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HYPERSTABILITY OF GENERALISED LINEAR FUNCTIONAL EQUATIONS IN SEVERAL VARIABLES

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Abstract

Zhang ['On hyperstability of generalised linear functional equations in several variables', *Bull. Aust. Math. Soc.* **92** (2015), 259–267] proved a hyperstability result for generalised linear functional equations in several variables by using Brzdęk's fixed point theorem. We complete and extend Zhang's result. We illustrate our results for general linear equations in two variables and Fréchet equations.

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1. Introduction

The stability of functional equations was proposed by Ulam [13] in 1940 and was partially solved by Hyers [9] in 1941. Since then many mathematicians have investigated this topic. We study hyperstability by asking: *When is it true that a function approximately satisfying a given functional equation must also be a solution of the equation?*

Throughout the paper, we assume that $\mathbb{F}, \mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, where $\mathbb{R}$ and $\mathbb{C}$ are the sets of all real numbers and complex numbers, respectively. We also assume that $\mathbb{N}$ and $\mathbb{R}_+$ are the sets of all positive integers and nonnegative real numbers, respectively. Suppose that X and Y are normed spaces over the scalar fields $\mathbb{F}$ and $\mathbb{K}$, respectively.

We are interested in the *generalised linear functional equation* (see [3, 4, 14])

$$\sum_{i=1}^{M} L_i f\Big(\sum_{j=1}^{N} a_{i,j} x_j\Big) = 0,$$

where $f : X \to Y$, $a_{i,j} \in \mathbb{F}$, $L_i \in \mathbb{K} \setminus \{0\}$ for $i = 1, \ldots, M$ and $j = 1, \ldots, N$. We assume that $M \geq 2$ and $N \geq 2$. This equation includes various well-known functional equations, including:

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- *the general linear equation*: $f(ax + by) = Af(x) + Bf(y)$;
- *the Fréchet equation*: $f(x + y + z) + f(x) + f(y) + f(z) = f(x + y) + f(x + z) + f(y + z)$. (A generalised version of the Fréchet equation with constant coefficients was studied by Brzdęk *et al.* [8] and Malejki [10].)

A further generalisation of the functional equation in this paper was proposed and studied by Bahyrycz *et al.* [1].

Stability and hyperstability results for the aforementioned equations have been investigated by many mathematicians (see, for example, [2, 12] and references therein). The starting point of this work is Zhang's result [14]. To state his result (Theorem Z below), we recall the following condition on the matrix $[a_{i,j}]_{M\times N}$.

Zhang's condition.

(1) *For each* $i \in \{1, \ldots, M\}$, *there exists* $j \in \{1, \ldots, N\}$ *such that* $a_{i,j} \neq 0$.
(2) *There exist* $i_0 \in \{1, \ldots, M\}$ *and two different indices* $j_1, j_2 \in \{1, \ldots, N\}$ *such that* $a_{i_0,j_1} \neq 0$, $a_{i_0,j_2} \neq 0$ *and, for any* $i \neq i_0$, $\gamma \neq 0$, *there is* $j \in \{1, \ldots, N\}$ *satisfying* $a_{i,j} \neq \gamma a_{i_0,j}$.

Theorem Z. *Suppose that* $[a_{i,j}]_{M\times N}$ *satisfies Zhang's condition. Suppose further that* $\varphi : (X \setminus \{0\})^N \to \mathbb{R}_+$ *and* $f : X \to Y$ *satisfy the inequality*

$$\bigg\| \sum_{i=1}^{M} L_i f\Big(\sum_{j=1}^{N} a_{i,j}x_j\Big)\bigg\| \le \varphi(x_1, \ldots, x_N) \quad \text{for all } x_1, \ldots, x_N \in X \setminus \{0\}.$$

Suppose that $C \ge 0$. *If either* $\varphi(x_1, \ldots, x_N) := C\sum_{j=1}^{N} \|x_j\|^p$ *where* $p < 0$ *or* $\varphi(x_1, \ldots, x_N) := C\prod_{j=1}^{N} \|x_j\|^{p_j}$ *where* $p_1 + \cdots + p_N < 0$, *then*

$$\sum_{i=1}^{M} L_i f\Big(\sum_{j=1}^{N} a_{i,j}x_j\Big) = 0 \quad \text{for all } x_1, \ldots, x_N \in X \setminus \{0\}.$$

Zhang's proof of Theorem Z is based on Brzdęk's fixed point theorem [7], but it is not complete. In fact, for a sufficiently large integer t, it follows from Brzdęk's fixed point theorem that there exists a function f_t defined on $X \setminus \{0\}$ (not on X) such that

$$\sum_{i=1}^{M} L_i f_t\Big(\sum_{j=1}^{N} a_{i,j}x_j\Big) = 0 \quad \text{for all } x_1, \ldots, x_N \in X \setminus \{0\}.$$

Since each f_t is defined for nonzero elements in X, it follows that

$$\sum_{i=1}^{M} L_i f_t\Big(\sum_{j=1}^{N} a_{i,j}x_j\Big) = 0 \quad \text{for all } x_1, \ldots, x_N \in X \setminus \{0\} \text{ with } \textstyle\sum_{j=1}^{N} a_{i,j}x_j \neq 0 \text{ for all } i = 1, \ldots, M.$$

By following the remaining part of Zhang's proof, we see that f satisfies

$$\sum_{i=1}^{M} L_i f\Big(\sum_{j=1}^{N} a_{i,j}x_j\Big) = 0 \quad \text{for all } x_1, \ldots, x_N \in X \setminus \{0\} \text{ with } \textstyle\sum_{j=1}^{N} a_{i,j}x_j \neq 0 \text{ for all } i = 1, \ldots, M.$$

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The aim of this paper is to give a different proof of Theorem Z. Moreover, we use a weaker assumption and obtain a hyperstability result for generalised linear functional equations in several variables. We illustrate our result with two concrete examples, namely, general linear equations and Fréchet equations.

2. Main results

To keep the notation simple, we allow the scalar multiplication from the right, that is, $xa = ax$, where a is a scalar and x is a vector. In particular, if $\boldsymbol{X} := [x_{i,j}]$ is an $n \times m$ matrix whose entries are vectors and $\boldsymbol{A} := [a_{j,k}]$ is an $m \times p$ matrix whose entries are scalars, then $\boldsymbol{XA} = [x'_{i,j}]$ is an $n \times p$ matrix whose entries are vectors such that $x'_{i,j} := x_{i,1}a_{1,j} + \cdots + x_{i,m}a_{m,j}$. We write $\operatorname{diag}(x_1, \ldots, x_N)$ to denote the $N \times N$ diagonal matrix $[x_{i,j}]$, where $x_{i,i} := x_i$ for $i = 1, \ldots, N$, and write $\mathbf{1}$ for the $N \times N$ matrix whose entries are all 1.

We first recall some facts from linear algebra.

Lemma 2.1. *A vector space over an infinite field cannot be a finite union of proper subspaces of itself.*

In particular, we obtain the following result.

Lemma 2.2. *Suppose that $f, g_1, \ldots, g_n : X \to \mathbb{F}$ are linear functionals with the property $\ker f \setminus \ker g_i \neq \varnothing$ for $i = 1, \ldots, n$. Then $\ker f \setminus \bigcup_{i=1}^{n} \ker g_i \neq \varnothing$.*

Without loss of generality, we always assume that $\boldsymbol{A} := [a_{i,j}]_{M\times N}$ satisfies Zhang's condition with $i_0 = 1$. (Otherwise, we can swap the first row and the (i_0)th row.) Then there exist $(k_1, \ldots, k_N), (c_1, \ldots, c_N) \in (\mathbb{F} \setminus \{0\})^N$ and $(b_1, \ldots, b_N) \in \mathbb{F}^N$ such that:

(Z1) $\sum_{j=1}^{N} a_{1,j}k_j = 0 \neq \sum_{j=1}^{N} a_{i,j}k_j$ for all $i = 2, \ldots, M$;

(Z2) $\sum_{j=1}^{N} a_{1,j}b_j = 1$ and $\sum_{j=1}^{N} a_{i,j}c_j \neq 0$ for all $i = 1, \ldots, M$.

To see this, for each $i \in \{1, \ldots, M\}$ and $j \in \{1, \ldots, N\}$, we define $\tau_i, \pi_j : \mathbb{F}^N \to \mathbb{F}$ by

$$\tau_i(t_1, \ldots, t_N) := a_{i,1}t_1 + \cdots + a_{i,N}t_N \quad \text{and} \quad \pi_j(t_1, \ldots, t_N) := t_j$$

for $(t_1, \ldots, t_N) \in \mathbb{F}^N$. By Zhang's condition:

- $\ker \tau_1 \setminus \ker \tau_i \neq \varnothing$ for $i = 2, \ldots, M$;
- $\ker \tau_1 \setminus \ker \pi_j \neq \varnothing$ for $j = 1, \ldots, N$.

By Lemma 2.2, we choose $(k_1, \ldots, k_N) \in \ker \tau_1 \setminus ((\bigcup_{j=1}^{N} \ker \pi_j) \cup (\bigcup_{i=2}^{M} \ker \tau_i))(\neq \varnothing)$. In particular, (Z1) holds. Since $a_{1,1}, \ldots, a_{1,N}$ are not all zero, there exist $b_1, \ldots, b_N \in \mathbb{F}$ such that $\sum_{j=1}^{N} a_{1,j}b_j = 1$. By Lemma 2.1, $\mathbb{F}^N \setminus ((\bigcup_{j=1}^{N} \ker \pi_j) \cup (\bigcup_{i=1}^{M} \ker \tau_i)) \neq \varnothing$. Then there exists $(c_1, \ldots, c_N) \in (\mathbb{F} \setminus \{0\})^N$ such that $\sum_{j=1}^{N} a_{i,j}c_j \neq 0$ for all $i = 1, \ldots, M$. That is, (Z2) holds.

From now on, we assume that the vectors $(k_1, \ldots, k_N), (c_1, \ldots, c_N) \in (\mathbb{F} \setminus \{0\})^N$ and $(b_1, \ldots, b_N) \in \mathbb{F}^N$ satisfy Conditions (Z1) and (Z2).

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Lemma 2.3. *Suppose that* $(x_1, \ldots, x_N) \in X^N$ *and* $y \in X \setminus \{0\}$. *For each* $n \in \mathbb{N}$, *let* $y_j^{(n)} := nc_j y$ *for* $j = 1, \ldots, N$ *and*

$$\begin{aligned} \boldsymbol{X}^{(n)} &:= \boldsymbol{A} \cdot \mathrm{diag}(b_1, \ldots, b_N) \cdot \mathbf{1} \cdot \mathrm{diag}(x_1, \ldots, x_N) \\ &\quad + \boldsymbol{A} \cdot \mathrm{diag}(k_1, \ldots, k_N) \cdot \mathbf{1} \cdot \mathrm{diag}(y_1^{(n)}, \ldots, y_N^{(n)}), \\ \boldsymbol{Z}^{(n)} &:= \boldsymbol{A} \cdot \mathrm{diag}(x_1, \ldots, x_N) \cdot \mathbf{1} \cdot \mathrm{diag}(b_1, \ldots, b_N) \\ &\quad + \boldsymbol{A} \cdot \mathrm{diag}(y_1^{(n)}, \ldots, y_N^{(n)}) \cdot \mathbf{1} \cdot \mathrm{diag}(k_1, \ldots, k_N). \end{aligned}$$

Then the following statements are true.

(1) *Both* $\boldsymbol{X}^{(n)} := [x_{i,j}^{(n)}]$ *and* $\boldsymbol{Z}^{(n)} := [z_{i,j}^{(n)}]$ *are* $M \times N$ *matrices.*
(2) *For each* $i = 2, \ldots, M$ *and* $j = 1, \ldots, N$, *there exist two elements* $u_{i,j} \in X$ *and* $v_{i,j} \in X \setminus \{0\}$ *such that* $x_{i,j}^{(n)} = u_{i,j} + nv_{i,j}$. *In particular,* $\lim_{n\to\infty} \|x_{i,j}^{(n)}\| = \infty$.
(3) *For each* $i = 1, \ldots, M$ *and* $j = 1, \ldots, N$, *there exist two elements* $u'_{i,j} \in X$ *and* $v'_{i,j} \in X \setminus \{0\}$ *such that* $z_{i,j}^{(n)} = u'_{i,j} + nv'_{i,j}$. *In particular,* $\lim_{n\to\infty} \|z_{i,j}^{(n)}\| = \infty$.
(4) $\boldsymbol{X}^{(n)}\boldsymbol{A}^T = (\boldsymbol{Z}^{(n)}\boldsymbol{A}^T)^T$, *where* $\boldsymbol{B}^T$ *is the transpose of* $\boldsymbol{B}$.
(5) *If* $\boldsymbol{X}^{(n)}\boldsymbol{A}^T := [s_{i,i'}^{(n)}]$, *then* $s_{1,i'}^{(n)} = \sum_{j=1}^N a_{i',j} x_j$ *for* $i' = 1, \ldots, M$.

Proof. Parts (1) and (4) are obvious. To see (2), we note that

$$\boldsymbol{X}^{(n)} = \begin{bmatrix} a_{1,1}b_1 & \cdots & a_{1,N}b_N \\ \vdots & \ddots & \vdots \\ a_{M,1}b_1 & \cdots & a_{M,N}b_N \end{bmatrix} \begin{bmatrix} x_1 & \cdots & x_N \\ \vdots & \ddots & \vdots \\ x_1 & \cdots & x_N \end{bmatrix} + \begin{bmatrix} a_{1,1}k_1 & \cdots & a_{1,N}k_N \\ \vdots & \ddots & \vdots \\ a_{M,1}k_1 & \cdots & a_{M,N}k_N \end{bmatrix} \begin{bmatrix} y_1^{(n)} & \cdots & y_N^{(n)} \\ \vdots & \ddots & \vdots \\ y_1^{(n)} & \cdots & y_N^{(n)} \end{bmatrix}.$$

In particular, for $i = 2, \ldots, M$ and $j = 1, \ldots, N$,

$$x_{i,j}^{(n)} = \sum_{q=1}^N a_{i,q} b_q x_j + \sum_{q=1}^N a_{i,q} k_q y_j^{(n)} = u_{i,j} + nv_{i,j},$$

where $u_{i,j} := \sum_{q=1}^N a_{i,q} b_q x_j$ and $v_{i,j} := \sum_{q=1}^N a_{i,q} k_q c_j y$. By Condition (Z1), $v_{i,j} \neq 0$.

To see (3), we note that

$$\boldsymbol{Z}^{(n)} = \begin{bmatrix} a_{1,1}x_1 & \cdots & a_{1,N}x_N \\ \vdots & \ddots & \vdots \\ a_{M,1}x_1 & \cdots & a_{M,N}x_N \end{bmatrix} \begin{bmatrix} b_1 & \cdots & b_N \\ \vdots & \ddots & \vdots \\ b_1 & \cdots & b_N \end{bmatrix} + \begin{bmatrix} a_{1,1}y_1^{(n)} & \cdots & a_{1,N}y_N^{(n)} \\ \vdots & \ddots & \vdots \\ a_{M,1}y_1^{(n)} & \cdots & a_{M,N}y_N^{(n)} \end{bmatrix} \begin{bmatrix} k_1 & \cdots & k_N \\ \vdots & \ddots & \vdots \\ k_1 & \cdots & k_N \end{bmatrix}.$$

In particular, for $i = 1, \ldots, M$ and $j = 1, \ldots, N$,

$$z_{i,j}^{(n)} = \sum_{q=1}^N a_{i,q} b_j x_q + \sum_{q=1}^N a_{i,q} k_j y_q^{(n)} = u'_{i,j} + nv'_{i,j},$$

where $u'_{i,j} := \sum_{q=1}^N a_{i,q} b_j x_q$ and $v'_{i,j} := \sum_{q=1}^N a_{i,q} k_j c_q y$. By Condition (Z2), $v'_{i,j} \neq 0$.

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For (5), it follows from Conditions (Z1) and (Z2) that

$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,N} \end{bmatrix} \cdot \operatorname{diag}(b_1, \ldots, b_N) \cdot \mathbf{1} = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix},$$
$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,N} \end{bmatrix} \cdot \operatorname{diag}(k_1, \ldots, k_N) \cdot \mathbf{1} = \begin{bmatrix} 0 & \cdots & 0 \end{bmatrix}.$$

Hence,

$$\begin{aligned} \begin{bmatrix} s_{1,1}^{(n)} & \cdots & s_{1,M}^{(n)} \end{bmatrix} &= \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} \cdot \operatorname{diag}(x_1, \ldots, x_N) \cdot \boldsymbol{A}^T \\ &\quad + \begin{bmatrix} 0 & \cdots & 0 \end{bmatrix} \cdot \operatorname{diag}(y_1^{(n)}, \ldots, y_N^{(n)}) \cdot \boldsymbol{A}^T \\ &= \begin{bmatrix} \sum_{j=1}^{N} a_{1,j} x_j & \cdots & \sum_{j=1}^{N} a_{M,j} x_j \end{bmatrix}. \end{aligned}$$ □

We are now ready to present the main result, which strengthens Theorem Z.

Definition 2.4. We say that a function $\varphi : (X \setminus \{0\})^N \to \mathbb{R}_+$ satisfies *Condition* $(*)$ if

$$\lim_{n\to\infty} \varphi(u_1 + nv_1, \ldots, u_N + nv_N) = 0$$

for all $u_1, \ldots, u_N \in X$ and $v_1, \ldots, v_N \in X \setminus \{0\}$.

Theorem 2.5. *Suppose that $\varphi : (X \setminus \{0\})^N \to \mathbb{R}_+$ and $f : X \to Y$ satisfy the inequality*

$$\Big\| \sum_{i=1}^{M} L_i f\Big(\sum_{j=1}^{N} a_{i,j} x'_j \Big) \Big\| \le \varphi(x'_1, \ldots, x'_N) \quad \text{for all } x'_1, \ldots, x'_N \in X \setminus \{0\}.$$

If φ satisfies Condition $(*)$, then

$$\sum_{i=1}^{M} L_i f\Big(\sum_{j=1}^{N} a_{i,j} x_j \Big) = 0 \quad \text{for all } x_1, \ldots, x_N \in X.$$

Proof. Let $x_1, \ldots, x_N \in X$. Define two $M \times N$ matrices $\boldsymbol{X}^{(n)} := [x_{i,j}^{(n)}]$ and $\boldsymbol{Z}^{(n)} := [z_{i,j}^{(n)}]$ as in Lemma 2.3. It follows from Condition $(*)$ and Lemma 2.3(2, 3) that:

- $\lim_{n\to\infty} \varphi(x_{r,1}^{(n)}, \ldots, x_{r,N}^{(n)}) = 0$ for $r = 2, \ldots, M$;
- $\lim_{n\to\infty} \varphi(z_{r,1}^{(n)}, \ldots, z_{r,N}^{(n)}) = 0$ for $r = 1, \ldots, M$.

In particular:

- $\lim_{n\to\infty} \sum_{i=1}^{M} L_i f(\sum_{j=1}^{N} a_{i,j} x_{r,j}^{(n)}) = 0$ for $r = 2, \ldots, M$;
- $\lim_{n\to\infty} \sum_{i=1}^{M} L_i f(\sum_{j=1}^{N} a_{i,j} z_{r,j}^{(n)}) = 0$ for $r = 1, \ldots, M$.

We write $\boldsymbol{X}^{(n)}\boldsymbol{A}^T := [s_{i,i'}^{(n)}]$ and $\boldsymbol{Z}^{(n)}\boldsymbol{A}^T := [t_{i,i'}^{(n)}]$. Suppose that $\boldsymbol{Y}^{(n)} := [f(s_{i,i'}^{(n)})]$ and $\overline{\boldsymbol{Y}}^{(n)} := [f(t_{i,i'}^{(n)})]$. Suppose that $\boldsymbol{L} := \begin{bmatrix} L_1 & \cdots & L_M \end{bmatrix}$. It follows that

$$\boldsymbol{L}\boldsymbol{Y}^{(n)}\boldsymbol{L}^T = \boldsymbol{L}(\boldsymbol{Y}^{(n)})^T\boldsymbol{L}^T \quad \text{and} \quad (\boldsymbol{Y}^{(n)})^T = \overline{\boldsymbol{Y}}^{(n)}.$$

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Note that

$$\lim_{n\to\infty} \boldsymbol{Y}^{(n)}\boldsymbol{L}^T = \begin{bmatrix} \sum_{i=1}^{M} L_i f\Big(\sum_{j=1}^{N} a_{i,j}x_j\Big) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{and} \quad \lim_{n\to\infty} (\boldsymbol{Y}^{(n)})^T\boldsymbol{L}^T = \lim_{n\to\infty} \overline{\boldsymbol{Y}}^{(n)}\boldsymbol{L}^T = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Hence,

$$\Big[L_1 \sum_{i=1}^{M} L_i f\Big(\sum_{j=1}^{N} a_{i,j}x_j\Big)\Big] = \lim_{n\to\infty} \boldsymbol{L}\boldsymbol{Y}^{(n)}\boldsymbol{L}^T = \lim_{n\to\infty} \boldsymbol{L}(\boldsymbol{Y}^{(n)})^T\boldsymbol{L}^T = \big[0\big].$$

This completes the proof. □

The following result strengthens Theorem Z.

Theorem 2.6. *Suppose that* $[a_{i,j}]_{M\times N}$ *satisfies Zhang's condition. Further suppose that* $\varphi : (X \setminus \{0\})^N \to \mathbb{R}_+$ *and* $f : X \to Y$ *satisfy the inequality*

$$\Big\| \sum_{i=1}^{M} L_i f\Big(\sum_{j=1}^{N} a_{i,j}x'_j\Big)\Big\| \le \varphi(x'_1, \dots, x'_N) \quad \text{for all } x'_1, \dots, x'_N \in X \setminus \{0\}.$$

Suppose that $C \ge 0$. *If either* $\varphi(x_1, \dots, x_N) := C\sum_{j=1}^{N} \|x_j\|^p$ *where* $p < 0$ *or* $\varphi(x_1, \dots, x_N) := C\prod_{j=1}^{N} \|x_j\|^{p_j}$ *where* $p_1 + \cdots + p_N < 0$, *then*

$$\sum_{i=1}^{M} L_i f\Big(\sum_{j=1}^{N} a_{i,j}x_j\Big) = 0 \quad \text{for all } x_1, \dots, x_N \in X.$$

Proof. It suffices to prove that φ satisfies Condition (∗). Assume that $x_1, \dots, x_N \in X$ and $y_1, \dots, y_N \in X \setminus \{0\}$.

(1) Suppose that $\varphi(x_1, \dots, x_N) := C\sum_{j=1}^{N} \|x_j\|^p$. For each $j = 1, \dots, N$, we note that $\lim_{n\to\infty} \|x_j + ny_j\|^p = \lim_{n\to\infty} |n|^p\|x_j/n + y_j\|^p = 0$. In particular,

$$\lim_{n\to\infty} \varphi(x_1 + ny_1, \dots, x_N + ny_N) = \lim_{n\to\infty} C\sum_{j=1}^{N} \|x_j + ny_j\|^p = 0.$$

(2) Suppose that $\varphi(x_1, \dots, x_N) := C\prod_{j=1}^{N} \|x_j\|^{p_j}$. It follows that

$$\lim_{n\to\infty} \varphi(x_1 + ny_1, \dots, x_N + ny_N) = C\lim_{n\to\infty} |n|^{p_1+\cdots+p_N} \prod_{j=1}^{N} \Big\|\frac{1}{n}x_j + y_j\Big\|^{p_j} = 0. \qquad \square$$

The next corollary follows directly from Theorem 2.6 with φ identically equal to zero. This corollary generalises [6, Lemma 4.7].

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COROLLARY 2.7. *Suppose that* $[a_{i,j}]_{M\times N}$ *satisfies Zhang's condition. If* $f : X \to Y$ *satisfies*

$$\sum_{i=1}^{M} L_i f\Big(\sum_{j=1}^{N} a_{i,j}x'_j\Big) = 0 \quad \textit{for all } x'_1, \ldots, x'_N \in X \setminus \{0\},$$

then

$$\sum_{i=1}^{M} L_i f\Big(\sum_{j=1}^{N} a_{i,j}x_j\Big) = 0 \quad \textit{for all } x_1, \ldots, x_N \in X.$$

3. Examples

We end the paper with some examples to illustrate our results and the proof technique.

EXAMPLE 3.1 (General linear equation). Let $a, b \in \mathbb{F} \setminus \{0\}$ and $A, B \in \mathbb{K} \setminus \{0\}$. Suppose that $f : X \to Y$ and $\varphi : (X \setminus \{0\})^2 \to \mathbb{R}_+$ satisfy

$$\|f(ax_1 + bx_2) - Af(x_1) - Bf(x_2)\| \le \varphi(x_1, x_2) \quad \text{for all } x_1, x_2 \in X \setminus \{0\}.$$

Let

$$\boldsymbol{A} := \begin{bmatrix} a & b \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This matrix satisfies Zhang's condition with $M = 3$, $N = 2$ and $i_0 = 1$. Put

$$(k_1, k_2) := (1/a, -1/b), \quad (b_1, b_2) := (1/a, 0) \quad \text{and} \quad (c_1, c_2) := (1, 1).$$

Let $x_1, x_2 \in X$, $y \in X \setminus \{0\}$ and $n \in \mathbb{N}$. Set $y_1^{(n)} := nc_1 y = ny$ and $y_2^{(n)} := nc_2 y = ny$. The two matrices in Lemma 2.3 are

$$\boldsymbol{X}^{(n)} := \begin{bmatrix} x_1 & x_2 \\ x_1/a + ny/a & x_2/a + ny/a \\ -ny/b & -ny/b \end{bmatrix} \quad \text{and} \quad \boldsymbol{Z}^{(n)} := \begin{bmatrix} x_1 + bx_2/a + ny + bny/a & -any/b - ny \\ x_1/a + ny/a & -ny/b \\ x_2/a + ny/a & -ny/b \end{bmatrix}.$$

Then

$$\boldsymbol{X}^{(n)}\boldsymbol{A}^T = (\boldsymbol{Z}^{(n)}\boldsymbol{A}^T)^T = \begin{bmatrix} ax_1 + bx_2 & x_1 & x_2 \\ x_1 + bx_2/a + ny + bny/a & x_1/a + ny/a & x_2/a + ny/a \\ -any/b - ny & -ny/b & -ny/b \end{bmatrix}.$$

If φ satisfies Condition $(*)$, then it follows from Theorem 2.5 that

$$f(ax_1 + bx_2) = Af(x_1) + Bf(x_2) \quad \text{for all } x_1, x_2 \in X.$$

REMARK 3.2. Piszczek [12] proved the hyperstability result for the function f in Example 3.1 where $\varphi(x_1, x_2) := \|x_1\|^p + \|x_2\|^p$ and $p < 0$ by using Brzdęk's fixed point theorem and concluded that

$$f(ax_1 + bx_2) = Af(x_1) + Bf(x_2) \quad \text{for all } x_1, x_2 \in X \setminus \{0\}.$$

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As already mentioned in the introduction of this paper in relation to the proof of Theorem Z, the application of Brzdęk's fixed point theorem can only conclude that

$$f(ax_1 + bx_2) = Af(x_1) + Bf(x_2) \quad \text{for all } x_1, x_2 \in X \setminus \{0\} \text{ with } ax_1 + bx_2 \neq 0.$$

Recently, the authors of the present paper applied Brzdęk's hyperstability result for a Cauchy functional equation on a restricted domain [5] to obtain the same conclusion as our Example 3.1 (see [11]).

Example 3.3 (Fréchet equation). Suppose that $f : X \to Y$ and $\varphi : (X \setminus \{0\})^3 \to \mathbb{R}_+$ satisfy

$$\|f(x_1 + x_2 + x_3) + f(x_1) + f(x_2) + f(x_3) - f(x_1 + x_2) - f(x_1 + x_3) - f(x_2 + x_3)\| \\ \leq \varphi(x_1, x_2, x_3) \quad \text{for all } x_1, x_2, x_3 \in X \setminus \{0\}.$$

Let

$$\boldsymbol{A} := \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

This matrix satisfies Zhang's condition with $M = 7$, $N = 3$ and $i_0 = 1$. Put

$$(k_1, k_2, k_3) := (2, -1, -1), \quad (b_1, b_2, b_3) := (1, 0, 0) \quad \text{and} \quad (c_1, c_2, c_3) := (1, 1, 1).$$

Let $x_1, x_2, x_3 \in X$, $y \in X \setminus \{0\}$ and $n \in \mathbb{N}$. Set $y_1^{(n)} := nc_1 y = ny$, $y_2^{(n)} := nc_2 y = ny$ and $y_3^{(n)} := nc_3 y = ny$. The two matrices in Lemma 2.3 are

$$\boldsymbol{X}^{(n)} := \begin{bmatrix} x_1 & x_2 & x_3 \\ x_1 + 2ny & x_2 + 2ny & x_3 + 2ny \\ -ny & -ny & -ny \\ -ny & -ny & -ny \\ x_1 + ny & x_2 + ny & x_3 + ny \\ x_1 + ny & x_2 + ny & x_3 + ny \\ -2ny & -2ny & -2ny \end{bmatrix} \quad \text{and} \quad \boldsymbol{Z}^{(n)} := \begin{bmatrix} x_1 + x_2 + x_3 + 6ny & -3ny & -3ny \\ x_1 + 2ny & -ny & -ny \\ x_2 + 2ny & -ny & -ny \\ x_3 + 2ny & -ny & -ny \\ x_1 + x_2 + 4ny & -2ny & -2ny \\ x_1 + x_3 + 4ny & -2ny & -2ny \\ x_2 + x_3 + 4ny & -2ny & -2ny \end{bmatrix}.$$

Then

$$\boldsymbol{X}^{(n)} \boldsymbol{A}^T = (\boldsymbol{Z}^{(n)} \boldsymbol{A}^T)^T \\ = \begin{bmatrix} x_1 + x_2 + x_3 & x_1 & x_2 & x_3 & x_1 + x_2 & x_1 + x_3 & x_2 + x_3 \\ x_1 + x_2 + x_3 + 6ny & x_1 + 2ny & x_2 + 2ny & x_3 + 2ny & x_1 + x_2 + 4ny & x_1 + x_3 + 4ny & x_2 + x_3 + 4ny \\ -3ny & -ny & -ny & -ny & -2ny & -2ny & -2ny \\ -3ny & -ny & -ny & -ny & -2ny & -2ny & -2ny \\ x_1 + x_2 + x_3 + 3ny & x_1 + ny & x_2 + ny & x_3 + ny & x_1 + x_2 + 2ny & x_1 + x_3 + 2ny & x_2 + x_3 + 2ny \\ x_1 + x_2 + x_3 + 3ny & x_1 + ny & x_2 + ny & x_3 + ny & x_1 + x_2 + 2ny & x_1 + x_3 + 2ny & x_2 + x_3 + 2ny \\ -6ny & -2ny & -2ny & -2ny & -4ny & -4ny & -4ny \end{bmatrix}.$$

If φ satisfies Condition $(*)$, then it follows from Theorem 2.5 that

$$f(x_1 + x_2 + x_3) + f(x_1) + f(x_2) + f(x_3) = f(x_1 + x_2) + f(x_1 + x_3) + f(x_2 + x_3)$$

for all $x_1, x_2, x_3 \in X$.

Bahyrycz *et al.* [2] gave the following hyperstability result for Fréchet equations.

THEOREM BBPS. *Suppose that* $f : X \to Y$ *and* $\varphi : (X \setminus \{0\})^3 \to \mathbb{R}_+$ *satisfy the condition*

$$\|f(x_1 + x_2 + x_3) + f(x_1) + f(x_2) + f(x_3) - f(x_1 + x_2) - f(x_1 + x_3) - f(x_2 + x_3)\| \\ \leq \varphi(x_1, x_2, x_3) \quad \text{for all } x_1, x_2, x_3 \in X \setminus \{0\}.$$

Suppose that $\omega : \mathbb{Z} \setminus \{0\} \to \mathbb{R}_+$, *where* $\mathbb{Z}$ *stands for the set of all integers, satisfies*

$$\varphi(kx_1, kx_2, kx_3) \leq \omega(k)\varphi(x_1, x_2, x_3)$$

for all $x_1, x_2, x_3 \in X \setminus \{0\}$. *Suppose in addition that*

$$\mathcal{M} := \{m \in \mathbb{Z} \setminus \{0\} : \omega(-2m) + 2\omega(m + 1) + 2\omega(-m) + \omega(2m + 1) < 1\} \neq \emptyset.$$

If $\inf_{m\in\mathcal{M}} \varphi((2m + 1)x, -mx, -mx) = 0$ *for all* $x \in X \setminus \{0\}$, *then* f *satisfies the equation*

$$f(x_1 + x_2 + x_3) + f(x_1) + f(x_2) + f(x_3) = f(x_1 + x_2) + f(x_1 + x_3) + f(x_2 + x_3)$$

for all $x_1, x_2, x_3 \in X$.

We now compare our Example 3.3 and Theorem BBPS. Set $X := \mathbb{R}$ and $\varphi(x_1, x_2, x_3) := |x_1|/(|x_2| + |x_3|^2)$ for $x_1, x_2, x_3 \in X \setminus \{0\}$. It is easy to see that φ satisfies Condition $(*)$. Now we show that Theorem BBPS is not applicable for φ. In fact, for $m \in \mathbb{Z} \setminus \{0\}$, suppose that $\omega(m)$ is a nonnegative real number such that

$$\varphi(mx_1, mx_2, mx_3) \leq \omega(m)\varphi(x_1, x_2, x_3)$$

for all $x_1, x_2, x_3 \in X \setminus \{0\}$. In particular, if we let $x_1 = x_2 := 1$ and $x_3 := 1/n$, where $n \in \mathbb{N}$, then

$$\frac{1}{1 + m/n^2} \leq \omega(m)\frac{1}{1 + 1/n^2}.$$

Taking $n \to \infty$ gives $\omega(m) \geq 1$ and hence $\mathcal{M} = \emptyset$.

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