

and note that k_n^a is non-decreasing in n.

Now let $Z = \{a \in \partial V : k_n^a \not\to \infty\}$. Then, for each $a \in \partial V$, since $k_n^a \not\to \infty$ and V is discrete, there is a finite set containing all v_n^a, so there is some $v \in V$ which is in T_n^a for infinitely many n. Thus, the geodesic class determined by the combinatorial geodesic starting at $\tilde{v}$ (which is determined by k_1^a) and following the colours of $\lim_n s_n^a \in C^{\mathbb{N}}$ is a Borel selector for E. Thus, E is smooth on Z and hence also on the saturation $[Z]_E$ (which may be larger than Z).

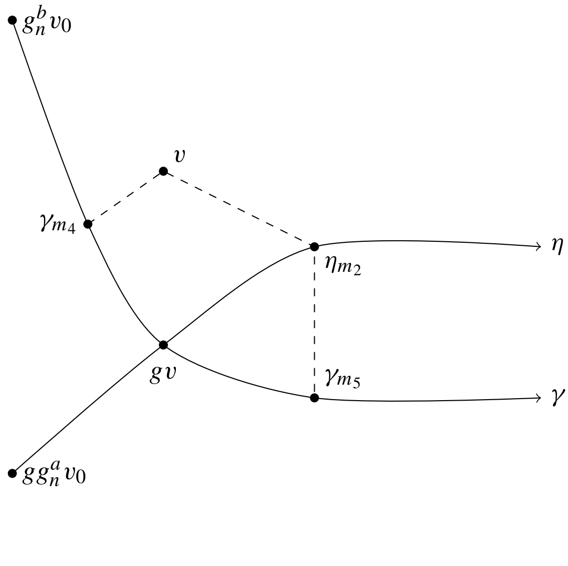
Now let $Y = \partial V \setminus [Z]_E = \{a \in \partial V : \forall b E a \;\; k_n^b \to \infty\}$. We will show that E is hyperfinite on Y. For each $n \in \mathbb{N}$, define $H_n : \partial V \to 2^V$ by

$$H_n(a) = g_n^a T_n^a,$$

where $g_n^a \in G$ is the unique element with $g_n^a v_n^a \in \tilde{V}$. Let F_n be the equivalence relation on im H_n which is the restriction of the shift action of G on 2^V. We have the following lemma.

LEMMA 5.2. *There exists $K \in \mathbb{N}$ such that on* im H_n, *the relation F_n has equivalence classes of size at most K.*

Proof. Let $a, b \in \partial V$ and suppose $g \in G$ is such that $gH_n(a) = H_n(b)$, i.e. $gg_n^a T_n^a = g_n^b T_n^b$. Since the vertices in both sets are in the same orbit, $g_n^a v_n^a$ and $g_n^b v_n^b$ are elements of $\tilde{V}$ which are in the same orbit, so they are equal, say to some $v \in \tilde{V}$. It suffices to show that $d(v, gv) \le 6\delta$, since then we can choose any $K \in \mathbb{N}$ larger than $\max_{v \in \tilde{V}} |\{g : d(v, gv) \le 6\delta\}|$.



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Note that since T_n^a and T_n^b are infinite, we have that $gg_n^a a = g_n^b b$, which we will call $c \in \partial X$. Let η be a geodesic from $gg_n^a v_0$ to c with $\eta_{m_1} = gv$. Now, $v \in gg_n^a T_n^a$, so there is some m_2 with $d(v, \eta_{m_2}) \leq 2\delta$. Note that by choice of v_n^a, we have $m_2 \geq m_1$. Now let γ be a geodesic from $g_n^b v_0$ to c with $\gamma_{m_3} = gv$. By the choice of v_n^b, there is some $m_4 \leq m_3$ such that $d(v, \gamma_{m_4}) \leq 2\delta$. Also, η and γ are 2δ-close after they go through gv, so since $m_2 \geq m_1$, there is some $m_5 \geq m_3$ such that $d(\eta_{m_2}, \gamma_{m_5}) \leq 2\delta$. Thus,

$$\begin{aligned} 2d(v, gv) &\leq d(v, \gamma_{m_4}) + d(\gamma_{m_4}, gv) + d(v, \eta_{m_2}) + d(\eta_{m_2}, \gamma_{m_5}) + d(\gamma_{m_5}, gv) \\ &= d(\gamma_{m_4}, \gamma_{m_5}) + d(v, \gamma_{m_4}) + d(v, \eta_{m_2}) + d(\eta_{m_2}, \gamma_{m_5}) \\ &\leq 2(d(v, \gamma_{m_4}) + d(v, \eta_{m_2}) + d(\eta_{m_2}, \gamma_{m_5})) \\ &\leq 2(6\delta), \end{aligned}$$

where the first equality follows from the fact that γ is a geodesic. □

Now, note that im H_n is analytic. To see this, note first that the set

$$C = \{(a, \eta) \in \partial V \times V^{\mathbb{N}} : \eta \text{ is a combinatorial geodesic from } v_0 \text{ to } a\}$$

is closed. Write

$$D = \{(a, (\eta_0, \ldots, \eta_n)) : a \in \partial V, \eta \text{ a combinatorial geodesic from } v_0 \text{ to } a\}.$$

CLAIM 5.3. *The set D is Borel in $\partial V \times V^{<\mathbb{N}}$.*

Proof. The set is clearly analytic, as for $a \in \partial V$ and $v \in V^n$ we have that $(a, v) \in D$ if and only if $\exists \eta \in V^{\mathbb{N}}$ such that $(a, \eta) \in C$ and $v = (\eta_0, \ldots, \eta_{n-1})$. To see that D is coanalytic, note that $(a, v) \in D$ if and only if $\forall m \geq n \exists w \in V^m$ (w is a finite geodesic extending v and $\forall \eta$, if $(a, \eta) \in C$, then $d(w, \eta) < 2\delta$). Here, by $d(w, \eta)$ we mean the max of $d(w_i, \eta_i)$ for $i < m$. In the latter equivalence, the 'if' part follows from a compactness argument and the 'only if' is obvious. □

Using the above claim and writing the definitions of the following sets (with formulas using only countable quantifiers and references to D) one can deduce that the sets $\{(a, S^a) : a \in \partial V\}$ and $\{(a, v_n^a) : a \in \partial V\}$, $\{(a, T_n^a) : a \in \partial V\}$ and $\{(a, H_n(a)) : a \in \partial V\}$ are also Borel, for every n. This implies that the set im H_n is analytic for every n.

By Lemma 4.1, there is a Borel equivalence relation F_n' on 2^V containing F_n whose classes are of size at most K. Every finite Borel equivalence relation is smooth (see [**Gao09**, Ch. 7.1]), so let $f_n : 2^V \to 2^{\mathbb{N}}$ be a reduction for $F_n' \leq_B \mathrm{id}_{2^{\mathbb{N}}}$, and define $f : \partial V \to (2^{\mathbb{N}})^{\mathbb{N}}$ by $f(a) = (f_n(H_n(a)) : n \in \mathbb{N})$. Write E' for the pullback of E_1 via f. Note that since each F_n' is finite, the relation E' is countable. As E' is clearly hypersmooth, we get that E' is hyperfinite by [**Gao09**, Theorem 8.1.5]. Now, note that f is a homomorphism from $E|Y$ to E_1. Indeed, if $a, b \in Y$ with aEb, then let g be such that $ga = b$. By our assumption, the sets $\{g\eta_m : m \in \mathbb{N}, \eta \text{ a combinatorial geodesic from } v_0 \text{ to } a\}$ and $\{\eta_m : m \in \mathbb{N}, \eta \text{ a combinatorial geodesic from } v_0 \text{ to } b\}$ differ by a finite set. Since both a, b are in Y and V is locally finite, there is $N \in \mathbb{N}$ such that $H_n(a) E_n H_n(b)$ for $n \geq N$, and thus $f(x) E_1 f(y)$. Thus, $E|Y \subseteq E'$ is a subrelation of a hyperfinite one, and hence it is hyperfinite as well. This ends the proof. □

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Proof of Theorem 1.1. Let G be a cubulated δ-hyperbolic group. Since hyperfiniteness passes to finite-index extensions [**JKL02**, Proposition 1.3], by Lemma 5.1, we can assume that G acts freely and cocompactly on a CAT(0) cube complex X. Let $V = X^{(0)}$ be the set of vertices of X. Now the statement follows from Theorem 1.4 and Lemma 1.3.

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