
Maximizing the Number of Independent Sets of a Fixed Size

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Let $i_t(G)$ be the number of independent sets of size t in a graph G . Engbers and Galvin asked how large $i_t(G)$ could be in graphs with minimum degree at least δ . They further conjectured that when $n \geq 2\delta$ and $t \geq 3$, $i_t(G)$ is maximized by the complete bipartite graph $K_{\delta, n-\delta}$. This conjecture has recently drawn the attention of many researchers. In this short note, we prove this conjecture.

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1. Introduction

Given a finite graph G , let $i_t(G)$ be the number of independent sets of size t in a graph, and let $i(G) = \sum_{t \geq 0} i_t(G)$ be the total number of independent sets. There are many extremal results on $i(G)$ and $i_t(G)$ over families of graphs with various degree restrictions. Kahn [6] and Zhao [12] studied the maximum number of independent sets in a d -regular graph. Relaxing the regularity constraint to a minimum degree condition, Galvin [5] conjectured that the number of independent sets in an n -vertex graph with minimum degree $\delta \leq n/2$ is maximized by a complete bipartite graph $K_{\delta, n-\delta}$. This conjecture was recently proved (in stronger form) by Cutler and Radcliffe [3] for all n and δ , and they characterized the extremal graphs for $\delta > n/2$ as well.

One can further strengthen Galvin's conjecture by asking whether the extremal graphs also simultaneously maximize the number of independent sets of size t , for all t . This claim unfortunately is too strong, as there are easy counterexamples for $t = 2$. On the

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other hand, no such examples are known for $t \geq 3$. Moreover, in this case Engbers and Galvin [4] made the following conjecture.

Conjecture 1.1. *For every $t \geq 3$ and $\delta \leq n/2$, the complete bipartite graph $K_{\delta, n-\delta}$ maximizes the number of independent sets of size t , over all n -vertex graphs with minimum degree at least δ .*

Engbers and Galvin [4] proved this for $\delta = 2$ and $\delta = 3$, and for all $\delta > 3$ they proved it when $t \geq 2\delta + 1$. Alexander, Cutler, and Mink [1] proved it for the entire range of t for bipartite graphs, but it appeared non-trivial to extend the result to general graphs. The first result for all graphs and all t was obtained by Law and McDiarmid [9], who proved the statement for $\delta \leq n^{1/3}/2$. This was improved by Alexander and Mink [2], who required that

$$\frac{(\delta + 1)(\delta + 2)}{3} \leq n.$$

In this short note, we completely resolve this conjecture.

Theorem 1.2. *Let $\delta \leq n/2$. For every $t \geq 3$, every n -vertex graph G with minimum degree at least δ satisfies $i_t(G) \leq i_t(K_{\delta, n-\delta})$, and when $t \leq \delta$, $K_{\delta, n-\delta}$ is the unique extremal graph.*

2. Proof

We will work with the complementary graph, and count cliques instead of independent sets. Cutler and Radcliffe [3] also discovered that the complement was more naturally amenable to extension; we will touch on this in our concluding remarks. Let us define some notation for use in our proof. A t -clique is a clique with t vertices. For a graph $G = (V, E)$, \bar{G} is its complement, and $k_t(G)$ is the number of t -cliques in G . For any vertex $v \in V$, $N(v)$ is the set of the neighbours of v , $d(v)$ is the degree of v , and $k_t(v)$ is the number of t -cliques which contain vertex v . Note that $\sum_{v \in V} k_t(v) = tk_t(G)$. We also define $G + H$ as the graph consisting of the disjoint union of two graphs G and H . By considering the complementary graph, it is clear that our main theorem is equivalent to the following statement.

Proposition 2.1. *Let $1 \leq b \leq \Delta + 1$. For all $t \geq 3$, $k_t(G)$ is maximized by $K_{\Delta+1} + K_b$, over $(\Delta + 1 + b)$ -vertex graphs with maximum degree at most Δ . When $t \leq b$, this is the unique extremal graph, and when $b < t \leq \Delta + 1$, the extremal graphs are $K_{\Delta+1} + H$, where H is an arbitrary b -vertex graph. \square*

Remark. When $b \leq 0$, the number of t -cliques in graphs with maximum degree at most Δ is trivially maximized by the complete graph. On the other hand, when $b > (\Delta + 1)$, the problem becomes much more difficult, and our investigation is still ongoing. This paper focuses on the first complete segment $1 \leq b \leq \Delta + 1$, which, as mentioned in the Introduction, was previously attempted in [2, 4, 9].

Although our result holds for all $t \geq 3$, it turns out that the main step is to establish it for the case $t = 3$ using induction and double-counting. Afterwards, a separate argument will reduce the general $t > 3$ case to this case of $t = 3$.

Lemma 2.2. *Proposition 2.1 is true when $t = 3$.*

Proof. We proceed by induction on b . The base case $b = 0$ is trivial. Now assume it is true for $b - 1$. Suppose first that $k_3(v) \leq \binom{b-1}{2}$ for some vertex v . Applying the inductive hypothesis to $G - v$, we see that

$$k_3(G) \leq k_3(G - v) + k_3(v) \leq \binom{\Delta + 1}{3} + \binom{b - 1}{3} + \binom{b - 1}{2} \leq \binom{\Delta + 1}{3} + \binom{b}{3},$$

and equality holds if and only if $G - v$ is optimal and $k_3(v) = \binom{b-1}{2}$. By the inductive hypothesis, $G - v$ is $K_{\Delta+1} + H'$, where H' is a $(b - 1)$ -vertex graph. The maximum degree restriction forces v 's neighbours to be entirely in H' , and so $G = K_{\Delta+1} + H$ for some b -vertex graph H . Moreover, since $k_3(v) = \binom{b-1}{2}$ we get that for $b \geq 3$, H is a clique.

This leaves us with the case where $k_3(v) > \binom{b-1}{2}$ for every vertex v , which forces $b \leq d(v) \leq \Delta$. We will show that here, the number of 3-cliques is strictly suboptimal. The number of triples (u, v, w) where uv is an edge and vw is not an edge is clearly $\sum_{i=1}^n d(v)(n - 1 - d(v))$. Also, every set of three vertices either contributes 0 to this sum (if either all or none of the three edges between them are present), or contributes 2 (if they induce exactly one or exactly two edges). Therefore, we obtain an equality which also appeared in [11]:

$$2 \left[\binom{n}{3} - (k_3(G) + k_3(\overline{G})) \right] = \sum_{v \in V} d(v)(n - 1 - d(v)).$$

Rearranging this equality and applying $k_3(\overline{G}) \geq 0$, we find

$$k_3(G) \leq \binom{n}{3} - \frac{1}{2} \sum_{v \in V} d(v)(n - 1 - d(v)). \tag{2.1}$$

Since we have already bounded $b \leq d(v) \leq \Delta$, and $b + \Delta = n - 1$ by definition, we have

$$d(v)(n - 1 - d(v)) \geq b\Delta.$$

Substituting this into (2.1) and using $n = (\Delta + 1) + b$,

$$k_3(G) \leq \binom{n}{3} - \frac{nb\Delta}{2} = \binom{\Delta + 1}{3} + \binom{b}{3} - \frac{b(\Delta + 1 - b)}{2} < \binom{\Delta + 1}{3} + \binom{b}{3},$$

because $b \leq \Delta$. This completes the case where every vertex has $k_3(v) > \binom{b-1}{2}$. □

We reduce the general case to the case of $t = 3$ via the following variant of the celebrated Kruskal–Katona theorem [7, 8], which appears as Exercise 31b in Chapter 13 of Lovász's book [10]. Here, the generalized binomial coefficient $\binom{x}{k}$ is defined to be the product

$$\frac{1}{k!} (x)(x - 1)(x - 2) \cdots (x - k + 1),$$

which exists for non-integral x .

Theorem 2.3. *Let $k \geq 3$ be an integer, and let $x \geq k$ be a real number. Then, every graph with exactly $\binom{x}{2}$ edges contains at most $\binom{x}{k}$ cliques of order k .*

We now use Lemma 2.2 and Theorem 2.3 to finish the general case of Proposition 2.1.

Lemma 2.4. *If Proposition 2.1 is true for $t = 3$, then it is also true for $t > 3$.*

Proof. Fix any $t \geq 4$. We proceed by induction on b . The base case $b = 0$ is trivial. For the inductive step, assume the result is true for $b - 1$. If there is a vertex v such that $k_3(v) \leq \binom{b-1}{2}$, then by applying Theorem 2.3 to the subgraph induced by $N(v)$, we find that there are at most $\binom{b-1}{t-1}$ cliques of order $t - 1$ entirely contained in $N(v)$. The t -cliques which contain v correspond bijectively to the $(t - 1)$ -cliques in $N(v)$, and so $k_t(v) \leq \binom{b-1}{t-1}$. The same argument used at the beginning of Lemma 2.2 then correctly establishes the bound and characterizes the extremal graphs.

If some $k_3(v) = \binom{\Delta}{2}$, then the maximum degree condition implies that the graph contains a $K_{\Delta+1}$ which is disconnected from the remaining $b \leq \Delta + 1$ vertices, and the result also easily follows. Therefore, it remains to consider the case where all $\binom{b-1}{2} < k_3(v) < \binom{\Delta}{2}$, in which we will prove that the number of t -cliques is strictly suboptimal. It is well known and standard that for each fixed k , the binomial coefficient $\binom{x}{k}$ is strictly convex and increasing in the real variable x on the interval $x \geq k - 1$. Hence, $\binom{k}{k} = 1$ implies that $\binom{x}{k} < 1$ for all $k - 1 < x < k$, and so Theorem 2.3 then actually applies for all $x \geq k - 1$. Thus, if we define $u(x)$ to be the positive root of $\binom{u}{2} = x$, that is,

$$u(x) = \frac{(1 + \sqrt{1 + 8x})}{2},$$

and let

$$f_t(x) = \begin{cases} 0 & \text{if } u(x) < t - 2, \\ \binom{u(x)}{t-1} & \text{if } u(x) \geq t - 2, \end{cases} \tag{2.2}$$

the application of the Kruskal–Katona theorem in the previous paragraph establishes that $k_t(v) \leq f_t(k_3(v))$.

We will also need that $f_t(x)$ is strictly convex for $x > \binom{t-2}{2}$. For this, observe that by the generalized product rule,

$$f'_t(x) = u' \cdot [(u - 1)(u - 2) \cdots (u - (t - 2)) + \cdots + u(u - 1) \cdots (u - (t - 3))],$$

which is $u'(x)$ multiplied by a sum of $t - 1$ products. Since

$$u'(x) = \frac{2}{\sqrt{1 + 8x}},$$

for any constant C ,

$$(u')(u - C) = 1 - \frac{(2C - 1)}{\sqrt{1 + 8x}}.$$

Note that this is a positive increasing function when $C \in \{1, 2\}$ and $x > \binom{t-2}{2}$. In particular, since $t \geq 4$, each of the $t - 1$ products contains a factor of $(u - 1)$ or $(u - 2)$, or possibly both; we can then always select one of them to absorb the (u') factor, and conclude that $f'_t(x)$ is the sum of $t - 1$ products, each of which is composed of $t - 2$ factors that are positive increasing functions on $x > \binom{t-2}{2}$. Thus $f_t(x)$ is strictly convex on that domain, and since $f_t(x) = 0$ for $x \leq \binom{t-2}{2}$, it is convex everywhere.

If $t = \Delta + 1$, there will be no t -cliques in G unless G contains a $K_{\Delta+1}$, which must be isolated because of the maximum degree condition; we are then finished as before. Hence we may assume $t \leq \Delta$ for the remainder, which in particular implies that $f_t(x)$ is strictly convex and strictly increasing in the neighbourhood of $x = \binom{\Delta}{2}$. Let the vertices be v_1, \dots, v_n , and define $x_i = k_3(v_i)$. We have

$$tk_t(G) = \sum_{v \in V} k_t(v) \leq \sum_{i=1}^n f_t(x_i),$$

and so it suffices to show that

$$\sum f_t(x_i) < t \binom{\Delta + 1}{t} + t \binom{b}{t}$$

under the following conditions, the latter of which comes from Lemma 2.2:

$$\binom{b-1}{2} < x_i < \binom{\Delta}{2}, \quad \sum_{i=1}^n x_i \leq 3 \binom{\Delta + 1}{3} + 3 \binom{b}{3}. \tag{2.3}$$

To this end, consider a tuple of real numbers (x_1, \dots, x_n) which satisfies the conditions. Although (2.3) constrains each x_i within an open interval, we will perturb the x_i within the closed interval which includes the endpoints, in such a way that the objective $\sum f_t(x_i)$ is non-decreasing, and we will reach a tuple which achieves an objective value of exactly $t \binom{\Delta+1}{t} + t \binom{b}{t}$. Finally, we will use our observation of strict convexity and monotonicity around $x = \binom{\Delta}{2}$ to show that one of the steps strictly increased $\sum f_t(x_i)$, which will complete the proof.

First, since the upper limit for $\sum x_i$ in (2.3) is achievable by setting $\Delta + 1$ of the x_i to $\binom{\Delta}{2}$ and b of the x_i to $\binom{b-1}{2}$, and $f_t(x)$ is non-decreasing, we may increase some of the x_i in the interval

$$\binom{b-1}{2} \leq x_i \leq \binom{\Delta}{2}$$

such that the inequality for $\sum x_i$ in (2.3) is tight. Next, by convexity of $f_t(x)$, we may push apart x_i and x_j while conserving their sum, and the objective is non-decreasing. After a finite number of steps, we arrive at a tuple in which all but at most one of the x_i is equal to either the lower limit $\binom{b-1}{2}$ or the upper limit $\binom{\Delta}{2}$, and

$$\sum x_i = 3 \binom{\Delta + 1}{3} + 3 \binom{b}{3}.$$

However, since this value of $\sum x_i$ is achievable by $\Delta + 1$ many $\binom{\Delta}{2}$ and b many $\binom{b-1}{2}$, this implies that in fact, the tuple of the x_i has precisely this form. (To see this, note that

by an affine transformation, the statement is equivalent to the fact that if n and k are integers, and $0 \leq y_i \leq 1$ are n real numbers which sum to k , all but one of which is at an endpoint, then exactly k of the y_i are equal to 1 and the rest are equal to 0.) Thus, our final objective is equal to

$$(\Delta + 1) \binom{\Delta}{t-1} + b \binom{b-1}{t-1} = t \binom{\Delta+1}{t} + t \binom{b}{t},$$

as claimed. Finally, since some x_i take the value $\binom{\Delta}{2}$, the strictness of $f_t(x)$'s monotonicity and convexity in the neighbourhood $x = \binom{\Delta}{2}$ implies that at some stage of our process, we strictly increased the objective. Therefore, in this case where all

$$\binom{b-1}{2} < k_3(v) < \binom{\Delta}{2},$$

the number of t -cliques is indeed sub-optimal, and our proof is complete. \square

3. Concluding remarks

The natural generalization of Proposition 2.1 considers the maximum number of t -cliques in graphs with maximum degree Δ and $n = a(\Delta + 1) + b$ vertices, where $0 \leq b < \Delta + 1$. In the language of independent sets, this question was also proposed by Engbers and Galvin [4]. The case $a = 0$ is trivial, and Proposition 2.1 completely solves the case $a = 1$. We believe that also for $a > 1$ and $t \geq 3$, $k_t(G)$ is maximized by $aK_{\Delta+1} + K_b$, over $(a(\Delta + 1) + b)$ -vertex graphs with maximum degree at most Δ .

An easy double-counting argument shows that it is true when $b = 0$. When $a \geq 2$ and $b > 0$, the problem seems considerably more delicate. Nevertheless, the same proof that we used in Lemma 2.4 (*mutatis mutandis*) shows that the general case $t > 3$ of this problem can be reduced to the case $t = 3$. Therefore, the most intriguing and challenging part is to show that $aK_{\Delta+1} + K_b$ maximizes the number of triangles over all graphs with $(a(\Delta + 1) + b)$ vertices and maximum degree at most Δ . We have some partial results on this main case, but our investigation is still ongoing.

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