COMPLETE REDUCIBILITY OF INFINITE GROUPS

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1. The theorems of the present paper deal with conditions which are necessary and sufficient in order that a solvable or nilpotent infinite group should have a completely reducible matrix representation over a given algebraically closed field.

It is known **(17)** that a locally nilpotent group of matrices is always solvable. Thus the first theorem of the present paper is a partial generalization of Theorem 1 of **(16)**, which states:

If G is a locally nilpotent subgroup of the full linear group $\mathrm{GL}(n, P)$ *over a perfect field P, then G is completely reducible if and only if each matrix of G is diagonizable* (by a similarity transformation over some extension field of P).

The remaining three theorems list necessary and sufficient conditions in order that solvable and nilpotent groups should have faithful, completely reducible representations over the given field. The first part of Theorem 3 is known **(14)**, but our proof is considerably shorter.

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Notation. The notation for groups is that of **(8**, §§57, 62**)**. In particular, $Z(x; G)$ denotes the centralizer of the element x in the group G, and $Z(G)$ denotes the centre of G. If H is a subgroup of G, then $|H|$ and $|G:H|$ denote the order and index of H respectively.

We shall suppose that the ground field F is a fixed algebraically closed field. A group G will be said to have a completely reducible representation if it has a faithful, completely reducible representation by finite matrices over F. In this case we shall write $G \in \mathfrak{R}$. If we wish to emphasize the degree of n of the representation, we shall write $G \in \mathfrak{R}_n$. Following **(16)** we call a matrix x over F a *d-matrix* when x can be diagonalized (by a similarity transformation over F). A group G will be called *non-modular* (with respect to F) if either the characteristic p of F is zero, or $p \neq 0$ and G has no subgroup of finite index h such that $p|h$.

Finally, we shall write $M < \beta(u, v)$ to mean that the variable quantity M is bounded by some function depending only on u and v.

We can then state the following theorems.

THEOREM 1. *If G is a non-modular solvable group of matrices over F, then*

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$G \in \mathfrak{N}$ if and only if every matrix $x \in G$ is a d-matrix, i.e. each cyclic subgroup $\langle x \rangle \in \mathfrak{N}$.

THEOREM 2. *If G is a non-modular solvable group, then $G \in \mathfrak{N}$ if and only if G contains an abelian subgroup A of finite index such that $A \in \mathfrak{N}$.*

Note. The condition that an abelian group A lies in $\mathfrak{N}$ is relatively simple. In fact, since A is completely reducible, the matrices of A may be simultaneously diagonalized over F. The condition then reduces to the following:

(*) *A is isomorphic to a subgroup of the direct product of a finite number of groups each of which is isomorphic to F^*, the multiplicative group of F.*

THEOREM 3. *If $G \in \mathfrak{N}_n$ and G is nilpotent of class k, then*
(*a*) $|G:Z(G)| < \beta(n, k)$
(*b*) $|G'| < \beta(n, k)$.

Combined with Theorem 2, this immediately gives the condition that a non-modular nilpotent group $G \in \mathfrak{N}$, if and only if $|G:Z(G)| < \infty$ and $Z(G) \in \mathfrak{N}$. An alternative condition is given by the following theorem.

THEOREM 4. *If G is a non-modular, nilpotent group, then $G \in \mathfrak{N}$ if and only if $|G:Z(G)| < \infty$ and $G/G' \in \mathfrak{N}$.*

2. We shall require the following well-known results.

THEOREM (Clifford). *If G is a completely reducible group of matrices over an arbitrary field, and if H is a normal subgroup of G, then H is completely reducible.*

Proof. See **(3)**.

THEOREM (Mal'cev). *If G is a solvable group of matrices of degree n over an algebraically closed field, then G has a subgroup H of finite index $h < \beta(n)$, such that the matrices of H may be put simultaneously into triangular form (by a similarity transformation).*

Proof. See **(9)**.

COROLLARY. *H may be taken to be normal in G. Furthermore, if G is completely reducible, then using Clifford's theorem, H may be put into diagonal form and hence H is abelian.*

The following is a very weak form of Burnside's theorem **(2)**, which is proved in considerable generality in **(9)**. In this case the proof is obvious, since the group A may be diagonalized.

THEOREM (Burnside). *If A is a completely reducible abelian group of matrices of degree n over an algebraically closed field, and if for some integer $m > 0$, $x^m = 1$ for all $x \in A$, then $|A| \leqslant m^n < \beta(m, n)$.*

3. In order to prove Theorems 1 and 2 we require two lemmas.

LEMMA 1. *Let G be a group of matrices over the field F, and let H be a subgroup of finite index h (such that $p \nmid h$ in the case that the characteristic p of F is non-zero). Then if H is completely reducible, G is completely reducible.*

Proof. (cf. **5**, p. 253). Consider G as operating on the underlying vector space $\mathfrak{W}$. We wish to show that if $\mathfrak{U} \subseteq \mathfrak{W}$ is an invariant subspace under G, then $\mathfrak{U}$ has a complementary invariant subspace.

Since H is completely reducible, $\mathfrak{U}$ has a complementary subspace $\mathfrak{V}$ which is invariant under H. Let π be the projection of $\mathfrak{W}$ onto $\mathfrak{V}$ along $\mathfrak{U}$. If R is a set of representatives for the right cosets of H in G, then we define a projection σ of $\mathfrak{W}$ by

$$\sigma = \frac{1}{h} \sum_{x \in R} x^{-1} \pi x.$$

$\mathfrak{V}_1 = \mathfrak{W}\sigma$ is independent of the choice of R. In fact,

$$\mathfrak{W}x^{-1}\pi x = \mathfrak{V}x = \mathfrak{V}z = \mathfrak{W}z^{-1}\pi z$$

when x and z are in the same right coset of H, because $\mathfrak{V}$ is an invariant subspace of H. Furthermore, for all $y \in G$,

$$\sigma y = y\left\{\frac{1}{h} \sum_{x \in R} (xy)^{-1}\pi(xy)\right\} = y\sigma,$$

and therefore $\mathfrak{V}_1 y = \mathfrak{W}y\sigma = \mathfrak{V}_1$, so $\mathfrak{V}_1$ is an invariant subspace of G.

Finally, since $\mathfrak{W}(1 - x^{-1}\pi x) = \mathfrak{W}(1 - \pi)x = \mathfrak{U}x = \mathfrak{U}$, it follows that $1 - \sigma$ is a projection of $\mathfrak{W}$ onto $\mathfrak{U}$ along $\mathfrak{V}_1$. Hence $\mathfrak{W} = \mathfrak{U} \oplus \mathfrak{V}_1$, where $\mathfrak{V}_1$ is an invariant subspace of G, as required.

LEMMA 2. *If x is a matrix over F, and $h > 0$ is an integer which is prime to the characteristic p of F if $p \neq 0$, then x is a d-matrix if x^h is a d-matrix.*

Proof. It is well known that a matrix is a d-matrix if and only if its minimal polynomial factors into distinct linear factors. Let $\phi(t)$ be the minimal polynomial for x^h. By hypothesis,

$$\phi(t) = (t - \alpha_1) \ldots (t - \alpha_k),$$

where $\alpha_1, \ldots, \alpha_k$ are distinct non-zero elements of F. Since the characteristic of the field does not divide h, it follows that $\phi(t^h)$ also has distinct linear factors. Since the minimal polynomial for x must divide $\phi(t^h)$, we may conclude that x is a d-matrix.

4. Proof of Theorem 1. By Mal'cev's theorem, G has a normal subgroup H of finite index $h < \beta(n)$, such that the matrices of H may be put simultaneously into triangular form. Hence the derived group H' consists of matrices whose eigenvalues are all 1.

(A) If $G \in \mathfrak{R}$, then by the corollary to Mal'cev's theorem, every matrix of H is a d-matrix. Since G is non-modular, h is prime to the characteristic p of F (if $p \neq 0$). Since $x^h \in H$ for all $x \in G$, it follows from Lemma 2 that every matrix $x \in G$ is a d-matrix.

(B) If every matrix $x \in G$ is a d-matrix, then only the identity matrix can be in H'. Therefore H is abelian. Since each matrix of H can be diagonalized, H can be put into simultaneous diagonal form. Therefore H is completely reducible and by Lemma 1, $G \in \mathfrak{R}$.

5. Proof of Theorem 2.

(A) If $G \in \mathfrak{R}$, then the existence of A follows from the corollary to Mal'cev's theorem.

(B) Conversely, if G has an abelian subgroup A of finite index h, then because G is non-modular, h is not divisible by the characteristic of F. Since $A \in \mathfrak{R}$, Schur's lemma implies that A can be diagonalized, and we shall suppose it to be in this form and of degree n.

There is a monomial representation for G in terms of $h \times h$ monomial matrices over the subgroup A. Therefore the given representation of A will give a representation for G of degree nh over F. In this representation, each matrix of A is diagonal and so A is completely reduced. Therefore by Lemma 1, G is completely reducible.

6. The proof of Theorem 3 requires three lemmas.

LEMMA 3. *Let G be a nilpotent group of class k, and let A be a normal abelian subgroup of G. If $x \in G$ and $x^h \in A$ for some integer $h > 0$, then for the commutators $[a, x] = a^{-1}x^{-1}ax$ we have*

$$[a, x]^{h^{k-1}} = 1 \qquad \textit{for all } a \in A.$$

Proof. We first note that, if $[b, x]$ commutes with x, then $[b, x]^h = [b, x^h] = 1$ for each $b \in A$.

Now consider the following sequence of commutators in A:

$$a_0 = a, \quad a_i = [a_{i-1}, x] \quad (\text{for } i = 1, 2, \ldots),$$

for a given element $a \in A$. We shall proceed by induction on s to show that

$$a_{k-s}^{h^s} = 1 \quad \text{for } 0 \leqslant s \leqslant k - 1.$$

The result is true for $s = 0$ because G is of class k. Assuming the result for $s - 1$, we have

$$[a_{k-s}, x]^{h^{s-1}} = [a_{k-s}^{h^{s-1}}, x] = 1,$$

because all the commutators commute with one another. Therefore x commutes with

$$[a_{k-s-1}, x]^{h^{s-1}} = [a_{k-s-1}^{h^{s-1}}, x]$$

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and so, by the remark at the beginning of the proof,

$$a_{k-s}^{h^s} = 1.$$

Thus the induction step is proved. In particular, for $s = k - 1$ we have the result stated.

LEMMA 4. *If $G \in \mathfrak{R}_n$ and G is a nilpotent group of class k, then for all $x \in G$, $|G:Z(x; G)| < \beta(n, k)$.*

Proof. By Mal'cev's theorem, G has a normal abelian subgroup A with index $h < \beta(n)$. Hence $x^h \in A$ for each $x \in G$. Therefore, by Lemma 3, $[a, x]^{h^{k-1}} = 1$ for all $a \in A$. Hence by Burnside's theorem, the group $C \subseteq A$ which is generated by all $[a, x]$ $(a \in A)$ has order $|C| < \beta(k, n)$. Because $a^{-1}xa \in Cx$ for all $a \in A$, x has at most $|C|$ conjugates with respect to A. Hence

$$|G:Z(x; G)| \leqslant |G:A||A:Z(x; A)| \leqslant h|C| < \beta(k, n).$$

We shall also require the following known result.

LEMMA 5. *If, in a group G, $|G:Z(G)| = h$, then $|G'| < \infty$ and $u^h = 1$ for all $u \in G'$.*

Proof. See **(10)** and **(1)**. A simple proof, basically due to Schur, is given in **(6)**. This last proof is based on the fact that the transfer of G into $Z(G)$ is just the mapping $x \rightarrow x^h$ and that the kernel of the transfer contains G'.

7. The proof of Theorem 3.

(*a*) The matrices of G span a vector space of dimension $s \leqslant n^2$ over F, and there is a basis of this space consisting of elements $u_1, u_2, \ldots, u_s$ of G. Then $Z(G) = \bigcap_i Z(u_i; G)$ and so, by Lemma 4:

$$|Z:Z(G)| \leqslant \Pi_i \ |G:Z(u_i; G)| < \beta(n, k)^s < \beta(n, k).$$

(*b*) Let $|G:Z(G)| = h$. From Lemma 5, we have that $u^h = 1$ for all

$$u \in Z(G) \cap G'.$$

Therefore by Burnside's theorem, $|Z(G) \cap G'| < \beta(h, n)$. Since

$$|G':Z(G) \cap G'| \leqslant |G:Z(G)| = h,$$

the result follows using Part (*a*).

8. We require three final lemmas to prove Theorem 4.

LEMMA 6. *Let G be a group and K and L be normal subgroups of G, each containing G'. If $G/K \in \mathfrak{R}_m$ and $G/L \in \mathfrak{R}_n$, then $G/(K \cap L) \in \mathfrak{R}_{m+n}$.*

Proof. As is well known, $G/(K \cap L)$ is isomorphic to a subgroup of the direct product $G/K \times G/L$ under the mapping: $(K \cap L)u \rightarrow (Ku, Lu)$ $(u \in G)$. Therefore $G/(K \cap L)$ has a representation of degree $m + n$ over F, and this representation is completely reducible from Clifford's theorem.

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LEMMA 7. *Let A be an abelian group and B be a subgroup. If for some subgroup K of B, $B/K \in \mathfrak{N}_n$, then for some subgroup N of A, $A/N \in \mathfrak{N}_n$, and $N \cap B = K$.*

Proof. Every completely reducible representation of an abelian group is similar to a group of diagonal matrices of the same degree. Let D be the (abelian) group of all diagonal matrices of degree n over F. Because F is algebraically closed, every element in D has an rth root in D for each integer $r > 0$. Thus D is a *divisible* group **(4**, Chapter 3**)**. Let σ denote the given homomorphism of B into D with kernel $\sigma = K$. Then by **(4**, Theorem 16.1**)**, there is a homomorphism τ of A into D with kernel $\tau = N$ such that the restriction $\tau|B = \sigma$. In particular, $K = \text{kernel } \tau|B = N \cap B$.

LEMMA 8. *Let $A \in \mathfrak{N}$ and A be an abelian group. If H is a finite subgroup of A such that the characteristic of F does not divide $|H|$, then $A/H \in \mathfrak{N}$.*

Proof. For each integer $m > 0$, we define $A_m = \{x \in A | x^m = 1\}$. By Burnside's theorem, A_m is a finite group. The homomorphism $x \rightarrow x^m$ of A into itself has kernel A_m and therefore $A/A_m \in \mathfrak{N}$.

Now take $m = |H|$. Then A_m/H is a finite non-modular group, and therefore $A_m/H \in \mathfrak{N}$. Hence, by Lemma 7, $A/N \in \mathfrak{N}$ for some subgroup N of A such that $N \cap A_m = H$. Therefore, by Lemma 6, $A/H \in \mathfrak{N}$.

9. Proof of Theorem 4. We shall write $Z(G) = Z$. We note that, since G is non-modular, if $|G:Z| < \infty$, then G' is a finite non-modular subgroup.

(A) If $G \in \mathfrak{N}$, then from Theorem 3, $|G:Z| < \infty$. Therefore, by Lemma 2, $Z \in \mathfrak{N}$. Since $ZG'/G' \simeq Z/G' \cap Z$, it folllws from Lemma 8 that $ZG'/G' \in \mathfrak{N}$. Then Lemma 1 shows that $G/G' \in \mathfrak{N}$.

(B) If $|G:Z| < \infty$ and $G/G' \in \mathfrak{N}$, then by Clifford's theorem, $Z/G' \cap Z \simeq ZG'/G' \in \mathfrak{N}$. Since $G' \cap Z$ is a finite non-modular group, $G' \cap Z \in \mathfrak{N}$. Therefore, by Lemma 7, there is a subgroup N of Z such that $Z/N \in \mathfrak{N}$ and

$$N \cap G' \cap Z = 1.$$

Then, by Lemma 6, $Z \in \mathfrak{N}$, and the result follows from Theorem 2.

Note (added June 4, 1963). The referee has pointed out that in **(15)** D. A. Suprunenko has given results corresponding to Theorem 1, but under slightly different hypotheses. It is noted in **(15)** that the direct part of Theorem 1 holds even when the condition of non-modularity is dropped. Specifically, if G is any solvable group of matrices over F and if every $x \in G$ is a d-matrix, then $G \in \mathfrak{N}$. This also follows from the proof given in Section 4 of the present paper provided it can be shown that, when F has characteristic $p \neq 0$, there is some abelian subgroup H of G whose index $|G:H|$ is finite and not divisible by p. In fact, it is easily shown that this condition holds for each maximal abelian subgroup H of finite index in G. For suppose the contrary; then for some $x \in G$, $x \notin H$ and $x^p \in H$. Since x is a d-matrix, we may transform the group so that x is diagonal: $x = \text{diag}(\alpha_1, \ldots, \alpha_n)$. But $\alpha_i^p = \alpha_j^p$ implies that $\alpha_i = \alpha_j$

(since F has characteristic p); therefore $Z(x; G) = Z(x^p; G) \supseteq H$. Hence H and x generate an abelian subgroup of G, contrary to the hypothesis that H is maximal.

On the other hand, the following examples show that there is some necessity for the conditions of non-modularity and solvability.

Example 1. A solvable group $G \in \mathfrak{R}$ for which not every element is a d-matrix.

Let F be any field of characteristic $p \neq 0$ and containing more than two elements. We define G as the monomial group of $p \times p$ matrices generated by a p-cycle permutation matrix u and the group of all non-singular diagonal matrices over F. Since the latter group is an abelian normal subgroup of index p in G, G is solvable. Furthermore, u is not a d-matrix. In fact, the minimal polynomial of u is $t^p - 1 = (t - 1)^p$, which does not have simple roots. We finally show that G is irreducible.

Let $e_1, \ldots, e_p$ be the standard basis of the vector space $\mathfrak{B}$ on which G acts. Since F has at least two non-zero elements, it is easily verified that any invariant subspace for H must contain at least one of these basis elements. Since the basis elements are permuted transitively by the powers of u, it therefore follows that the only non-zero invariant subspace of G is the whole space $\mathfrak{B}$. Thus G is irreducible.

Example 2. A solvable group G with an abelian subgroup A of finite index such that $A \in \mathfrak{R}$, but $G \notin \mathfrak{R}$.

Let F be a field of characteristic $p \neq 0$. Let P be the prime subfield of F generated by 0, 1. We define G as the group of 2×2 matrices over F which is generated by

$$u = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \text{(for all non-zero } \alpha, \beta \in F \text{ with } \alpha^{-1}\beta \in P).$$

The subgroup U which is generated by u is normal in G since

$$\begin{bmatrix} \alpha^{-1} & 0 \\ 0 & \beta^{-1} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} = \begin{bmatrix} 1 & \alpha^{-1}\beta \\ 0 & 1 \end{bmatrix}$$

and $\alpha^{-1}\beta \in P$. Hence the diagonal matrices in G form an abelian subgroup A of index p. $G/U \simeq A$, so G is solvable, and clearly $A \in \mathfrak{R}$, but $G \notin \mathfrak{R}$.

Example 3. A (non-solvable) group $G \in \mathfrak{R}$, over any field whose elements are not all d-matrices.

We define G as the group generated by

$$u = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and } v = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Since $\det(uv - vu) \neq 0$, these matrices have no common eigenvector. Thus G is irreducible, although u is not a d-matrix.

References

1. R. Baer, *Finiteness properties of groups*, Duke Math. J., *15* (1948), 1021–1032.
2. W. Burnside, *On criteria for the finiteness of the order of a group of linear substitutions*, Proc. Lond. Math. Soc. (2), *3* (1905), 435–440.
3. A. H. Clifford, *Representations induced in an invariant subgroup*, Ann. Math., *38* (1937), 533–550.
4. L. Fuchs, *Abelian groups* (Budapest, 1958).
5. M. Hall, *Theory of groups* (New York, 1959).
6. P. Hall, *Nilpotent groups*, Can. Math. Congress, Summer Seminar (1957).
7. B. Huppert, *Lineare auflösbare Gruppen*, Math. Z., *67* (1957), 479–518.
8. A. G. Kurosh, *Theory of groups* (New York, 1955).
9. A. I. Mal'cev, *On some classes of infinite solvable groups*, Mat. Sborn, N.S. *28*(70) (1951), 567–588 [Am. Math. Soc. Translations (2), *2* (1956), 1–22].
10. B. H. Neumann, *Groups with finite classes of conjugate elements*, Proc. Lond. Math. Soc. (3), *1* (1951), 178–187.
11. D. A. Suprunenko, *Solvable groups of matrices*, Doklady Akad. Nauk, SSSR, *83* (1952), 183–186 (Russian).
12. ——— *Locally nilpotent irreducible subgroups of the full linear group*, Doklady Akad. Nauk SSSR, *102* (1955), 41–44 (Russian).
13. ——— *On linear solvable groups*, Mat. Sborn., N.S. *41* (82) (1957), 317–322 (Russian).
14. ——— *Solvable and nilpotent linear groups* (Minsk, 1958) (Russian). [Transl. Math. Monographs, vol. 9 (Providence, 1963).]
15. ——— *The condition of complete reducibility of a solvable linear group*, Doklady Akad. Nauk BSSR, *5* (1961), 321–323 (Russian) (see Math. Rev., *25* (1963), No. 125).
16. D. A. Suprunenko and R. Tyshkevich, *Reducible locally nilpotent linear groups*, Izv. Akad. Nauk SSSR, *24* (1960), 787–806 (Russian).
17. H. Zassenhaus, *Beweis eines Satzes über diskrete Gruppen*, Abh. Math. Sem. Hamburg, *12* (1938), 289–312.

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