Can. J. Math., Vol. XXXIV, No. 4, 1982, pp. 996–1010

BRAUER GROUPS, CLASS GROUPS AND MAXIMAL ORDERS FOR A KRULL SCHEME

HEISOOK LEE AND MORRIS ORZECH

Introduction. In a previous paper [**13**] one of us considered Brauer groups $Br(\mathbf{C})$ and class groups $Cl(\mathbf{C})$ attached to certain monoidal categories $\mathbf{C}$ of divisorial R-lattices. That paper showed that the groups arising for a suitable pair of categories $\mathbf{C}_1 \subseteq \mathbf{C}_2$ could be related by a tidy exact sequence

$$1 \to Cl(\mathbf{C}_1) \to Cl(\mathbf{C}_2) \to BCl(\mathbf{C}_1, \mathbf{C}_2) \to Br(\mathbf{C}_1) \to Br(\mathbf{C}_2).$$

It was shown that this exact sequence specializes to a number of exact sequences which had formerly been handled separately. At the same time the conventional setting of noetherian normal domains was replaced by that of Krull domains, thus generalizing previous results while also simplifying the proofs. This work was carried out in an affine setting, and one aim of the present paper is to carry these results over to Krull schemes. This will enable us to recover the non-affine version of an exact sequence obtained by Auslander [**1**, p. 261], as well as to introduce a new, non-affine version of a different sequence derived by the same author [**2**, Theorem 1]. There are further advantages in our broadened setting. After following up on Fossum's work [**9**] on maximal orders over Krull domains we shall be able to study the map from $Br(X)$ to $Br(K)$, where X is a Krull scheme with function field K, and obtain information such as when it is one-one.

Our first section sets the foundation for the study of Krull schemes and lattices over them. Section 2 introduces the concept of the Brauer group for a category of divisorial lattices over a scheme. The development parallels that in the affine case [**13**] and is therefore sketchy. In Section 3 we examine how the Brauer groups we are studying behave under base change. This enables us to study the map $Br(X) \to Br(K)$ and some others, which we do in Section 5 after laying out in Section 4 what we need about maximal orders.

Part of this work was done when the first author was a Post-doctoral fellow at Queen's University. The author would like to thank the Department of Mathematics and Statistics for its hospitality and Queen's University for the financial support which made the stay possible.

Received November 26, 1981. The research of the second author was partially supported by N.S.E.R.C. grant A7202.

1. Divisorial lattices over a Krull scheme. Our terminology relating to schemes and modules over schemes will generally be that of Hartshorne [**12**]. A scheme $(X, \mathcal{O}_X)$ is called a *Krull scheme* if it is quasi-compact, integral and satisfies the following three conditions:

(K1) Let Z be the set of points x of X for which $\mathcal{O}_{X,x}$ has Krull dimension one. Let K be the function field of the scheme $(X, \mathcal{O}_X)$. Then for each x in Z, $\mathcal{O}_{X,x}$ is a discrete valuation ring (D.V.R.), for which the corresponding valuation will be called v_x.

(K2) For every non-zero element f in K, $v_x(f) = 0$ for all but finitely many x in Z.

(K3) If U is an open set in X and f is a non-zero element of K such that $v_x(f) \geqq 0$ for all x in $U \cap Z$, then f is in $\mathcal{O}_X(U)$.

We shall talk about the scheme X rather than carrying the full notation $(X, \mathcal{O}_X)$ throughout our discussion. When we refer to $\mathcal{M}$ as an $\mathcal{O}_X$-module it will be understood that $\mathcal{M}$ is a sheaf of modules over the sheaf of rings $\mathcal{O}_X$. For an integral scheme X the $\mathcal{O}_X$-module $\mathcal{M}$ is called *torsion-free* if it is quasi-coherent and for any open set U in X, $\mathcal{M}(U)$ is torsion-free as a module over $\mathcal{O}_X(U)$. For such $\mathcal{M}$ the presheaf given by the assignment $U \rightarrow \mathcal{K}(U) \otimes \mathcal{M}(U)$ is in fact a sheaf, where $\mathcal{K}(U)$ is the field of fractions of $\mathcal{O}_X(U)$ and the tensor product is taken over $\mathcal{O}_X(U)$. Because $(X, \mathcal{O}_X)$ is integral we may identify $\mathcal{K}(U)$ with K, and we write $U \rightarrow K \otimes \mathcal{M}(U)$ for the assignment referred to above, $K \otimes_{\mathcal{O}_X} \mathcal{M}$ or just $K \otimes \mathcal{M}$ for the resulting sheaf and $K\mathcal{M}$ for the vector space $K \otimes \mathcal{M}(U)$ (tensor is over $\mathcal{O}_X(U)$) where U is any open set in X. The *rank* of $\mathcal{M}$ is defined to be the dimension over K of $K\mathcal{M}$. Since X is connected the rank is well-defined.

For X a Krull scheme, a torsion-free $\mathcal{O}_X$-module $\mathcal{M}$ is said to be *divisorial* if for any open set U in X we have

$$\mathcal{M}(U) = \bigcap_{x \in U'} \mathcal{M}_x, \quad U' = U \cap Z,$$

with the intersection being taken in $K\mathcal{M}$. Using the definition of divisoriality and of morphisms of sheaves it is easy to prove the following useful result:

LEMMA 1.1. *Let $f : \mathcal{M} \rightarrow \mathcal{N}$ be a morphism of $\mathcal{O}_X$-modules, with $\mathcal{M}$ divisorial and $\mathcal{N}$ torsion-free. Then f is an isomorphism if and only if $f_x : \mathcal{M}_x \rightarrow \mathcal{N}_x$ is an isomorphism for all x in Z.*

For X a Krull scheme we shall define an $\mathcal{O}_X$-*lattice* to be a torsion-free $\mathcal{O}_X$-module $\mathcal{L}$ such that for every open set U in X there exists a finitely generated $\mathcal{O}_X(U)$-module M_U such that $\mathcal{L}(U) \subseteq M_U \subseteq K\mathcal{L}$. Such an $\mathcal{L}$ is necessarily of finite rank. If X is an affine scheme, with $X = \text{Spec}\ (R)$ and R a Krull domain, $\mathcal{L}$ is an $\mathcal{O}_X$-lattice if and only if $\mathcal{L} = \tilde{L}$ for L an R-lattice in the usual sense [**10**, § 1.2]. If X is noetherian in addition to being a Krull scheme, then the $\mathcal{O}_X$-lattices are precisely the torsion-free coherent $\mathcal{O}_X$-modules.

The basic properties of lattices over a Krull domain carry over to a Krull scheme. The facts noted in the next result will be used repeatedly in later discussion, often implicitly. Affine versions of these facts are to be found in [**10**, Propositions 2.2 and 5.2]. We shall omit the proofs, which involve only a reduction to the affine case and use of the definitions.

PROPOSITION 1.1. *Let X be a Krull scheme, $\mathcal{M}$ and $\mathcal{N}$ two $\mathcal{O}_X$-lattices.*

(a) *If $K \otimes \mathcal{M} = K \otimes \mathcal{N}$ then $\mathcal{M}_x = \mathcal{N}_x$ for all but finitely many x in Z.*

(b) *For each x in Z let $L(x)$ be an $\mathcal{O}_{X,x}$-lattice in the K-vector-space $K \otimes \mathcal{M}_x$ ($\otimes$ is over $\mathcal{O}_{X,x}$) and assume $L(x) = \mathcal{M}_x$ for all but finitely many x in Z. Then the correspondence*

$$U \rightarrow \bigcap_{x \in Z \cap U} L(x)$$

defines an $\mathcal{O}_X$-lattice $\mathcal{L}$ for which $\mathcal{L}_x = L(x)$ whenever x is in Z.

(c) *$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ is an $\mathcal{O}_X$-lattice. If $\mathcal{N}$ is divisorial so is $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ and for any x in Z we have*

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})_x = \mathcal{H}om_{\mathcal{O}_{X,x}}(\mathcal{M}_x, \mathcal{N}_x).$$

Let $\mathcal{M}$ and $\mathcal{N}$ be torsion-free $\mathcal{O}_X$-modules. If we view $\mathcal{M}_x \otimes \mathcal{N}_x$ ($\otimes$ over $\mathcal{O}_{X,x}$) as lying in the constant sheaf $K \otimes \mathcal{M} \otimes \mathcal{N}$, the assignment

$$U \rightarrow \bigcap_{x \in Z \cap U} (\mathcal{M}_x \otimes \mathcal{N}_x),$$

for U open in X, is a sheaf and an $\mathcal{O}_X$-module, which we denote by $\mathcal{M} \tilde{\otimes}_{\mathcal{O}_X} \mathcal{N}$, or by $\mathcal{M} \tilde{\otimes} \mathcal{N}$ if the scheme $(X, \mathcal{O}_X)$ is fixed and clear from the context. We write $\tilde{\mathcal{M}}$ for $\mathcal{M} \tilde{\otimes} \mathcal{O}_X$. In the case of an affine scheme, say with $X = \text{Spec}(R)$, the notation $\tilde{M}$, where M is an R-module, is ambiguous. It can denote the $\mathcal{O}_X$-module associated to M, or the R-module $\bigcap M_p$ (the intersection being taken over p in Z). We shall attempt to specify which use is intended when the ambiguity arises by specifying whether M is an $\mathcal{O}_X$-module or an R-module. The second interpretation of M is consistent with the terminology of [**13**].

If $\mathcal{M}$ is an $\mathcal{O}_X$-module then the natural inclusion $\mathcal{M} \rightarrow \tilde{\mathcal{M}}$ is an equality if and only if $\mathcal{M}(U) \rightarrow \widetilde{\mathcal{M}(U)}$ is an equality of $\mathcal{O}_X$-modules for each affine open set U in X, i.e., if and only if $\mathcal{M}$ is a divisorial $\mathcal{O}_X$-module. If $\mathcal{M}$ is an $\mathcal{O}_X$-lattice then for each affine open set U, $\mathcal{M}(U)$ is an $\mathcal{O}_X(U)$-lattice, and $\mathcal{O}_X(U)$ is a Krull domain. Over a Krull domain a lattice is divisorial if and only if it is reflexive [**10**, Corollary 4.2]. It follows that an $\mathcal{O}_X$-lattice $\mathcal{M}$ is divisorial if and only if the canonical map from $\mathcal{M}$ to $\mathcal{M}^{**}$ is an isomorphism.

The operations of forming $\tilde{\mathcal{M}}$ and $\mathcal{M} \tilde{\otimes} \mathcal{N}$ are well-behaved in several respects, detailed below in two propositions which we will use repeatedly. Some of the proofs reduce to invoking the corresponding affine results [**13**, Theorem 1.1], [**14**, Proposition 2].

PROPOSITION 1.2. *Let X be a Krull scheme. Let $\mathcal{L}, \mathcal{M}, \mathcal{N}$ and $\mathcal{M}_i$ (i in some index set) be torsion-free $\mathcal{O}_X$-modules.*

(a) *$\mathcal{M} \tilde{\otimes} \mathcal{N}$ is a divisorial $\mathcal{O}_X$-module and is an $\mathcal{O}_X$-lattice if $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{O}_X$-lattices.*

(b) $(\mathcal{L} \tilde{\otimes} \mathcal{M}) \tilde{\otimes} \mathcal{N} = \mathcal{L} \tilde{\otimes} (\mathcal{M} \tilde{\otimes} \mathcal{N})$.

(c) $\mathcal{L} \tilde{\otimes} (\bigoplus_i \mathcal{M}_i) = \bigoplus_i (\mathcal{L} \tilde{\otimes} \mathcal{M}_i)$.

(d) $\mathcal{M} \tilde{\otimes} \mathcal{N} = \mathcal{N} \tilde{\otimes} \mathcal{M}$.

(e) $(\mathcal{M} \tilde{\otimes} \mathcal{N})_x = \mathcal{M}_x \tilde{\bigotimes}_{\mathcal{O}_{X,x}} \mathcal{N}_x$ *for x in X.*

(f) *There is a canonical map $\alpha : \mathcal{M} \otimes \mathcal{N} \to \mathcal{M} \tilde{\otimes} \mathcal{N}$ having the following properties:*

(i) *α is an isomorphism if $\mathcal{M}$ is a flat $\mathcal{O}_X$-module (i.e., if $\mathcal{M}_x$ is a flat $\mathcal{O}_{X,x}$-module for each x in X) and $\mathcal{N}$ is divisorial.*

(ii) *For any divisorial $\mathcal{O}_X$-module $\mathcal{L}$ and any $\mathcal{O}_X$-module map $f : \mathcal{M} \otimes \mathcal{N} \to \mathcal{L}$, there exists a unique $\mathcal{O}_X$-module map $g : \mathcal{M} \tilde{\otimes} \mathcal{N} \to \mathcal{L}$ such that $\alpha f = g$.*

Proof. To prove (a) it suffices to reduce to the case where $X = \operatorname{Spec}(R)$, R a Krull domain. If M is any torsion-free R-module then $\tilde{M}$ is clearly divisorial. If M is also an R-lattice, then it is easy to see [**5**, Section III.8, p. 148] that $M \subseteq F \subseteq KM$ with F free. Since $\tilde{F} = F$, it follows that M is an R-lattice. This proves (a).

To prove (f) we may describe α on the pieces $(\mathcal{M} \tilde{\otimes} \mathcal{N})(U)$, U open in X. Hence we may again reduce to the affine case, and there (f) holds by [**14**, Proposition 2]. Once (f) is proved, repeated applications of assertion (ii) can be used to construct the isomorphisms needed to prove (b)–(e). The technique is straightforward and we omit the details.

PROPOSITION 1.3. *Let X be a Krull scheme. Let $\mathcal{M}$ and $\mathcal{N}$ be divisorial $\mathcal{O}_X$-lattices.*

(a) *If $\mathcal{A}$ is an $\mathcal{O}_X$-algebra and $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{A}$-modules then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ is divisorial $\mathcal{O}_X$-lattice, and for each x in X we have*

$$\mathcal{H}om_{\mathcal{A}}(\mathcal{M}, \mathcal{N})_x = \mathcal{H}om_{\mathcal{A}_x}(\mathcal{M}_x, \mathcal{N}_x).$$

(b) *There are natural isomorphisms*

$$\mathcal{M} \tilde{\bigotimes}_{\mathcal{O}_X} \mathcal{M}^* \to \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{M})$$

$$\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{M}) \tilde{\bigotimes}_{\mathcal{O}_X} \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{N}) \to \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{M} \tilde{\bigotimes}_{\mathcal{O}_X} \mathcal{N}).$$

(c) *If $\mathcal{M}$ has rank one then the natural map from $\mathcal{O}_X$ to* $\operatorname{End}_{\mathcal{O}_X}(\mathcal{M})$ *is an isomorphism.*

Proof. Because localizations such as $\mathcal{M}_x$ depend only on the local nature of X and $\mathcal{M}$, to prove (a) we may assume $X = \text{Spec}(R)$, R a Krull domain. In this case (a) holds by [**13**, Lemma 1.3]. To prove (b) first note that if the operation $\tilde{\otimes}$ is replaced by $\otimes$ there are natural homomorphisms as indicated by the arrows. Using these maps and assertion (ii) of Proposition 1.2, part (f), we can obtain homomorphisms on the corresponding constructs with $\tilde{\otimes}$. When localized at each x in Z, these homomorphisms become isomorphisms (use (a) and the fact that for x in Z, $\mathcal{M}_x$ and $\mathcal{N}_x$ are projective $\mathcal{O}_x$-modules of finite type). By Lemma 1.1, the maps constructed are isomorphisms. A similar application of Lemma 1.1 may be used to prove (c).

2. Brauer groups and class groups. Let X be a Krull scheme. Proposition 1.3 indicates that divisorial $\mathcal{O}_X$-lattices behave with respect to $\tilde{\otimes}$ much as locally free modules of finite rank do with respect to $\otimes$. This will allow us to associate a Brauer group and a class group to the category of divisorial $\mathcal{O}_X$-lattices, and more generally, to any subcategory satisfying a few axioms. These axioms, together with the results of Section 1, suffice to make the subcategory being considered a monoidal category [**16**]. For a pair of subcategories $\mathbf{C}_1 \subseteq \mathbf{C}_2$ we shall construct a Brauer-class group, which will act as a link between the groups defined for $\mathbf{C}_1$ and those defined for $\mathbf{C}_2$. The approach followed is that of [**13**], and the proofs there can be invoked for filling in the details that are omitted below.

Let X be a fixed Krull scheme and let $\mathbf{D}$ denote the category of divisorial $\mathcal{O}_X$-lattices (view all categories as being full subcategories of quasi-coherent $\mathcal{O}_X$-modules). Let $\mathbf{C}$ be a subcategory of $\mathbf{D}$ satisfying the following axioms:

(A1) $\mathcal{O}_X$ is in $\mathbf{C}$.

(A2) For any $\mathcal{M}$ and $\mathcal{N}$ in $\mathbf{C}$ each of $\mathcal{M} \tilde{\otimes}_{\mathcal{O}_X} \mathcal{N}$ and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ is in $\mathbf{C}$.

For any sheaf $\mathcal{A}$ of algebras over $(X, \mathcal{O}_X)$ with $\mathcal{A}$ in $\mathbf{D}$ there is, by (f) of Proposition 1.2, a morphism of $\mathcal{O}_X$-algebras

$$\eta_A : \mathcal{A} \tilde{\otimes} \mathcal{A}^0 \rightarrow \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{A}).$$

Let $\mathbf{Az}(\mathbf{C})$ be the set of isomorphism classes of $\mathcal{O}_X$-algebras which are in $\mathbf{C}$ as $\mathcal{O}_X$-modules, for which the map $\mathcal{O}_X \rightarrow \mathcal{C}enter(\mathcal{A})$ is an isomorphism, and for which $\eta_{\mathcal{A}}$ is an isomorphism as well. Except for not being necessarily locally free these algebras are like the Azumaya $\mathcal{O}_X$-algebras considered by Auslander or Grothendieck [**1**, **11**] in defining the usual Brauer group of a scheme, and we define our generalized Brauer group similarly. Let the relation $\sim$ on $\mathbf{Az}(\mathbf{C})$ be given by setting

$$\mathcal{A} \sim \mathcal{B} \text{ if } \mathcal{A} \tilde{\otimes}_{\mathcal{O}_X} \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{M}) \cong \mathcal{B} \tilde{\otimes}_{\mathcal{O}_X} \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{N})$$

for some $\mathcal{M}$ and $\mathcal{N}$ in $\mathbf{C}$. This gives an equivalence relation and the equivalence classes form a group $Br(\mathbf{C})$ relative to the operations

$$[\mathscr{A}] \cdot [\mathscr{B}] = [\mathscr{A} \tilde{\otimes}_{\mathscr{O}_X} \mathscr{B}], \quad [\mathscr{A}]^{-1} = [\mathscr{A}^0].$$

For algebras $\mathscr{A}$ over a ringed space $(X, \mathscr{O}_X)$ one can prove using [**1**, Theorem III.2] that the definition of *Azumaya* in terms of $\mathscr{A}$ being $\mathscr{O}_X$-central, locally free of finite rank and with $\eta_{\mathscr{A}}$ an isomorphism, is equivalent to $\mathscr{A}_x$ being Azumaya over the ring $\mathscr{O}_{X,x}$ for each x in X. A corresponding result holds in our setting, and its proof is straightforward using Lemma 1.1. However, the statement of this result requires some explanation as to terminology. For each x in X let $\mathbf{C}_x$ denote the category of $\mathscr{O}_{X,x}$-modules whose objects are the modules $\mathscr{M}_x$ for $\mathscr{M}$ in $\mathbf{C}$. By (e) of Proposition 1.1 and (b) of Proposition 1.2 it follows that if $\mathbf{C}$ satisfies axioms (A1) and (A2) then $\mathbf{C}_x$ satisfies the corresponding axioms for the affine case.

PROPOSITION 2.1. *Let X be a Krull scheme and $\mathbf{C}$ a subcategory of $\mathbf{D}$ satisfying axioms* (A1) *and* (A2). *Let $\mathscr{A}$ be an $\mathscr{O}_X$-algebra which is in $\mathbf{C}$ as an $\mathscr{O}_X$-module. Then the isomorphism class of $\mathscr{A}$ is in* $\mathbf{Az}(\mathbf{C})$ *if and only if for each x in Z the isomorphism class of $\mathscr{A}_x$ is in* $\mathbf{Az}(\mathbf{C}_x)$.

When $\mathbf{C}$ is a subcategory of $\mathbf{D}$ satisfying axioms (A1) and (A2) we define the class group $Cl(\mathbf{C})$ as the set of isomorphism classes $\{\mathscr{I}\}$ of rank one $\mathscr{O}_X$-modules $\mathscr{I}$ which are in $\mathbf{C}$. It can be seen easily that $Cl(\mathbf{C})$ is an abelian group with operation induced by $\tilde{\otimes}$ (over $\mathscr{O}_X$). The inverse of $\{\mathscr{I}\}$ is given by $\{\mathscr{I}^*\}$; the natural map

$$\mathscr{I} \tilde{\otimes}_{\mathscr{O}_X} \mathscr{I}^* \rightarrow \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{I})$$

is an isomorphism since $\mathscr{I}$ is in $\mathbf{D}$, and $\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{I})$ is isomorphic to $\mathscr{O}_X$ by (c) of Proposition 1.2.

Let $\mathbf{C}_1 \subseteq \mathbf{C}_2$ be two subcategories of $\mathbf{D}$ satisfying axioms (A1) and (A2). We define a group $BCl(\mathbf{C}_1, \mathbf{C}_2)$ as follows: Let $\mathscr{B}$ be the set of isomorphism classes of objects $\mathscr{M}$ in $\mathbf{C}_2$ for which $\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{M})$ is in $\mathbf{C}_1$. The relation $\sim$ is defined on $\mathbf{B}$ by

$$\mathscr{M} \sim \mathscr{N} \quad \text{if} \quad \mathscr{M} \tilde{\otimes}_{\mathscr{O}_X} \mathscr{P} \simeq \mathscr{N} \tilde{\otimes}_{\mathscr{O}_X} \mathscr{Q}$$

for some $\mathscr{P}$ and $\mathscr{Q}$ in $\mathbf{C}_1$. The relation thus defined is an equivalence relation and the set $BCl(\mathbf{C}_1, \mathbf{C}_2)$ of equivalence classes of objects in $\mathbf{B}$ is an abelian group under the multiplication induced by $\tilde{\otimes}$, with the inverse of the class $\langle \mathscr{M} \rangle$ being given by $\langle \mathscr{M}^* \rangle$.

To relate the various groups defined we construct maps i, j, α and β which fit into a sequence

$$(1) \quad 1 \rightarrow Cl(\mathbf{C}_1) \xrightarrow{i} Cl(\mathbf{C}_2) \xrightarrow{j} BCl(\mathbf{C}_1, \mathbf{C}_2) \xrightarrow{\alpha} Br(\mathbf{C}_1) \xrightarrow{\beta} Br(\mathbf{C}_2)$$

whose exactness will be proved under suitable hypotheses. The definitions of the maps involved are: for $\{\mathscr{I}\}$ in $Cl(\mathbf{C}_1)$, $i\{\mathscr{I}\} = \{\mathscr{I}\}$ (the latter class being taken in $Cl(\mathbf{C}_2)$); for $\{\mathscr{J}\}$ in $Cl(\mathbf{C}_2)$, $j\{\mathscr{J}\} = \langle\mathscr{J}\rangle$; for $\langle\mathscr{M}\rangle$ in $BCl(\mathbf{C}_1, \mathbf{C}_2)$, $\alpha\langle\mathscr{M}\rangle = [\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{M})]$ in $Br(\mathbf{C}_1)$; for $[\mathscr{A}]$ in $Br(\mathbf{C}_1)$, $\beta[\mathscr{A}] = [\mathscr{A}]$ (the latter class being taken in $Br(\mathbf{C}_2)$).

THEOREM 2.1. *Let* $\mathbf{C}_1 \subseteq \mathbf{C}_2$ *be two subcategories of* $\mathbf{D}$ *satisfying axioms* (A1) *and* (A2). *Then in the sequence* (1) *above the composite of any two maps is trivial, i is one-one and* Image (α) = Kernel (β). *The sequence is exact if the following axioms holds with* $\mathbf{C} = \mathbf{C}_1$ *and with* $\mathbf{C} = \mathbf{C}_2$*:*
(A3) *if* $\mathscr{M}$ *is in* $\mathbf{C}$, $\mathscr{N}$ *is in* $\mathbf{D}$ *and* $\mathscr{M} \tilde{\otimes}_{\mathscr{O}_X} \mathscr{N}$ *is in* $\mathbf{C}$ *then* $\mathscr{N}$ *is in* $\mathbf{C}$.

Proof. That our constructions give well-defined homomorphisms is an easy consequence of the definitions for the groups involved. That the maps are homomorphisms follows from (b) of Proposition 1.3. The exactness properties can be proved as in the affine case [**13**, Theorem 3.1]; our results in the previous sections generalize to our scheme-theoretic setting the facts on which the proof for the affine case depends, and the slightly more precise form of our theorem (vis-a-vis when exactness holds) is implicit in [**13**].

Remarks. (a) The exact sequence (1) is in appearance reminiscent of an exact sequence from algebraic K-theory which looks like

(2) $\quad K_1\mathbf{C} \to K_1\mathbf{C}' \to K_0\Phi F \to K_0\mathbf{C} \to K_0\mathbf{C}'$

for $F : \mathbf{C} \to \mathbf{C}'$ a cofinal product-preserving functor between categories with product (see [**5**, Chapter VII, Theorem 5.3, p. 375]). However, because in our situation the inclusion functor from $\mathbf{C}_1$ to $\mathbf{C}_2$ is not cofinal, there seems to be an essential barrier to interpreting our sequence (1) as a special case of a sequence (2) with $\mathbf{C}$, $\mathbf{C}'$ and F natural.

(b) If the category $\mathbf{C}$ satisfies axiom (A3) of Theorem 2.1, then an algebra $\mathscr{A}$ in $\mathbf{Az}(\mathbf{C})$ represents the trivial element of $Br(\mathbf{C})$ if and only if $\mathscr{A} \simeq$ End $(\mathscr{M})$ with $\mathscr{M}$ in $\mathbf{C}$. This can be proved as in the affine case (see [**13**, Theorem 3.1, concluding remarks of the proof]).

3. Change of base scheme. Let $(f, f^{\#}) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ be a morphism of schemes (we will abbreviate this by saying that $f : X \to Y$ is a morphism of schemes). The assignment $\mathscr{F} \to f^{-1}(\mathscr{F}) \otimes \mathscr{O}_X$ (tensor over $f^{-1}(\mathscr{O}_Y)$) gives a way of associating to a locally free $\mathscr{O}_Y$-module of finite rank a locally free $\mathscr{O}_X$-module of finite rank. If X and Y are Krull schemes and $\mathscr{F}$ is a divisorial $\mathscr{O}_X$-lattice rather than a locally free one, the construction given may not produce a divisorial $\mathscr{O}_Y$-lattice. To construct a functor $\mathbf{f}$ from the category $\mathbf{D}(Y)$ of divisorial $\mathscr{O}_Y$-lattices to the category $\mathbf{D}(X)$ we can proceed as follows: Let $Z(X)$ be the set of height one primes of X. For any $\mathscr{M}$ in $\mathbf{D}(Y)$ and any open set U in X set

$$\mathbf{f}(\mathscr{M})(U) = \bigcap_{x \in Z(X) \cap U} (f^{-1}(\mathscr{M}) \otimes_{f^{-1}(\mathscr{O}_Y)} \mathscr{O}_X)_x.$$

This gives a divisorial $\mathcal{O}_X$-module, but the correspondence is not necessarily functorial with respect to arbitrary morphisms of Krull schemes. The considerations below are aimed at overcoming these difficulties by restricting our attention to particularly nice morphisms of Krull schemes.

Definition. Let $f : X \to Y$ be a morphism of Krull schemes. We will say f is a *Krull morphism* if the generic point of X is mapped to the generic point η of Y and $f(Z(X)) \subseteq Z(Y) \cup \{\eta\}$.

PROPOSITION 3.1. *Let $f : X \to Y$ be a morphism of Krull schemes.*

(a) *f is a Krull morphism if and only if $f_*(\mathcal{O}_X)$ is divisorial as an $\mathcal{O}_Y$-module.*

(b) *If f is a Krull morphism then for any divisorial $\mathcal{O}_Y$-lattice $\mathcal{M}$, the $\mathcal{O}_X$-module $\mathbf{f}(\mathcal{M})$ defined by*

$$\mathbf{f}(\mathcal{M}) = f^{-1}(\mathcal{M}) \tilde{\otimes}_{f^{-1}(\mathcal{O}_Y)} \mathcal{O}_X$$

is a divisorial $\mathcal{O}_X$-lattice.

(c) *If $g : W \to X$ is another morphism of Krull schemes and $h = fg$ then $\mathbf{h} = \mathbf{f} \circ \mathbf{g}$ (up to natural equivalence of functors from $\mathbf{D}(W)$ to $\mathbf{D}(Y)$).*

Proof. Since the generic point of X maps to the generic point of Y, the map $\mathcal{O}_Y \to f_*(\mathcal{O}_X)$ is an inclusion. To prove (a) we may reduce to the affine case, and what we then need to know is that if $R \to S$ is an inclusion of Krull domains, this inclusion yields a Krull morphism if and only if S is divisorial as an R-module. This is indeed the case [**14**, Theorem 1].

Our notation $\mathbf{f}(\mathcal{M})$ for the module defined in (b) seems to conflict with the notation introduced at the beginning of this section. In fact there is no conflict when $f : X \to Y$ is a Krull morphism. The two constructions give the same result. A proof for the affine case is given in Proposition 4 of [**14**].

The proof of the next result is straightforward and will be omitted.

PROPOSITION 3.2. *Let $f : X \to Y$ and $g : W \to X$ be Krull morphisms of Krull schemes. Let $\mathbf{C}(W)$, $\mathbf{C}(X)$ and $\mathbf{C}(Y)$ be subcategories of $\mathbf{D}(W)$, $\mathbf{D}(X)$ and $\mathbf{D}(Y)$ respectively, such that for $\mathcal{M}$ in $\mathbf{C}(Y)$ we have that $\mathbf{f}(\mathcal{M})$ is in $\mathbf{C}(X)$, and for $\mathcal{N}$ in $\mathbf{C}(X)$ we have that $\mathbf{g}(\mathcal{N})$ is in $\mathbf{C}(W)$.*

(a) *The correspondences $\{\mathcal{M}\} \to \{\mathbf{f}(\mathcal{M})\}$ and $[\mathcal{A}] \to [\mathbf{f}(\mathcal{A})]$ define group homomorphisms*

$$Cl(f) : Cl(\mathbf{C}(Y)) \to Cl(\mathbf{C}(X)),$$

$$Br(f) : Br(\mathbf{C}(Y)) \to Br(\mathbf{C}(X)).$$

(b) *$Cl(fg) = Cl(f)Cl(g)$ and $Br(fg) = Br(f)Br(g)$.*

(c) *Suppose that for $V = X$ and for $V = Y$ we have $\mathbf{C}_1(V) \subseteq \mathbf{C}_2(V)$, and that $\mathbf{C}_1(V)$ and $\mathbf{C}_2(V)$ are subcategories of $\mathbf{D}(V)$ satisfying axioms*

https://doi.org/10.4153/CJM-1982-072-4 Published online by Cambridge University Press

(A1), (A2) *and* (A3) *of Section* 3. *Then there is a commutative diagram with exact rows:*

$$\begin{array}{ccccccccccc}
1 \to & Cl(\mathbf{C}_1(Y)) & \xrightarrow{i} & Cl(\mathbf{C}_2(Y)) & \xrightarrow{j} & BCl(\mathbf{C}_1(Y), \mathbf{C}_2(Y)) & \xrightarrow{\alpha} & Br(\mathbf{C}_1(Y)) & \xrightarrow{\beta} & Br(\mathbf{C}_2(Y)) \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 \to & Cl(\mathbf{C}_1(X)) & \xrightarrow[i]{} & Cl(\mathbf{C}_2(X)) & \xrightarrow[j]{} & BCl(\mathbf{C}_1(X), \mathbf{C}_2(X)) & \xrightarrow[\alpha]{} & Br(\mathbf{C}_1(X)) & \xrightarrow[\beta]{} & Br(\mathbf{C}_2(X)).
\end{array}$$

Remark. It is possible to have a non-Krull morphism $f : X \to Y$ which induces a functor from $\mathbf{C}(Y)$ to $\mathbf{C}(X)$, but does not induce a group homomorphism between Brauer groups, from $Br(\mathbf{C}(Y))$ to $Br(\mathbf{C}(X))$. We mention one example of this phenomenon. Let R be a complete local Krull domain, with maximal ideal M. Let $\beta(R)$ denote $Br(\mathbf{D}(\text{Spec }(R)))$. Suppose there existed a homomorphism from $\beta(R)$ to $\beta(R/M)$ (i.e., to $Br(R/M)$) which made the following diagram commutative:

$$\begin{array}{ccc}
Br(R) & \longrightarrow & \beta(R) \\
& \searrow \quad \swarrow & \\
& Br(R/M) &
\end{array}$$

The horizontal map arises from the inclusion of categories $\mathbf{P} \subseteq \mathbf{D}$, where $\mathbf{P}$ is the category of locally free (projective) R-lattices. The map from $Br(R)$ to $Br(R/M)$ is an isomorphism. Let K be the field of fractions of R. We shall see later (Proposition 5.1) that $\beta(R) \subseteq Br(K)$. If the map in question existed, then $Br(R) \to Br(K)$ would necessarily be one-one. But this is not always the case, as can be seen from examples presented in [**6**].

4. Maximal orders over a Krull scheme. Let X, K, $\mathbf{C}$, etc., continue to be as in the previous section. There is a canonical map $Br(\mathbf{C}) \to Br(K)$, where $Br(K)$ is the Brauer group of the field K. To study this map we shall consider orders and maximal orders over a Krull scheme in a central simple finite dimensional K-algebra Σ. Given such a Σ we may view it, and K as well, as a constant sheaf on X. An $\mathcal{O}_X$-algebra $\mathcal{A}$ is called an *$\mathcal{O}_X$-order* in Σ if $\mathcal{A}$ is quasi-coherent as an $\mathcal{O}_X$-module and the following properties hold:

(01) $\mathcal{O}_X \subseteq \mathcal{A} \subseteq \Sigma$.

(02) $\mathcal{A} \otimes_{\mathcal{O}_X} K = \Sigma$.

(03) For an open set U in X, every element of $\mathcal{A}(U)$ is integral over $\mathcal{O}_X(U)$.

We note that Σ always contains an $\mathcal{O}_X$-order. Let $\mathcal{L}$ be an $\mathcal{O}_X$-lattice in Σ. Then the presheaf defined by

$$U \rightarrow \mathcal{O}_l(\mathcal{L})(U) = \{x \text{ in } \Sigma | x\mathcal{L}(U) \subseteq \mathcal{L}(U)\}$$

for any open set U in X is a quasi-coherent sheaf. It can be seen easily [**15**, Example 6.15] that this sheaf is in fact an $\mathcal{O}_X$-order.

For a finite dimensional central simple K-algebra Σ, the reduced trace $\mathrm{Tr} : \Sigma \rightarrow K$ induces an isomorphism t from Σ to $\mathrm{Hom}_K(\Sigma, K)$, defined by $t(x)(y) = \mathrm{Tr}(xy)$. Following the notation of [**9**], if $\{x_1, \ldots, x_n\}$ is a basis of Σ over K let the set $\{x_1^*, \ldots, x_n^*\}$ be a basis satisfying $\mathrm{Tr}(x_i^* x_j) = \delta_{ij}$ (Kronecker's delta). If U is an affine open set in X and L is the free $\mathcal{O}_X(U)$-module on the x_i, $i = 1, \ldots, n$, write L^c for the free $\mathcal{O}_X(U)$-module on the x_i^*, $i = 1, \ldots, n$. Since L^c is the inverse image under t of the conductor $(\mathcal{O}_X(U) : L)$, it is independent of the basis chosen for L.

The next several results are non-affine counterparts of statements to be found in [**9**], and we omit proofs which are to be found there in the affine case.

PROPOSITION 4.1. *Let X and Σ be as above, with $\dim_K \Sigma = n$. Let $\mathcal{A}$ be an $\mathcal{O}_X$-order in Σ and let $\{U_i\}$ be an affine open cover of X. Let L_i be a free $\mathcal{O}_X(U_i)$-submodule of $\mathcal{A}(U_i)$ of rank n. If $\mathcal{B}$ is any $\mathcal{O}_X$-order in Σ containing $\mathcal{A}$ then*

$$L_i \subseteq \mathcal{B}(U_i) \subseteq L_i^c \text{ for all } i.$$

COROLLARY. *Every $\mathcal{O}_X$-order in Σ is an $\mathcal{O}_X$-lattice in Σ.*

An $\mathcal{O}_X$-order in Σ is called *maximal* if it is not a proper subsheaf of another $\mathcal{O}_X$-order in Σ. The affine version of the next result on maximal orders is well known [**3**, Theorem 1.5].

PROPOSITION 4.2. *Let $\mathcal{A}$ be an $\mathcal{O}_X$-order in Σ. Then $\mathcal{A}$ is a maximal $\mathcal{O}_X$-order if and only if*

(i) *$\mathcal{A}$ is a divisorial $\mathcal{O}_X$-lattice and*

(ii) *$\mathcal{A}_x$ is a maximal $\mathcal{O}_{X,x}$-order for each x in Z.*

Proof. As noted following Corollary 1.2 of [**9**], if we have an affine situation and A is an order in Σ then so is $\bigcap_{p \in Z} A_p$. Thus in our situation if $\mathcal{A}$ is an $\mathcal{O}_X$-order in Σ so is the $\mathcal{O}_X$-module $\tilde{\mathcal{A}}$ constructed in Section 1. If $\mathcal{A}$ is maximal then we have $\mathcal{A} = \tilde{\mathcal{A}}$, so that (i) holds (Proposition 1.2, (a)). That (ii) holds follows by a reduction to the affine case.

Suppose, conversely, that (i) and (ii) hold and that $\mathcal{B}$ is an $\mathcal{O}_X$-order containing $\mathcal{A}$. Then $\mathcal{B}_x = \mathcal{A}_x$ for all x in Z, by (ii). By Lemma 1.1 we get that $\mathcal{A} = \mathcal{B}$.

https://doi.org/10.4153/CJM-1982-072-4 Published online by Cambridge University Press

THEOREM 4.1. *Let X and Σ be as above. Then every $\mathcal{O}_X$-order $\mathcal{A}$ in Σ is contained in a maximal $\mathcal{O}_X$-order.*

Proof. Let D be the set of divisorial $\mathcal{O}_X$-orders in $\sum$ containing $\mathcal{A}$. D is not empty, since we have noted that $\tilde{\mathcal{A}}$ is in D. Over a Krull domain the divisorial lattices in any fixed lattice satisfy the ascending chain condition [**5**, Chapter 3, Corollary 8.3]. Since X is quasi-compact, D has a maximal element $\mathcal{B}$. Using our Proposition 4.2, the argument in [**9**, Theorem 1.4] shows that $\mathcal{B}$ is a maximal $\mathcal{O}_X$-order in Σ.

We now focus temporarily on the affine case $X = \text{Spec}(R)$, where R is a Krull domain, since a proof of the next result about orders over commutative rings does not seem to appear in the literature. The proof is due to M. Chamarie.

PROPOSITION 4.3. *Let R be a Krull domain with field of fractions K. Let M be an R-lattice. Then* $\text{End}_R(M)$ *is an R-order in* $\text{End}_K(K \otimes_R M)$.

Proof. Let F be a free R-module of finite type for which $M \subseteq F \subseteq K \otimes_R M$, and let $\text{rank}_R F = n$. Identify $\text{End}_R(M)$ (respectively $\text{End}_K(K \otimes_R M)$) with the matrix ring $(R)_n$ (respectively $(K)_n$) and write A for $\text{End}_R(M)$. We must show each element of A is integral over R. Each of A and $(R)_n$ is an R-lattice in $(K)_n$, so that for a suitable non-zero element r of R we have $rA \subseteq (R)_n$. Thus, for α in A the elements $r\alpha^j$, $j = 0, 1, \ldots,$ are in $(R)_n$. Let $p(X)$ be the minimal polynomial of α over K, and let $p(X)$ factor as $(X - a_1) \ldots (X - a_m)$, with a_i in the algebraic closure of K, $i = 1, \ldots, m$. The K-algebra $K[\alpha]$ admits a K-algebra homomorphism to $K[a_i]$ for each $i = 1, \ldots, m$, and since for each $j = 0, 1, \ldots$ we have that $r\alpha^j$ is integral over R, the same is true for ra_i^j. Let a be one of the coefficients of $p(X)$. Then $a = f(a_1, \ldots, a_m)$ for some symmetric polynomial $f(X_1, \ldots, X_m)$ in $\mathbf{Z}[X_1, \ldots, X_m]$. Because ra_i^j is integral over R, it follows that $r^m a^j$ is integral over R for each $j = 1, 2, \ldots.$ Because R is completely integrally closed, a is in R. Thus $p(X)$ is in $R[X]$, so α is integral over R.

COROLLARY. *Let X be a Krull scheme, $\mathcal{L}$ an $\mathcal{O}_X$-lattice. Then $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{L})$ is an $\mathcal{O}_X$-order in* $\text{End}_K(K \otimes_{\mathcal{O}_X} \mathcal{L})$.

THEOREM 4.2. *Let X be a Krull scheme, K its function field, V a finite dimensional K-space. Then $\mathcal{A}$ is a maximal $\mathcal{O}_X$-order in* $\text{End}_K(V)$ *if and only if $\mathcal{A} = \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{L})$ for some divisorial $\mathcal{O}_X$-lattice $\mathcal{L}$ in V.*

Proof. The proof of Theorem 6.19 of [**15**] shows that $\mathcal{A} \simeq \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$ for an $\mathcal{O}_X$-lattice $\mathcal{E}$. Let $\mathcal{L} = \tilde{\mathcal{E}}$. Then $\mathcal{L}$ is divisorial, hence so is $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{L})$ (see Section 1). Also $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}) \subseteq \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{L})$. If $\mathcal{A}$ is a maximal $\mathcal{O}_X$-order then by the corollary above $\mathcal{A}$ must equal $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{L})$.

That $\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{L})$ is maximal for $\mathscr{L}$ a divisorial $\mathscr{O}_X$-lattice follows from Proposition 4.2 and from the relation

$$\mathscr{E}nd_{\mathscr{O}_X}(\mathscr{L})_x = \mathscr{E}nd_{\mathscr{O}_{K,x}}(\mathscr{L}_x) \quad \text{for } x \text{ in } Z.$$

5. Applications. In this section we shall consider specific categories **C** to which the discussion of Sections 2 and 3 applies, and obtain some exact sequences which will be seen to yield information about the kernels of the maps from $Br(X)$ to $Br(K)$ and to $\Pi Br(\mathscr{O}_{X,p})$, where p ranges over Z. Using the results of Section 4 we shall also show that the Brauer group $Br(\mathbf{D})$ based on the divisorial lattices (a group we called $\beta(R)$ in Section 3, when X was Spec (R)) is equal to $\cap\, Br(\mathscr{O}_{X,p})$, the intersection being taken over all p in Z. We shall also show in a non-cohomological way that the map $Br(X) \rightarrow Br(K)$ is one-one when X has a suitable local behavior. Most of these results are known either in the affine case or for noetherian normal schemes, and we shall refer to the previous versions during our discussion.

As usual X will denote a Krull scheme with function field K. The category of divisorial $\mathscr{O}_X$-lattices will be denoted by **D**, that of locally free $\mathscr{O}_X$-lattices by **P**. Let Y be a subset of X containing Z. Let $\mathbf{P}_Y$ be the category of divisorial $\mathscr{O}_X$-lattices $\mathscr{M}$ such that $\mathscr{M}_y$ is a free $\mathscr{O}_{X,y}$-module for all y in Y. Let $\mathbf{I}_Y$ be the category of divisorial $\mathscr{O}_X$-lattices $\mathscr{M}$ such that for each y in Y there is an ideal $I(y)$ in $\mathscr{O}_{X,y}$ such that $\mathscr{M}_y \simeq I(y)^n$, the n-fold direct sum of $I(y)$ with itself, where $n = \text{rank}\,\mathscr{M}$. It is clear that the categories defined satisfy the closure axioms (A1) and (A2) considered in Section 2, as well as the functoriality hypothesis of Proposition 3.2. Moreover, they all satisfy the axiom (A3) given in the statement of Theorem 2.1. For **P**, **D** and $\mathbf{P}_Y$ this is easy to verify, while for $\mathbf{I}_Y$ it follows from Proposition 5.1 of [**13**]. We shall write $Pic(X)$ for $Cl(\mathbf{P})$, $Br(X)$ for $Br(\mathbf{P})$, $Cl(X)$ for $Cl(\mathbf{D})$, $BCl(X)$ for $BCl(\mathbf{P}, \mathbf{D})$, $\beta(X)$ for $Br(\mathbf{D})$, $Pic_Y(X)$ for $Cl(\mathbf{P}_Y)$ and $Br_Y(X)$ for $Br(\mathbf{P}_Y)$. If $Y = Z$ then $\mathbf{P}_Y = \mathbf{D}$.

The morphism of Krull schemes $K \rightarrow X$ is a Krull morphism, since K is flat as an $\mathscr{O}_X$-module (see Proposition 1.2). Hence there is a homomorphism $Br(\mathbf{C}) \rightarrow Br(K)$ whenever **C** is a subcategory of **D** for which (A1) and (A2) hold.

PROPOSITION 5.1. *Let X and K be as above. Then the map $\beta(X) \rightarrow Br(K)$ is one-one.*

Proof. Let $\mathscr{A}$ in $\beta(X)$ be such that $[K \otimes \mathscr{A}] = [K]$. Then $K \otimes \mathscr{A} \simeq \text{End}_K(V)$ for some finite dimensional vector space V over K. Clearly $\mathscr{A}$ is a divisorial $\mathscr{O}_X$-lattice in $\text{End}_K(V)$. For each x in Z, $\mathscr{A}_x$ is a maximal $\mathscr{O}_{X,x}$-order in $\text{End}_K(V)$, since it is an Azumaya $\mathscr{O}_{X,x}$-algebra. By Proposition 4.2 $\mathscr{A}$ is a maximal $\mathscr{O}_X$-order in $\text{End}_K(V)$. By Theorem 4.2 $\mathscr{A} \simeq \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{L})$ for some $\mathscr{L}$ in **D**. Hence $[\mathscr{A}] = 1$ in $\beta(X)$.

https://doi.org/10.4153/CJM-1982-072-4 Published online by Cambridge University Press

PROPOSITION 5.2. *Let X be a Krull scheme. Then the map*

$$Br(\mathbf{I}_Y) \to \prod_{y \in Y} Br(\mathcal{O}_{X,y})$$

is one-one.

Proof. Let $[\mathcal{A}]$ in $Br(\mathbf{I}_Y)$ be such that

$$\mathcal{A}_y \simeq \mathcal{E}nd_{\mathcal{O}_{X,y}}(F_y)$$

for each y in Y, with F_y a free $\mathcal{O}_{X,y}$-module of rank n. Then $K \otimes \mathcal{A} = 1$ in $Br(K)$, hence by Proposition 5.1 we have that

$$\mathcal{A} \simeq \mathrm{End}_{\mathcal{O}_X}(\mathcal{L})$$

with $\mathcal{L}$ a divisorial $\mathcal{O}_X$-lattice. Moreover, $\mathcal{L}_y \simeq (\mathcal{O}_{X,y})_n$, the ring of $n \times n$ matrices over $\mathcal{O}_{X,y}$. It follows that $\mathcal{L}_y \simeq I(y)^n$ with $I(y)$ an ideal of $\mathcal{O}_{X,y}$; for details see Proposition 5.3 of [**13**].

THEOREM 5.1. *Let X be a Krull scheme with function field K. Let Y be a subset of X containing the set Z of height one primes. Then there are exact sequences*

(a) $1 \to Pic(X) \to Pic_Y(X) \to BCl(\mathbf{P}, \mathbf{P}_Y) \to Br(X) \to Br_Y(X)$

(b) $1 \to Pic(X) \to Cl(X) \to BCl(X) \to Br(X) \to Br(K)$

(c) $1 \to Pic(X) \to Cl(X) \to BCl(\mathbf{P}, \mathbf{I}_Y) \to Br(X) \to \prod_{Y \in y} Br(\mathcal{O}_{X,y})$.

Proof. These sequences are obtained from Theorem 2.1 by choosing various categories for $\mathbf{C}_1$ and $\mathbf{C}_2$: for (a) take $\mathbf{P}$ for $\mathbf{C}_1$ and $\mathbf{P}_Y$ for $\mathbf{C}_2$; for (c) take $\mathbf{P}$ for $\mathbf{C}_1$ and $\mathbf{I}_Y$ for $\mathbf{C}_2$ and use Proposition 5.2 plus the equality $Cl(\mathbf{I}_Y) = Cl(X)$; to get (b) let $Y = Z$ in (a) and use Proposition 5.1.

Auslander [**1**] obtained the exact sequence (b) for X a noetherian normal scheme, as well as the sequence (c) for $X = \mathrm{Spec}\ (R)$ with R a noetherian normal domain [**2**]. The latter sequence was obtained by a direct computation not based on the results of [**1**]. Thus, our Theorem 2.1 can be viewed as providing a unification of these results. The corollary below is also modeled on a result of [**1**], and follows easily from the exactness of sequence (b) in the theorem above.

COROLLARY. *Let X be a Krull scheme with function field K. Assume every divisorial $\mathcal{O}_X$-lattice contained in K is a locally free $\mathcal{O}_X$-module. Then $B(X) \to B(K)$ is one-one if and only if every divisorial $\mathcal{O}_X$-lattice $\mathcal{L}$ for which $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{L})$ is a locally free $\mathcal{O}_X$-module is itself a locally free $\mathcal{O}_X$-module.*

The next result is proved in [**1**] for $X = \mathrm{Spec}\ (R)$, with R a noetherian normal domain.

https://doi.org/10.4153/CJM-1982-072-4 Published online by Cambridge University Press

THEOREM 5.2. *Let X be a Krull scheme with function field K. Assume that for each x in X every Galois extension of $\mathcal{O}_{X,x}$ is a factorial domain. Then $B(X) \to B(K)$ is one-one.*

Proof. Since each $\mathcal{O}_{X,x}$ is factorial, every divisorial $\mathcal{O}_X$-lattice contained in K is a locally free $\mathcal{O}_X$-module. By the corollary above it suffices to show that $\mathcal{L}$ is a locally free $\mathcal{O}_X$-module whenever it is a divisorial lattice such that $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{L})$ is locally free. By (a) of Proposition 1.3 we have that for each x in X,

$$\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{L})_x \simeq \mathrm{End}_{\mathcal{O}_{X,x}}(\mathcal{L}_x).$$

Because $\mathcal{O}_{X,x}$ is local there exists a Galois extension $S(x)$ of $\mathcal{O}_{X,x}$ such that for x in X we have

$$S(x) \otimes_{\mathcal{O}_{X,x}} \mathrm{End}_{\mathcal{O}_{X,x}}(\mathcal{L}_x) \simeq (S(x))_n, \quad n = \mathrm{rank}\,\mathcal{L}$$

(see [**15**, Theorem 11.3 and Remark 1, p. 128], [**7**, Theorem 2.9]). Thus $\mathrm{End}_{S(x)}(S(x) \otimes \mathcal{L}_x)$ is a matrix ring, from which it follows that

$$S(x) \otimes \mathcal{L}_x \simeq I^n,$$

with I an ideal of $S(x)$ (this follows from Proposition 5.2; a direct argument is to be found in [**13**, Proposition 5.3]). The ideal I is necessarily divisorial, hence is projective over $S(x)$ since the latter is factorial. But $S(x) \otimes \mathcal{L}_x$ projective over $S(x)$ and $S(x)$ faithfully flat over $\mathcal{O}_{X,x}$ implies $\mathcal{L}_x$ is free over $\mathcal{O}_{X,x}$. Hence we are done.

The result above was proved cohomologically by Grothendieck [**11**]. The hypotheses of the foregoing theorem hold if for each x in X, the strict henselization $\mathcal{O}_{X,x}{}^{sh}$ of $\mathcal{O}_{X,x}$ is factorial.

THEOREM 5.3. *Let X be a Krull scheme. Then*

$$\beta(X) = \bigcap_{x \in Z} Br(\mathcal{O}_{X,x}).$$

Proof. The inclusion of $\beta(X)$ in the given intersection holds in view of Proposition 5.1. For the reverse inclusion one can use the proof of Theorem VII.7 of [**1**], which proves the equality when $X = \mathrm{Spec}\,(R)$ with R noetherian and normal, substituting as needed our results in Section 4 on orders over a Krull scheme X for the corresponding results on orders over R. We remark that a proof of the affine case is also to be found in [**17**], again for R noetherian and normal.

REFERENCES

1. B. Auslander, *The Brauer group of a ringed space*, J. Algebra *4* (1966), 220–273.
2. ——— *Central separable algebras which are locally endomorphism rings of free modules*, Proc. Amer. Math. Soc. *30* (1971), 395–404.
3. M. Auslander and O. Goldman, *Maximal orders*, Trans. Amer. Math. Soc. *97* (1960), 1–24.

https://doi.org/10.4153/CJM-1982-072-4 Published online by Cambridge University Press

4. ——— *The Brauer group of a commutative ring*, Trans. Amer. Math. Soc. *97* (1960), 367–409.
5. H. Bass, *Algebraic K-theory* (W. A. Benjamin, N.Y., 1968).
6. L. N. Childs, G. Garfinkel and M. Orzech, *On the Brauer group and factoriality of normal domains*, J. Pure and Applied Algebra *6* (1975), 111–123.
7. F. DeMeyer and E. Ingraham, *Separable algebras over commutative rings*, Lecture Notes in Mathematics *181* (Springer-Verlag, Berlin, 1971).
8. A. Fröhlich and C. T. C. Wall, *Graded monoidal categories*, Composito Math. (1974), 229–285.
9. R. Fossum, *Maximal orders over Krull domains*, J. Algebra *10* (1968), 321–332.
10. ——— *The divisor class group of a Krull domain* (Springer-Verlag, New York, 1973).
11. A. Grothendieck, *Le groupe de Brauer, II*, in *Dix exposees sur la cohomologie des schemas* (North-Holland, 1968).
12. R. Hartshorne, *Algebraic geometry* (Springer-Verlag, N.Y., 1977).
13. M. Orzech, *Brauer groups and class groups for a Krull domain*, in *Brauer groups in ring theory and algebraic geometry*, Lecture Notes in Mathematics *917* (Springer–Verlag, Berlin, 1981).
14. ——— *Divisorial modules and Krull morphisms*, J. Pure and Applied Algebra *25* (1982), to appear.
15. M. Orzech and C. Small, *The Brauer group of commutative rings* (Marcel Dekker, N.Y., 1975).
16. B. Pareigis, *The Brauer group of a monoidal category*, in *Brauer groups*, Lecture Notes in Mathematics *549* (Springer-Verlag, Berlin, 1976).
17. S. Yuan, *Reflexive modules and algebra class groups over Noetherian integrally closed domains*, J. Algebra *32* (1974), 405–417.

Ewha Woman's University,
Seoul, South Korea;
Queen's University,
Kingston, Ontario