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Volume, energy and generalized energy of unit vector fields on Berger spheres: stability of Hopf vector fields

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We study to what extent the known results concerning the behaviour of Hopf vector fields, with respect to volume, energy and generalized energy functionals, on the round sphere are still valid for the metrics obtained by performing the canonical variation of the Hopf fibration.

1. Introduction

Let $V : M \to TM$ be a smooth vector field on a manifold. For a given Riemannian metric g on M, the tangent manifold can be endowed with a natural metric g^{S} known as the Sasaki metric. The volume of V is the volume of $V(M)$ considered as a submanifold of (TM, g^{S}). Analogously, we can define the energy of V as the energy of the map $V : (M, g) \to (TM, g^{\mathrm{S}})$ and, more generally, if $\tilde{g}$ is another metric on M, we can define the generalized energy of V as the energy of $V : (M, \tilde{g}) \to (TM, g^{\mathrm{S}})$. On each manifold, these functionals have a lower bound and then a natural problem arises, namely that of determining the infimum of their values when acting on vector fields such that $g(V, V) = 1$ and finding the minimizers, or at least a minimizing sequence. It is easy to see that if M admits unit parallel vector fields, these should be exactly the minimizers, and so volume and energy can be seen as a measure of how much the vector field deviates from being parallel. The geometrically simplest manifolds admitting unit vector fields but not parallel ones are odd-dimensional round spheres, and Hopf vector fields on them are very special unit vector fields. They are tangent to the fibres of the Hopf fibration $\pi : S^{2m+1} \to \mathbb{C}P^m$. When both manifolds are endowed with their usual metrics, this map is a Riemannian submersion with totally geodesic fibres whose tangent space is generated by the unit vector field $V = JN$, where N is the unit normal to the sphere and J is the usual complex structure of $\mathbb{R}^{2m+2}$. It is also usual to call a Hopf vector field any vector field obtained as the image of N by any complex structure; they can be characterized as the unit Killing vector fields of the sphere. In [9], Gluck and Ziller showed that Hopf vector fields on the three-dimensional round sphere are the absolute minimizers of the volume and the analogous result for the energy was shown by Brito [3]. For spheres of higher dimension, they are unstable critical points of the energy (see [7, 13, 14]) and critical points of the volume, which is equivalent to defining a minimal immersion in the unit tangent bundle, as has been shown in [8].

The results quoted above are independent of the radius of the sphere. Nevertheless, concerning the stability of Hopf vector fields as critical points of the volume it has been shown [1,7] that for $m > 1$ they are unstable if and only if the curvature is lower than $2m-3$. The infimum of the volume of unit vector fields (as well as the regularity and properties of minimizers) appears to be a very sensitive geometrical invariant that enables us to detect a variation of the metric by homotheties.

In order to better understand these phenomena, we study the behaviour of the Hopf vector field with respect to the volume and the energy when we consider another variation of the standard metric on the sphere (which is a little more complicated but also very natural): the canonical variation of the Riemannian submersion given by the Hopf fibration. The metrics so constructed are known as Berger metrics; they consist in a one-parameter variation g_μ for $\mu > 0$. In the last section, we will also consider the Lorentzian Berger metrics, i.e. when $\mu < 0$. This paper is organized as follows. In § 2 we recall the definitions and state the results we will need in the remainder of the paper, and show that, for all $\mu \neq 0$, the unit Hopf vector field V^μ defines a harmonic map $V^\mu : (S^{2m+1}, g_\lambda) \to (T^1 S^{2m+1}, g_\mu^{\mathrm{S}})$ for all $\lambda \neq 0$ and that consequently it is a critical point for the generalized energy E_{g_λ}. Moreover, V^μ defines a minimal immersion.

In § 3, we study the special case of the three-dimensional sphere and show that *the unit Hopf vector field on (S^3, g_μ) is the only absolute minimizer of the energy, and of the volume, if and only if $\mu \leqslant 1$. For $\mu > 1$, we show that it is not even a local minimum, since it is unstable.*

So, the minimizing properties of Hopf vector fields on the round S^3 can be extended to Berger 3-spheres if $\mu \leqslant 1$, but not otherwise. It is worthwhile to recall here that with these metrics the sphere can be isometrically immersed as a geodesic sphere in the complex projective space and that, in contrast, for $\mu > 1$ it can be identified with a geodesic sphere of the complex hyperbolic space.

For higher-dimensional spheres, we have determined the values of μ for which the Hopf vector field is stable as a critical point of the energy and as a critical point of the volume. More precisely *the Hopf vector field on (S^{2m+1}, g_μ), with $m > 1$, is energy stable if and only if $(2m-2)\mu^2 \leqslant 1$ and it is volume stable if and only if* $(2m-2)\mu^3 - \mu \leqslant 1$. This is done in § 5, by using the methods developed in [1,7] for the round sphere and the various expressions of the Hessians computed in § 4.

We have used the same ideas to study the subset $\mathcal{E}$ of $\mathbb{R}^+ \times \mathbb{R}^+$ of pairs (μ, λ) such that V^μ is stable as a critical point of the generalized energy E_{g_λ}. Although a complete description of $\mathcal{E}$ is still unavailable, we can show, for example, that if $(2m-1)\mu \leqslant 2$, then $(\mu, \lambda) \in \mathcal{E}$, for all $\lambda > 0$, and that if $(2m-1)\mu > 2$ and $\mu \leqslant \frac{3}{2}$, then $(\mu, \lambda) \in \mathcal{E}$ if and only if $((2m-1)\mu - 2)\lambda \leqslant (\mu - 1)^2$. As a consequence, Hopf vector fields of the round sphere S^{2m+1}, with $m > 1$, are unstable as critical points of the generalized energy E_{g_λ}, for all $\lambda > 0$.

Section 6 is devoted to the study of the behaviour of Hopf vector fields on Lorentzian Berger spheres with respect to energy and volume functionals. Obtaining the corresponding expressions for the Hessians is straightforward: one should only pay attention to the timelike character of the Hopf vector field in these metrics. We have shown that *on (S^{2m+1}, g_μ), with $\mu < 0$, if $(2m-2)\mu^2 < 1$, the unit Hopf vector* *field is an unstable critical point of the energy and if* $(2-2m)\mu^3+(4m-4)\mu^2+\mu<1$, *it is an unstable critical point of the volume.* In contrast, neither the stability results nor the minimizing properties for the three-dimensional case have Lorentzian analogues; in fact we have shown that *on* (S^3, g_μ)*, for all* $\mu<0$*, the unit Hopf vector field is unstable.* These kind of difficulties which we met when trying to determine the stability are not exclusive to Berger spheres and, moreover, since on Lorentzian manifolds the energy of unit timelike vector fields is not bounded below, it is not natural to talk about absolute minimizers. These facts led us to define in [6] a new functional on the space of unit timelike vector fields of a Lorentz manifold, which we called spacelike energy and which is given by the integral of the square norm of the projection of the covariant derivative of the vector field onto its orthogonal complement. In [6] we have shown that Hopf vector fields are stable critical points of the spacelike energy. We finish this paper by showing that *on any Lorentzian Berger* 3*-sphere, the Hopf vector field is, up to sign, the only minimizer of the spacelike energy.*

2. Definitions and first results

2.1. Energy and volume of vector fields

Given a Riemannian manifold (M,g), the Sasaki metric g^{S} on the tangent bundle TM is defined, using g and its Levi-Civita connection ∇, as follows:

$$g^{\mathrm{S}}(\zeta_1,\zeta_2)=g(\pi_*\circ\zeta_1,\pi_*\circ\zeta_2)+g(\kappa\circ\zeta_1,\kappa\circ\zeta_2),$$

where $\pi: TM\to M$ is the projection and κ is the connection map of ∇. We also consider its restriction to the tangent sphere bundle, obtaining the Riemannian manifold (T^1M, g^{S}).

As in [5], for each metric $\tilde g$ on M we can define the generalized energy of the vector field V, denoted $E_{\tilde g}(V)$, as the energy of the map $V:(M,\tilde g)\to(TM,g^{\mathrm{S}})$ that is given by

$$E_{\tilde g}(V)=\frac{1}{2}\int_M \operatorname{tr} L_{(\tilde g,V)}\,\mathrm{d}v_{\tilde g},$$

where $L_{(\tilde g,V)}$ is the endomorphism determined by $V^*g^{\mathrm{S}}(X,Y)=\tilde g(L_{(\tilde g,V)}(X),Y)$. This energy can also be written as

$$E_{\tilde g}(V)=\frac{1}{2}\int_M\sqrt{\det P_{\tilde g}}\,\operatorname{tr}(P_{\tilde g}^{-1}\circ L_V)\,\mathrm{d}v_g, \tag{2.1}$$

where $P_{\tilde g}$ and L_V are defined by

$$\tilde g(X,Y)=g(P_{\tilde g}(X),Y)\quad\text{and}\quad V^*g^{\mathrm{S}}(X,Y)=g(L_V(X),Y),$$

respectively. By the definition of the Sasaki metric, $L_V=\mathrm{Id}+(\nabla V)^{\mathrm{T}}\circ\nabla V$. In particular, for $\tilde g=g$,

$$E_g(V)=\frac{1}{2}\int_M\operatorname{tr} L_V\,\mathrm{d}v_g=\tfrac{1}{2}n\operatorname{vol}(M,g)+\frac{1}{2}\int_M\|\nabla V\|^2\,\mathrm{d}v_g. \tag{2.2}$$

This functional is known as the energy and will be represented by E. Its relevant part, $B(V)=\frac{1}{2}\int_M\|\nabla V\|^2\,\mathrm{d}v_g$, is known as the total bending of V and its restriction to unit vector fields has been thoroughly studied by Wiegmink in [13] (see also [14]).

On the other hand, the volume of a vector field V is defined as the volume of the submanifold $V(M)$ of (TM, g^{S}). It is given by

$$F(V) = \int_M \sqrt{\det L_V}\, \mathrm{d}v_g. \tag{2.3}$$

Since for $\tilde{g} = V^* g^{\mathrm{S}}$ we have $P_{\tilde{g}} = L_V$, (2.1) and (2.3) give

$$F(V) = \frac{2}{n} E_{V^* g^{\mathrm{S}}}(V).$$

The first variation of the generalized energy was computed in [5]. It has also been shown there that V is a critical point of F if and only if V is a critical point of $E_{V^* g^{\mathrm{S}}}$ and that, on a compact M, a critical vector field of any of these generalized energies should be parallel. This is one of the reasons why it is usual to restrict the functionals to the submanifold of unit vector fields, and so critical points are those V which are stationary for variations consisting of unit vector fields, or equivalently with variational field orthogonal to V. *From now on, we consider the restriction of these functionals to the submanifold of unit vector fields.* The following proposition, shown in [5], generalizes the characterization of critical points of the total bending in [13] and of the volume in [8].

Proposition 2.1 (Gil-Medrano [5]). *Let (M, g) be a Riemannian manifold. A unit vector field V is a critical point of $E_{\tilde{g}}$ if and only if*

$$\omega_{(V,\tilde{g})}(V^\perp) = \{0\},$$

with $\omega_{(V,\tilde{g})} = C_1^1 \nabla K_{(V,\tilde{g})}$ and $K_{(V,\tilde{g})} = \sqrt{\det P_{\tilde{g}}}\, P_{\tilde{g}}^{-1} \circ (\nabla V)^{\mathrm{T}}$.

Remark 2.2. For a $(1,1)$-tensor field K, if $\{E_i\}$ is a g-orthonormal local frame, we have

$$C_1^1 \nabla K(X) = \sum_i g((\nabla_{E_i} K)X, E_i).$$

As a particular case of proposition 2.1, for $\tilde{g} = g$, a unit vector field V is a critical point of the energy (or of the total bending) if and only if

$$\omega_{(V,g)}(V^\perp) = \{0\}, \quad \text{with } \omega_{(V,g)} = C_1^1 \nabla (\nabla V)^{\mathrm{T}}.$$

Furthermore, if we put $\tilde{g} = V^* g^{\mathrm{S}}$, we find that critical points of the volume are characterized by the condition

$$\omega_V(V^\perp) = \{0\}, \quad \text{where } \omega_V = C_1^1 \nabla K_V \quad \text{and} \quad K_V = \sqrt{\det L_V}\, L_V^{-1} \circ (\nabla V)^{\mathrm{T}}.$$

In [8] it was proved that a unit vector field is a critical point of F if and only if it defines a minimal immersion in (T^1M, g^{S}). Nevertheless, as shown in [5], for a critical point V of $E_{\tilde{g}}$ to determine a harmonic map of $(M, \tilde{g})$ in (T^1M, g^{S}), V has to satisfy the condition

$$\sum_i R((\nabla V)\tilde{E}_i, V, \tilde{E}_i) + \sum_i (\nabla_{\tilde{E}_i} \tilde{E}_i - \tilde{\nabla}_{\tilde{E}_i} \tilde{E}_i) = 0, \tag{2.4}$$

where $\{\tilde{E}_i\}$ is a $\tilde{g}$-orthonormal frame and R represents the curvature operator of the metric g, that is

$$R(X,Y,Z) = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z.$$

THEOREM 2.3 (Gil-Medrano and Llinares-Fuster [7]). *Let V be a unit vector field on the Riemannian manifold (M,g).*

(a) *If V is a critical point of $E_{\tilde{g}}$, the Hessian of $E_{\tilde{g}}$ at V acting on $A \in V^\perp$ is given by*

$$(\mathrm{Hess}\, E_{\tilde{g}})_V(A) = \int_M \|A\|^2 \omega_{(V,\tilde{g})}(V)\, \mathrm{d}v_g + \int_M \sqrt{\det P_{\tilde{g}}}\, \mathrm{tr}(P_{\tilde{g}}^{-1} \circ (\nabla A)^{\mathrm{T}} \circ \nabla A)\, \mathrm{d}v_g.$$

(b) *If V is a critical point of the energy, the Hessian of E at V acting on $A \in V^\perp$ is given by*

$$(\mathrm{Hess}\, E)_V(A) = \int_M \|A\|^2 \omega_{(V,g)}(V)\, \mathrm{d}v_g + \int_M \|\nabla A\|^2\, \mathrm{d}v_g.$$

(c) *For a unit vector field V defining a minimal immersion, the Hessian of F at V acting on $A \in V^\perp$ is given by*

$$\begin{aligned}(\mathrm{Hess}\, F)_V(A) = &\int_M \|A\|^2 \omega_V(V)\, \mathrm{d}v_g + \int_M \frac{2}{\sqrt{\det L_V}} \sigma_2(K_V \circ \nabla A)\, \mathrm{d}v_g \\ &- \int_M \mathrm{tr}(L_V^{-1} \circ (\nabla A)^{\mathrm{T}} \circ \nabla V \circ K_V \circ \nabla A)\, \mathrm{d}v_g \\ &+ \int_M \sqrt{\det L_V}\, \mathrm{tr}(L_V^{-1} \circ (\nabla A)^{\mathrm{T}} \circ \nabla A)\, \mathrm{d}v_g,\end{aligned}$$

where σ_2 is the second elementary symmetric polynomial function. In particular, $\sigma_2(K_V \circ \nabla A) = \frac{1}{2}(\mathrm{tr}(K_V \circ \nabla A))^2 - \mathrm{tr}(K_V \circ \nabla A)^2$.

REMARK 2.4. The Hessian of the volume at a vector field V defining a minimal immersion can be simplified if V is assumed to be a Killing vector field. Using [7, lemma 9], we obtain

$$\begin{aligned}(\mathrm{Hess}\, F)_V(A) = &\int_M \|A\|^2 \omega_V(V)\, \mathrm{d}v_g + \int_M \frac{2}{\sqrt{\det L_V}} \sigma_2(K_V \circ \nabla A)\, \mathrm{d}v_g \\ &+ \int_M \sqrt{\det L_V}\, \mathrm{tr}(L_V^{-1} \circ (\nabla A)^{\mathrm{T}} \circ L_V^{-1} \circ \nabla A)\, \mathrm{d}v_g. \end{aligned} \tag{2.5}$$

2.2. Berger spheres

Hopf vector fields on odd-dimensional spheres are tangent to the fibres of the Hopf fibration $\pi : (S^{2m+1}, g) \to (\mathbb{C}P^m, \bar{g})$, where g is the usual metric of curvature 1 and $\bar{g}$ is the Fubini–Study metric with sectional curvatures between 1 and 4. This map is a Riemannian submersion with totally geodesic fibres whose tangent

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space is generated by the unit vector field $V = JN$, where N is the unit outward normal to the sphere and J is the usual complex structure of $\mathbb{R}^{2m+2}$; in other words, $V(p) = \mathrm{i}p$.

The canonical variation of the submersion is the one-parameter family of metrics (S^{2m+1}, g_μ), $\mu \neq 0$, defined by

$$g_\mu|_{V^\perp} = g|_{V^\perp}, \qquad g_\mu(V,V) = \mu g(V,V), \qquad g_\mu(V,V^\perp) = 0, \tag{2.6}$$

where $V^\perp$ denotes the orthogonal with respect to metric g of the one-dimensional distribution generated by V. When $\mu > 0$, the new metric is Riemannian and if $\mu < 0$, the metric is Lorentzian and V is timelike.

For all $\mu \neq 0$, the map $\pi : (S^{2m+1}, g_\mu) \to (\mathbb{C}P^m, \bar{g})$ is a semi-Riemannian submersion with totally geodesic fibres. (S^3, g_μ), with $\mu > 0$, is known as a Berger sphere. We will use the same name for all dimensions and we will call $V^\mu = (1/\sqrt{|\mu|})V$ the Hopf vector field. It is a unit Killing vector field with geodesic flow.

We denote by $\bar{\nabla}$ the Levi-Civita connection on $\mathbb{R}^{2m+2}$. The Levi-Civita connection ∇ on (S^{2m+1}, g) is $\nabla_X Y = \bar{\nabla}_X Y - \langle \bar{\nabla}_X Y, N\rangle N$ and $\bar{\nabla}_X V = J\bar{\nabla}_X N = JX$. Therefore, $\nabla_V V = 0$ and if $\langle X, V\rangle = 0$, then $\nabla_X V = JX$.

Using the Koszul formula, one obtains the relation of ∇^μ, the Levi-Civita connection of the metric g_μ, with ∇,

$$\nabla^\mu_V X = \nabla_V X + (\mu - 1)\nabla_X V, \qquad \nabla^\mu_X V = \mu \nabla_X V, \qquad \nabla^\mu_X Y = \nabla_X Y, \tag{2.7}$$

for all $X, Y \in V^\perp$.

By straightforward computations it can be seen that the sectional curvature K_μ of (S^{2m+1}, g_μ) takes the value

$$K_\mu(\sigma) = 1 + (1-\mu)g(X, JY)^2,$$

if $\sigma \subset V^\perp$ and $\{X, Y\}$ is an orthonormal basis, and it takes the value $K_\mu(\sigma) = \mu$ if the plane σ contains the vector V^μ. Consequently, the Ricci tensor has the form

$$\left.\begin{aligned} \mathrm{Ric}_\mu(V^\mu, V^\mu) &= 2m|\mu|, \\ \mathrm{Ric}_\mu(X, V^\mu) &= 0, \\ \mathrm{Ric}_\mu(X, Y) &= 2(1 - \mu + m)g(X, Y), \end{aligned}\right\} \tag{2.8}$$

for all $X, Y \in V^\perp$, and the scalar curvature is given by

$$S_\mu = 2m(2 + 2m - \mu).$$

It has been shown in [5] that, for all $\lambda > 0$, the map

$$V : (S^{2m+1}, g_\lambda) \to (T^1(S^{2m+1}), g^{\mathrm{S}})$$

is harmonic. More generally we have the following.

Proposition 2.5. *For all $\mu, \lambda \neq 0$, the map $V^\mu : (S^{2m+1}, g_\lambda) \to (T^1(S^{2m+1}), g^{\mathrm{S}}_\mu)$ is harmonic.*

Proof. According to proposition 2.1 and condition (2.4), we need to show that

$$\omega_{(V^\mu, g_\lambda)}(X) = 0 \quad \text{for all } X \in V^\perp \tag{2.9}$$

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and

$$\sum_i R_\mu((\nabla^\mu V^\mu)\tilde{E}_i, V^\mu, \tilde{E}_i) + \sum_i (\nabla^\mu_{\tilde{E}_i}\tilde{E}_i - \nabla^\lambda_{\tilde{E}_i}\tilde{E}_i) = 0, \tag{2.10}$$

where $\{\tilde{E}_i\}$ is a g_λ-orthonormal frame with $\tilde{E}_{2m+1} = V^\lambda$.

Using (2.7), for $i = 1, \dots, 2m$, we have

$$\nabla^\mu_{\tilde{E}_i}\tilde{E}_i - \nabla^\lambda_{\tilde{E}_i}\tilde{E}_i = 0$$

and it is easy to see that $R_\mu(X, V^\mu, Y) = \mu g(X, Y)V^\mu$, for all $X, Y \in V^\perp$, and then

$$R_\mu((\nabla^\mu V^\mu)\tilde{E}_i, V^\mu, \tilde{E}_i) = \mu g((\nabla^\mu V^\mu)\tilde{E}_i, \tilde{E}_i)V^\mu = 0.$$

For the last equality we use the fact that V^μ is a Killing vector field. Since it is also geodesic, we get (2.10).

The endomorphism P_{g_λ} relating the metrics g_μ and g_λ is the identity on $V^\perp$ and $P_{g_\lambda}(V) = (\lambda/\mu)V$. On the other hand, for $X \in V^\perp$,

$$(\nabla^\mu V^\mu)(X) = \frac{\mu}{\sqrt{|\mu|}}JX.$$

Then $K_{(V^\mu, g_\lambda)}(V^\mu) = 0$ and

$$K_{(V^\mu, g_\lambda)}(X) = -\frac{\mu}{|\mu|}\sqrt{|\lambda|}JX.$$

Therefore, when either $Y \in V^\perp$ or $Y = V$,

$$(\nabla^\mu_Y K_{(V^\mu, g_\lambda)})X = \frac{\mu}{|\mu|}\sqrt{|\lambda|}g(X, Y)V,$$

and then

$$g_\mu((\nabla^\mu_Y K_{(V^\mu, g_\lambda)})X, Y) = 0,$$

from which we get (2.9). □

Since $(V^\mu)^* g^{\mathrm{S}}_\mu = (1 + |\mu|)g_\lambda$, where $\lambda = \mu/(1 + |\mu|)$, as a consequence of the proposition above, we have the following result.

Corollary 2.6. *For all $\mu \neq 0$, the Hopf vector field V^μ is a critical point of the generalized energy E_{g_λ}, for all $\lambda \neq 0$, and it defines a minimal immersion.*

Remark 2.7. Although we have stated proposition 2.1 and condition (2.4) only for Riemannian metrics, it is easy to see that for Lorentzian metrics the analogous result also holds, up to the sign of the terms involving V^λ, which does not appear in this case because V^λ is geodesic.

Let us end this section by describing the holomorphic and anti-holomorphic derivatives. Since a vector field on S^{2m+1} can be seen as a map on $\mathbb{C}^{m+1}$, apart from the covariant derivatives ∇^μ we will use other differential operators that take into account the complex structure. Although it turns out that these operators are independent of μ, and so the description is identical to the corresponding one in [1], we find it convenient to reproduce it here.

Let $W : \mathcal{U} \subset \mathbb{C}^{m+1} \to \mathbb{C}^{m+1}$ be a vector field. We put $D^{\mathbb{C}}_X W = \bar\nabla_{JX} W - J\bar\nabla_X W$ and $\bar D^{\mathbb{C}}_X W = \bar\nabla_{JX} W + J\bar\nabla_X W$. Recall that W is *holomorphic* (respectively *anti-holomorphic*) if, for all X, $D^{\mathbb{C}}_X W = 0$ (respectively $\bar D^{\mathbb{C}}_X W = 0$).

Let $V^\perp$ be the distribution $\operatorname{span}(x, Jx)^\perp$ on $\mathbb{C}^{m+1} \setminus \{0\}$ and $\pi : T(\mathbb{C}^{m+1} \setminus \{0\}) \to V^\perp$ be the natural projections $\{x\} \times \mathbb{C}^{m+1} \to V^\perp_x$. We denote by $\|\pi \circ D^{\mathbb{C}} W\|_{V^\perp}$ the norm of $\pi \circ D^{\mathbb{C}} W_{|V^\perp} : V^\perp \to V^\perp$, that is

$$\|\pi \circ D^{\mathbb{C}} W\|^2_{V^\perp} = \sum_{i=1}^{2m} \|\pi \circ D^{\mathbb{C}}_{E_i} W\|^2,$$

where $E_1, \dots, E_{2m}$ is a local orthonormal frame of $V^\perp$. Similarly,

$$\|\pi \circ \bar D^{\mathbb{C}} W\|^2_{V^\perp} = \sum_{i=1}^{2m} \|\pi \circ \bar D^{\mathbb{C}}_{E_i} W\|^2,$$

but in that case

$$\pi \circ \bar D^{\mathbb{C}} W_{|V^\perp} = \bar D^{\mathbb{C}} W_{|V^\perp} : V^\perp \to V^\perp$$

so that

$$\|\pi \circ \bar D^{\mathbb{C}} W\|^2_{V^\perp} = \|\bar D^{\mathbb{C}} W\|^2_{V^\perp}.$$

We compute $\|\pi \circ D^{\mathbb{C}} A\|^2_{V^\perp}$ and $\|\bar D^{\mathbb{C}} A\|^2_{V^\perp}$ in terms of the matrix B of ∇A in a local frame, i.e. $B^j_i = \langle \nabla_{E_i} A, E_j\rangle$, to obtain

$$\begin{aligned}\tfrac{1}{2}\|\pi \circ D^{\mathbb{C}} A\|^2_{V^\perp} &= \sum_{i,j=1}^{m} (B^{j*}_{i*} - B^j_i)^2 + (B^j_{i*} + B^{j*}_i)^2 \\ &= \sum_{i,j=1}^{2m} (B^j_i)^2 + 2\sum_{i,j=1}^{m} (B^j_{i*} B^{j*}_i - B^{j*}_{i*} B^j_i) \end{aligned} \tag{2.11}$$

and

$$\begin{aligned}\tfrac{1}{2}\|\bar D^{\mathbb{C}} A\|^2_{V^\perp} &= \sum_{i,j=1}^{m} (B^{j*}_{i*} + B^j_i)^2 + (B^j_{i*} - B^{j*}_i)^2 \\ &= \sum_{i,j=1}^{2m} (B^j_i)^2 - 2\sum_{i,j=1}^{m} (B^j_{i*} B^{j*}_i - B^{j*}_{i*} B^j_i). \end{aligned} \tag{2.12}$$

In the remainder of the paper, if not otherwise stated, we will assume that the parameters μ and λ are positive; the study of Lorentzian Berger metrics will be performed in the final section.

3. The special case of S^3

The aim of this section is to show that the unit Hopf vector field on (S^3, g_μ) is the only absolute minimizer of the energy, and of the volume, if and only if $\mu \leqslant 1$. For $\mu > 1$, we will show that it is not even a local minimum.

Since $S^3 \subset \mathbb{R}^4 = \mathbb{H}$, we can define on S^3 a global g-orthonormal frame $\{V = J_0 N, E_1 = J_1 N, E_2 = J_2 N\}$, where $\{J_0, J_1, J_2\}$ denote the three standard complex structures defining the quaternionic structure of $\mathbb{R}^4$. Then $\{V^\mu, E_1, E_2\}$ is a g_μ-orthonormal frame.

Lemma 3.1. *Let X be a unit vector field which is an element of the two-dimensional space generated by $\{E_1, E_2\}$ and let W be of the form $W = \cos(t)V^\mu + \sin(t)X$. Then*

$$\|\nabla^\mu W\|^2 = 2\mu + 4\sin^2(t)\frac{1-\mu}{\mu}.$$

In particular,

$$\|\nabla^\mu X\|^2 = 2\mu + 4\frac{1-\mu}{\mu} \quad \textit{and} \quad \|\nabla^\mu V^\mu\|^2 = 2\mu.$$

Moreover,

$$\det L_W = (1+\mu)^2 + 4\sin^2(t)(1+\mu)\frac{1-\mu}{\mu}.$$

In particular

$$\det L_X = (1+\mu)^2 + 4(1+\mu)\frac{1-\mu}{\mu} \quad \textit{and} \quad \det L_{V^\mu} = (1+\mu)^2.$$

Proof. Since

$$\begin{aligned} \nabla_V V = \nabla_{E_1} E_1 = \nabla_{E_2} E_2 = 0, &\qquad \nabla_{E_1} E_2 = -\nabla_{E_2} E_1 = -V, \\ \nabla_V E_1 = -\nabla_{E_1} V = -E_2, &\qquad \nabla_{E_2} V = -\nabla_V E_2 = -E_1, \end{aligned}$$

using (2.7), we then find

$$\begin{aligned} \nabla^\mu_{V^\mu} V^\mu = \nabla^\mu_{E_1} E_1 = \nabla^\mu_{E_2} E_2 = 0, &\qquad \nabla^\mu_{E_1} E_2 = -\nabla^\mu_{E_2} E_1 = -\sqrt{\mu} V^\mu, \\ \nabla^\mu_{V^\mu} E_1 = \frac{\mu-2}{\sqrt{\mu}} E_2, &\qquad \nabla^\mu_{E_1} V^\mu = \sqrt{\mu} E_2, \\ \nabla^\mu_{V^\mu} E_2 = -\frac{\mu-2}{\sqrt{\mu}} E_1, &\qquad \nabla^\mu_{E_2} V^\mu = -\sqrt{\mu} E_1. \end{aligned}$$

For a unit vector field $X = a_1 E_1 + a_2 E_2$ with $a_i \in C^\infty(S^3)$, if we take $Y = -a_2 E_1 + a_1 E_2$, then

$$\left.\begin{aligned} \nabla^\mu_X X &= X(a_1)E_1 + X(a_2)E_2, \\ \nabla^\mu_{V^\mu} X &= V^\mu(a_1)E_1 + V^\mu(a_2)E_2 + \frac{\mu-2}{\sqrt{\mu}} Y, \\ \nabla^\mu_Y X &= Y(a_1)E_1 + Y(a_2)E_2 + \sqrt{\mu} V^\mu. \end{aligned}\right\} \tag{3.1}$$

If we assume that the functions a_1 and a_2 are constant, then

$$\nabla^\mu_X X = 0, \quad \nabla^\mu_{V^\mu} X = \frac{\mu-2}{\sqrt{\mu}} Y \quad \text{and} \quad \nabla^\mu_Y X = \sqrt{\mu} V^\mu.$$

Therefore, if $W = \cos(t)V^\mu + \sin(t)X$, since $\nabla^\mu W = \cos(t)\nabla^\mu V^\mu + \sin(t)\nabla^\mu X$, it is not difficult to see that

$$\begin{aligned} L_W(X) &= (1 + \mu\cos^2(t))X + (\mu - 2)\sin(t)\cos(t)V^\mu, \\ L_W(V^\mu) &= (\mu - 2)\sin(t)\cos(t)X + \left(1 + \sin^2(t)\frac{(\mu - 2)^2}{\mu}\right)V^\mu, \\ L_W(Y) &= (1 + \mu)Y, \end{aligned}$$

from whence

$$\operatorname{tr} L_W = 3 + \|\nabla^\mu W\|^2 = 3 + 2\mu + 4\sin^2(t)\frac{1 - \mu}{\mu},$$

and

$$\det L_W = (1 + \mu)^2 + 4\sin^2(t)(1 + \mu)\frac{1 - \mu}{\mu}.$$

□

For $\mu = 1$, all the elements of the unit sphere of the three-dimensional vector space generated by $\{V^\mu, E_1, E_2\}$ are also called Hopf vector fields; they can be characterized as the unit Killing vector fields. It is known [3, 9] that Hopf vector fields have all the same volume and the same energy and that they are the only minimizers of both functionals. For $\mu \neq 1$, the situation is quite different, as shown in the next result.

Theorem 3.2. *Let (S^3, g_μ) be the three-dimensional Berger sphere.*

(a) *If $\mu < 1$, V^μ is, up to sign, the only minimizer of the energy and of the volume of unit vector fields; the minima of the functionals are*

$$E(V^\mu) = (\tfrac{3}{2} + \mu)\operatorname{vol}(S^3, g_\mu) \quad \textit{and} \quad F(V^\mu) = (1 + \mu)\operatorname{vol}(S^3, g_\mu),$$

respectively.

(b) *If $\mu > 1$, for all unit vector fields A in the two-dimensional space generated by $\{E_1, E_2\}$ we have*

$$E(A) = \left(\frac{3}{2} + \mu + 2\left(\frac{1 - \mu}{\mu}\right)\right)\operatorname{vol}(S^3, g_\mu) < E(V^\mu)$$

and

$$F(A) = \sqrt{(1 + \mu)^2 + 4(1 + \mu)\frac{1 - \mu}{\mu}}\operatorname{vol}(S^3, g_\mu) < F(V^\mu).$$

In fact V^μ is not even a local minimum. Moreover, for all unit vector fields X,

$$E(X) > (\tfrac{7}{2} - \mu)\operatorname{vol}(S^3, g_\mu) \quad \textit{and} \quad F(X) > (3 - \mu)\operatorname{vol}(S^3, g_\mu).$$

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Proof. For any three-dimensional compact manifold, the energy and the volume of unit vector fields are related with the integral of the Ricci tensor, as shown in [3]. In this particular case the inequalities are written as

$$\left.\begin{aligned} E(X) &\geqslant \tfrac{3}{2}\operatorname{vol}(S^3, g_\mu) + \frac{1}{2}\int_{S^3} \operatorname{Ric}_\mu(X,X)\,\mathrm{d}v_\mu, \\ F(X) &\geqslant \operatorname{vol}(S^3, g_\mu) + \frac{1}{2}\int_{S^3} \operatorname{Ric}_\mu(X,X)\,\mathrm{d}v_\mu. \end{aligned}\right\} \tag{3.2}$$

In both cases, the equality holds if and only if $\nabla^\mu_X X = 0$, $h_{11} = h_{22}$ and $h_{12} = -h_{21}$, where $h_{ij} = g_\mu(\nabla^\mu_{\bar{E}_i} X, \bar{E}_j)$ and $\{X, \bar{E}_1, \bar{E}_2\}$ is a g_μ-orthonormal frame.

Using (2.8), we find that if $\mu < 1$, then

$$\operatorname{Ric}_\mu(X,X) \geqslant \operatorname{Ric}_\mu(V^\mu, V^\mu) = 2\mu$$

for all unit X, with equality if and only if $X = \pm V^\mu$ and therefore $E(X) \geqslant E(V^\mu)$ and $F(X) \geqslant F(V^\mu)$. The Hopf vector field is then, up to sign, the only minimizer and we have shown (a).

The first sentence of (b) is a direct consequence of lemma 3.1. To see that, for $\mu > 1$, the Hopf vector field is not a local minimum, we need only to consider the curve of unit vector fields $W(t) = \cos(t)V^\mu + \sin(t)A$, where $A = a_1E_1 + a_2E_2$, with $a_i \in \mathbb{R}$, is a unit vector field. In lemma 3.1 we have computed the value of the functions $E(t) = E(W(t))$ and $F(t) = F(W(t))$, from which we observe that for $t = 0$ both functions reach their maximum.

Moreover, (2.8) gives us that if $\mu > 1$, then

$$\operatorname{Ric}_\mu(X,X) \geqslant \operatorname{Ric}_\mu(A,A) = 2(2-\mu),$$

for all unit X and all unit $A \in V^\perp$, with equality if and only if $X \in V^\perp$. Consequently, if we use (3.2),

$$E(X) \geqslant (\tfrac{7}{2} - \mu)\operatorname{vol}(S^3, g_\mu) \quad \text{and} \quad F(X) \geqslant (3-\mu)\operatorname{vol}(S^3, g_\mu), \tag{3.3}$$

with equality if and only if $X \in V^\perp$, $\nabla^\mu_X X = 0$, $h_{11} = h_{22}$ and $h_{12} = -h_{21}$.

Let us assume that a unit vector field X satisfies the four conditions above. Firstly $X = a_1E_1 + a_2E_2$ with $a_i \in C^\infty(S^3)$ and if we take $Y = -a_2E_1 + a_1E_2$, then, by (3.1), the other three conditions become

$$X(a_1) = X(a_2) = 0, \tag{3.4}$$

$$-a_2Y(a_1) + a_1Y(a_2) = 0, \tag{3.5}$$

$$a_2V^\mu(a_1) - a_1V^\mu(a_2) = \frac{2\mu - 2}{\sqrt{\mu}}. \tag{3.6}$$

If a_2 vanishes identically, then a_1 should be constant and (3.6) give us a contradiction. So, the open set where $a_2 \neq 0$ is not empty, and (3.4) and (3.5) then imply that on this set we have

$$X\left(\frac{a_1}{a_2}\right) = Y\left(\frac{a_1}{a_2}\right) = 0.$$

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But, by the above choice of X and Y, this is equivalent to

$$E_1\left(\frac{a_1}{a_2}\right) = E_2\left(\frac{a_1}{a_2}\right) = 0,$$

which, using the relation between V^μ and $[E_1, E_2]$, implies $V^\mu(a_1/a_2) = 0$, which is again in contradiction to (3.6). Therefore, the lower bounds in (3.3) are never reached. □

Remark 3.3. Since on any three-dimensional manifold the functional E (respectively F) is bounded below by $\frac{3}{2}$ times the volume (respectively by the volume) of the manifold, the lower bounds appearing in the above theorem are relevant only for $\mu < 2$.

Part (a) of the theorem is a particular case of a result of [10] concerning unit Killing vector fields on a three-dimensional compact manifold.

A relation between the energy and the integral of the Ricci tensor similar to the one quoted in (3.2) is valid for any compact manifold (see [3]) and then we have the following result.

Proposition 3.4. *For all unit vector fields X on (S^{2m+1}, g_μ),*

$$E(X) \geqslant \frac{2m+1}{2}\,\mathrm{vol}(S^{2m+1}, g_\mu) + \frac{1}{2(2m-1)}\int_{S^{2m+1}} \mathrm{Ric}_\mu(X, X)\,\mathrm{d}v_\mu,$$

with equality if and only if $\nabla^\mu_X X = 0$, and the distribution $X^\perp$ determines a foliation with umbilical leaves.

Since $\mathrm{Ric}_\mu(V^\mu, V^\mu) = 2m\mu$ and $\mathrm{Ric}_\mu(A, A) = 2(1-\mu+m)\|A\|^2$ for all $A \in V^\perp$, if $\mu < 1$,

$$E(X) \geqslant \left(\frac{2m+1}{2} + \frac{m\mu}{2m-1}\right)\mathrm{vol}(S^{2m+1}, g_\mu).$$

Moreover, if $m \neq 1$, equality never holds because this will imply that $X = V^\mu$ and

$$E(V^\mu) = \frac{2m+1+2m\mu}{2}\,\mathrm{vol}(S^{2m+1}, g_\mu).$$

If $\mu > 1$, then

$$E(X) \geqslant \left(\frac{2m+1}{2} + \frac{1-\mu+m}{2m-1}\right)\mathrm{vol}(S^{2m+1}, g_\mu),$$

with equality if and only if $X \in V^\perp$, $\nabla^\mu_X X = 0$, and the distribution $X^\perp$ determines a foliation with umbilical leaves. As for the three-dimensional sphere, this lower bound is relevant for $1 < \mu < m + 1$.

In the case of the round sphere, $\mu = 1$, the bound

$$\left(\frac{2m+1}{2} + \frac{m}{2m-1}\right)\mathrm{vol}(S^{2m+1})$$

is the value of the energy of radial vector fields defined on the complement of two antipodal points. Moreover, it has been shown in [2] that it is the infimum of the

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energy. In contrast with this situation, for $\mu > 1$ we do not know if any unit vector field which has this energy exists, even if we allow singularities.

Concerning the volume, the difference between the case $\mu = 1$ and the general one is greater. In fact, it is shown in [4] that for the round spheres the volume of radial vector fields is also a lower bound of F, but the proof is based on an inequality relating the volume of a unit vector field with the curvature of the manifold that is only valid for constant curvature spaces.

4. Second variation of the generalized energy and of the volume at Hopf vector fields

For fixed $\mu \neq 0$, V^μ is a critical point of E and F and is also a critical point of E_{g_λ}, for all $\lambda \neq 0$. We can then compute the Hessians of these functionals at V^μ. For simplicity we give the proof only for positive values of the parameter.

Proposition 4.1. *Let V^μ be the Hopf unit vector field on (S^{2m+1}, g_μ). For each vector field A orthogonal to V^μ we have*

$$(\mathrm{Hess}\, E_{g_\lambda})_{V^\mu}(A) = \int_{S^{2m+1}} (-2m\sqrt{\lambda\mu}\|A\|^2 + \sqrt{\lambda/\mu}\|\nabla^\mu A\|^2 + (\sqrt{\mu/\lambda} - \sqrt{\lambda/\mu})\|\nabla^\mu_{V^\mu} A\|^2)\,\mathrm{d}v_\mu, \tag{4.1}$$

$$(\mathrm{Hess}\, E)_{V^\mu}(A) = \int_{S^{2m+1}} (-2m\mu\|A\|^2 + \|\nabla^\mu A\|^2)\,\mathrm{d}v_\mu, \tag{4.2}$$

$$(\mathrm{Hess}\, F)_{V^\mu}(A) = (1+\mu)^{m-2} \int_{S^{2m+1}} (\mu(-2m\mu + 2(1-\mu))\|A\|^2 + \|\nabla^\mu A\|^2 + \mu\|\nabla^\mu_{V^\mu} A + \sqrt{\mu} J A\|^2)\,\mathrm{d}v_\mu. \tag{4.3}$$

Proof. We need only to compute the elements appearing in theorem 2.3 for this particular case. Since $L_{g_\lambda} = \sqrt{\lambda/\mu}(g_\lambda^{-1} g_\mu)$ and $\nabla^\mu V^\mu = \sqrt{\mu} J$, by direct computation we obtain

$$\omega_{(V^\mu, g_\lambda)}(V^\mu) = -2m\sqrt{\lambda\mu},$$

and

$$\begin{aligned}\mathrm{tr}(L_{g_\lambda} \circ (\nabla^\mu A)^{\mathrm{T}} \circ (\nabla^\mu A)) &= \sqrt{\lambda/\mu} \sum_{i=1}^{2m} g_\mu(\nabla^\mu_{E_i} A, \nabla^\mu_{E_i} A) + \sqrt{\lambda/\mu}\, g_\mu(\nabla^\mu_{V^\lambda} A, \nabla^\mu_{V^\lambda} A) \\ &= \sqrt{\lambda/\mu} \sum_{i=1}^{2m} g_\mu(\nabla^\mu_{E_i} A, \nabla^\mu_{E_i} A) + \sqrt{\mu/\lambda}\, g_\mu(\nabla^\mu_{V^\mu} A, \nabla^\mu_{V^\mu} A),\end{aligned}$$

from whence (4.1) and (4.2) hold.

Since $\nabla^\mu V^\mu = \sqrt{\mu} J$, on $(V^\mu)^\perp$ and V^μ is geodesic, and we find that $L_{V^\mu}(V^\mu) = V^\mu$ and $L_{V^\mu} = (1+\mu)\mathrm{Id}$ on $(V^\mu)^\perp$. Then,

$$f(V^\mu) = (1+\mu)^m \quad \text{and} \quad K_{V^\mu} = -(1+\mu)^{m-1} \nabla^\mu V^\mu.$$

By direct computation we obtain

$$\omega_{V^\mu}(V^\mu) = -2m\mu(1+\mu)^{m-1},$$

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and

$$(K_{V^\mu} \circ \nabla^\mu A)(X) = -\sqrt{\mu}(1+\mu)^{m-1}(\nabla^\mu_X JA + g(X, A)V),$$

from whence

$$\frac{2}{f(V^\lambda)}\sigma_2(K_{V^\mu} \circ \nabla^\mu A) = 2\mu(1+\mu)^{m-2}(\sigma_2(\nabla^\mu JA) - \sqrt{\mu} g(\nabla^\mu_{V^\mu} JA, A)).$$

Using the fact that, on any Riemannian manifold, $2\sigma_2(\nabla X)$ and $\mathrm{Ric}(X, X)$ differ in a divergence (see, for example, [12, p. 170]) and using the value of the Ricci tensor of g_μ (2.8), we have

$$\int_{S^{2m+1}} \frac{1}{f(V^\lambda)}\sigma_2(K_{V^\mu} \circ \nabla^\mu A)\,\mathrm{d}v_\mu \\ = \mu(1+\mu)^{m-2}\int_{S^{2m+1}}((m-\mu+1)\|A\|^2 - \sqrt{\mu} g(\nabla^\mu_{V^\mu} JA, A))\,\mathrm{d}v_\mu.$$

Finally,

$$\mathrm{tr}(L^{-1}_{V^\mu} \circ (\nabla^\mu A)^{\mathrm{T}} \circ L^{-1}_{V^\mu} \circ \nabla^\mu A) \\ = (1+\mu)^{-2}\sum_{i,j=1}^{2m} g_\mu(\nabla^\mu_{E_i} A, E_j)^2 + (1+\mu)^{-1}(\mu\|JA\|^2 + \|\nabla^\mu_{V^\mu} A\|^2) \\ = (1+\mu)^{-2}(\|\nabla^\mu A\|^2 + \mu^2\|JA\|^2 + \mu\|\nabla^\mu_{V^\mu} A\|^2).$$

For the last equality we have used the fact that

$$\|\nabla^\mu A\|^2 = \sum_{i,j=1}^{2m} g_\mu(\nabla^\mu_{E_i} A, E_j)^2 + \mu\|JA\|^2 + \|\nabla^\mu_{V^\mu} A\|^2.$$

Since V^μ is a Killing vector field, we can use (2.5) to compute the Hessian and then we get (4.3). □

In order to study the stability of the Hopf vector field it will be useful to find new expressions of the Hessians. We will proceed following closely the arguments used in [1], for the volume functional in the case of the round spheres. There, the key was to relate the integral of $\|\nabla A\|^2$ with the integral of $\|\pi \circ D^C A\|^2_{V^\perp}$ and that of $\|\bar{D}^C A\|^2_{V^\perp}$.

Firstly, since

$$\sum_{i,j=1}^{m}(B^j_{i*}B^{j*}_i - B^{j*}_{i*}B^j_i) = -\sum_{i=1}^{m} g_\mu(\nabla^\mu_{JE_i} A, J\nabla^\mu_{E_i} A),$$

equations (2.11) and (2.12) can be written as

$$\|\nabla^\mu A\|^2 = \tfrac{1}{2}\|\pi \circ D^C A\|^2_{V^\perp} + \|\nabla^\mu_{V^\mu} A\|^2 + \mu\|A\|^2 + 2\sum_{i=1}^{m} g_\mu(\nabla^\mu_{JE_i} A, J\nabla^\mu_{E_i} A)$$

and

$$\|\nabla^\mu A\|^2 = \tfrac{1}{2}\|\bar{D}^C A\|^2_{V^\perp} + \|\nabla^\mu_{V^\mu} A\|^2 + \mu\|A\|^2 - 2\sum_{i=1}^{m} g_\mu(\nabla^\mu_{JE_i} A, J\nabla^\mu_{E_i} A),$$

respectively.

The second step is the following lemma, the proof of which is very similar to the corresponding one in [1] and will be omitted.

LEMMA 4.2. *For all $\mu \neq 0$ we have*

$$2m\sqrt{|\mu|}V^{\mu} = -\sum_{i=1}^{m}[E_i, JE_i] + \sum_{i=1}^{m}\operatorname{div}^{\mu}(JE_i)E_i - \sum_{i=1}^{m}\operatorname{div}^{\mu}(E_i)JE_i$$

and

$$\begin{aligned} m\sqrt{|\mu|}\int_{S^{2m+1}} g_{\mu}(\nabla^{\mu}_{V^{\mu}}A, JA)\,\mathrm{d}v_{\mu} \\ = (m\mu - m - 1)\int_{S^{2m+1}} \|A\|^2\,\mathrm{d}v_{\mu} + \int_{S^{2m+1}}\sum_{i=1}^{m} g_{\mu}(\nabla^{\mu}_{JE_i}A, J\nabla^{\mu}_{E_i}A)\,\mathrm{d}v_{\mu}. \end{aligned}$$

Now, as a consequence, we have the following lemma.

LEMMA 4.3. *For all $\mu > 0$,*

$$\begin{aligned} &\int_{S^{2m+1}} \|\nabla^{\mu}A\|^2\,\mathrm{d}v_{\mu} \\ &\quad = \int_{S^{2m+1}} (\tfrac{1}{2}\|\pi \circ D^C A\|^2_{V^{\perp}} + \|\nabla^{\mu}_{V^{\mu}}A\|^2 + (2 + \mu + 2m(1-\mu))\|A\|^2 \\ &\qquad + 2m\sqrt{\mu}g_{\mu}(\nabla^{\mu}_{V^{\mu}}A, JA))\,\mathrm{d}v_{\mu} \\ &\int_{S^{2m+1}} \|\nabla^{\mu}A\|^2\,\mathrm{d}v_{\mu} \\ &\quad = \int_{S^{2m+1}} (\tfrac{1}{2}\|\bar{D}^C A\|^2_{V^{\perp}} + \|\nabla^{\mu}_{V^{\mu}}A\|^2 + (\mu - 2 + 2m(\mu - 1))\|A\|^2 \\ &\qquad - 2m\sqrt{\mu}g_{\mu}(\nabla^{\mu}_{V^{\mu}}A, JA))\,\mathrm{d}v_{\mu}. \end{aligned}$$

If we use these values on the corresponding expressions of proposition 4.1, we obtain the following proposition.

PROPOSITION 4.4. *Let V^{μ} be the Hopf unit vector field on (S^{2m+1}, g_{μ}). For each vector field A orthogonal to V^{μ} we have*

$$\begin{aligned} &(\operatorname{Hess} E_{g_{\lambda}})_{V^{\mu}}(A) \\ &\quad = \int_{S^{2m+1}} \biggl((\sqrt{\lambda\mu}(1-4m) + \sqrt{\lambda/\mu}(2m+2-\lambda m^2))\|A\|^2 \\ &\qquad + \sqrt{\mu/\lambda}\Bigl\|\nabla^{\mu}_{V^{\mu}}A + \frac{\lambda m}{\sqrt{\mu}}JA\Bigr\|^2 + \tfrac{1}{2}\sqrt{\lambda/\mu}\|\pi \circ D^C A\|^2_{V^{\perp}}\biggr)\,\mathrm{d}v_{\mu}, \quad (4.4) \\ &(\operatorname{Hess} E_{g_{\lambda}})_{V^{\mu}}(A) \\ &\quad = \int_{S^{2m+1}} \biggl((\sqrt{\lambda\mu} - \sqrt{\lambda/\mu}(2m+2+\lambda m^2))\|A\|^2 \\ &\qquad + \sqrt{\mu/\lambda}\Bigl\|\nabla^{\mu}_{V^{\mu}}A - \frac{\lambda m}{\sqrt{\mu}}JA\Bigr\|^2 + \tfrac{1}{2}\sqrt{\lambda/\mu}\|\bar{D}^C A\|^2_{V^{\perp}}\biggr)\,\mathrm{d}v_{\mu}, \quad (4.5) \end{aligned}$$

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$$
\begin{aligned}
(\operatorname{Hess} E)_{V^\mu}(A) \\
&= \int_{S^{2m+1}} ((2m+2-\mu(m^2+4m-1))\|A\|^2 \\
&\qquad + \|\nabla^\mu_{V^\mu} A + m\sqrt{\mu} JA\|^2 + \tfrac{1}{2}\|\pi \circ D^C A\|^2_{V^\perp})\,\mathrm{d}v_\mu, \qquad (4.6)
\end{aligned}
$$

$$
\begin{aligned}
(\operatorname{Hess} E)_{V^\mu}(A) \\
&= \int_{S^{2m+1}} ((-2m-2-\mu(m^2-1))\|A\|^2 \\
&\qquad + \|\nabla^\mu_{V^\mu} A - m\sqrt{\mu} JA\|^2 + \tfrac{1}{2}\|\bar{D}^C A\|^2_{V^\perp})\,\mathrm{d}v_\mu, \qquad (4.7)
\end{aligned}
$$

$$
\begin{aligned}
(\operatorname{Hess} F)_{V^\mu}(A) \\
&= (1+\mu)^{m-2} \int_{S^{2m+1}} \bigg(f_1(m,\mu)\|A\|^2 + (1+\mu)\|\nabla^\mu_{V^\mu} A \\
&\qquad + \frac{\sqrt{\mu}(m+\mu)}{1+\mu} JA\|^2 + \tfrac{1}{2}\|\pi \circ D^C A\|^2_{V^\perp} \bigg)\,\mathrm{d}v_\mu, \qquad (4.8)
\end{aligned}
$$

$$
\begin{aligned}
(\operatorname{Hess} F)_{V^\mu}(A) \\
&= (1+\mu)^{m-2} \int_{S^{2m+1}} \bigg(f_2(m,\mu)\|A\|^2 + (1+\mu)\|\nabla^\mu_{V^\mu} A \\
&\qquad + \frac{\sqrt{\mu}(\mu-m)}{1+\mu} JA\|^2 + \tfrac{1}{2}\|\bar{D}^C A\|^2_{V^\perp} \bigg)\,\mathrm{d}v_\mu, \qquad (4.9)
\end{aligned}
$$

where

$$f_1(m,\mu) = \mu(3-\mu-2m-2m\mu) + (2m+2) - \frac{\mu(m+\mu)^2}{1+\mu}$$

and

$$f_2(m,\mu) = \mu(3-\mu+2m-2m\mu) - (2m+2) - \frac{\mu(\mu-m)^2}{1+\mu}.$$

5. Stability of Hopf vector fields on S^{2m+1} with $m > 1$

The instability results for the round spheres have been obtained by showing that the Hessian is negative when it acts on the vector fields $A_a = a - \langle a, V\rangle V - \langle a, N\rangle N = a - \bar{f}_a V - f_a N$ for all $a \in \mathbb{R}^{2m+2}$, $a \neq 0$. A geometrical description of these vector fields can be seen in [7], as well as the following lemma.

Lemma 5.1.

$$\int_{S^{2m+1}} \bar{f}_a^2 \,\mathrm{d}v = \int_{S^{2m+1}} f_a^2 \,\mathrm{d}v = \frac{|a|^2}{2m+2} \operatorname{vol}(S^{2m+1}).$$

If we use proposition 4.1 to compute the value of the Hessian acting on these particular vector fields, we obtain the following lemma.

Lemma 5.2. *Let V^μ be the Hopf unit vector field on (S^{2m+1}, g_μ). For each $a \in \mathbb{R}^{2m+2}$, $a \neq 0$ we have*

$$(\text{Hess}\, E_{g_\lambda})_{V^\mu}(A_a) = \frac{\sqrt{\lambda}m}{m+1}|a|^2\left((1-2m)\mu + 2 + \frac{(\mu-1)^2}{\lambda}\right)\text{vol}(S^{2m+1}), \quad (5.1)$$

$$(\text{Hess}\, E)_{V^\mu}(A_a) = \frac{\sqrt{\mu}m}{m+1}|a|^2\left((1-2m)\mu + 2 + \frac{(\mu-1)^2}{\mu}\right)\text{vol}(S^{2m+1}), \quad (5.2)$$

$$(\text{Hess}\, F)_{V^\mu}(A_a) = (1+\mu)^{m-2}\frac{\sqrt{\mu}m}{m+1}|a|^2 f(m,\mu)\,\text{vol}(S^{2m+1}), \quad (5.3)$$

where $f(m,\mu) = ((1-2m)\mu(1+\mu) + 2m\mu + 2 + (1+\mu)(\mu-1)^2/\mu)$.

Proof. We need to compute all the elements appearing in the formulae given in proposition 4.1. Since A_a is orthogonal to V we have, as in the case $\mu = 1$ computed in [7],

$$\|A\|^2 = |a|^2 - \bar{f}_a^2 - f_a^2 \quad \text{and} \quad \sum_{i,j=1}^{2m}(B_i^j)^2 = 2m(\bar{f}_a^2 + f_a^2).$$

But now

$$\nabla^\mu_{V^\mu}A = -\frac{\mu-1}{\sqrt{\mu}}\sum_{j=1}^{m}(E_j(\bar{f}_a)E_j + E_{j*}(\bar{f}_a)E_{j*}),$$

and then

$$\|\nabla^\mu_{V^\mu}A\|^2 = \frac{(\mu-1)^2}{\mu}\sum_{j=1}^{m}((E_j(\bar{f}_a))^2 + (E_{j*}(\bar{f}_a))^2) = \frac{(\mu-1)^2}{\mu}(|a|^2 - \bar{f}_a^2 - f_a^2),$$

$$g(\nabla^\mu_{V^\mu}A, JA) = \frac{\mu-1}{\sqrt{\mu}}(|a|^2 - \bar{f}_a^2 - f_a^2).$$

The integrands of the Hessians are obtained by straightforward computation and then we use lemma 5.1 to conclude. □

It is an immediate consequence of (5.1) that, if $m > 1$, Hopf vector fields of the round sphere are unstable when considered as critical points of all the energy functionals E_{g_λ}, thus generalizing the corresponding result for the usual energy. But lemma 5.2 gives us the instability of V^μ in many other cases, which we summarize in the following proposition.

Proposition 5.3. *Let V^μ be the Hopf unit vector field on (S^{2m+1}, g_μ) with $m > 1$.*

(a) *If $(2m-1)\mu > 2$, and $((2m-1)\mu - 2)\lambda > (\mu-1)^2$, then V^μ is an unstable critical point of the energy E_{g_λ}.*

(b) *If $(2m-2)\mu^2 > 1$, then V^μ is energy unstable.*

(c) *If $(2m-2)\mu^3 - \mu > 1$, then V^μ is volume unstable.*

In all cases the index is at least $2m + 2$.

Proof. To show (b) and (c) we only need to use lemma 5.2 to write, respectively, the conditions

$$(\text{Hess}\, E)_{V^\mu}(A_a) < 0 \quad \text{and} \quad (\text{Hess}\, F)_{V^\mu}(A_a) < 0.$$

Analogously, from (5.1), we find that

$$(\text{Hess}\, E_{g_\lambda})_{V^\mu}(A_a) < 0$$

if $(1-2m)\mu + 2 + (\mu-1)^2/\lambda < 0$, which is equivalent to the condition stated in (a). □

We will show that as concerns volume and energy the sufficient condition for instability is also necessary. For the other functionals the situation is more complicated and is still open for some values of (μ, λ).

In order to obtain the stability results, it is convenient to see a vector field A on S^{2m+1}, orthogonal to the Hopf vector field, as a map $A : S^{2m+1} \to V^\perp \subset \mathbb{C}^{m+1}$, where $V^\perp$ represents the distribution $V_x^\perp = \text{span}\{x, Jx\}^\perp$. For such a map A, we write

$$A_l(p) = \frac{1}{2\pi}\int_0^{2\pi} A(e^{i\theta}p)e^{-il\theta}\, d\theta \in V_p^\perp$$

so that

$$A(p) = \sum_{l\in Z} A_l(p)$$

is the Fourier series of A. Since $A_l(e^{i\theta}p) = e^{il\theta}A_l(p)$,

$$\nabla_V A = \bar{\nabla}_V A = \sum_{l\in Z} ilA_l = \sum_{l\in Z} lJA_l$$

and

$$\|\nabla^\mu_{V^\mu} A_l + \alpha JA_l\|^2 = \frac{1}{\mu}(l - 1 + \mu + \alpha\sqrt{\mu})^2\|A_l\|^2.$$

If $\mathcal{C}(p)$ denotes the fibre of the Hopf fibration $\pi : S^{2m+1} \to \mathbb{C}P^m$ passing through p and, for $l \neq q$,

$$\int_{\mathcal{C}(p)} \langle A_l, A_q\rangle = 0.$$

By the construction of the Berger metrics, this fact is independent of μ and so the essential following lemma, shown in [1] for the volume functional in the case $\mu = 1$, remains valid.

Lemma 5.4.

$$(\text{Hess}\, E_{g_\lambda})_{V^\mu}(A) = \sum_{l\in Z}(\text{Hess}\, E_{g_\lambda})_{V^\mu}(A_l), \tag{5.4}$$

$$(\text{Hess}\, E)_{V^\mu}(A) = \sum_{l\in Z}(\text{Hess}\, E)_{V^\mu}(A_l), \tag{5.5}$$

$$(\text{Hess}\, F)_{V^\mu}(A) = \sum_{l\in Z}(\text{Hess}\, F)_{V^\mu}(A_l). \tag{5.6}$$

We can now show the following theorem.

THEOREM 5.5. *On (S^{2m+1}, g_μ), with $m > 1$, the Hopf unit vector field V^μ is stable as a critical point of the energy if and only if $(2m-2)\mu^2 \leqslant 1$, and it is stable as a critical point of the volume if and only if $(2m-2)\mu^3 - \mu \leqslant 1$.*

Proof. We need only to show that under the hypothesis on μ, the corresponding Hessians are non-negative, when acting on any vector field A orthogonal to V^μ.

By (4.6),

$$(\mathrm{Hess}\, E)_{V^\mu}(A_l) \geqslant e_1(m,\mu,l) \int_{S^{2m+1}} \|A_l\|^2 \,\mathrm{d}v_\mu,$$

with

$$\begin{aligned} e_1(m,\mu,l) &= \mu(1-m^2-4m) + 2m + 2 + \frac{1}{\mu}(l-1+\mu(m+1))^2 \\ &= \mu(2-2m) + 2l(m+1) + \frac{1}{\mu}(l-1)^2. \end{aligned}$$

Therefore, if $(2m-2)\mu^2 \leqslant 1$,

$$e_1(m,\mu,l) \geqslant 2l(m+1) + \sqrt{2m-2}((l-1)^2 - 1).$$

Consequently, $(\mathrm{Hess}\, E)_{V^\mu}(A_l) \geqslant 0$ for all $l \geqslant 0$. If we use now (4.7),

$$(\mathrm{Hess}\, E)_{V^\mu}(A_l) \geqslant e_2(m,\mu,l) \int_{S^{2m+1}} \|A_l\|^2 \,\mathrm{d}v_\mu,$$

with

$$\begin{aligned} e_2(m,\mu,l) &= \mu(1-m^2) - 2m - 2 + \frac{1}{\mu}(l-1+\mu(1-m))^2 \\ &= \mu(2-2m) + 2l(1-m) - 4 + \frac{1}{\mu}(l-1)^2. \end{aligned}$$

If we assume again $(2m-2)\mu^2 \leqslant 1$, we obtain

$$e_2(m,\mu,l) \geqslant 2l(1-m) - 4 + \sqrt{2m-2}((l-1)^2 - 1).$$

Since $\sqrt{2m-2}((l-1)^2-1) \geqslant 4$, for all $l < 0$, we have $(\mathrm{Hess}\, E)_{V^\mu}(A_l) \geqslant 0$.

Equation (5.5) gives us that V^μ is energy stable. The corresponding result for the volume can be established in a similar way.

By (4.8),

$$(\mathrm{Hess}\, F)_{V^\mu}(A_l) \geqslant (1+\mu)^{m-2} f_1(m,\mu,l) \int_{S^{2m+1}} \|A_l\|^2 \,\mathrm{d}v_\mu,$$

with

$$f_1(m,\mu,l) = f_1(m,\mu) + \frac{(1+\mu)}{\mu}\left(l - 1 + \mu + \frac{\mu(m+\mu)}{1+\mu}\right)^2.$$

Developing the right-hand side, we have

$$f_1(m,\mu,l) = \mu^2(2-2m) + \mu 4l + 2l(m+1) + (l-1)^2 + \frac{1}{\mu}(l-1)^2.$$

In particular, $\mu f_1(m,\mu,0)=\mu^3(2-2m)+\mu+1$, and the condition $(2m-2)\mu^3-\mu\leqslant 1$ then implies that $(\operatorname{Hess} F)_{V^\mu}(A_0)\geqslant 0$.

Let us point out that if μ verifies the condition above, then it should also verify $\mu\leqslant 1$, and then for $l\geqslant 1$ we have $f_1(m,\mu,l)>(2-2m)+2(m+1)$ and then $(\operatorname{Hess} F)_{V^\mu}(A_l)>0$.

Let us use now (4.9):

$$(\operatorname{Hess} F)_{V^\mu}(A_l)\geqslant (1+\mu)^{m-2}f_2(m,\mu,l)\int_{S^{2m+1}}\|A_l\|^2\,\mathrm{d}v_\mu,$$

with

$$f_2(m,\mu,l)=f_2(m,\mu)+\frac{(1+\mu)}{\mu}\biggl(l-1+\mu+\frac{\mu(\mu-m)}{1+\mu}\biggr)^2.$$

Developing the right-hand side, we have

$$f_2(m,\mu,l)=\mu^2(2-2m)+\mu 4l+l^2-3-2ml+\frac{1}{\mu}(l-1)^2.$$

So, under the hypothesis, $f_2(m,\mu,l)\geqslant -1+\mu 4l+l^2-3-2ml+\mu^{-1}(l^2-2l)$ and, since $\mu\leqslant 1$, for all $l<0$ we have $f_2(m,\mu,l)\geqslant 2l+2l^2-4-2ml\geqslant 0$, and then $(\operatorname{Hess} F)_{V^\mu}(A_l)\geqslant 0$. □

For the generalized energy we can use the same arguments to obtain the stability of Hopf vector fields under some conditions.

Proposition 5.6. *On (S^{2m+1},g_μ), with $m>1$, the Hopf unit vector field V^μ is stable as a critical point of the energy E_{g_λ} in the following cases:*

(a) *if $(2m-1)\mu\leqslant 2$, for all $\lambda>0$;*

(b) *if $2/(2m-1)<\mu\leqslant\frac{3}{2}$, for*

$$\lambda\leqslant\frac{(\mu-1)^2}{(2m-1)\mu-2};$$

(c) *if $\mu\geqslant 2m+2$, for*

$$\lambda\leqslant\frac{\mu-2m-2}{m^2};$$

(d) *if $m>2$ and $\frac{3}{2}<\mu$, for λ such that*

$$\frac{2\mu-3}{2m-4}\leqslant\lambda\leqslant\frac{(\mu-1)^2}{(2m-1)\mu-2}.$$

Proof. Let us show that under the hypothesis the corresponding Hessian is non-negative, when acting on any vector field A orthogonal to V^μ.

The fact that this is the case for (μ,λ) as in (c) is a direct consequence of (4.5). If instead we use (4.4),

$$(\operatorname{Hess} E_{g_\lambda})_{V^\mu}(A_l)\geqslant e_1(m,\mu,\lambda,l)\sqrt{\lambda/\mu}\int_{S^{2m+1}}\|A_l\|^2\,\mathrm{d}v_\mu,$$

with

$$e_1(m,\mu,\lambda,l) = \mu(1-4m) + 2m + 2 - \lambda m^2 + \frac{1}{\lambda}(l-1+\mu+\lambda m)^2.$$

In particular, if we assume (a), (b) or (d),

$$e_1(m,\mu,\lambda,0) = \frac{(\mu-1)^2}{\lambda} + \mu(1-2m) + 2 \geqslant 0.$$

Moreover,

$$e_1(m,\mu,\lambda,l) = e_1(m,\mu,\lambda,0) + \frac{1}{\lambda}(l^2 + 2l(-1+\mu+\lambda m)) \geqslant 0,$$

provided $l > 1$ or $l = 1$ and $\mu \geqslant 1$. But

$$e_1(m,\mu,\lambda,1) = \frac{\mu^2}{\lambda} + \mu + 2 + 2m(1-\mu) \geqslant 0,$$

when $\mu \leqslant 1$. Consequently, $(\mathrm{Hess}\, E_{g_\lambda})_{V^\mu}(A_l) \geqslant 0$ for all $l \geqslant 0$. If we now use (4.5),

$$(\mathrm{Hess}\, E_{g_\lambda})_{V^\mu}(A_l) \geqslant e_2(m,\mu,\lambda,l)\sqrt{\lambda/\mu}\int_{S^{2m+1}} \|A_l\|^2 \,\mathrm{d}v_\mu,$$

with

$$e_2(m,\mu,\lambda,l) = \mu - (2m+2+\lambda m^2) + \frac{1}{\lambda}(l-1+\mu-\lambda m)^2.$$

Under (a), (b) or (d) of the hypothesis,

$$e_2(m,\mu,\lambda,l) \geqslant -4 - 2lm + \frac{1}{\lambda}(l^2 - 2l + 2l\mu).$$

Therefore, if $\mu \leqslant \frac{3}{2}$, then $e_2(m,\mu,\lambda,l) \geqslant 0$ for all $l < 0$.

We get the same result if we assume that $m > 2$ and

$$\lambda \geqslant \frac{2\mu - 3}{2m - 4},$$

since

$$\frac{2\mu-3}{2m-4} \geqslant \frac{-2l\mu - l^2 + 2l}{-2lm - 4},$$

for all $l \leqslant -1$. Equation (5.4) gives us that V^μ is stable. □

The proposition above, together with proposition 5.3, solves completely the problem of the stability of Hopf vector fields as critical points of the generalized energy for $\mu \leqslant \frac{3}{2}$. For other values of μ we have only a partial answer. It is also worthwhile to point out that, depending on μ, the set of values of λ for which condition (d) is fulfilled can be empty.

6. The Lorentzian case

In this section we will consider the sphere endowed with a Berger metric g_μ with $\mu < 0$. Then $\|V^\mu\|^2 = -1$ and it is a critical point of the energy restricted to unit timelike vector fields. Using the definition of the Sasaki metric in terms of horizontal and vertical lifts, it is easy to see that g_μ^{S} is a metric of index 2. The restriction of it to the bundle of vectors of square -1, $T^{-1}S^{2m+1}$, has index 1. So $(T^{-1}S^{2m+1}, g_\mu^{\mathrm{S}})$ is a Lorentzian manifold. These facts are true for any Lorentz manifold (M, g).

In contrast with the energy, that is defined for all vector fields, the volume of a unit timelike vector field V will be defined only if V is an element of the open subset consisting in the sections of $T^{-1}M$ such that V^*g^{S} is non-degenerate.

Now, since g^{S} is Lorentzian, this subset has exactly two connected components corresponding to unit timelike vector fields for which V^*g^{S} is Riemannian and those for which V^*g^{S} is Lorentzian. Variational calculus has to be done separately in each component.

In particular, Hopf vector fields on Berger Lorentzian spheres induce Lorentzian metrics $(V^\mu)^*g_\mu^{\mathrm{S}}$ on the sphere and V^μ is critical for the volume restricted to the open set of unit timelike vector fields having this property, which we will denote by $\Gamma^-(T^{-1}S^{2m+1})$.

On a Lorentzian manifold, if V is a unit timelike vector field and $\{V, E_i\}_{i=1}^{2m}$ is an adapted orthonormal local frame, then the vector fields E_i are spacelike for all $1 \leqslant i \leqslant 2m$ and all vector fields X can be written as $X = -g(X, V)V + \sum_i g(X, E_i)E_i$. Then we can state the following proposition.

Proposition 6.1 (Hurtado [11]). *Let V be a unit timelike vector field on the compact Lorentzian manifold (M, g).*

(a) *If V is a critical point of the energy, the Hessian of E at V acting on $A \in V^\perp$ is given by*

$$(\operatorname{Hess} E)_V(A) = -\int_M \|A\|^2 \omega_{(V,g)}(V)\,\mathrm{d}v_g + \int_M \|\nabla A\|^2\,\mathrm{d}v_g.$$

(b) *For a unit timelike vector field $V \in \Gamma^-(T^{-1}M)$ defining a minimal immersion, the Hessian of F at V acting on $A \in V^\perp$ is given by*

$$\begin{aligned}(\operatorname{Hess} F)_V(A) = &-\int_M \|A\|^2 \omega_V(V)\,\mathrm{d}v_g + \int_M \frac{2}{\sqrt{\det L_V}}\sigma_2(K_V \circ \nabla A)\,\mathrm{d}v_g \\ &-\int_M \operatorname{tr}(L_V^{-1} \circ (\nabla A)^{\mathrm{T}} \circ \nabla V \circ K_V \circ \nabla A)\,\mathrm{d}v_g \\ &+\int_M \sqrt{\det L_V}\operatorname{tr}(L_V^{-1} \circ (\nabla A)^{\mathrm{T}} \circ \nabla A)\,\mathrm{d}v_g.\end{aligned}$$

In a similar way to that described in proposition 4.1, we can show, by straightforward computation, the following proposition.

Proposition 6.2. *Let V^μ be the Hopf unit vector field on (S^{2m+1}, g_μ), where $\mu < 0$, for each vector field A orthogonal to V^μ we have*

$$(\operatorname{Hess} E)_{V^\mu}(A) = \int_{S^{2m+1}} (-2m\mu\|A\|^2 + \|\nabla^\mu A\|^2)\,\mathrm{d}v_\mu, \tag{6.1}$$

$$(\text{Hess}\, F)_{V^\mu}(A) = (1-\mu)^{m-2} \int_{S^{2m+1}} (\mu(2m\mu + 2\mu - 4m - 2)\|A\|^2 + \|\nabla^\mu A\|^2 + \mu \|\nabla^\mu_{V^\mu} A - \sqrt{-\mu} JA\|^2)\, \mathrm{d}v_\mu. \tag{6.2}$$

Using these expressions to compute the Hessian in the direction of the vector fields A_a, as in lemma 5.2, we obtain the following lemma.

LEMMA 6.3. *Let V^μ be the Hopf unit vector field on (S^{2m+1}, g_μ), with $\mu < 0$. For each $a \in \mathbb{R}^{2m+2}$, $a \neq 0$ we have*

$$(\text{Hess}\, E)_{V^\mu}(A_a) = \frac{\sqrt{-\mu} m}{m+1} |a|^2 \left((1-2m)\mu + 2 + \frac{(\mu-1)^2}{\mu} \right) \text{vol}(S^{2m+1}),$$

$$(\text{Hess}\, F)_{V^\mu}(A_a) = (1-\mu)^{m-2} \frac{\sqrt{-\mu} m}{m+1} |a|^2 f(m,\mu)\, \text{vol}(S^{2m+1}).$$

where

$$f(m,\mu) = \left((2m-1)\mu^2 + (1-4m)\mu + 2 + (1-\mu)\frac{(\mu-1)^2}{\mu} \right).$$

From here, an immediate consequence is the following proposition.

PROPOSITION 6.4. *Let V^μ be the Hopf unit vector field on (S^{2m+1}, g_μ), with $\mu < 0$. If $(2m-2)\mu^2 < 1$, it is energy unstable, and if $(2-2m)\mu^3 + (4m-4)\mu^2 + \mu < 1$, then it is volume unstable. In particular, on (S^3, g_μ) the Hopf vector field is unstable, for all $\mu < 0$.*

The alternative expressions of the Hessian (see proposition 4.4), used to show stability results in the Riemannian case, can be extended without difficulty to include negative values of μ.

PROPOSITION 6.5. *Let V^μ be the Hopf unit vector field on (S^{2m+1}, g_μ), with $\mu < 0$. For each vector field A orthogonal to V^μ we have*

$$(\text{Hess}\, E)_{V^\mu}(A) = \int_{S^{2m+1}} \Big((2m + 2 - \mu(m^2 + 4m - 1))\|A\|^2 - \|\nabla^\mu_{V^\mu} A - m\sqrt{-\mu} JA\|^2 + \tfrac{1}{2}\|\pi \circ D^C A\|^2_{V^\perp} \Big)\, \mathrm{d}v_\mu, \tag{6.3}$$

$$(\text{Hess}\, E)_{V^\mu}(A) = \int_{S^{2m+1}} ((-2m - 2 - \mu(m^2 - 1))\|A\|^2 - \|\nabla^\mu_{V^\mu} A + m\sqrt{-\mu} JA\|^2 + \tfrac{1}{2}\|\bar{D}^C A\|^2_{V^\perp})\, \mathrm{d}v_\mu, \tag{6.4}$$

$$(\text{Hess}\, F)_{V^\mu}(A) = (1-\mu)^{m-2} \int_{S^{2m+1}} \Big(f_1(m,\mu)\|A\|^2 - (1-\mu)\Big\|\nabla^\mu_{V^\mu} A - \frac{\sqrt{-\mu}(m-\mu)}{1-\mu} JA\Big\|^2 + \tfrac{1}{2}\|\pi \circ D^C A\|^2_{V^\perp} \Big)\, \mathrm{d}v_\mu, \tag{6.5}$$

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$$(\operatorname{Hess} F)_{V^\mu}(A) = (1-\mu)^{m-2} \int_{S^{2m+1}} \biggl(f_2(m,\mu)\|A\|^2 - (1-\mu)\|\nabla^\mu_{V^\mu} A + \frac{\sqrt{-\mu}(\mu+m)}{1-\mu} JA\|^2 + \tfrac{1}{2}\|\bar{D}^C A\|^2_{V^\perp} \biggr) \, \mathrm{d}v_\mu, \quad (6.6)$$

where

$$f_1(m,\mu) = \mu(-1+\mu-6m+2m\mu) + (2m+2) - \frac{\mu(m-\mu)^2}{1-\mu}$$

and

$$f_2(m,\mu) = \mu(1+\mu+2m\mu) - (2m+2)(\mu+1) - \frac{\mu(\mu+m)^2}{1-\mu}.$$

Nevertheless, the arguments used in theorem 5.5 do not allow us to draw any definitive conclusions and thus the stability question is open.

All these facts led us to consider in [6] a new functional $\tilde{B}$, better adapted to the Lorentzian situation, which we called the spacelike energy. It is defined on the manifold of unit timelike vector fields and it is related to the energy by

$$\tilde{B}(X) = E(X) - \int_{S^{2m+1}} \biggl(\frac{2m+1}{2} - \|\nabla^\mu_X X\|^2 \biggr) \, \mathrm{d}v_\mu.$$

Since the Hopf vector field is geodesic,

$$\tilde{B}(V^\mu) = E(V^\mu) - \tfrac{1}{2}(2m+1)\operatorname{vol}(S^{2m+1}, g_\mu) = B(V^\mu).$$

We have shown in [6] that it is also a critical point of the spacelike energy but, in contrast to proposition 6.4, *for any odd-dimensional sphere, endowed with a Lorentzian Berger metric, the Hopf vector field is stable as a critical point of the spacelike energy.* The proof is obtained using (6.3) and the fact that

$$(\operatorname{Hess} \tilde{B})_{V^\mu}(A) = \int_{S^{2m+1}} \|\nabla^\mu_A V^\mu + \nabla^\mu_{V^\mu} A\|^2 \, \mathrm{d}v_\mu + (\operatorname{Hess} E)_{V^\mu}(A).$$

For the three-dimensional sphere we can do better because, although the inequality (3.2) fails on a Lorentzian manifold, we have shown in [6] that

$$\tilde{B}(X) \geqslant \frac{1}{2} \int_{S^3} \operatorname{Ric}_\mu(X,X) \, \mathrm{d}v_\mu,$$

for all timelike unit vector fields, with equality if and only if $h_{11} = h_{22}$ and $h_{12} = -h_{21}$. The Ricci tensor verifies $\operatorname{Ric}_\mu(X,X) \geqslant -2\mu = \operatorname{Ric}_\mu(V^\mu, V^\mu)$ for all unit timelike vector fields X, with equality if and only if $X = V^\mu$. Therefore, we have shown the following result.

Proposition 6.6. *On any Lorentzian Berger* 3*-sphere, the Hopf vector field is, up to sign, the only minimizer of the spacelike energy.*

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