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Creating context for social influence processes in multiplex networks

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Abstract

This paper elaborates on two theories of social influence processes to multiplex network structures. First, cohesion influence is based on mutual communication made by different types of relations, and second comparison influence that is built on contrasting types of tie. While a system of bundles with a mutual character constitutes the setting for a multiplex network exposure measure within cohesion, comparison influence is defined algebraically through classes of actors in terms of a weakly balanced semiring structure that considers positive, negative, and also ambivalent types of tie. A case study with these approaches is made on an entrepreneurial community network with formal business relations, informal friendship ties, and perceived competition among the firms, and the methods are validated with the Sampson Monastery data set.

Keywords: *multiplex networks, network exposure, structural balance, bundle patterns, semirings, innovation adoption*

Highlights

- Social influence theories for innovation adoption are applied to multiplex network structures.
- Mutually bundled patterns provide the setting for the analysis of cohesion influence.
- The network exposure measure is generalized to multivariate structures.
- Classes of correspondent actors related by contrasting ties are obtained via semiring structures.
- The structural balance theory is generalized to include ambivalent patterned ties.

1 Introduction

Networks made of organizations are complex and dynamic systems, particularly when they are located in economical environments. The system complexity lies in the web of relations among the members that operate at different levels and continually evolve over time. Although such evolution marks the system dynamics, it is not exclusively concerned with the structure of relations, however, but also with the actors' changing attributes. Most social actors face constraints and opportunities imposed on them by the social setting (Wellman, 1988) and this is especially true for actors such as start-up companies, which must have the capability to value, assimilate, and apply new information into its practices in order to survive in the business sector (McKnight et al., 2002). The adoption of new ideas and products provides competitive advantage to companies, and therefore it is essential to understand the context for the decision to adopt a novelty.

Although social influence can generate a contrast (Barrat et al., 2008), many studies of innovation assert that persuasion through interpersonal relations has convinced people to adopt a new product and that there are key actors responsible for the effective diffusion of the innovation (Ryan & Gross, 1943; Coleman et al., 1966; Strang & Tuma, 1993; Valente, 2010). Thus, innovations are inherently social by nature and this implies that social relations are indispensable in order to get access to additional resources and to the new information that the innovations bring. In this sense, the analysis of the arrangement of relations between a set of actors serves to distinguish different types of influence processes in the network and the potential agents that channel the dissemination of the given innovation.

This paper elaborates on theories of social influence in a delimited system and the consequences of multiplex network structures—aka *multiple* networks in the literature—for the adoption of innovative products. The methodological framework is given by social network analysis, and as an illustration we make an empirical analysis of social influence and innovation adoption where the contexts are structural settings in which the actors share ties with different relational contents.

2 Multiple networks and influence

An especially fruitful schema to describe the social environments of the actors and for tracing the channels along which new information may flow is found in social network analysis (Wasserman & Faust, 1994). A social network X is a setting of n social entities $N = \{i \mid i \text{ is an entity, for } i = 1, 2, \ldots, n\}$ measured under a collection of social ties $R = \{(i, j) \mid i \text{ "has a tie to" } j\}$ where for individuals $i, j \in X$, $X_R(i, j) = 1$ represents a tie R , and $X_R(i, j) = 0$ denotes the lack of a tie between them. A network is directed when (i, j) is an ordered pair, and the pairs on X_R are usually recorded in an adjacency matrix A with size $n \times n$.

Human social networks are typically complex systems having a combination of different types of tie. These are represented in a multiplex network $\mathbf{X}$ with a collection $\mathbf{R}$ of r different kinds of relations, $\mathbf{R} = \{R_1, R_2, \ldots, R_r\}$ measured over N. Each relational type is then stored in a separate adjacency matrix $A_1, A_2, \ldots, A_r$, which are stacked together into a single array $\mathbf{A}$ with size $n \times n \times r$ that can be regarded as an adjacency tensor of a third-order with n horizontal and vertical slices, and r slices that are frontal. A mathematical foundation of multi-layer structures that is applicable to multiplex networks is found in De Domenico et al. (2013), whereas Kivelä et al. (2014) deliver a review on multiplex networks and other complex structures.

Even though complex structures are gaining increasing attention within network analysis, Lazega & Pattison (1999) and Lazega (2001) already made an empirical analysis on multiplexity involving competition and cooperation in a social setting, where these types for collective action are shaped by dyadic and structural forms of exchange including reciprocity. A comprehensive treatment of the dynamical processes in multiplex networks that includes diffusion and social influence processes is found in Barrat et al. (2008), whereas the exposure through social media in large-scale networks is reported by Bakshy et al. (2015) with a cross-cutting content at each stage in the diffusion process, and measurements for ideological homophily that is applicable to 84 interpersonal ties. Besides, Salehi et al. (2015) make a survey of diffusion processes in multi-layered networks where influence in static structures is based on diverse threshold models.

Statistical analyses of global multiplex patterns typically use maximum likelihood for parameter estimation, and are based on exponential family distributions (Holland & Leinhardt, 1981; Fienberg & Wasserman, 1981, cf. Lusher et al. (2013) for recent developments). In fact, Lazega & Pattison (1999) applied p^* models to identify local regularities in the interplay between exchanges and transfers of three kinds of social relations, whereas Snijders et al. (2006) produced maximum likelihood estimators of higher level structural properties through Markov chain Monte Carlo procedures. However, Gallagher & Robins (2015) warn that fitting exponential random graph models to multiplex networks can prove difficult in terms of estimation and model convergence (p. 954).

With respect to signed structures, there is not yet available a mature implementation of statistical models to fit these structures, even without ambivalent relations, and building stochastic models that are able to handle signed networks in a satisfactory manner still today an open challenge. Doreian & Mrvar (2015) generate pre-specified signed blockmodeling to characterize multiplex networks with contrasting relations, whereas Estrada & Benzi (2014) provide measurements of equilibrium in signed networks, as well as community structure detection through random walks with applications in large online social networks. We extend in this paper the work on signed networks with the incorporation of ambivalent relations and by relaxing the measurement of balance in such structures with sequences of ties that do not have to be necessarily closed.

2.1 Social influence

In graph theoretical terms, nodes represent the actors and edges between the nodes are the corresponding relations, and a *multigraph* is a graph depicting a multiplex network with differentiated edges. Figure 1 show both a graph and a multigraph representing a network of relations made of three connected actors (a triad) and an isolated node. Notice that the nodes in the multigraph are labeled, and nodes 70 and 89 represent actors having some characteristic that the other two individuals do not possess.[1]

If, for example, such characteristic is an acquired attribute of the two actors, say they adopted an innovation, and the ties represent the flow of information in the network, then a social influence process is inferred from the triad. Such influence is a result of the direct ties of non-adopters in the system, and also of the indirect link through an intermediary actor.

In the network analysis literature (Rogers, 2003; Burt, 1987; Coleman et al., 1966), a social influence process is also known as contagion or persuasion, and it occurs when a social actor adapts his behavior, attitude, or belief to that of other

[1] All multigraphs and outcomes in the tables are produced with the *R* (R Core Team, 2015) packages *multigraph* (devel version 0.40) (`http://CRAN.R-Project.org/package=multigraph`) and *multiplex* v.2.4.1 (`http://CRAN.R-Project.org/package=multiplex`), and the layouts correspond to the forced directed algorithm described in Fruchterman & Reingold (1991), except for Figures 1 and 7 that are made with a stress majorization scheme (Gansner et al., 2005).

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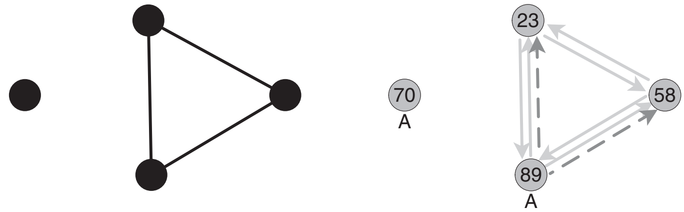

Fig. 1. Graph and multigraph representations of a small configuration. *Edges and arcs stand for undirected and directed ties among unlabeled and labeled social actors, respectively.*

individuals in the social system through a direct communication between the actors. Direct communication corresponds to a body of theories dealing with *cohesion* in the social system where the main task is typically to find cohesive subgroups of actors in the network.

However, an indirect tie between the actors through a common neighbor in the network also represents a social influence process. In this case, what is important is that the two actors are related in a similar way in the system, and this means that they are regarded as equal in a structural manner. Actors who are structurally alike in a network of relations occupy the same position and they are meant to play a similar role in the relational system. In this case, the social influence process is based on *comparison* rather than direct communication, and the main task is to make operational the notion of network positions according to a chosen notion of equivalence among the actors.

Thus, comparison and direct communication are two underlying mechanisms in a social influence process that represent two different contextual settings for a given actor. With contagion through direct communication, the neighborhood of the actor constitutes the frame of reference, whereas the comparison mechanism takes the class membership of the actors in terms of their structural correspondence as the frame of reference. In this latter case, there is no need to have a direct tie between pairs of actors in order to influence on each other.

2.2 Business community network and Web innovations

To make a structural analysis of the two social influence mechanisms and their related theory, we are going to consider entrepreneurial firms located in a Danish business community center and their adoption of two popular social network services from the World Wide Web. A survey was conducted in year 2010 through a self-reporting questionnaire to the firms about different types of ties among them, and the methodological aspects of the survey are given elsewhere (Ostoic, 2013). However, it is important to mention that the business center is a physically closed setting where the social network operates, and the actors considered here are those firms that participated in the survey and therefore this is a restricted version of the entire center.[2]

[2] Additional information suggests that the respondents of the survey have been typically the small start-up companies, while the large companies with a registered address in the center abstained from participating.

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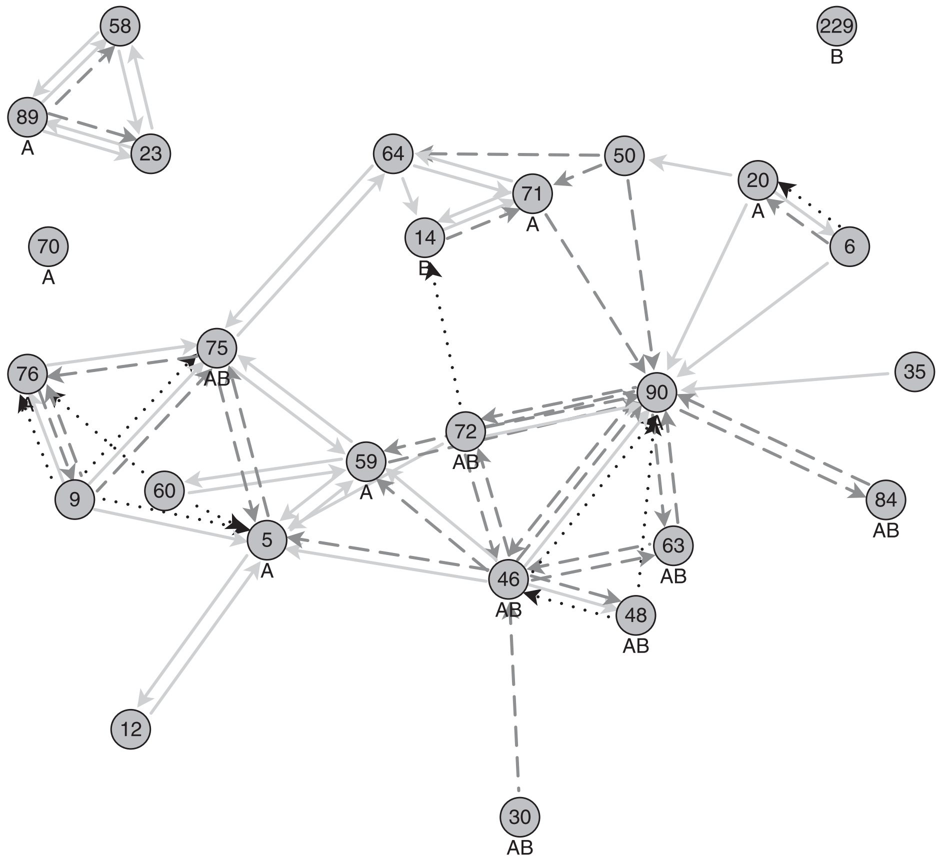

Fig. 2. Labeled multigraph of the Business community network with actor attributes. *Plot made with multigraph (devel version). Collaboration: dashed, Friendship: solid, Competition dotted arcs.*

There are three types of relations measured in the network: business collaboration between entrepreneurs, friendship ties that are informal talks with peer organizations in the center, and perceived competition among the firms. Thus, while cooperation has a formal character, friendship is an informal type of tie, and competition is the polar opposite of cooperation. With respect to the innovations, these are the social network services Linkedin and Facebook that the firms adopted at that time to promote their products and services.[3] For purposes of the analysis, Linkedin is called innovation A, whereas Facebook is denoted innovation B.

The empirical network is depicted in Figure 2 as a multigraph where each firm is represented by a labeled node with the type of innovation(s) the actor acquired at that time, and directed edges representing friendship, business collaboration, and perceived competition ties. The visualization of the graph is itself a great way for making an initial analysis of the network, and a special algorithm has been applied

[3] Linkedin, with URL `http://www.linkedin.com`, is a professionally oriented Web service launched in year 2003, whereas Facebook with URL `https://www.facebook.com` was founded in year 2004 and has a more general purpose, although it is currently used in business and in private contexts as well.

Table 1. *Descriptive statistics of the empirical network*. Relations and Actors are broken down according to their type. Bundles are without null patterns (cf. Figure 3).

Number of actors	26	Number of relations	78
Adopters innovation A	8	Cooperation	32
Adopters innovation B	2	Friendship	36
Adopters innovation A & B	7	Competition	10
Non-adopters	9	Total bundles	41

to avoid edge crossing and also to arrange of the actors such that highly connected nodes are placed in the middle while the others are placed on the periphery.

The graph clearly shows that the network has two components and a pair of isolated actors, which means that the information flow through indirect ties cannot occur in the entire network because the system is disconnected. Besides, a large part of the big component appears to be centered around actor 90, who is adopter of A and has a direct link with three firms that are adopters of innovation B as well, and also actor 90 has an indirect influence link from actors 30 and 14. However, actor 90 may influence the non-adopters in the component through communication, and firms with a structural location similar to this actor are expected to adopt innovation A in the future.

Although a similar exercise could be made with the rest of the potential adopters in the network, the graph visualization simply gives an initial flavor of the data. To obtain deeper insights into the structural characteristics of the network and its relationship to innovation adoption, we need to rely on more precise information about the system. In this sense, initial descriptive statistics of the network are given in Table 1, and we can see on the left panel that the system has 26 actors of which one third are not adopters, more than half adopted innovation A, and only 6 actors adopted B.

With respect to the relations, there are 78 ties in total among the actors in the network, but many of these ties connect the same individuals. In fact, there are just 41 pairwise configurations made of one or several types of tie and, depending on the type and direction of the ties involved, a particular "bundle" pattern exists in the system. Such patterns play an important role in the analysis of one type of social influence in multiplex networks, and therefore we look at them in a more detail.

3 Bundle patterns

When analyzing social influence processes in multiplex networks, it is important to preserve the multiplicity of ties. This means that separate analyses of each type of tie must be avoided or that the different levels of the relationship must be collapsed into a multiplex bond. Hence, we detect the possible patterns at the dyadic level to be found in the multiplex network structure that includes the well-known patterns for single networks like the *null*, *asymmetric*, and *reciprocal* dyads, but also others that have the multiplexity property.

For instance, if two different types of tie occur simultaneously in a dyad, they can have either the same direction or rather opposite courses. A pattern of ties pointing in the same direction is called *tie entrainment*, whereas two ties with a crossing direction form a pattern known as *tie exchange*. From a modeling perspective, these patterns clearly represent two fundamentally distinct realities and they can have a different character as well. Although tie exchange can occur only with two types of relation, it is certainly possible to have an entrainment of three or more kinds of tie.

An example of a tie exchange pattern is found in the empirical network given in Figure 2 where actor 75 reported a collaboration relation with actor 76 who in turn perceives this actor as a friend. On the other hand, actor 5 is involved in two tie entrainment patterns with actors 9 and 46 who pointed this actor as "friend and competitor" and as "friend and collaborator," respectively.

Both tie entrainment and tie exchange are extensions of the asymmetric and reciprocal dyads to different levels of the relationship. However, a combination of an asymmetric and a reciprocal dyad leads to a *mixed* pattern where there is both an entrainment and exchange of ties in the dyadic bond, such as the links of actor 89 in the empirical network.

All these dyadic patterns are *relational bundles*, and they serve to characterize the type of relational system operating in multiplex networks. When the relational content of the ties is positive, the reciprocal, tie exchange, and mixed bundle patterns have a mutual character since ties are pointing in both directions of the dyad, whereas asymmetric and tie entrainment bundle patterns represent an unequal link between pairs of actors without a mutual character.

Formally, a relational bundle B of a class $\mathbf{B}$ involves multiple relations, and it is defined in an ordered pair of adjacent actors i and j as

$$B_{ij}^{\mathbf{B}} = \langle\ R_r \ \mid\ (i,j) \in R_r \text{ and } r \geqslant 2\ \rangle$$

For the definition of the different bundle classes, all types of tie are considered as absent unless otherwise stated. This means that such statements are the minimal requirements for the classes of bundle properties.

We start with the patterns having just one type of tie:

$$B_{ij}^{N}, \text{ Null}: \qquad (i,j) \notin R_{1,\dots,r} \ \wedge\ (j,i) \notin R_{1,\dots,r}$$

$$B_{ij}^{A}, \text{ Asymmetric}: \qquad \big(\ (i,j) \in R_p \ \wedge\ (j,i) \notin R_p\ \big) \quad \wedge \quad \big(\ (i,j) \notin R_{r-p} \ \wedge\ (j,i) \notin R_{r-p}\ \big)$$

$$B_{ij}^{R}, \text{ Reciprocal}: \qquad \big(\ (i,j) \in R_p \ \wedge\ (j,i) \in R_p\ \big) \quad \wedge \quad \big(\ (i,j) \notin R_{r-p} \ \wedge\ (j,i) \notin R_{r-p}\ \big)$$

And then the bundle classes with the multiplexity property:

$$B_{ij}^{E}, \text{ Tie Entrainment}: \qquad (i,j) \in R_{1,\dots,s} \ \wedge\ (j,i) \notin R_{1,\dots,s}, \quad \wedge \quad \big(\ (i,j) \notin R_{r-s)} \ \wedge\ (j,i) \notin R_{r-s}\ \big) \quad \text{for } 1 < s \leqslant r$$

$$B_{ij}^{X}, \text{ Tie Exchange}: \qquad (i,j) \in R_p \ \wedge\ (j,i) \in R_q, \quad \wedge \quad \big(\ (i,j) \notin R_{r-(p,q)} \ \wedge\ (j,i) \notin R_{r-(p,q)}\ \big) \quad \text{for } R_p \neq R_q$$

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Asymmetric: 15

$\overrightarrow{\mathsf{C}}$ (30, 46)
$\overrightarrow{\mathsf{C}}$ (50, 64)
$\overrightarrow{\mathsf{C}}$ (50, 71)
$\overrightarrow{\mathsf{C}}$ (50, 90)
$\overrightarrow{\mathsf{C}}$ (71, 90)
$\overrightarrow{\mathsf{F}}$ (6, 90)
$\overrightarrow{\mathsf{F}}$ (20, 50)
$\overrightarrow{\mathsf{F}}$ (20, 90)
$\overrightarrow{\mathsf{F}}$ (35, 90)
$\overrightarrow{\mathsf{F}}$ (64, 14)
$\overrightarrow{\mathsf{F}}$ (72, 5)
$\overrightarrow{\mathsf{K}}$ (48, 90)
$\overrightarrow{\mathsf{K}}$ (60, 5)
$\overrightarrow{\mathsf{K}}$ (60, 76)
$\overrightarrow{\mathsf{K}}$ (72, 14)

Full: 0

Reciprocal: 13

$\overleftrightarrow{\mathsf{C}}$ (46, 63) (AB, AB)
$\overleftrightarrow{\mathsf{C}}$ (46, 72) (AB, AB)
$\overleftrightarrow{\mathsf{C}}$ (5, 75) (A, AB)
$\overleftrightarrow{\mathsf{C}}$ (59, 90) (A, A)
$\overleftrightarrow{\mathsf{C}}$ (63, 90) (AB, A)
$\overleftrightarrow{\mathsf{C}}$ (84, 90) (AB, A)
$\overleftrightarrow{\mathsf{F}}$ (12, 5) $(A, -)$
$\overleftrightarrow{\mathsf{F}}$ (23, 58) $(-, -)$
$\overleftrightarrow{\mathsf{F}}$ (5, 59) (A, A)
$\overleftrightarrow{\mathsf{F}}$ (59, 60) $(A, -)$
$\overleftrightarrow{\mathsf{F}}$ (59, 75) (A, AB)
$\overleftrightarrow{\mathsf{F}}$ (64, 71) $(-, A)$
$\overleftrightarrow{\mathsf{F}}$ (64, 75) $(-, AB)$

Null: 284

. . .
. . .

Tie Entrainment: 4

$\overrightarrow{\mathsf{C}}\overrightarrow{\mathsf{F}}$ (46, 5)
$\overrightarrow{\mathsf{C}}\overrightarrow{\mathsf{F}}$ (46, 59)
$\overrightarrow{\mathsf{F}}\overrightarrow{\mathsf{K}}$ (9, 5)
$\overrightarrow{\mathsf{C}}\overrightarrow{\mathsf{F}}\overrightarrow{\mathsf{K}}$ (9, 75)

Tie Exchange: 1

$\overrightarrow{\mathsf{C}}\overleftarrow{\mathsf{F}}$ (75, 76) (AB, A)

Mixed: 8

$\overrightarrow{\mathsf{C}}\overleftrightarrow{\mathsf{F}}$ (14, 71) (B, A)
$\overleftarrow{\mathsf{C}}\overrightarrow{\mathsf{F}}\overleftarrow{\mathsf{K}}$ (20, 6)* $(A, -)$
$\overleftarrow{\mathsf{C}}\overleftrightarrow{\mathsf{F}}$ (23, 89) $(-, A)$
$\overrightarrow{\mathsf{C}}\overrightarrow{\mathsf{F}}\overleftarrow{\mathsf{K}}$ (46, 48)*
$\overleftrightarrow{\mathsf{C}}\overrightarrow{\mathsf{F}}\overrightarrow{\mathsf{K}}$ (46, 90) (AB, A)
$\overleftarrow{\mathsf{C}}\overleftrightarrow{\mathsf{F}}$ (58, 89) $(-, A)$
$\overleftrightarrow{\mathsf{C}}\overrightarrow{\mathsf{F}}$ (72, 90) (AB, A)
$\overleftrightarrow{\mathsf{C}}\overleftarrow{\mathsf{F}}\overleftarrow{\mathsf{K}}$ (76, 9) $(A, -)$

Fig. 3. Bundle census of the business network and related actor attributes. *$\overrightarrow{R}$ and $\overleftarrow{R}$ are asymmetric relations, while $\overleftrightarrow{R}$ represents a mutual tie. * Mixed pattern that belongs to another class when* K *is disregarded.*

$$B_{ij}^{M}, \text{Mixed}: \qquad B_{ij}^{A} \wedge B_{ij}^{R}, \qquad r = 2$$
$$B_{ij}^{A} \wedge B_{ij}^{X}, \qquad r > 2$$
$$B_{ij}^{F}, \text{Full}: \qquad (i, j) \in R_{1,\dots,r} \wedge (j, i) \in R_{1,\dots,r}$$

In this case, the complement of the null dyad is another class named *full* bundle where all types of tie occur in both directions and hence this pattern constitutes a especial case of a mixed bundle.

3.1 Bundle census

With the definition of the bundle patterns, we are able to perform a *bundle census* of the network, which is an enumeration of the different classes of bundles that exist in the system. Recall that the empirical network is a business center where three types of relations were measured: Collaboration, Friendship, and perceived Competition that, in order to make the analysis more clear, will be represented as C, F, and K, respectively.

Figure 3 provides the bundle census of the business center network where the existing dyadic patterns are given as ordered pairs of actors together with the relations involved in the bundle. Besides, there is an arc on the top of each type of tie pointing the direction of the relationship, and with a single exception each pattern having a mutual character shows as well whether the corresponding actor is an adopter of innovation A, of B, of both A and B, or none of these novelties.

The exception lies in the fact that the analysis of cohesion influence is based only on the collaboration and friendship ties reported. The reason why we have excluded the competition ties from the analysis of cohesion is that this type of relation is a perceived contest where no social interaction is taking place, and the mechanism for a competition relation corresponds to comparison rather than direct communication.

As a result, bundles (20, 6) and (46, 48) from the mixed class, which are marked with an asterisk in the picture, become tie exchange and tie entrainment, respectively, with C and F. Since an entrainment of ties does not imply a mutual communication between actors, then this latter bundle will be disregarded from the analysis of cohesion influence.

4 Cohesion influence

Direct communication is a valuable mechanism for understanding a social influence process in a system of relations as the direct communication allows two actors to share attitudes toward a social phenomenon which increases the possibility of similar behavior. Hence, if one network member is an adopter, then there is a great chance that the others will follow soon because they communicate with each other through direct contacts. This constitutes cohesion influence.

The importance of communication in social influence processes has been supported by many authors. For instance, Lazarsfeld and colleagues (1948) studied political preferences in presidential campaigns and they pointed out that personal influence is more powerful than the media. Evidences from field and laboratory studies posits that communication is a locomotive force in the social structure and a source of pressure toward uniformity (Festinger, 1950) and an influence to reach agreements (Back, 1951). A systematic theoretical foundation for social influence through communication was provided by Homans (1961) in his series of studies of small group dynamics, and in the context of social networks, Friedkin (1984) points out that structural cohesion models based on direct and short indirect communication channels assist in predicting social homogeneity. Similarly, there are multiple sources of persuasion that are based on verbal and non-verbal communicative activity (Berger & Burgoon, 1998).

In social network analysis, cohesion is a concept that is defined as the strength of the tie density among a subset of actors where cohesive subgroups are those actors in the system among which there are relatively strong, direct, frequent, or positive relations (Wasserman & Faust, 1994, pp. 249). Social influence via communication most easily takes place within cliques and cohesive subgroups, as there is a uniformity of action among the actors in this configuration type and hence higher chances of a similar behavior eventually.

A clique is a maximally connected graph in which all nodes are connected with each other, and it constitutes one of the earliest conceptualizations of group structure. For this reason, the identification of cliques in social systems has been an important task in early social network analysis. Already in year 1949, Luce & Perry provided a method to identify cliques and *n*-cliques in networks that is based on the matrix multiplication operation for each type of relation, which means that social influence can be extended from direct communication to chains of relations. However, this method has a significant drawback, namely that by disregarding the multiplexity in

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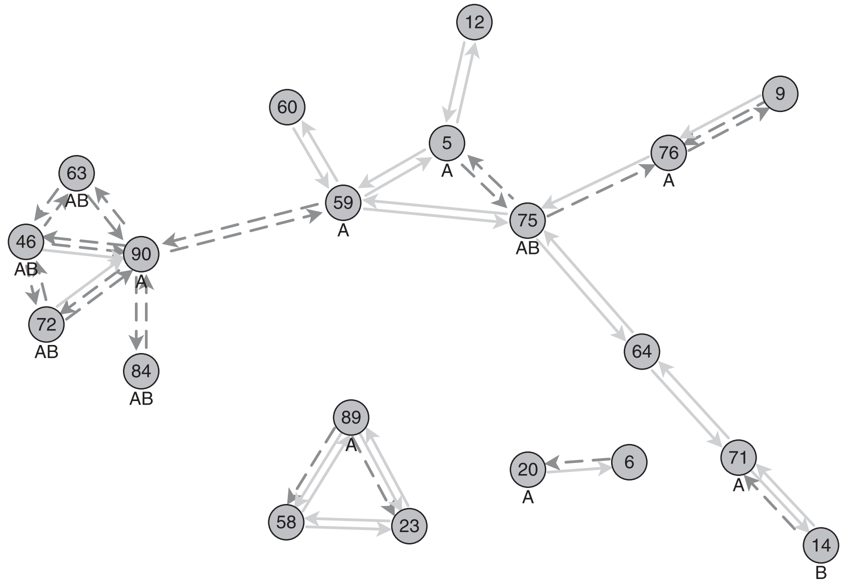

Fig. 4. System of strong bonds in the business network, Z. *These are existing bundles with a mutual character in* $\mathbf{X}$ *of collaboration and friendship ties.*

the ties, there is an important loss of information in the analysis.[4] Thus, rather than focus on individual relationships separately, we need to look at the combination of multiple ties when searching for cohesive subgroups in the empirical system.

4.1 System of strong bonds

The analysis of the possible channels of influence in the empirical network depends on the definition of the different types of bundle patterns. Since social influence via communication is more likely to occur in configurations having reciprocal ties than in asymmetric patterns, then the fundamental structural tendency for cohesion influence is *mutuality* and we look at the directionality of each of the ties in the bundle pattern.

In this sense, to operationalize cohesion influence for multiplex networks, we differentiate two types of bonds among the actors: weak bonds $\mathbf{B}_W = \{B^A \text{ and } B^E\}$ that are asymmetric by nature; whereas the strong bonds $\mathbf{B}_S = \{B^R, B^X, B^M, B^F\}$ are the patterns in the network having a mutual character. Thus, strong bonds are crucial patterns for the analysis of cohesion influence in multiple structures and the assumption is that these relational bundles have made a *system* $Z \subseteq \mathbf{X}$ inside the network, which constitutes the setting where cohesion influence is more likely to occur.

Figure 4 depicts the system of strong bonds of the business community network with bundle patterns involving C and F since—as said before—perceived competition is disregarded from the analysis of cohesion influence and innovation adoption in the empirical network. We can see in the picture that because the unidirectional

[4] Besides, when cliques actually occur with separate analyses, the length of the chains of relations will mostly not be the same for each type of tie.

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bundle patterns are disregarded, the network become less transitive, and therefore it is a more open structure. Besides it is easier to distinguish the maximal connected subgraphs in this system than with κ included.[5]

4.2 Exposure in multiplex networks

The results of the bundle census plus the establishment of the system of strong bonds constitute the basis for the analysis of cohesion influence in multiplex networks, and by fusing these two notions together we can establish how an actor is "exposed" to a certain innovation or other type of behavior. Hence, the potential influence pressure to adapt the social conduct varies for each actor in the system according to the characteristics of its *personal network*, which consists of the members in the system adjacent to this actor.

In this sense, the *network direct exposure* of a reference actor $i \in Z$ is the proportion of adopters in the individual's personal network (Burt, 1987; Valente, 2010, cf. also Rogers, 2003):

$$E_i = \frac{P_i^*}{P_i}, \qquad P_i^* \in [a]$$

where P_i is the personal network of i, and P_i^* represents the set of neighbor nodes of i that adopted a certain behavior a.

As a result, the potential influence pressure to adapt the social conduct varies for each actor in the system according to the characteristic of its personal network, and the concept of network exposure characterizes the threshold for adoption of the actors in the relational system. Since Z is defined in terms of bundle patterns, then the network exposure is generalized to multiplex network structures, and what is important in this case is to consider the bundles that constitute strong bonds.

4.3 Cohesion and adoption results

With the establishment of the system of strong bonds, we are able to calculate measures of the network direct exposure for particular actors in multiplex structures. This means that the results of cohesion influence are based on Z rather than $\mathbf{X}$, and most of the times such configuration is a reduced version of the network. In its upper panel, Table 2 provides the intensities of network exposure measures corresponding to the two types of innovations, whereas the lower panel of this table indicates the new composition of the adopter categories inside the strong bonds system of the business center.

If we take a look at the classes of adopters first, we observe that all categories have fewer members than when considering the entire network whose numbers are in parentheses. This means that some actors are not linked with a mutual bundle pattern, and they remain unaffected by cohesion influence through personal contacts. In addition to the two isolated members of the network, actors 30, 35, 48, and 50 do not belong to any cohesive subgroup because they do not have any formal business

[5] For instance, the four cliques in the system are more evident now than when looking at the complete multigraph in Figure 2.

Table 2. *Network exposure and adopter categories in the systems of strong bonds Z of the empirical network*. Upper panel: Exposure for each actor in Z in percentages. Lower panel: Labeled actors per category in Z with totals, and in parentheses totals in **X**.

To innovation A (Linkedin)	To innovation B (Facebook)
100 pct. = { 6, 9 (C and F); 14 (mostly F); 12, 60, 64 (F) } 50 pct. = { 23, 58 (mostly F) }	80 pct. = { 90 (mostly C) } 50 pct. = { 64 (F); 71 (mostly F); 76 (C and F) } 33 pct. = { 5 (C and F) } 25 pct. = { 59 (mostly F) }
Non-adopters	Adopters A
7(9): { 6, 9, 12, 23, 58, 60, 64 }	7(8): { 5, 20, 59, 71, 76, 89, 90 }
Adopters B	Adopters A&B
1(2): { 14 }	5(7): { 46, 63, 72, 75, 84 }

collaboration ties or informal friendship relations with a mutual character. The analysis of the network exposure measure in the entrepreneurial center therefore shows 20 out of the 26 actors in the network that are specified in Table 2 with the two types of tie involved in the social influence process.

The results of the network exposure measure given in Table 2 are followed by the potential adopters and the relational content in the channel of influence they are linked to. More details about the bundle pattern are given in the bundle census of Figure 3, and the next points give an exhaustive analysis of the cohesion influence by the different categories in the system of strong bonds of the empirical network.

- *Non-adopters*: Seven out of nine firms that are non-adopters in the networks are part of the strong bonds system (only firms 35 and 50 do not belong to any cohesive subgroup because they do not have any ties with mutual character). All non-adopters are only exposed to innovation A and, except for actors 23 and 58 (which are linked together) all other non-adopters have a 100 pct. network exposure to this innovation type (i.e. Linkedin). The potential cohesion influence for non-adopters is mostly through informal relationships, but for actors six and nine the cohesion influence is based on a combination of collaboration and friendship ties. The only network member that is linked by more than one strong bond to the two types of innovation is actor 64 with a 100 pct. exposure measure to innovation A and 50 pct. exposure to innovation B.

- *Adopters*: Adopters of just one type of innovation yet have the chance to adopt the other innovation type through persuasion by means of a direct communication mechanism. There are six adopters of innovation A that are exposed to innovation B (i.e. Facebook) through direct cohesion influence. In this case, the most evident actor is 90 who has 80 pct. exposure measure to innovation B through collaboration ties as the main channels of influence. Actors 71 and 76 have 50 pct. exposure to the other innovation type via informal ties, even though the last actor exchanges a formal collaboration tie with actor 75, which is a full adopter of both innovations. On the other hand, actors 5 and 59 have a network exposure of 33 pct. and 25 pct., respectively.

For actor 5, the exposure is through a combination of formal and informal ties, whereas actor 59 is mostly influenced by informal friendship bonds.

The only adopter of just innovation B in the entrepreneurial network is actor 14 that in the system of strong bonds is related solely to actor 71, who is an adopter of innovation A. This means that this particular actor has a 100 pct. exposure measure to the other innovation type predominately through an informal bond.

4.4 Cohesion influence: a summary

The first part of the analysis of social influence has focused on cohesion for innovation adoption in an empirical multivariate network that consists of formal collaboration ties and informal friendship relations. The essential mechanism for cohesion influence is direct communication and the context for the analysis relies on a system of dyadic bundle patterns with a mutual character or strong bonds inside the network, which differ from the asymmetric bundles regarded as weak bonds.

The adoption results within cohesion are based on different categories of adopters for the two innovation types in the entrepreneurial center and on an exposure measure that is generalized to multiplex network structures. This quantity is estimated for each actor with the predominant types of tie in the bond that channels the influence.

5 Comparison and influence

The other topic of this paper is to look at the social influence process in multiplex networks by means of a comparison mechanism. Comparison influence differs from cohesion, the context for persuasion we addressed in the previous section, especially by the type of structural effect on which it is based. Cohesion requires that each bond in the network has a mutual character, but in the case of comparison, mutuality is not a requisite at all: Quite on the contrary, comparison usually occurs in hierarchical systems where mutuality is absent by definition.

A hierarchy is a gradation where superordinate and subordinate elements occupy different levels in the system, and in the case of a hierarchical social network, the subordinates compare themselves with their superordinates typically by matching achievements. Leenders (2002) points out that in case of comparison, actors use other individual actors that they “feel similar” as their frame of reference and that the higher the alignment of these individuals in the system, the higher the probability of analogous behavior. This is because the actors are facing fundamentally similar constrains and opportunities in the near future within the social structure, and hence have comparable immediate needs.

5.1 Signed networks

The comparison mechanism is naturally related to the third type of measured relation in the empirical network, namely perceived competition among the firms. By combining this type of tie with formal collaboration ties, it is possible to see how the adoption of innovations is driven by the interplay of cooperation and

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competition among the actors. Hence, the influence process is not only impregnated by cohesive forces, but is also based on a system of classes of actors related in a similar structural way.

The fact that collaboration and competition have opposite meanings in an ontological sense implies that these types of tie can be contrasted to each other. For this reason, the effects related to the comparison mechanism are the structural tendencies associated with a *signed network* $\mathbf{X}^\sigma = (\mathbf{X}, \sigma)$, which is a special case of a multiplex network with a *sign function* σ on the relations that usually considers either a positive or a negative valence, v. That is $\sigma \rightarrow \{\mathsf{p}, \mathsf{n}\}$. Note that the characterization of "positive" and "negative" is not made at all in a moral sense, but only means that the ties have opposite signs.

Furthermore, even though signed networks are typically characterized with undirected ties with either p or n as a sign, in the context of human social networks, it is plausible to have an entrainment of positive and negative ties. For instance, in the case of the business center network, the collaboration relations are represented as edges with a positive sign whereas the competition ties are considered to be edges having a negative valence. We see, however, that some firms in the empirical network consider some of the other actors as collaborators and competitors at the same time, and these types of links in which directed edges have opposite valences are regarded as *ambivalent* relations or a.[6] Finally, there are cases where network members are not connected such as in incomplete networks or systems with components and isolates, and in such cases, the lack of relationship constitutes another valence type denoted as *absent*, o.

Putting into the language of relational bundles, the assignment of the four valences by the sign function in the empirical network is

$$\begin{aligned}
\sigma \rightarrow \mathsf{p} \quad &\Leftrightarrow \quad v_1 = \{\, B^A_{ij}, B^R_{ij}, B^X_{ij} \,\}, \text{ for } \mathsf{C} \\
\sigma \rightarrow \mathsf{n} \quad &\Leftrightarrow \quad v_2 = \{\, B^A_{ij}, B^R_{ij}, B^X_{ij} \,\}, \text{ for } \mathsf{K} \\
\sigma \rightarrow \mathsf{a} \quad &\Leftrightarrow \quad v_3 = \{\, B^E_{ij} \,\}, \text{ for } \mathsf{C} \text{ and } \mathsf{K} \\
\sigma \rightarrow \mathsf{o} \quad &\Leftrightarrow \quad v_4 = \{\, B^N_{ij} \,\}
\end{aligned}$$

Hence, the relational content of the asymmetric and reciprocal dyads determines whether the valence is positive or negative, and the null dyad corresponds to the absence sign for both parts. Regarding the patterns with a multiple character, the tie entrainment bundle represents an ambivalent relation, whereas the tie exchange pattern is just that, namely an exchange of a positive tie with a negative relation with no ambivalence. Mixed patterns with $r = 2$ are an interchange of an entrainment of ties either with an asymmetric relation or with another tie entrainment, and hence an ambivalent tie "crosses" with p, n, or else with another a. Finally, all mixed patterns with more than two types of relations are ambivalent by nature.

5.2 Structural balance

The building blocks for the study of cohesive influence in the empirical network were mutual formal and informal patterns at the dyadic level, but the analysis of indirect

[6] This is the case of pairs (9, 75) and (6, 20) in the empirical network having a mixed pattern with friendship included.

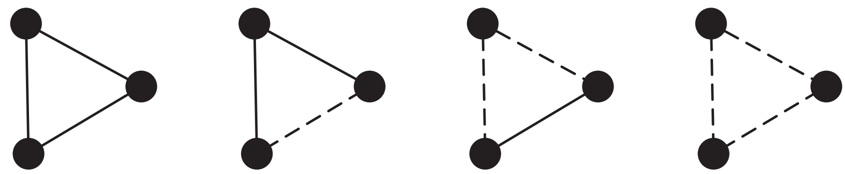

Fig. 5. Four complete triads with signed undirected relations. *Solid edges represent positive ties and dashed edges represent negative relations.*

persuasion through comparison requires higher level structures like triads and so on. In this sense, even though a social relation starts with the dyad, a structure that has a true character of society has the triad as its smallest configuration (Simmel & Wolff, 1950, (1st edn. 1908)). It is through relations between three elements that we can deduce phenomena like *transitivity*, which is the tendency that relates two actors that have a common neighbor.

Transitivity has the potential to play a considerable role in the collaboration relations among firms and in the adoption of innovations as an indirect effect of the ties in the persuasion process. But transitivity is a generalization of another structural tendency known as *structural balance*, which can be established by means of signed graphs. A structural balanced system of relations has significant consequences for the social behavior of the actors such as the willingness to adopt an innovation via the comparison mechanism in this case.

The theory of structural balance was proposed by Heider in 1958 (2013, (1st edn. 1958)) as the theory of cognitive dissonance when he studied diverse situations that arise when positive and negative ties are found in small configurations, especially in triads. In his original formulation, Heider considered both shared and individual attitudes to a certain object, but the principles can also be applied as well to structures with three actors with affective relations such as "like" and "dislike," and to instrumental ties such as the collaboration and competition among actors such as the entrepreneurial network. This theory suggests that structures that are imbalanced have an inherent tension and are prone to change, whereas balanced configurations have a more steady structure, which implies that network structures have a tendency to be balanced over time.

In order to illustrate the principle of the structural balance theory, we start with the smallest structure made of three connected actors with undirected ties that are either positive or negative, but not both. By combining these two valences in a triad, then four possible situations can arise in the system, and these are depicted as signed graphs in Figure 5. In case that the positive and negative edges correspond to the cooperation ties and perceived competition, respectively, i.e. two of the measured relations in the study, then the configuration triad to the left tells us that all actors cooperate with each other, whereas in the triad to the right every actor regards their neighbors as competitors. Between these opposites, there are two configurations with both cooperation and competition relations among the actors.

Cartwright & Harary's Structure Theorem (1956, also Harary et al., 1965) states that a signed graph is *balanced* if and only if its nodes can be divided into two mutually exclusive groups of actors—although a group can be empty—where all positive edges are within each group and all negative edges are between the groups.

This means that when a negative tie exists in a network of relations, then a balanced structure is obtained with the polarization of the population system. Davis (1967) extended the structural balance theory by considering more than two groups of actors, and he called the different subsets *plus-sets* where a system is said to be *clusterable* if and only if the ties within plus-sets are positive and the ties between the subsets are negative.

According to the structural balance theory, configurations with only positive ties are balanced, but with the systems having just negative ties the situation is less clear. Heider asserts that a balanced triad with two negative ties is obtained with *either* a positive *or* a negative third relation, but there is a strong tendency toward a closure with a positive relation (Heider, 2013, pp. 206; cf. also Davis, 1967). Thus, the triad at the right-hand side of Figure 5 implies that the group is imbalanced, even though it is clusterable in terms of Davis' formulation.

Now, if we look at the rest of configurations with C and K, the second triad from the left in Figure 5 tells us the following story: "Two actors collaborate with a third one while they are competing with each other" or, equally, "an actor collaborates with two competing neighbors." This type of structure represents a stressed situation among the actors and therefore the system is considered as imbalanced. Lastly, the triad with a single collaboration tie says that "two collaborating actors have a common competitor," which implies that there is no tension in the group and the structure is regarded as balanced.When we follow the structural balance theory, we expect to have more triads like the first and third ones and fewer of the other two configurations in this picture.

The fact that a complete triad in the network has one or two negative ties is determinant for the whole system in terms of structural balance. Flament (1963) asserted that a complete graph is balanced if and only if all its triangles are balanced and also that it is possible to evaluate higher level structures by looking at the valence type in sequences of edges or arcs. Within this vein of thought, Cartwright & Harary (1956) observed that paths in a signed graph are positive if they have an even number of negative edges and are negative otherwise (in this case, zero is considered to be an even number).

The structural balance theory has implication for the analysis of comparison influence in the empirical network, because any alteration in the network structure may be interrelated with the changing attributes of the actors. Besides, balanced configurations without inherent tension foster communication among its members and hence they are a potential source for social influence. It is therefore perfectly justifiable to see the correspondence between the implied type of structure of the network with contrasting relations and the changing attributes of the actors in terms of structural balance.

The Structure Theorem of Cartwright & Harary plays a fundamental role in evaluating the empirical network in terms of structural balance. Recall that this proposition states that what makes a signed network balanced or not is that all relations within the group members are positive, and all relations between groups are negative. This aspect can be evaluated in the signed network in terms of paths, semipaths, or even chains.

An alternating sequence of nodes, n_i, and directed edges, e_i, in a graph is called a *path*, $s = n_0, e_1, n_1, e_2, \ldots, e_k, n_k$, of length $k \geqslant i \geqslant 1$, where the order of the initial

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Table 3. *Valence matrix of the large component in* **X**. Relations C and K have positive and negative signs, respectively, whereas B^E and B^N represent ambivalent and absent valences in the system.

	5	9	14	30	46	48	50	59	60	63	64	71	72	75	76	84	90
5	o	o	o	o	o	o	o	o	o	o	o	o	o	p	o	o	o
9	n	o	o	o	o	o	o	o	o	o	o	o	o	a	a	o	o
14	o	o	o	o	o	o	o	o	o	o	o	p	o	o	o	o	o
30	o	o	o	o	p	o	o	o	o	o	o	o	o	o	o	o	o
46	p	o	o	o	o	p	o	p	o	p	o	o	p	o	o	o	a
48	o	o	o	o	n	o	o	o	o	o	o	o	o	o	o	o	n
50	o	o	o	o	o	o	o	o	o	o	p	p	o	o	o	o	p
59	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o	p
60	n	o	o	o	o	o	o	o	o	o	o	o	o	o	n	o	o
63	o	o	o	o	p	o	o	o	o	o	o	o	o	o	o	o	p
64	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o
71	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o	p
72	o	o	n	o	p	o	o	o	o	o	o	o	o	o	o	o	p
75	p	o	o	o	o	o	o	o	o	o	o	o	o	o	p	o	o
76	o	p	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o
84	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o	p
90	o	o	o	o	p	o	o	p	o	p	o	o	p	o	o	p	o

and terminal endpoints in each link of the sequence is important, which means that $s = (n_0, n_k)$; otherwise, the sequence is a *semipath* in which s equals either (n_0, n_k) or (n_k, n_0). Moreover, a closed path and a closed semipath make a cycle and a semicycle respectively, and in both cases this implies that $n_0 = n_k$.

Because we are just interested in the cooperation ties and perceived competition of the empirical system, actors who have only friendship ties like 12 and 35 (cf. Figure 2) become isolated and are therefore disregarded from the analysis of comparison influence. On the other hand, the pair $(6, 20)$ gets separated from the main component of the network, and it forms a configuration dyad with an ambivalent relationship. As a result, checking for balance in the empirical network involves working particularly with the large component of the system, and for this we benefit from an algebraic analysis which will be explained below.

5.3 Semiring structures

In order to check for balance in the system, we need to find groups or clusters of actors that are located within positive relations and between negative ties. This is typically represented by a valence matrix that for the main component of the empirical network is given in Table 3 for collaboration and competition ties representing positive and negative signs, respectively. The matrix shows that there are three ambivalent relations in the system, and this constitutes an important challenge for the analysis because this type of valence is not considered in the Structure Theorem, which is used to evaluate the valence that the paths have in the system.

The problem lies in the fact that a path is considered either as positive or negative if and only if there is no mix of these valences in the signature of the involved ties, and the presence of an ambivalent relation signs the entire path as ambivalent. In this sense, an *extended* structural balance considers the following rules of associations among the four valences considered in the signed network.

+	o	n	p	a
o	o	n	p	a
n	n	n	a	a
p	p	a	p	a
a	a	a	a	a

The addition operation corresponds to the concatenation of element ties that produces a compound relation with a valence attached to it. When we look at the products of the different valence types, we verify that the ambivalent relation is an absorbing element, whereas an absent tie acts as the neutral valence in this particular system.

In the case of the business network, this system has different components or separate parts with absent "relations" among them. However, the absence of a tie does not prevent a relational system to be structurally balanced as long as there is some type of relation, and this is in accordance with the role of o as neutral element. Hence, with incomplete graphs such as the empirical network, the assessment of structural balance is then established separately for each component of the network with the isolated nodes disregarded.

In fact, the products of the addition of signed relations depend on matrix products of the elements in the valence matrix. Although we can produce all paths in the system with matrix multiplication, we are able to obtain a closed system from the signed trails with the addition operation, and this is a necessary condition to make an algebraic structure that allows us to evaluate the signed network in terms of structural balance.

The operation of matrix multiplication on signed relations for the extended structural balance follows specific rules as well:

·	o	n	p	a
o	o	o	o	o
n	o	p	n	a
p	o	n	p	a
a	o	a	a	a

In the case of multiplication, the absence of a tie between two nodes prevents the existence of a path, and this is expressed in the absorbing character of o in the matrix product, whereas positive ties constitute the neutral elements in this case.

The set of valences in the signed network together with the two operations of addition and multiplication make the algebraic structure *semiring* Q, in which the neutral elements in addition and multiplication are specified as 0 and 1, respectively.

$$\langle\, Q, +, \;\cdot\;, 0, 1 \,\rangle$$

Both operations are closed and associative, and the elements under addition are also commutative. Besides multiplication distributes over addition, which means that $\mathsf{p} \cdot (\mathsf{n} + \mathsf{a}) = (\mathsf{p} \cdot \mathsf{n}) + (\mathsf{p} \cdot \mathsf{a})$ and $(\mathsf{p} + \mathsf{n}) \cdot \mathsf{a} = (\mathsf{p} \cdot \mathsf{a}) + (\mathsf{n} \cdot \mathsf{a})$, for all $\mathsf{p}, \mathsf{n}, \mathsf{a} \in Q$.

Table 4. *Balance semiring structures within a signed triadic configuration, t^{α}, for the large component in* $\mathbf{X}^{\alpha}$. All paths larger than one are invariant, whereas semipaths of lengths 3 or more lack structure.

(a) t^{α}

	5	9	75
5	o	o	p
9	n	o	a
75	p	o	o

(b) t^{α} paths, $k > 1$

	5	9	75
5	p	o	o
9	a	o	n
75	o	o	p

(c) t^{α} semipaths, $k = 2$

	5	9	75
5	p	a	a
9	a	a	n
75	a	n	a

(d) t^{α} semipaths, $k > 2$

	5	9	75
5	a	a	a
9	a	a	a
75	a	a	a

Harary et al. (1965), Batagelj (1994), Doreian et al. (2004) provide a more extensive definition of the semiring object with the clustering semiring operations as well. Empirical applications of balance and clustering semirings with both paths and semipaths are given in Ostoic (2013).

5.4 Weakly balanced structures

The main benefit of the semiring structure is that it allows us to find structurally balanced structures in signed networks. Ideally, the ties in the system are either positive or negative since ambivalent relations prevent us to have both positive and negative paths of relations as they characterize balanced and clustered systems. Nevertheless, with the operations involved in the extended version of structural balance that consider ambivalent relations as well, we are able to evaluate systems with entrainments of positive and negative ties such as the empirical network with bundles of collaboration and competition ties.

To demonstrate how the semiring operations work in the evaluation process of structural balance, let us consider the triangle made by actors 5, 9, and 75, whose valence matrix t^{α} is represented in Table 4. Although actor 5 is not connected by any ambivalent relation here and in the network, the fact that the neighboring pair $(9, 75)$ is related simultaneously through C and K eventually affects this actor when considering paths and especially the semipaths. For instance, any path of $k > 1$ between $(9, 5)$ becomes ambivalent because of the product of $(9, 75)$ and $(5, 75)$ is $\mathsf{a} \cdot \mathsf{p} = \mathsf{a}$. However, for the resulting valence matrix in Table 4b such product has been added to the other possible paths with similar lengths, which for $k = 2$ are $\mathsf{n} \cdot \mathsf{o}$ and $\mathsf{o} \cdot \mathsf{n}$ representing a loop in the respective node that is preceded and followed by the negative link from 9 to 5. Since o constitutes an absorbing element in the multiplication operation, at the end these two products result in absent relations with no effect in the addition operation.

The other entries in the valence matrix result from a process similar to the one just described; for example, pair $(9, 75)$, which is linked by an ambivalent relation, is just connected through a negative 2-path product of $\mathsf{n} \cdot \mathsf{p}$ from $(9, 5)$ and $(5, 75)$ because the other products involve absent relations and are neutral under addition. Likewise, longer paths follow the same logic as for $k = 2$ as well and they are based on the outcomes from shorter paths; however, the structure results are invariant in this case and hence the semiring structure with paths of length 2 serves to evaluate this particular system in terms of balance.

Table 5. *Weakly balanced semiring structure in the large component of* $\mathbf{X}^{\alpha}$. Permuted and partitioned matrix for paths of length 4 with relations C and K.

	5	9	75	76	14	46	48	59	63	71	72	84	90	30	50	60	64
5	a	a	a	a	o	o	o	o	o	o	o	o	o	o	o	o	o
9	a	a	a	a	o	o	o	o	o	o	o	o	o	o	o	o	o
75	a	a	a	a	o	o	o	o	o	o	o	o	o	o	o	o	o
76	a	a	a	a	o	o	o	o	o	o	o	o	o	o	o	o	o
14	a	a	a	a	a	a	a	a	a	a	a	a	a	o	o	o	o
46	a	a	a	a	a	a	a	a	a	a	a	a	a	o	o	o	o
48	a	a	a	a	a	a	a	a	a	a	a	a	a	o	o	o	o
59	a	a	a	a	a	a	a	a	a	a	a	a	a	o	o	o	o
63	a	a	a	a	a	a	a	a	a	a	a	a	a	o	o	o	o
71	a	a	a	a	a	a	a	a	a	a	a	a	a	o	o	o	o
72	a	a	a	a	a	a	a	a	a	a	a	a	a	o	o	o	o
84	a	a	a	a	a	a	a	a	a	a	a	a	a	o	o	o	o
90	a	a	a	a	a	a	a	a	a	a	a	a	a	o	o	o	o
30	a	a	a	a	a	a	a	a	a	a	a	a	a	o	o	o	o
50	a	a	a	a	a	a	a	a	a	a	a	a	a	o	o	o	o
60	a	a	a	a	o	o	o	o	o	o	o	o	o	o	o	o	o
64	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o

Undirected ties imply a symmetric closure of the valence matrix and for this small example, it implies that pairs (5, 9) and (75, 9) become negative and ambivalent, respectively. In practice, however, semipaths tend to have an increased ambivalent character in this case as the length of the implied sequences grows, and the entire structure becomes ambivalent for semipaths longer than 2 (cf. semipaths in Tables 4c and d). At some point, in fact, the entire large component of the signed network to which this triangle belongs becomes ambivalent with semipaths of length 4, and there is no structural differentiation to work with in the analysis. Thus, we concentrate the study of structural balance in this network component on the paths made from cooperation and competition relations.

As with the triadic configuration, it is possible to obtain stable structures when checking for balance with paths in larger structures that are not entirely ambivalent. Table 5 provides the balanced semiring structure of the main component of the empirical network that is permuted in a way such that the different subsets in this part of the network are differentiated from each other. Although such structure is not strictly balanced, it may still hold important characteristics of the signed network by making a partition of the system according to the presence or the lack of path. Such procedure is just like blockmodeling with structural equivalence (Lorrain & White, 1971), and by assigning binary values to the valences with the rule:

$$a_{ij}^{\sigma} = \begin{cases} 0 & \text{iff } (i, j) \in \mathsf{o}, \\ 1 & \text{otherwise.} \end{cases}$$

In this case, after applying the operation rules corresponding to balance semiring of paths, one is assigned to the only existing ambivalent ties, whereas absent relations are zero.

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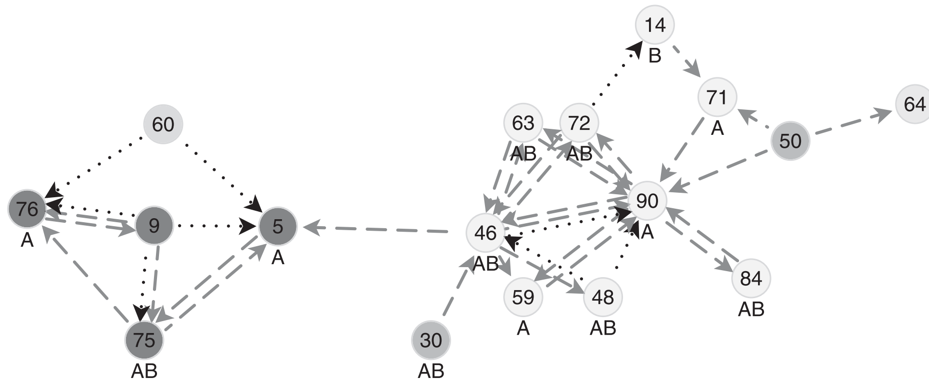

Fig. 6. Balance semiring structure for the large component in **X**, $k = 4$. *Graph representation without inconsistences with factions* $\{5, 9, 75, 76\}$, $\{14, 46, 48, 50, 59, 63, 71, 72, 84, 90\}$, $\{30, 50\}$, $\{60\}$, *and* $\{64\}$ *based on collaboration and competition relations.*

A partition of the main component in terms of structurally equivalence leads to subsets of actors or *factions* that, although they are not fully related structurally balanced structures, yet have the capability to reveal important structural features of the signed network regardless of the relational content of the path relation. The resulting system of factions correspond in this sense to an "approximately" or *weakly* balanced structure, which serves as the setting or context for the analysis of comparison influence in the entrepreneurial network.

5.5 Adoption results with comparison

The context for the analysis of comparison is given by the factional system produced by the weakly balanced semiring structure given in Table 5. Here, we can see, for instance, that in the empirical network actor 64 remains apart from the factional scheme, because there is no outgoing tie from this actor to the rest of the component, and hence there is no possibility to obtain a path from this particular node. On the other hand, actor 30 who has one outgoing tie is equivalent in structural terms to actor 50 and they make a class of their own, whereas actor 60 is linked in a singular way to actors in another subset.

Figure 6 depicts the main component of the empirical network according to the identified factions that are produced through the rules of the balance semiring structure, and clearly there are two separated subgraphs in this configuration and four particular actors. However, it might be possible to further relax the criterion for the assignment of actors to a subset and actor 60 for instance could be part of Faction 1 whereas actors 30 and 50 could join Faction 2 which is the largest subset of actors. In this way, the factional structure would constitute a weakly balanced structure that is made basically by two classes of actors based on paths with ambivalent character. It is evident from the picture that the asymmetric tie between actors 46 and 5 acts as a bridge that connects these two classes of actors, and the lack of mutuality in this dyad pattern prevents the formation of a cycle between these parts. Hence, in this case, a social influence process for innovation B occurs just from Faction 2 to Faction 1 through a business collaboration tie, since

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the structurally equivalent actors that could imitate each other are most related to the larger faction.

As opposed to collaboration, the influence of perceived competition is rather from the recipient of the tie to the sender because the object of comparison is a frame of reference that the subject is predisposed to imitate. This means, for instance, that actor 60, who is a non-adopter in Faction 4, is likely to adopt the innovations that the actors in Faction 1 have already embraced even if there is no social interaction among them. Likewise, actor 72 that, although it is an adopter of innovation A only, has as a frame of reference actor 14 who has adopted the other type of innovation.

Finally, there is the case in which the social influence is determined solely by the class membership of the actors and in this case such similarity is based on the combination of these two contrasting kinds of tie. Because structurally alike actors are related in a similar way to the system, we expect them to behave similarly and hence to adopt the same innovations. As a result, there is a high chance that actor 50 who is not even directly related by perceived competition to actor 30 will end up adopting the innovations this actor has, just by the fact that these actors are structurally similar and together they make a single class.[7]

5.6 Comparison and influence: a summary

The study of the comparison influence complements the previous results from direct communication as a persuasion mechanism. Within cohesion influence, actors are exposed to innovations through their direct contacts and even though they might be at different levels, they are all positively valuated. On the other hand, with comparison influence, the implicated ties have a divergent sign, which is the case of the existing collaboration and competition ties, and the analysis requires a radically different approach than with direct communication, namely these types of tie are combined into an algebraic system for analysis.

With comparison, a weakly balanced semiring structure based on paths lays the foundations for a system of classes of actors or a factional system that serves as the context for the interpretation of the innovation adoption results in the empirical network. Hence, in this case, social influence is likely to occur among structurally equivalent actors facing similar constraints and opportunities in the network even if they do not share direct contacts of social interaction as we found with cohesion influence.

6 Validation

For the validation of the methods developed in the paper, we need to take into account that the two approaches are based on information that is selected rather than sampled as is the case for the statistical analyses. For instance, the network exposure measure for multiplex networks is merely the degree of mutual contact with other adopters in the actor's personal network, and a straightforward validation of this form of cohesion influence process requires the existence of panel data of

[7] Certainly, the assumption for establishing this class is that the link from actor 50 to actor 64 does not affect the structure of paths.

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the network. Hence, in case of a posterior adoption of the innovations, one needs to ask the adopters to state why they accepted them and one also needs to measure the network exposure of persisting non-adopters in an equivalent system of strong bonds.

In contrast, the stable weakly balanced structure in comparison influence is a configuration where actors who belong to the same faction are prone to behave in the same way, which in our case means to acquire or not acquire an innovation, but it can be as well another type of ascribed attribute. One form of verification of a posterior behavior is to apply a goodness-of-fit measure to the partition of the signed structure of the network data; however, since statistical methods are not used from the beginning of the analysis, there is a limitation in using standard and parametric statistical tests and measures in positional analyses (Wasserman & Faust, 1994, pp. 676).

6.1 Sampson Monastery

Innovation adoption via social influence is just a scenario where context settings such as the system of strong bonds and the weakly balance semiring structure are useful. Next, we are going to perform similar analyses with the Sampson Monastery classic data set (Sampson, 1969) and look at the structures obtained both from the bundle patterns with a mutual character, and the ones resulting from the balance semiring operations to the contrasting relations. Significant positional analyses of this data set have been made by Breiger et al. (1975), White et al. (1976), and Fienberg et al. (1985), whereas de Nooy et al. (2005) detected structural balance and clusterability on this signed network as well.

The Monastery data set is a rich ethnographic collection of 10 ranked relationships (strengths 1 to 3) with several dualities of positive and negative signs among novices in a cloister. Sampson himself identified four groups of actors and labeled them Young Turks (YT), Loyal Opposition (LO), Outcasts (OC), and an In-Between group (IB), and four pairwise valences: "like vs dislike," "esteem vs disesteem," "positive vs. negative influence," and "praise vs blame." The relation "like" was measured over three points of time (T1-3) and, if we contrast such episodes with "dislike," we end up with six different signed networks to work with.[8]

We start by creating the strong bonds system of this multiplex network, which is made up of the positive-valued relations with a mutual character. In case we consider the bonds irrespectively of their strength, we end up with a single component depicted in Figure 7(a) in which all actors are included except for Amand. It is evident from the multigraph that the three clusters of densely intra-connected actors correspond to the categorization made by Sampson, LO, YT, and OC, with a pair of actors making an interstitial group.[9] The exception is Ambrose who acts more as a bridge than as a member of LO, and this is also true with the four components given in Figure 7(b) that results when we consider the highest weight of the ties as the cut-off value for the dichotomization of the network. In fact, the system of strong bonds not only corroborates the substantial observations made by Sampson in this

[8] The data for the analysis comes from http://moreno.ss.uci.edu/data#sampson.
[9] This system even has a full bundle pattern between Berthold and Peter.

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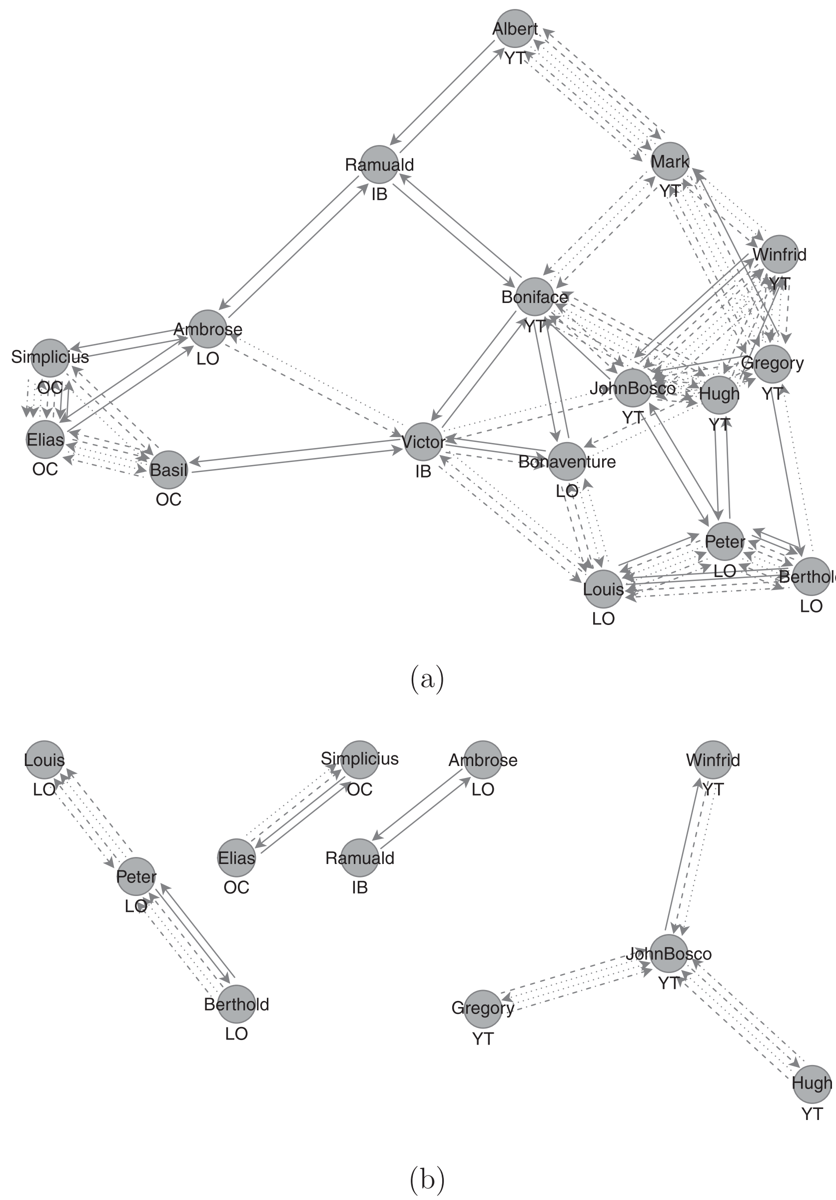

Fig. 7. Systems of strong bonds of the Monastery data with two strength values for the dichotomization. *Relations Like (T3), Esteem, Positive Influence, and Praise are depicted as solid, dashed, dotted, and dot-dash arcs, respectively*. (a) Cut-off value 1. (b) Cut-off value 3.

singular network, but also provides in detail the bundle patterns as the channels where cohesion influence is more likely to occur.

With respect to comparison, a system of different classes of actors is found in this data set with the sparsest versions of the signed networks. This means that all these arrangements are dichotomized with the highest rank as a cut-off measure in order to obtain a stable balance system via semiring operations; the exception is positive vs. negative influence where the sparsest network resulted in a single faction. Figure 8 depicts for each signed network the different factions obtained from a

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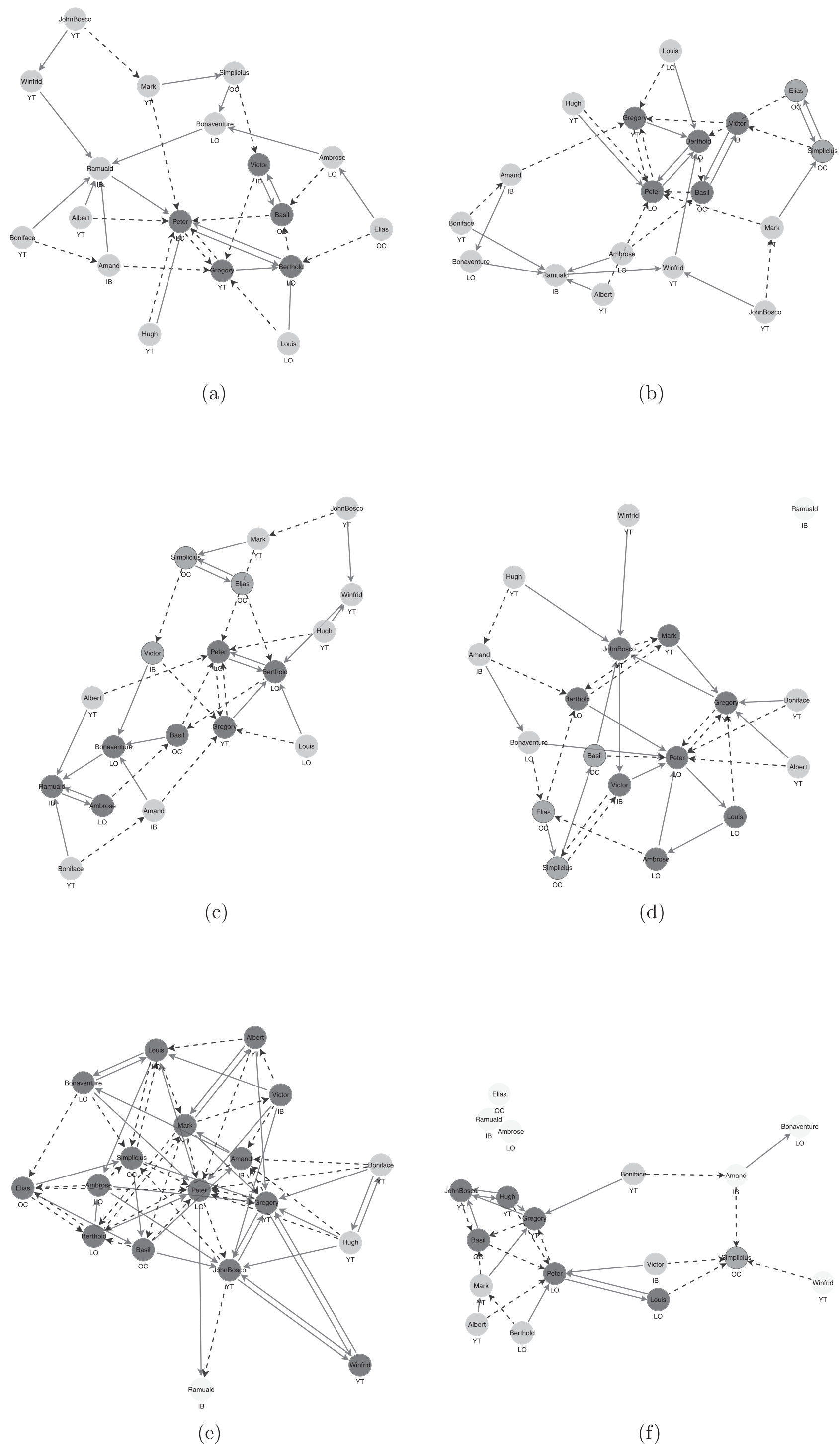

Fig. 8. Factions in the weakly balanced semiring structures of the Monastery data. *All cut-off values are* 3 *except for* (e)*, which is* 2*. Image matrices with path lengths are given in Table 6.* (a) Like (T1) vs Dislike (b) Like (T2) vs. Dislike. (c) Like (T3) vs. Dislike. (d) Esteem vs. Disesteem. (e) Positive vs. Negative Influence (f) Praise vs. Blame.

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Table 6. *Image matrices of the weakly balanced semiring structures in the Monastery data.* Class membership of the different factions is according to the partition given in Figure 8.

a	o
a	o

(a) Like (T1) vs Dislike, $k = 4$

a	o	o
a	p	o
a	?	o

(b) Like (T2) vs Dislike, $k = 4$

a	o	o
a	?	o
a	?	o

(c) Like (T3) vs Dislike, $k = 5$

a	a	o	o
a	a	o	o
a	a	o	o
o	o	o	o

(d) Esteem vs Disesteem, $k = 4$

a	a	a
a	p	a
o	o	a

(e) Positive vs Negative Influence, $k = 3$

a	a	o	o
o	o	o	o
a	a	o	o
o	o	o	o

(f) Praise vs Blame, $k = 3$

weakly balanced semiring structure, and Table 6 provides the corresponding image matrices with the minimum stable length path value. We note that even though only the first two signed systems have an entrainment of positive and negative ties, most relations in the factional system end up having paths of an ambivalent character.

The semiring structures result in two to four classes of actors with a different centrality location in the network depending on the relational content of the ties; three relations among factions have a question mark in the image matrices of Table 6 because they combine positive and negative ties but no ambivalence. Interestingly, if we take a look at the signed graphs of Figure 8, the most central faction (densest gray color) in all networks mixes LO and YT with outsider members as described by Sampson. In the case of the OC, except for Basil, they are recurrent members of another faction of mostly positive ties in the semiperiphery. Besides, we note in the praise-blame network that three non-isolated actors end up in the unconnected faction of the semiring structure, because they are unable to send or receive paths with $k \geqslant 3$. This latter circumstance and others in structurally balanced systems is not merely a product of chance, but it reflects the actors' relatedness with the other network members having particular structural characteristics including the type of bundle pattern that links them.

One significant difference between prior analyses performed on the Monastery network, including the categorization proposed by Sampson, and the analyses made in this paper is that our study is the first to be based on balanced semiring structures that are able to handle ambivalent relations. In this case, weakly balanced semiring structures are capable to reflect structurally correspondent actors linked by contrasting relations also in different structures from this particular network.

7 Conclusions

Cohesion and comparison are two forms for social influence that have been operationalized through relational settings made of different types of ties among actors in a social network. While cohesion influence has direct communication between actors as the primary persuasion mechanism, comparison influence is based

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on the structural similarity among the network members. In this sense, we look at the consistent bonds in the network to trace the potential channels of social influence, which can be pairwise relations or the co-occurrence of actors in a structurally similar class in the system.

More specifically, cohesion influence takes the different relational patterns at the dyadic level with a mutual character found through bundle census of the network. Such patterns are the building blocks of a system of strong bonds that constitutes the context on which individual network exposure measures are based, and which serves for the analysis of potential persuasion processes. One benefit of the strong bonds system is that it includes different levels in the relationship that are not constrained in principle except that the ties must be positively valued.

We deal with another situation in the analysis of social influence when assessing with contrasting relations. When we looked at the combination of cooperation and competition as drivers for innovation adoption for instance, we benefited from signed networks and the structural balance theory. These techniques turned out to be useful for the establishment of a semiring balanced system based on paths through an algebraic procedure that considers all the possible signs that the relations in the empirical networks can have.

Although the entrainment of ties with a positive and a negative relational content constitutes a link of an ambiguous character that prevents us to have a strictly balanced system; an ambivalent structure, however, does produce a factional system of classes of actors from a weakly balanced semiring structure that allows us to make a qualified interpretation of comparison influence and the acquiring of an attribute. Sources of influence are therefore not limited to direct contacts of social interaction or cognitive types of tie, but also include similar structural conditions that some of the actors in the network may have.

The circumstance that the factional system is based on paths rather than semipaths makes a lot of sense since what characterizes the comparison mechanism is the asymmetry in the relationship, which is in contrast with direct communication where mutuality is the key condition for the influence process. Nevertheless, both the system of strong bonds and the factional semiring balanced system have been correlated in a descriptive manner with the actor attributes that refers to innovation adoption in this case. Another possibility is to integrate the ties and actor attributes into a single relational system in which the acquired attributes are operationalized as the corresponding actor's self-ties represented by diagonal matrices. This will make possible to discover the algebraic constraints of the network as significant elements for the substantive interpretation of cohesion and comparison influence and the logic of interlock of the relational system.

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