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ON INTERVAL AND INSTANT AVAILABILITY OF THE SYSTEM

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This article considers the interval availability and instant availability of the k-system. A certain relationship between the two types of availability is established. Some lower and upper bounds to interval availability are derived. It also provides a couple of conditions under which the availability of two systems can be compared. Several examples are given to show the complexity of comparisons of availability.

Keywords: applied probability, reliability theory

1. INTRODUCTION

In engineering, it is common to consider various repair policies in order to keep the system highly reliable. One of these policies is perfect repair that repairs a failed system as good as new. However, it is often practically infeasible to implement perfect repair due to technical limitations. Moreover, comparing with other imperfect repair policies, the perfect repair usually takes longer time and is more costly. On the other hand, imperfect repair may reduce the lifetime of the system even though it takes shorter time and fewer resources. Taking care of these issues, Dorado et al. [3] proposed a *better-than-minimal repair model*. Biswas and Sarkar [2] further considered an intermediate repair policy in which several imperfect repairs on successive failures are allowed before a replacement with a new system or a perfect repair is performed. A more general model considered in Huang and Mi [5] combines the two models studied in Biswas and Sarkar [2] and can be described as follows.

Consider systems each of which has only two states at any given time, namely "up" if it is properly functioning and "down" if it is in the process of replacing or repairing. These systems will perform the same job one after one consecutively. Precisely, let $k \geq 0$ be a predetermined integer and $\{S_i, 1 \leq i \leq k+1\}$ be a sequence of independent systems. System S_1 starts working at time $t = 0$ until it fails and it will be replaced by system S_2. This pattern will be continued until the failure of system S_{k+1}. Denote the lifetime of S_i as U_i and the needed replacement time at its failure as D_i. Obviously, $(U_i, D_i),\ i \geq 1$ are independent since the systems $\{S_i, 1 \leq i \leq k+1\}$ are independent. We also assume that U_i and D_i are independent of each other. The time interval $(0, U_1 + D_1]$ is called as cycle $\mathbb{C}_1$, and time interval $[\sum_{i=1}^{j-1}(U_i + D_i), \sum_{i=1}^{j}(U_i + D_i)]$ is called as the jth cycle $\mathbb{C}_j, 2 \leq j \leq k+1$. Finally, we assume that the process mentioned above will be renewed at the end of cycle

$\mathbb{C}_{k+1}$, i.e., iid copies of consecutive cycles $\mathbb{C}_j, 1 \le j \le k+1$ will be continued accordingly at time $\sum_{i=1}^{k+1}(U_i + D_i)$, and the process develops in the same manner again and again as time goes. Note that the word "replace(ment)" is used in the above description, but it can readily be interpreted as "repair(ing)". The two words will be used alternately whenever it is convenient.

Denote the collection of the systems $\{S_i, 1 \le i \le k+1\}$ along with their corresponding lifetimes and repair times as $\mathbb{S}^{(k)}$ and call it as a k-system. The progress of $\mathbb{S}^{(k)}$ can be characterized by a stochastic process that is renewed after the completion of $k+1$ cycles. A special case of the k-system is the 0-system $\mathbb{S}^{(0)}$ which is actually characterized by the usual alternative renewal process $\{(U^{(n)}, D^{(n)})\}$ where $\{(U^{(n)}, D^{(n)})\}$ are iid copies of (U, D) that are the lifetime and repair time of system S_1.

At any time t, let binary random variable $X(t)$ denotes the state of k-system $\mathbb{S}^{(k)}$, that is $X(t) = 1$ means that a certain system S_i or its an iid copy is in the "up" state, and $X(t) = 0$ otherwise. The performance of $\mathbb{S}^{(k)}$ can be measured by two types of availability. The *instant availability* of $\mathbb{S}^{(k)}$ at time t is $A(t) \equiv P(X(t) = 1)$, and the *interval availability* of $\mathbb{S}^{(k)}$ at time t is $A_w(t) \equiv P(X(s) = 1, t \le s \le t+w)$ where $w \ge 0$ is a given constant. Clearly, $A_0(t) = A(t)$. Some recent works relevant to reliability engineering applications of system availability can be found in Levitin et al. [8], Liu et al. [9], Naseri et al. [11], Sabri-Laghaie and Noorossana [13], Zio [15], Du et al. [4], and references therein among others.

Throughout this article, we further assume that for any k-system $\mathbb{S}^{(k)}$, (i) U_i has CDF F_i and pdf f_i and (ii) D_i has CDF G_i but could be either a discrete or continuous random variable. Denote the CDF of $U_i + D_i$ as H_i. It is easy to see that each H_i must have density due to the property of convolution and the assumption that F_i has density. In the case of $k = 0$, we simply use the notation F, G, and H since subscript is unnecessary.

The article is organized as follows. Section 2 will explore the relationship between interval availability and instant availability of certain k-systems. Some bounds to the 0-system are studied in Section 3. Section 4 compares availabilities of two systems. Several examples focusing on comparisons of system availabilities are given in Section 5. The last section summarizes the results obtained in the article.

2. RELATIONSHIP BETWEEN INTERVAL AVAILABILITY AND INSTANT AVAILABILITY

It is obvious that $A_w(t) \le A(t)$ for any $t \ge 0$. Moreover, the following result is true.

THEOREM 2.1: *It holds for any k-system* $\mathbb{S}^{(k)}$ *that* $\lim_{w \to 0+} A_w(t) = A(t),\ \forall t \ge 0$ *and the convergence is monotonically increasing.*

PROOF: For any $0 \le w_1 < w_2$ clearly $(X(s) = 1, t \le s \le t + w_2) \subseteq (X(s) = 1, t \le s \le t + w_1)$ hence $A_{w_2}(t) \le A_{w_1}(t)$. Therefore, $A^*(t) \equiv \lim_{w \to 0+} A_w(t)$ must exist.

The interval availability of $\mathbb{S}^{(k)}$ was derived in Huang and Mi [5] as follows:

$$A_w(t) = c_w(t) + \int_0^t A_w(t-s)d\left(*_{j=1}^{k+1} H_j\right)(s) \tag{2.1}$$

where $H_j(t) \equiv (F_j * G_j)(t)$ is the convolution of F_j and G_j,

$$c_w(t) = \sum_{i=0}^{k} \left[\left(*_{j=0}^{i} H_j\right)(t) - \left(\left(*_{j=0}^{i} H_j\right) * F_{i+1}(w + \cdot)\right)(t)\right]. \tag{2.2}$$

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and

$$\left(\left(*_{j=0}^{i} H_j\right) * F_{i+1}(w+\cdot)\right)(t) \equiv \int_0^t F_{i+1}(w+t-s)d\left(*_{j=0}^{i} H_j\right)(s). \tag{2.3}$$

Particularly, the instant availability of the system is given as follows:

$$A(t) = c(t) + \int_0^t A(t-s)d\left(*_{j=1}^{k+1} H_j\right)(s) \tag{2.4}$$

where

$$\begin{aligned} c(t) &= \sum_{i=0}^{k}\left[\left(*_{j=0}^{i} H_j\right)(t) - \left(\left(*_{j=0}^{i} H_j\right) * F_{i+1}\right)(t)\right] \\ &= \overline{F}_1(t) + \sum_{i=1}^{k}\left[\left(*_{j=0}^{i} H_j\right)(t) - \left(\left(*_{j=0}^{i} H_j\right) * F_{i+1}\right)(t)\right]. \end{aligned} \tag{2.5}$$

From the above, we have

$$\begin{aligned} \lim_{w\to 0+}\left(\left(*_{j=0}^{i} H_j\right) * F_{i+1}(w+\cdot)\right)(t) &= \lim_{w\to 0+}\int_0^t F_{i+1}(w+t-s)d\left(*_{j=0}^{i} H_j\right)(s) \\ &= \int_0^t F_{i+1}(t-s)d\left(*_{j=0}^{i} H_j\right)(s) \\ &= \left(\left(*_{j=0}^{i} H_j\right) * F_{i+1}\right)(t) \end{aligned}$$

and consequently,

$$A^*(t) = c(t) + \int_0^t A^*(t-s)d\left(*_{j=1}^{k+1} H_j\right)(s). \tag{2.6}$$

However, Eq. (2.4) has a unique solution, so it must be true that $A^*(t) = A(t)$, $\forall t \geq 0$. The desired result thus follows. ■

THEOREM 2.2: *For any integer $k \geq 0$, let k-system $\mathbb{S}^{(k)}$ satisfies $F \equiv F_1 = F_2 = \ldots = F_{k+1}$. Then, the following are true for any $t \geq 0$.*

- *(a) If $F \in \mathrm{NBU}$ then $A_w(t) \leq A(t)\bar{F}(w)$;*
- *(b) If $F \in \mathrm{NWU}$ then $A_w(t) \geq A(t)\bar{F}(w)$;*
- *(c) If F has the exponential distribution with a failure rate λ, then $A_w(t) = A(t)\bar{F}(w) = A(t)e^{-\lambda w}$.*

PROOF: Under the assumption $F \equiv F_1 = F_2 = \ldots = F_{k+1}$, the functions $c_w(t)$ and $c(t)$ given in Eqs. (2.2) and (2.5) can be rewritten as follows:

$$\begin{aligned} c_w(t) &= [1 - F(t+w)] + \sum_{i=1}^{k}\left[\left(*_{j=1}^{i} H_j\right)(t) - \left(\left(*_{j=1}^{i} H_j\right) * F(w+\cdot)\right)(t)\right] \\ &= \bar{F}(t+w) + \sum_{i=1}^{k}\int_0^t [1 - F(t+w-s)]\, d\left(*_{j=1}^{i} H_j\right)(s) \\ &= \bar{F}(t+w) + \sum_{i=1}^{k}\int_0^t \bar{F}(t+w-s)d\left(*_{j=1}^{i} H_j\right)(s) \end{aligned} \tag{2.7}$$

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and

$$c(t) = \bar{F}(t) + \sum_{i=1}^{k} \int_0^t \bar{F}(t-s) d\big(*_{j=1}^{i} H_j \big)(s) \tag{2.8}$$

To prove result (a), note that $F \in \text{NBU}$ means $\bar{F}(x+y) \leq \bar{F}(x)\bar{F}(y),\ \forall x, y \geq 0$. The definitions of classes of lifetime distributions NBU (new better than used) and NWU (new worse than used) can be found, for example, in Barlow and Proschan [1].

The NBU assumption as well as Eqs. (2.7) and (2.8) immediately imply that

$$\begin{aligned} c_w(t) &\leq \bar{F}(t)\bar{F}(w) + \sum_{i=1}^{k} \int_0^t \bar{F}(t-s)\bar{F}(w) d\big(*_{j=1}^{i} H_j \big)(s) \\ &= \big[\bar{F}(t) + \sum_{i=1}^{k} \int_0^t \bar{F}(t-s) d\big(*_{j=1}^{i} H_j \big)(s) \big] \bar{F}(w) \\ &= c(t)\bar{F}(w). \end{aligned} \tag{2.9}$$

As shown in Eqs. (2.1) and (2.4), $A_w(t)$ is the unique solution of the equation

$$A_w(t) = c_w(t) + \int_0^t A_w(t-s) d\left(*_{i=1}^{k+1} H_i \right)(s) \tag{2.10}$$

and $A(t)$ is the unique solution of the following equation

$$A(t) = c(t) + \int_0^t A(t-s) d\left(*_{i=1}^{k+1} H_i \right)(s). \tag{2.11}$$

Consider the equation

$$u(t) = c(t)\bar{F}(w) + \int_0^t u(t-s) d\left(*_{i=1}^{k+1} H_i \right)(s), \quad \forall t \geq 0. \tag{2.12}$$

Clearly, $A(t)\bar{F}(w)$ is the unique solution of Eq. (2.12) since according to Eq. (2.11) it holds that

$$A(t)\bar{F}(w) = c(t)\bar{F}(w) + \int_0^t A(t-s)\bar{F}(w) d\left(*_{i=1}^{k+1} H_i \right)(s), \quad \forall t \geq 0. \tag{2.13}$$

Let $A_w^*(t) = A(t)\bar{F}(w)$ and $\Delta(t) \equiv A_w(t) - A_w^*(t)$. From Eqs. (2.10) and (2.13), we have

$$\Delta(t) = \delta(t) + \int_0^t \Delta(t-s) d\left(*_{i=1}^{k+1} H_i \right)(s), \quad \forall t \geq 0, \tag{2.14}$$

where $\delta(t) \equiv c_w(t) - c(t)\bar{F}(w)$. Obviously, $\delta(t) \leq 0, \forall t \geq 0$ (see Eq. (2.9)). The solution for the renewal Eq. (2.14) is given by (see, for example, [7])

$$\Delta(t) = \delta(t) + \int_0^t \delta(t-s) dM(s)$$

where $M(s) = \sum_{n=1}^{\infty} (*_{i=1}^{k+1} H_i)^{(n)}$ and $(*_{i=1}^{k+1} H_i)^{(n)}$ is the nth fold convolution of $*_{i=1}^{k+1} H_i$. Therefore, $\Delta(t) \leq 0, \forall t \geq 0$, and consequently $A_w(t) \leq A(t)\bar{F}(w)$.

If $F \in \text{NWU}$, then $\delta(t) \geq 0$, and consequently, $A_w(t) \geq A(t)\bar{F}(w)$, i.e., (b) is true.

Finally, if F is exponential, then F is both NBU and NWU, i.e., $\delta(t) \equiv \bar{F}(t+w) - \bar{F}(t)\bar{F}(w) = 0,\ \forall t, w \geq 0$. Therefore, $\Delta(t) = 0,\ \forall t \geq 0$. That is, $A_w(t) = A(t)\bar{F}(w)$. ■

Remark 2.3: In addition to the NBU and NWU classes, there are increasing (decreasing) failure rate IFR (DFR) class, increasing (decreasing) failure rate average IFRA (DFRA) class etc. Among these classes of lifetime distributions, the following inclusion relationship holds

$$\text{IFR} \subset \text{IFRA} \subset \text{NBU} \quad \text{and} \quad \text{DFR} \subset \text{DFRA} \subset \text{NWU}.$$

Moreover, the IFRA property is preserved under formation of the coherent system, both IFR and IFRA properties are preserved under convolution of distribution functions, and both DFR and DFRA properties are preserved under mixture of distributions. Hence, the NBU and NWU classes are huge and in fact most commonly applied lifetime distributions belong to the two classes. The readers are referred to Barlow and Proschan [1] for the details.

Remark 2.4: A special case was mentioned in Mathew and Balakrishna [10] that when both the lifetime U and repair time D are exponential, say $U \sim \text{Exp}(\lambda_1)$ and $D \sim \text{Exp}(\lambda_2)$ where λ_1 and λ_2 are hazard rates, then

$$A_w(t) = \left[\frac{\lambda_2}{\lambda_1+\lambda_2} + \frac{\lambda_1}{\lambda_1+\lambda_2} e^{-(\lambda_1+\lambda_2)t}\right] e^{-\lambda_1 w} = \left[\frac{\lambda_2}{\lambda_1+\lambda_2} + \frac{\lambda_1}{\lambda_1+\lambda_2} e^{-(\lambda_1+\lambda_2)t}\right] \bar{F}(w) = A(t)\bar{F}(w).$$

3. LOWER AND UPPER BOUNDS TO INTERVAL AVAILABILITY

For the rest of this article, we focus on the 0-system $\mathbb{S}^{(0)}$ and will just call it as system $\mathbb{S}$ for simplicity. In this case, Eqs. (2.1) and (2.4) are reduced to

$$A_w(t) = \bar{F}(t+w) + \int_0^t A_w(t-s)dH(s) \tag{3.1}$$

and

$$A(t) = \bar{F}(t) + \int_0^t A(t-s)dH(s) \tag{3.2}$$

where $H = F * G$.

THEOREM 3.1: *Suppose that F and $H = F * G$ have density functions f and h, and for a given $T > w$, the following conditions are true*

(a) $f(t)$ increases in $t \in [0, T-w]$;
(b) $h(t) \leq f(t), \forall t \in [0, T-w]$.

Then, $A_w(t)$ decreases in $t \in [0, T-w]$.

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Proof: Taking derivative with respect to t on both the sides of Eq. (3.1), we have

$$a_w(t) \equiv \frac{dA_w(t)}{dt} = -f(t+w) + A_w(0)h(t) + \int_0^t a_w(t-s)h(s)ds$$
$$= [\bar{F}(w)h(t) - f(t+w)] + \int_0^t a_w(t-s)h(s)ds.$$

From the assumptions, it is easy to see that for any $t \in [0, T-w]$

$$c(t) \equiv \bar{F}(w)h(t) - f(t+w) \leq \bar{F}(w)h(t) - f(t) \leq 0, \quad \forall t \in [0, T-w].$$

From Karlin and Taylor [7], we have

$$a_w(t) = \int_0^t c(t-s)dM(s) \leq 0, \quad \forall t \in [0, T-w],$$

where $M(s) \equiv \sum_{i=1}^{\infty} H^{(i)}(s)$ and $H^{(i)}(\cdot)$ is the ith fold convolution of $H(\cdot)$. This immediately implies $a_w(t) \leq 0,\ \forall t \in [0, T-w]$ and consequently $A_w(t)$ decreases in $t \in [0, T-w]$. ■

Remark 3.2: Let both $f(\cdot)$ and $h(\cdot)$ be continuous on $(0, \infty)$. Denote $T_0 \equiv \inf\{t > 0 :\ f(t) = h(t)\}$. It was shown in Huang and Mi [6] that if $T_0 > 0$, then $f(t) > h(t),\ \forall t \in (0,\ T_0)$. Moreover, it is easy to see that $h(0) = 0$ is a sufficient condition for $f(t) > h(t)$ to be true in a neighborhood of $t = 0$.

As a matter of fact, due to the obvious order $U \leq_{\rm st} U + D$ so if there is only one crossing of the two density functions $f(t)$ and $h(t)$, then it must be true that $f(t) > h(t)$ before crossing.

Theorem 3.3: *For any given $T > 0$, it holds that*

$$\min_{0 \leq x \leq T} \frac{\bar{F}(x+w)}{\bar{H}(x)} \leq A_w(t) \leq \max_{0 \leq x \leq T} \frac{\bar{F}(x+w)}{\bar{H}(x)}, \quad \forall t \in [0, T].$$

Proof: Let constant c be defined as follows:

$$c \equiv \min_{0 \leq x \leq T} \frac{\bar{F}(x+w)}{\bar{H}(x)}.$$

From the fact that the following equation has a unique solution which is finite on any finite interval

$$u(t) = c\bar{H}(t) + \int_0^t u(t-s)dH(s) \tag{3.3}$$

We see that the unique solution of Eq. (3.3) is $u(t) = c,\ \forall t \geq 0$. Denote $\delta(t) \equiv A_w(t) - u(t)$. It follows that

$$\delta(t) = [\bar{F}(t+w) - c\bar{H}(t)] + \int_0^t \delta(t-s)dH(s).$$

This immediately implies $\delta(t) \geq 0,\ \forall t \in [0, T]$ since $\bar{F}(t+w) - c\bar{H}(t) \geq 0,\ \forall t \in [0, T]$. That is $A_w(t) \geq c,\ \forall t \in [0, T]$.

The another claimed inequality can be shown in a similar way. ■

COROLLARY 3.4: *For any given $T > 0$, it holds that*

$$\min_{0\le x\le T}\frac{\bar F(x)}{\bar H(x)} \le A(t) \le \max_{0\le x\le T}\frac{\bar F(x)}{\bar H(x)}, \quad \forall t\in[0,T].$$

The usual stochastic ordering $\le_{\mathrm{st}}$ and hazard rate ordering $\le_{\mathrm{hr}}$ will be used in this and later sections. Their definitions and properties can be found, for instance, in Ross [12] and Shaked and Shanthikumar [14].

LEMMA 3.5: *Suppose that non-negative random variable X has CDF and pdf and $F \in$ IFR. Then, for any non-negative random variable Y, it holds that $X \le_{\mathrm{hr}} X + Y$, where $\le_{\mathrm{hr}}$ is the hazard rate order.*

PROOF: Let $Z = X + Y$. We will denote the CDF, pdf, and the hazard rate of X as F_X, f_X, and r_X, respectively. Similarly, we let F_Y and F_Z be the CDFs of Y and Z, respectively.

In order to show that $X \le_{\mathrm{hr}} Z$, it suffices to show that for any $0 \le t_1 \le t_2$, it holds that

$$\bar F_Z(t_2)\bar F_X(t_1) - \bar F_Z(t_1)\bar F_X(t_2) \ge 0. \tag{3.4}$$

We have

$$\begin{aligned}
&\bar F_Z(t_2)\bar F_X(t_1) - \bar F_Z(t_1)\bar F_X(t_2)\\
&\quad= \int_0^\infty \bar F_X(t_1)\bar F_X(t_2-u)dF_Y(u) - \int_0^\infty \bar F_X(t_2)\bar F_X(t_1-u)dF_Y(u)\\
&\quad= \int_0^\infty \bar F_X(t_1-u)\bar F_X(t_2-u)\left[\frac{\bar F_X(t_1)}{\bar F_X(t_1-u)} - \frac{\bar F_X(t_2)}{\bar F_X(t_2-u)}\right]dF_Y(u).
\end{aligned} \tag{3.5}$$

Note that

$$\frac{\bar F_X(t_1)}{\bar F_X(t_1-u)} = \exp\left\{-\int_{t_1-u}^{t_1} r_X(s)ds\right\}$$

and

$$\frac{\bar F_X(t_2)}{\bar F_X(t_2-u)} = \exp\left\{-\int_{t_2-u}^{t_2} r_X(s)ds\right\}.$$

Hence, in order to show Eq. (3.4), we need only to show

$$\exp\left\{\int_{t_1-u}^{t_1} r_X(s)ds\right\} \le \exp\left\{\int_{t_2-u}^{t_2} r_X(s)ds\right\}$$

or equivalently

$$\int_0^u r_X(w+t_1-u)dw \le \int_0^u r_X(w+t_2-u)dw. \tag{3.6}$$

The inequality Eq. (3.6) is certainly true since $F \in$ IFR. Therefore, Eq. (3.4) is true and consequently the desired result follows. ■

THEOREM 3.6: *Suppose that $F \in$ IFR. Then, for any $T \ge 0$, the following holds*

$$\frac{\bar F(T+w)}{\bar H(T)} \le A_w(t) \le \bar F(w), \quad \forall t\in[0,T].$$

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PROOF: We have

$$\frac{d}{dx}\left(\frac{\bar{F}(x+w)}{\bar{H}(x)}\right) = \frac{-f(x+w)\bar{H}(x)+\bar{F}(x+w)h(x)}{\bar{H}^2(x)} = \frac{\bar{F}(x+w)\bar{H}(x)[r_H(x)-r_F(x+w)]}{\bar{H}^2(x)},$$

where $r_F(x)$ and $r_H(x)$ are the hazard rate functions of F (or U) and H (or $U+D$), respectively.

Clearly, $r_F(x) \leq r_F(x+w)$ since $F \in$ IFR. By Lemma 1, it holds that $r_H(x) \leq r_F(x)$. Thus, we obtain $r_H(x) \leq r_F(x+w)$. This implies that

$$\frac{d}{dx}\left(\frac{\bar{F}(x+w)}{\bar{H}(x)}\right) \leq 0, \quad \forall x \geq 0.$$

That is, $\bar{F}(x+w)/\bar{H}(x)$ decreases in $x \geq 0$. Therefore,

$$\min_{0\leq x\leq T}\frac{\bar{F}(x+w)}{\bar{H}(x)} = \frac{\bar{F}(T+w)}{\bar{H}(T)} \quad \text{and} \quad \max_{0\leq x\leq T}\frac{\bar{F}(x+w)}{\bar{H}(x)} = \bar{F}(w).$$

Therefore, the result of the theorem follows. ■

Remark 3.7: The upper bound to $A_w(t)$ claimed in Theorem 5 is actually true for any $F \in$ NBU. To see this recall that by Theorem 4

$$A_w(t) \leq \max_{0\leq x\leq T}\frac{\bar{F}(x+w)}{\bar{H}(x)}, \quad \forall t \in [0,T].$$

Note that $\bar{F}(x+w) \leq \bar{F}(x)\bar{F}(w)$ since $F \in$ NBU. Furthermore, $\bar{F}(x)/\bar{H}(x) \leq 1, \forall x \geq 0$ since $U \leq_{\text{st}} U+D$. Therefore,

$$A_w(t) \leq \max_{0\leq x\leq T}\frac{\bar{F}(x+w)}{\bar{H}(x)} \leq \max_{0\leq x\leq T}\frac{\bar{F}(x)\bar{F}(w)}{\bar{H}(x)} \leq \bar{F}(w), \quad \forall t \in [0,T].$$

4. COMPARISONS OF INTERVAL AVAILABILITY

Let $\mathbb{S}$ and $\mathbb{S}^*$ be two repairable systems. Denote the lifetimes and repair times of $\mathbb{S}$ and $\mathbb{S}^*$ as U, D and U^*, D^*, respectively. Denote the interval availability of $\mathbb{S}$ and $\mathbb{S}^*$ as $A_w(t)$ and $A_w^*(t)$. We will compare these interval availability functions under certain conditions in this subsection.

The following two results can be shown in the similar way as Theorems 6 and 7 in Huang and Mi [6] and so their proofs are omitted.

THEOREM 4.1: *Suppose $U+D =_{\text{st}} U^*+D^*$ and $U \leq_{\text{st}} U^*$, here $\leq_{\text{st}}$ is the usual stochastic ordering. Then $A_w(t) \leq A_w^*(t)$, $\forall t \geq 0$.*

THEOREM 4.2: *Let $T \geq w$ be such that $\bar{F}(t) \leq \bar{F}^*(t)$ and $h(t) \leq h^*(t)$ for any $t \in [0,T)$, where $h(t)$ and $h^*(t)$ are the density functions of convolutions $H = F * G$ and $H = F^* * G^*$. Then $A_w(t) \leq A_w^*(t), \forall t \in [0, T-w)$.*

DEFINITION 4.3: *Let $A_w(t)$ be the interval availability of repairable system $\mathbb{S}$. The function defined by*

$$K_w(x) \equiv \int_0^x A_w(t)dt, \quad x \geq 0$$

is called the cumulative interval availability of $\mathbb{S}$.

THEOREM 4.4: *Suppose that $F \leq_{\rm st} F^*$ and $H \geq_{\rm st} H^*$. Then, the cumulative interval availability of $\mathbb{S}$ is less than or equal to that of $\mathbb{S}^*$ on any interval $(0, x)$, that is $K_w(x) \leq K_w^*(x), \forall x \in [0, \infty)$.*

PROOF: From Eq. (3.1), we have

$$\begin{aligned} K_w(x) &= \int_0^x \bar{F}(t+w)dt + \int_0^x \left(\int_0^t A_w(t-s)dH(s)\right)dt \\ &= \int_0^x \bar{F}(t+w)dt + \int_0^x \left(\int_s^x A_w(t-s)dt\right)dH(s) \\ &= \int_0^x \bar{F}(t+w)dt + \int_0^x \left(\int_0^{x-s} A_w(u)du\right)dH(s) \\ &= \int_0^x \bar{F}(t+w)dt + \int_0^x K_w(x-s)dH(s) \\ &= \int_0^x \bar{F}(t+w)dt + (H * K_w)(x). \end{aligned} \tag{4.1}$$

Let $c(x) = \int_0^x \bar{F}(t+w)dt$, Again, from Karlin and Taylor [7], we have

$$K_w(x) = c(x) + \big(M * c\big)(x) = c(x) + \sum_{i=1}^{\infty} \big(H^{(i)} * c\big)(x) \tag{4.2}$$

where $H^{(i)}$ is the ith convolution of H. Similarly, we can obtain

$$K_w^*(x) = c^*(x) + \big(M^* * c^*\big)(x) = c^*(x) + \sum_{i=1}^{\infty} \big(H^{*(i)} * c^*\big)(x), \tag{4.3}$$

where $c^*(x) = \int_0^x \bar{F}^*(t+w)dt$.

Note that

$$\big(H^{(i)} * c\big)(x) = \int_0^x c(x-s)dH^{(i)}(s) = \int_0^{\infty} c(x-s)I_{(0,x)}(s)dH^{(i)}(s)$$

where $I_B(t)$ is the indicator function of set $B \subset (0, \infty)$. Clearly, $H^{(i)} \geq_{\rm st} H^{*(i)}$ for any $i \geq 1$ since $H \geq_{\rm st} H^*$. Moreover, $c(x-s)I_{(0,x)}(s)$ is a decreasing function of s since $c(t)$ is an increasing function of $t \geq 0$. Hence,

$$\big(H^{(i)} * c\big)(x) \leq \big(H^{*(i)} * c\big)(x), \quad \forall x \geq 0, i \geq 1.$$

Therefore, from Eqs. (4.2) and (4.3), we see that

$$K_w(x) \leq K_w^*(x), \quad \forall x \geq 0$$

since $c(x) = \int_0^x \bar{F}(t+w)dt \leq \int_0^x \bar{F}^*(t+w)dt = c^*(x)$. ■

5. EXAMPLES

In the following examples, we assume that the system lifetime $U \sim \text{Gamma}(\alpha, \lambda)$ has density function

$$f(t; \alpha, \lambda) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} t^{\alpha-1} e^{-\lambda t}, \quad t \geq 0$$

with various shape parameter α and scale parameter λ. Obviously, the $\text{Gamma}(1, \lambda)$ distribution is exactly the exponential distribution $\text{Exp}(\lambda)$ with density function

$$f(t; \lambda) = \lambda e^{-\lambda t}, \quad t \geq 0.$$

Note that even the facts revealed by the examples in this section are given in terms of instant availability they are also valid for interval availability with sufficient small $w > 0$ due to Theorem 1.

EXAMPLE 5.1: *Let system* $\mathbb{S}^*$ *have lifetime* $U^* \sim \text{Exp}(\lambda_1^*)$ *and repair time* $D^* \sim \text{Exp}(\lambda_2^*)$, *and system* $\mathbb{S}$ *have lifetime* $U \sim \text{Exp}(\lambda_1)$ *and repair time* $D \sim \text{Exp}(\lambda_2)$. *We claim that if* $\lambda_1^* < \lambda_1$ *and* $\lambda_2^* > \lambda_2$, *then it holds that* $A^*(t) > A(t),\ \forall t > 0$.

We have

$$A_w(t) = \frac{\lambda_2}{\lambda_1 + \lambda_2} + \frac{\lambda_1}{\lambda_1 + \lambda_2} e^{-(\lambda_1+\lambda_2)t}.$$

So, if we define $\delta(t) \equiv A^*(t) - A(t)$, *then* $\delta(0) = 0$. *Note that* $\lambda_1^*/\lambda_2^* < \lambda_1/\lambda_2$, *hence*

$$A^*(\infty) = \frac{\lambda_2^*}{\lambda_1^* + \lambda_2^*} > \frac{\lambda_2}{\lambda_1 + \lambda_2} = A(\infty).$$

The derivative of $\delta(t)$ *with respect to time* t *is*

$$\delta'(t) = -\lambda_1^* e^{-(\lambda_1^*+\lambda_2^*)t} + \lambda_1 e^{-(\lambda_1+\lambda_2)t}.$$

Hence, $\delta'(t) > 0$ *if and only if*

$$\lambda_1 e^{-(\lambda_1+\lambda_2)t} > \lambda_1^* e^{-(\lambda_1^*+\lambda_2^*)t}$$

or

$$e^{[(\lambda_1+\lambda_2)-(\lambda_1^*+\lambda_2^*)]t} < \frac{\lambda_1}{\lambda_1^*}.$$

Case 1: $\lambda_1 + \lambda_2 \leq \lambda_1^* + \lambda_2^*$

In this case, obviously $\delta'(t) > 0,\ \forall t \geq 0$ *since* $\lambda_1/\lambda_1^* > 1$. *It certainly implies that* $\delta(t) > 0$ *or* $A^*(t) > A(t), \forall t > 0$.

Case 2: $\lambda_1 + \lambda_2 > \lambda_1^* + \lambda_2^*$

Under this condition, we see that $\delta'(t) > 0$ *if and only if*

$$t < \frac{\ln(\lambda_1/\lambda_1^*)}{(\lambda_1+\lambda_2) - (\lambda_1^*+\lambda_2^*)} \equiv t_0.$$

Obviously, $t_0 > 0$. *Hence, we have*

$$\delta'(t) \quad \begin{cases} > 0 & \text{if } 0 < t < t_0 \\ = 0 & \text{if } t = t_0 \\ < 0 & \text{if } t_0 < t < \infty \end{cases}$$

Note that $\delta(\infty) = A^*(\infty) - A(\infty) > 0$, *so it must be true that* $\delta(t) > 0$ *or* $A^*(t) > A(t),\ \forall t > 0$.

In Example 5.1, when both lifetime and repair time are exponential, we see that $U^* \geq_{\text{st}} U$ and $D^* \leq_{\text{st}} D$. Intuitively, one believes that the system $\mathbb{S}^*$ is "stronger" than system $\mathbb{S}$ in certain sense and so it is expected that $A^*(t) > A(t), \forall t > 0$ and this is actually true theoretically. However, if the conditions $U^* \geq_{\text{st}} U$ and $D^* \leq_{\text{st}} D$ do not hold simultaneously, then the result in Example 5.1 will no longer be true as shown in the examples below.

Example 5.2: *Let system* $\mathbb{S}^*$ *has lifetime* $U^* \sim \text{Exp}(1.4)$ *and repair time* $D^* \sim \text{Exp}(1.6)$*, and system* $\mathbb{S}$ *has lifetime* $U \sim \text{Exp}(1)$ *and repair time* $D \sim \text{Exp}(1)$*. From these, we see that* $U^* \leq_{\text{st}} U$ *and* $D^* \leq_{\text{st}} D$*. It means that the lifetime of system* $\mathbb{S}^*$ *is shorter than that of system* $\mathbb{S}$*, and the repair time of* $\mathbb{S}^*$ *is also shorter than that of* $\mathbb{S}$ *in the sense of the usual stochastic ordering. We will show that* $A^*(t)$ *is not always greater or smaller than* $A(t)$ *in this case. Clearly,*

$$A^*(t) = \frac{1.6}{3} + \frac{1.4}{3} e^{-3t}$$

and

$$A(t) = \frac{1}{2} + \frac{1}{2} e^{-2t}.$$

Define $\delta(t) \equiv A^*(t) - A(t)$*. We have* $\delta(0) = 0$ *and*

$$\delta'(t) = -1.4 e^{-3t} + e^{-2t}.$$

It is easy to see that $\delta'(t) \leq 0$ *if and only if*

$$e^{-2t} \leq 1.4 e^{-3t}, \quad \text{or} \quad e^t \leq 1.4, \quad \text{or} \quad t \leq \ln 1.4 \approx 0.3364.$$

It can also be verified that $\delta(t) = 0$ *has a unique solution* $t_0 \approx 1.1863$ *in* $(0, \infty)$*. Therefore,* $\delta(0) = \delta(t_0) = 0$,

$$\delta(t) < 0 \quad \forall t \in (0, t_0) \quad \text{and} \quad \delta(t) > 0 \quad \forall t \in (t_0, \infty).$$

The graph of the function $\delta(t)$ *is shown in Figure 1. Hence,* $A^*(0) = A(0)$, $A^*(t_0) = A(t_0) \approx 0.5466$,

$$A^*(t) < A(t) \quad \forall t \in (0, t_0) \quad \text{and} \quad A^*(t) > A(t) \quad \forall t \in (t_0, \infty).$$

By the way $A^*(\ln 1.4) = 0.7034$, $A(\ln 1.4) = 0.7551$, *and* $\delta(\ln 1.4) = -0.05170 < 0$.

Example 5.3: $U \sim \text{Gamma}(2, \lambda)$ *and* $D \sim \text{Exp}(\lambda)$*.*

$$A(t; \lambda) = \frac{2}{3} + \left[\frac{\sqrt{3}}{3} \sin\left(\frac{\sqrt{3}}{2} \lambda t \right) + \frac{1}{3} \cos\left(\frac{\sqrt{3}}{2} \lambda t \right) \right] e^{-\frac{3}{2}\lambda t}.$$

$U^* \sim \text{Gamma}(3, \lambda)$ *and* $D^* \sim \text{Exp}(\lambda)$*.*

$$A^*(t; \lambda) = \frac{3}{4} + \frac{1}{4} e^{-2\lambda t}.$$

In this example, clearly $U^* \geq_{\text{st}} U$, $D^* =_{\text{st}} D$, *and* $A^*(\infty; \lambda) = \frac{3}{4} > \frac{2}{3} = A(\infty; \lambda)$*. However, these do not guarantee that* $A^*(t; \lambda)$ *is always higher than* $A(t; \lambda)$ *for all* t *as shown below.*

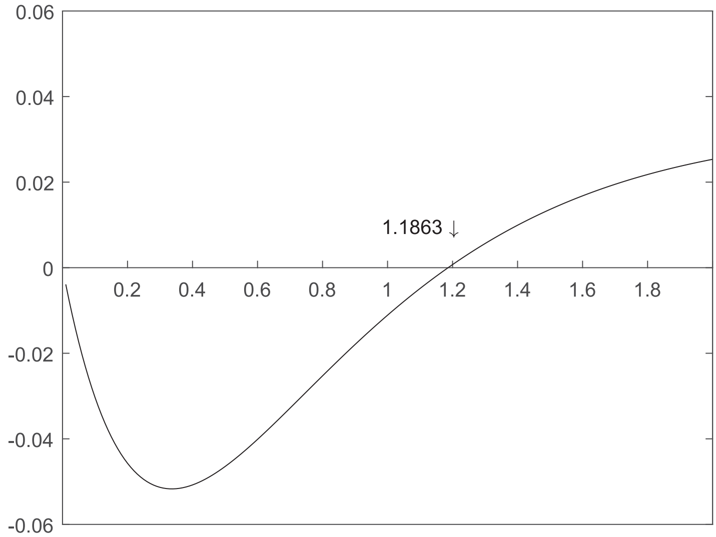

FIGURE **1.** Example 5.2.

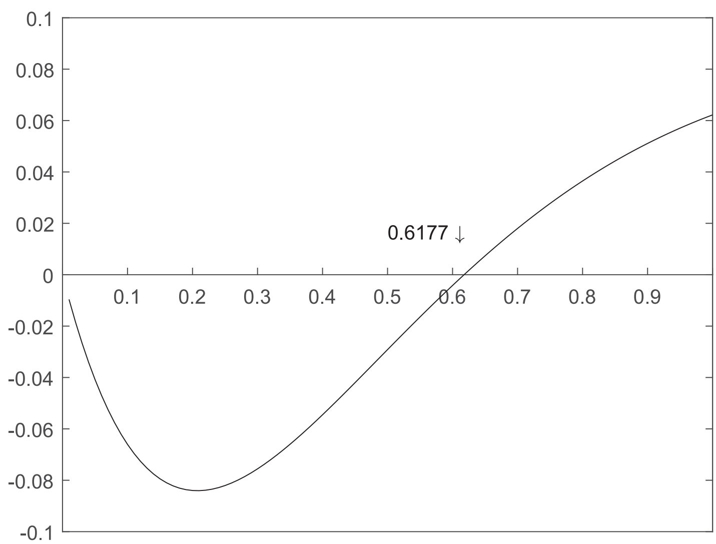

FIGURE **2.** Example 5.3.

Suppose $\lambda = 2$. *Define*

$$\Delta(t) \equiv A^*(t;2) - A(t;2).$$

Based on numerical computation, we can obtain the graph of $\Delta(t)$ *displayed in Figure 2. Let* $t_0 = 0.6177$, *then the graph shows that*

$$A^*(t;2) \begin{cases} < A(t;2) & \text{if } 0 < t < t_0 \\ = A(t;2) & \text{if } t = t_0 \\ > A(t;2) & \text{if } t_0 < t < \infty \end{cases}$$

Example 5.4: *Let*

$$U \sim \text{Gamma}(3, \lambda) \quad \textit{and} \quad D \sim \text{Exp}(\lambda).$$

It can be obtained that

$$A(t; \lambda) = \frac{3}{4} + \frac{1}{4} e^{-2\lambda t}.$$

Obviously, for any fixed $t > 0$*, the availability* $A(t; \lambda)$ *is a strictly decreasing function of* $\lambda > 0$.

Now let

$$U^* \sim \text{Gamma}(3, \lambda^*) \quad \textit{and} \quad D^* \sim \text{Exp}(\lambda^*),$$

$t = 1$, $\lambda = 1$*, and* $\lambda^* = 2$*. Then, we have*

$$A(1; 1) = \frac{3}{4} + \frac{1}{4} e^{-2} > \frac{3}{4} + \frac{1}{4} e^{-4} = A^*(1; 2).$$

This example indicates that even the repair time D *of* $\mathbb{S}$ *is longer than that of* $\mathbb{S}^*$*, the availability of* $\mathbb{S}$ *can still be higher than that of* $\mathbb{S}^*$ *for some time* t *if the lifetime* U *of* $\mathbb{S}$ *is longer than that of* $\mathbb{S}^*$.

6. SUMMARY

The interval availability of k-system $\mathbb{S}^{(k)}$ and particularly system $\mathbb{S}^{(0)}$ is studied in this article.

It is first established that the instant availability $A(t)$ of any k-system is the limit of its interval availability $A_w(t)$ when $w \to 0+$. This result allows one to derive various properties of $A_w(t)$ based on properties of $A(t)$ and so provides a lot of convenience of the studies in this article. Furthermore, the relationship between $A_w(t)$ and $A(t)$ is also obtained in the case of identical lifetime distributions of all systems S_i, $1 \leq k+1$. It is still an open question that whether the relationship in Theorem 2 holds when the F_i's are not all the same.

Both the lower and upper bounds of interval availability $A_w(t)$ of system $\mathbb{S}^{(0)}$ are derived in Section 3. It may provide certain useful or interesting information for practitioners. However, it is unclear that if similar bounds can be derived for more general k-system $\mathbb{S}^{(k)}$ with $k \geq 1$.

Some comparison results of instant and interval availabilities are obtained in terms of the usual stochastic order for different systems. Several examples are also presented for illustrating the theorems.

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