Euro. Jnl of Applied Mathematics (2018), *vol.* 29, *pp.* 226–252.
doi:10.1017/S0956792517000092

Stability of weakly dissipative Reissner–Mindlin–Timoshenko plates: A sharp result†

A. D. S. CAMPELO, D. S. ALMEIDA JÚNIOR and M. L. SANTOS

Department of Mathematics – Federal University of Pará
Augusto Corrêa Street, 01, 66075-110, Belém, Pará, Brazil
emails: campelo.ufpa@gmail.com, dilberto@ufpa.br, ls@ufpa.br

(*Received 12 November 2015; revised 13 March 2017; accepted 30 March 2017; first published online 27 April 2017*)

In the present article, we show that there exists a critical number that stabilizes the Reissner–Mindlin–Timoshenko system with frictional dissipation acting on rotation angles. We identify two speed characteristics $v_1^2 := K/\rho_1$ and $v_2^2 := D/\rho_2$, and we show that the system is exponentially stable if and only if

$$v_1^2 = v_2^2.$$

For $v_1^2 \neq v_2^2$, we prove that the system is polynomially stable and determine an optimal estimate for the decay. To confirm our analytical results, we compute the numerical solutions by means of several numerical experiments by using a finite difference method.

Key words: Reissner-Mindlin-Timoshenko system, wave propagation speed, exponential stability, optimal decay, finite difference.

Dedicated to Prof. Marcelo Moreira Cavalcanti on the occasion of his 60th Birthday.

1 Introduction

The main objective of this article is to prove a result that characterizes exponential stability for the Reissner–Mindlin–Timoshenko system taking into account two frictional damping terms acting on the equations of the rotational angles.

† The first author thanks to PNPD/CAPES for his financial support. The second and third authors thank to IMPA for its hospitality during their stage visiting professor. The second author is supported by CNPq Grant 311553/2013-3 and by CNPq Grant 458866/2014-8 (Universal Project -2014). The third author is supported by CNPq Grant 302899/2015-4 and by CNPq Grant 401769/2016-0 (Universal Project -2016).

We investigate at theoretical and numerical levels questions related to the exponential and polynomial stability of the Reissner–Mindlin–Timoshenko system given by

$$\rho_1\omega_{tt} - K(\psi+\omega_x)_x - K(\varphi+\omega_y)_y = 0, \tag{1.1}$$

$$\rho_2\psi_{tt} - D\psi_{xx} - D\left(\frac{1-\mu}{2}\right)\psi_{yy} - D\left(\frac{1+\mu}{2}\right)\varphi_{xy} + K(\psi+\omega_x) + d_1\psi_t = 0, \tag{1.2}$$

$$\rho_2\varphi_{tt} - D\varphi_{yy} - D\left(\frac{1-\mu}{2}\right)\varphi_{xx} - D\left(\frac{1+\mu}{2}\right)\psi_{xy} + K(\varphi+\omega_y) + d_2\varphi_t = 0, \tag{1.3}$$

in $\Omega \times \mathbb{R}^+$. Here $\rho_1 = \rho h$, $\rho_2 = \frac{\rho h^3}{12}$, where ρ is the (constant) mass per unit of surface area, h is the (uniform) plate thickness, μ is Poisson's ratio ($0 < \mu < 1/2$), $D = \frac{Eh^3}{12(1-\mu^2)}$ is the modulus of flexural rigidity, $K = \frac{kEh}{2(1+\mu)}$ is the shear modulus, where E is the Young's modulus and k is the shear correction. Moreover, $d_i,\ i = 1,2$, are positive constants and the functions ω, ψ and φ depend on $(x,y,t) \in \Omega \times \mathbb{R}^+$ and denote the transverse displacement of the plate and the rotational angles of a filament of the plate, respectively. More precise details of the physical derivation of this hyperbolic system (for the undamped case) can be found in [17, 18]. In this case, in contrast to the analogous 1-D case, two rotational angles as well as the transverse displacement of a filament of the elastic structure are included in the mathematical modelling of a thin plate.

The initial data is given by

$$\omega(x,y,0) = \omega_0(x,y), \quad \omega_t(x,y,0) = \omega_1(x,y), \text{ in } \Omega, \tag{1.4}$$

$$\psi(x,y,0) = \psi_0(x,y), \quad \psi_t(x,y,0) = \psi_1(x,y), \text{ in } \Omega, \tag{1.5}$$

$$\varphi(x,y,0) = \varphi_0(x,y), \quad \varphi_t(x,y,0) = \varphi_1(x,y), \text{ in } \Omega, \tag{1.6}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with boundary $\Gamma = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$, where Γ_1 and Γ_2 are non–empty. We adopt the following boundary conditions as in [9]:

$$\omega = 0, \text{ on } \Gamma \times \mathbb{R}^+, \tag{1.7}$$

$$\psi = 0, \ \left(\frac{1-\mu}{2}(\varphi_x+\psi_y), \varphi_y + \mu\psi_x\right)\cdot\nu = 0, \text{ on } \Gamma_1 \times \mathbb{R}^+, \tag{1.8}$$

$$\varphi = 0, \ \left(\psi_x + \mu\varphi_y, \frac{1-\mu}{2}(\varphi_x+\psi_y)\right)\cdot\nu = 0, \text{ on } \Gamma_2 \times \mathbb{R}^+, \tag{1.9}$$

and throughout this work, we will consider $\Omega \subset \mathbb{R}^2$ as the rectangular configuration given by

$$\Omega := [0,L_1] \times [0,L_2], \quad \text{with} \quad L_1, L_2 > 0,$$

with boundary given by

$$\Gamma_1 := \{(x,y) : 0 < x < L_1, \ y = 0, \ L_2\},$$
$$\Gamma_2 := \{(x,y) : 0 < y < L_2, \ x = 0, \ L_1\},$$

satisfying $\Gamma := \overline{\Gamma}_1 \cup \overline{\Gamma}_2$.

https://doi.org/10.1017/S0956792517000092 Published online by Cambridge University Press

There is a substantial literature concerning the mathematical models of oscillations in elastic structures of plates type. The most well known is due to Lagnese [18]. In his monograph, he addressed the question of uniform and strong stability of purely elastic plates due to boundary feedback. Lagnese proved that problem (1.1)–(1.3) with $d_i = 0$, $i = 1, 2$ and boundary feedback conditions is exponentially stable, without any restrictions on the coefficients of the system. Analogous results were also obtained by Muñoz Rivera and Portillo Oquendo [20], where they considered boundary conditions of memory type. In this case, they proved that the solutions of the system is exponentially stable provided that the kernels have exponential behaviour, and are polynomially stable for kernels of polynomial type. Similar dissipations have been used by M. Santos [28], where the author considered a Timoshenko model in $\Omega \subset \mathbb{R}^n$.

For the cases when damping mechanisms act on the whole domain, we should note the work of Fernándes Sare [9], where he considered the equations (1.1)–(1.3). He proved, using a resolvent criterion that the Reissner–Mindlin–Timoshenko system is not exponentially stable independent of any relations between the coefficients of the system, making this case different from the analogous 1-D case. However, in a recent work due to Campelo *et al.* [8], the authors showed that the Reissner–Mindlin–Timoshenko system has two speeds of wave propagation, which play an important role in the stabilization of this system (see also Section 3).

On the other hand, the impact of thermal coupling on the strong stability of a Reissner–Mindlin–Timoshenko plate has been studied by Grobbelaar-Van Dalsen in her papers [10–12] and references therein. In [10], she considered a structural 3-D acoustic model with a 2-D plate interface and proved strong asymptotic stability for the radially symmetric case. A similar result was later obtained in [12] for a rotationally symmetric Reissner–Mindlin–Timoshenko plate with hyperbolic heat conduction due to Cattaneo. To this end, both articles employed Benchimol's spectral criterion. In [11], she proved a polynomial decay rate of $t^{-\frac{1}{4}}$ in the rotationally symmetric case for the Reissner–Mindlin–Timoshenko system coupled to the classical Fourier heat conduction under Dirichlet boundary conditions on ω and θ (θ temperature given by the Fourier law), as well as free boundary conditions on ψ and φ. Other important contribution in linear thermo-elasticity for the Reissner–Mindlin–Timoshenko system was given by Pokojovy [25]. He considered the stability of his model by incorporating various kinds of damping in the interior of the plate both with and without the radial symmetry assumption.

Here, we ask

- Is it possible to get the exponential decay when the Reissner–Mindlin–Timoshenko system given by (1.1)–(1.9) is damped by two feedback laws?
- Do speeds of wave propagations of the Reissner–Mindlin–Timoshenko system play some role in getting the exponential decay with only two dissipative mechanisms?

It is well known that in the 1-D case, if there exist two dissipative mechanisms, we always get exponential stability, whether the speeds of wave propagation are equal or not (see [16, 27]). Of course, since the Reissner–Mindlin–Timoshenko equations (1.1)–(1.3) constitute a conservative three-by-three hyperbolic system [13], if we consider three frictional damping terms, that is, we consider terms $d_0\omega_t$, $d_1\psi_t$ and $d_2\varphi_t$, we always get the exponential stability, irrespective of any relation between the coefficients. Moreover, numerical experiments indicate that on combining one dissipation on the transverse displacement (denoted by ω) and one dissipation on a rotational angle (φ or ψ) we get the exponential stability without any dependence between the coefficients. In this case, the mathematical proofs remain to be done.

Inspired by several results in the literature on stability of the 1-D dissipative Timoshenko system by taking only one dissipative mechanism (see, for example, [1, 3, 21, 22, 29, 30]) and also in recent work due to Campelo *et al.* [8], the main objectives of this paper are to answer the above questions. To the best of our knowledge, Campelo *et al.* [8] were the first one to show that the classical Reissner–Mindlin–Timoshenko system has two speeds of wave propagations (see calculation on page 159 [8]). In that case, the equality between speeds constitutes a new paradigm, which plays an important role in stabilization of the Reissner–Mindlin–Timoshenko system when few dissipative mechanisms act [8]. Of course, there are several criticisms about the non-physicality of the equality of wave speeds in order to get the exponential decay in the cited cases. Indeed, with $v_1^2 = K/\rho_1$ and $v_2^2 = D/\rho_2$ and taking them to be equal we arrive at

$$k = \frac{2}{1-\mu}, \tag{1.10}$$

and this equality is not feasible because $k < 1$ and $\mu < 1/2$. However, this condition constitutes an important mathematically sound condition for stabilization issues of beams and plates.

In a previous analysis, through several numerical experiments using finite differences methods (see Section 6), we reached positive results confirming our conjectures. Motivated by this setting, we study the asymptotic behaviour of the system (1.1)–(1.9).

Concerning the mathematical analysis, the method that we use to determine the asymptotic behaviour is based on *Gearhart–Herbst–Prüss–Huang* Theorem for dissipative systems [14, 26].

Theorem 1.1 *Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup of contractions on a Hilbert space $\mathcal{H}$. Then $S(t)$ is exponentially stable if and only if*

$$\rho(\mathcal{A}) \supseteq \{i\lambda : \lambda \in \mathbb{R}\} \equiv i\mathbb{R}, \tag{1.11}$$

and

$$\overline{\lim}_{|\lambda|\to\infty} ||(i\lambda I - \mathcal{A})^{-1}||_{\mathcal{L}(\mathcal{H})} < \infty, \tag{1.12}$$

hold, where $\rho(\mathcal{A})$ is the resolvent set of the differential operator $\mathcal{A}$.

On the other hand, to show the polynomial stability and the optimality of its rate, we use the result due to Borichev and Tomilov [6].

Theorem 1.2 *Let $S(t)$ be a bounded C_0-semigroup on a Hilbert space $\mathcal{H}$ with generator $\mathcal{A}$ such that $i\mathbb{R} \subset \rho(\mathcal{A})$. Then*

$$\frac{1}{|\lambda|^{\alpha}}\|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leqslant C, \quad \forall\lambda \in \mathbb{R} \quad \Leftrightarrow \quad \|S(t)\mathcal{A}^{-1}\|_{\mathcal{L}(\mathcal{H})} \leqslant \frac{c}{t^{1/\alpha}}. \tag{1.13}$$

The mathematical structure of the article is organized as follows. In Section 2, we discuss the existence, regularity and uniqueness of global solutions of (1.1)–(1.9). To for this, we use the semigroup techniques. In Section 3, we identify the speeds of wave propagation and we show that if these speeds are different, then the semigroup associated with the system (1.1)–(1.9) loses exponential stability. In Section 4, we study the exponential decay of the semigroup associated with the system (1.1)–(1.9). At this point, it is important to emphasize that exponential stability occurs under the assumption of equality between the speeds of wave propagation. In Section 5, we show that in general the semigroup associated with the system (1.1)–(1.9) is polynomially stable and we present an optimal decay rate. Finally, in Section 6, numerical results by using a finite difference method on a rectangular domain are obtained to confirm our analytical results.

2 Semigroup setting

In this section, we will show that the system (1.1)–(1.9) is well posed using the semigroup techniques. Let us denote by

$$\mathcal{H} := H_0^1(\Omega) \times L^2(\Omega) \times H_{\Gamma_1}^1(\Omega) \times L^2(\Omega) \times H_{\Gamma_2}^1(\Omega) \times L^2(\Omega), \tag{2.1}$$

the Hilbert space with internal product given by

$$\begin{aligned}(U,V)_{\mathcal{H}} = {} & \rho_1 \int_\Omega u^2\overline{v^2}\, d(x,y) + \rho_2 \int_\Omega u^4\overline{v^4}\, d(x,y) + \rho_2 \int_\Omega u^6\overline{v^6}\, d(x,y)\\ & + K\int_\Omega (u^3 + u_x^1)(\overline{v^3 + v_x^1})\, d(x,y) + K\int_\Omega (u^5 + u_y^1)(\overline{v^5 + v_y^1})\, d(x,y)\\ & + D\int_\Omega u_x^3\overline{v_x^3}\, d(x,y) + D\int_\Omega u_y^5\overline{v_y^5}\, d(x,y)\\ & + D\left(\frac{1-\mu}{2}\right)\int_\Omega (u_y^3 + u_x^5)(\overline{v_y^3 + v_x^5})\, d(x,y)\\ & + D\mu\int_\Omega u_x^3\overline{v_y^5}\, d(x,y) + D\mu\int_\Omega u_y^5\overline{v_x^3}\, d(x,y),\end{aligned} \tag{2.2}$$

and norm given by

$$\begin{aligned}||U||_{\mathcal{H}}^2 = ||(u^1,u^2,u^3,u^4,u^5,u^6)^T||_{\mathcal{H}}^2 = {} & \rho_1\int_\Omega |u^2|^2\, d(x,y) + \rho_2\int_\Omega |u^4|^2\, d(x,y)\\ & + \rho_2\int_\Omega |u^6|^2\, d(x,y) + K\int_\Omega |u^3 + u_x^1|^2\, d(x,y) + K\int_\Omega |u^5 + u_y^1|^2\, d(x,y)\end{aligned}$$

https://doi.org/10.1017/S0956792517000092 Published online by Cambridge University Press

$$+ D\int_\Omega |u_x^3|^2\,d(x,y) + D\int_\Omega |u_y^5|^2\,d(x,y) + D\left(\frac{1-\mu}{2}\right)\int_\Omega |u_y^3+u_x^5|^2\,d(x,y)$$
$$+ D\mu\int_\Omega u_x^3\overline{u_y^5}\,d(x,y) + D\mu\int_\Omega u_y^5\overline{u_x^3}\,d(x,y), \tag{2.3}$$

where $U=(u^1,u^2,u^3,u^4,u^5,u^6)'$, $V=(v^1,v^2,v^3,v^4,v^5,v^6)'$ and

$$H^1_{\Gamma_i}(\Omega) := \{\psi\in H^1(\Omega)\mid \psi=0 \text{ on } \Gamma_i\},\quad i=1,2.$$

Then using Korn and Poincaré inequalities it follows that $||\cdot||_{\mathcal{H}}$ is equivalent to the usual norm in $\mathcal{H}$ (see [9] for details). If we write $U=(\omega,\omega_t,\psi,\psi_t,\varphi,\varphi_t)'$ and $U_0=(\omega_0,\omega_1,\psi_0,\psi_1,\varphi_0,\varphi_1)'$ then the equations (1.1)–(1.9) can be rewritten as follows:

$$\frac{dU}{dt}=\mathcal{A}U,\ \text{for } t>0, \tag{2.4}$$
$$U(0)=U_0, \tag{2.5}$$

where $\mathcal{A}:\mathcal{D}(\mathcal{A})\subset\mathcal{H}\to\mathcal{H}$ is the operator defined formally by

$$\mathcal{A} := \begin{pmatrix} 0 & Id & 0 & 0 & 0 & 0\\ \frac{K}{\rho_1}\Delta & 0 & \frac{K}{\rho_1}\partial_x & 0 & \frac{K}{\rho_1}\partial_y & 0\\ 0 & 0 & 0 & Id & 0 & 0\\ -\frac{K}{\rho_2}\partial_x & 0 & \mathcal{B}_1 & -\frac{d_1}{\rho_2}Id & \frac{D}{\rho_2}\left(\frac{1+\mu}{2}\right)\partial^2_{xy} & 0\\ 0 & 0 & 0 & 0 & 0 & Id\\ -\frac{K}{\rho_2}\partial_y & 0 & \frac{D}{\rho_2}\left(\frac{1+\mu}{2}\right)\partial^2_{xy} & 0 & \mathcal{B}_2 & -\frac{d_2}{\rho_2}Id \end{pmatrix},$$

where the operators $\mathcal{B}_i(i=1,2)$ are given by

$$\mathcal{B}_1=\frac{D}{\rho_2}\left[\partial_x^2+\left(\frac{1-\mu}{2}\right)\partial_y^2\right]-\frac{K}{\rho_2}Id,$$
$$\mathcal{B}_2=\frac{D}{\rho_2}\left[\left(\frac{1-\mu}{2}\right)\partial_x^2+\partial_y^2\right]-\frac{K}{\rho_2}Id,$$

Id is the identity operator, and

$$\mathcal{D}(\mathcal{A})=\big\{U=(\omega,W,\psi,\Psi,\varphi,\Phi)'\in\mathcal{H}\mid \omega,\psi,\varphi\in H^2(\Omega),\ W\in H^1_0(\Omega),\ \Psi\in H^1_{\Gamma_1}(\Omega),$$
$$\times\,\Phi\in H^1_{\Gamma_2}(\Omega)$$
$$\text{with } \left(\frac{1-\mu}{2}(\varphi_x+\psi_y),\varphi_y+\mu\psi_x\right)\cdot\nu=0,\ \text{on } \Gamma_1$$
$$\text{and } \left(\psi_x+\mu\varphi_y,\frac{1-\mu}{2}(\varphi_x+\psi_y)\right)\cdot\nu=0,\ \text{on } \Gamma_2\Big\},$$

where $\nu=(\nu_1,\nu_2)'$ denotes the outer unit normal vector to Γ.

In the next theorem, we assume that Γ_1 is non-empty, $\overline{\Gamma_1}\cap\overline{\Gamma_2}$ is empty and $\partial\Omega$ is either C^2 or piecewise C^2 with all of the cusps being of angle of at least $\pi/2$ [18].

https://doi.org/10.1017/S0956792517000092 Published online by Cambridge University Press

Theorem 2.1 *The operator $\mathcal{A}$ generates a C_0-semigroup $S(t)$ of contraction on $\mathcal{H}$. Thus, for any initial data $U_0 \in \mathcal{H}$, the problem* (1.1)–(1.9) *has a unique mild solution $U \in C^0([0,\infty),\mathcal{H})$. Moreover, if $U_0 \in \mathcal{D}(\mathcal{A})$, then U is a classical solution of* (1.1)–(1.9), *i.e., $U \in C^1([0,\infty),\mathcal{H}) \cap C^0([0,\infty),\mathcal{D}(\mathcal{A}))$.*

Proof Simple calculations give us

$$\mathcal{R}e(\mathcal{A}U,U)_{\mathcal{H}} = -d_1 \int_\Omega |\Psi|^2 \, d(x,y) - d_2 \int_\Omega |\Phi|^2 \, d(x,y) \leqslant 0, \tag{2.6}$$

from where it follows that $\mathcal{A}$ is a dissipative operator. Thus, thanks to the Lax–Milgram Theorem (see [7]), resolvent equation $\mathcal{A}U = F$, for any $F \in \mathcal{H}$, admits a unique solution $U \in \mathcal{D}(\mathcal{A})$. Therefore, we deduce that $0 \in \rho(\mathcal{A})$, where $\rho(\mathcal{A})$ is the resolvent set of $\mathcal{A}$. Then by the resolvent identity, for small $\lambda > 0$, we have $R(\lambda I - \mathcal{A}) = \mathcal{H}$ (see Theorem 1.2.4 in [19]). Finally, thanks to the Lumer–Phillips Theorem (see [24], Theorem 1.4.3), the operator $\mathcal{A}$ generates a C_0-semigroup of contractions $e^{\mathcal{A}t}$ on $\mathcal{H}$.

□

Now, we introduce the energy functional of system (1.1)–(1.3) that plays an important role in stabilization setting as

$$\begin{aligned} E(t) := \frac{1}{2} \int_\Omega & [\rho_1|\omega_t|^2 + \rho_2|\psi_t|^2 + \rho_2|\varphi_t|^2 + K|\psi + \omega_x|^2 + K|\varphi + \omega_y|^2 \\ & + D|\psi_x|^2 + D|\varphi_y|^2 + D\left(\frac{1-\mu}{2}\right)|\psi_y + \varphi_x|^2 + 2D\mu\psi_x\varphi_y] \, d(x,y), \quad \text{for } t \geqslant 0. \end{aligned} \tag{2.7}$$

It is immediate that the energy functional (2.7) is a monotone non-increasing function of the time t. Indeed, we have the following Proposition:

Proposition 2.2 *Let $(\omega,\omega_t,\varphi,\varphi_t,\psi,\psi_t)$ be the classical solution of* (1.1)–(1.9). *Then, the instantaneous rate of change of energy of the system with respect to time t is given by*

$$\frac{d}{dt}E(t) = -d_1 \int_\Omega \psi_t^2 \, d(x,y) - d_2 \int_\Omega \varphi_t^2 \, d(x,y) \leqslant 0, \quad \forall t \geqslant 0. \tag{2.8}$$

Proof As usual, we can find that if we multiply formally the equations in (1.1), (1.2) and (1.3) by ω_t, ψ_t and φ_t, respectively, then performing integration by parts one has the conclusion of the proposition. □

From (2.8), since $d_i > 0$ for some $i = 1, 2$, we obtain the energy dissipation law

$$E(t) \leqslant E(0), \quad \forall t \geqslant 0. \tag{2.9}$$

It is clear that if $d_1 = d_2 = 0$ we obtain the energy conservation law

$$E(t) = E(0), \quad \forall t \geqslant 0. \tag{2.10}$$

https://doi.org/10.1017/S0956792517000092 Published online by Cambridge University Press

3 Lack of exponential decay

In this section, we show the lack of exponential decay of the Reissner–Mindlin–Timoshenko system (1.1)–(1.9) for the rectangular domain under condition in (3.8). In order to do this, we use Theorem 1.1 in the following manner: we will argue by contradiction, that is, we will show that there exists a sequence of values $(\lambda_n) \subset \mathbb{R}$ with $\lim_{n\to\infty} |\lambda_n| = \infty$ and $U_n = (\omega_n, W_n, \psi_n, \Psi_n, \varphi_n, \Phi_n)' \in \mathcal{D}(\mathcal{A})$ for $F_n = (f_{1n}, f_{2n}, f_{3n}, f_{4n}, f_{5n}, f_{6n})' \in \mathcal{H}$, $n \in \mathbb{N}$, such that

$$(i\lambda_n I - \mathcal{A})U_n = F_n, \tag{3.1}$$

where $(F_n)_n$ is bounded in $\mathcal{H}$ but $\|U_n\|_{\mathcal{H}}$ tends to infinity. So, rewriting the spectral equation in terms of its components we have

$$i\lambda_n\omega_n - W_n = f_n^1, \tag{3.2}$$

$$i\lambda W_n - \frac{K}{\rho_1}(\psi_n + \omega_{nx})_x - \frac{K}{\rho_1}(\varphi_n + \omega_{ny})_y = f_n^2, \tag{3.3}$$

$$i\lambda_n\psi_n - \Psi_n = f_n^3, \tag{3.4}$$

$$i\lambda_n\Psi_n - \frac{D}{\rho_2}\left[\psi_{nxx} + \left(\frac{1-\mu}{2}\right)\psi_{nyy} + \left(\frac{1+\mu}{2}\right)\varphi_{nxy}\right] + \frac{K}{\rho_2}(\psi_n + \omega_{nx}) + \frac{d_1}{\rho_2}\Psi_n = f_n^4, \tag{3.5}$$

$$i\lambda_n\varphi_n - \Phi_n = f_n^5, \tag{3.6}$$

$$i\lambda_n\Phi_n - \frac{D}{\rho_2}\left[\left(\frac{1-\mu}{2}\right)\varphi_{nxx} + \varphi_{nyy} + \left(\frac{1+\mu}{2}\right)\psi_{nxy}\right] + \frac{K}{\rho_2}(\varphi_n + \omega_{ny}) + \frac{d_2}{\rho_2}\Phi_n = f_n^6. \tag{3.7}$$

Now, we are in a position to establish the principal result of this section.

Theorem 3.1 *Let us suppose that*

$$v_1^2 \neq v_2^2. \tag{3.8}$$

Then the semigroup associated with the system (1.1)–(1.9) *is not exponentially stable.*

Proof Let us take $F_n = (0, \sin(\delta\lambda_1 x)\sin(\delta\lambda_2 y), 0, 0, 0, 0)'$ with

$$\lambda_j = \lambda_{j,n} := \frac{n\pi}{\delta L_j}, \quad j = 1, 2, \quad n \in \mathbb{N}, \quad \delta := \sqrt{\frac{\rho_1}{K}}.$$

Finally, we define

$$\lambda_n := \sqrt{\lambda_1^2 + \lambda_2^2}. \tag{3.9}$$

https://doi.org/10.1017/S0956792517000092 Published online by Cambridge University Press

Taking into account the above notation, the equations (3.2)–(3.7) can be rewritten as

$$- \lambda_n^2\rho_1\omega_n - K(\psi_n+\omega_{nx})_x - K(\varphi_n+\omega_{ny})_y = \rho_1 \sin(\delta\lambda_1 x)\sin(\delta\lambda_2 y), \tag{3.10}$$

$$- \lambda_n^2\rho_2\psi_n - D\left[\psi_{nxx} + \left(\frac{1-\mu}{2}\right)\psi_{nyy} + \left(\frac{1+\mu}{2}\right)\varphi_{nxy}\right] + K(\psi_n+\omega_{nx}) + i\lambda_n d_1\psi_n = 0, \tag{3.11}$$

$$- \lambda_n^2\rho_2\varphi_n - D\left[\left(\frac{1-\mu}{2}\right)\varphi_{nxx} + \varphi_{nyy} + \left(\frac{1+\mu}{2}\right)\psi_{nxy}\right] + K(\varphi_n+\omega_{ny}) + i\lambda_n d_2\varphi_n = 0. \tag{3.12}$$

Because of the boundary conditions we can take solution of type

$$\omega_n(x,y) = A\sin(\delta\lambda_1 x)\sin(\delta\lambda_2 y),$$
$$\psi_n(x,y) = B\cos(\delta\lambda_1 x)\sin(\delta\lambda_2 y),$$
$$\varphi_n(x,y) = C\sin(\delta\lambda_1 x)\cos(\delta\lambda_2 y),$$

where A, B, C depend on λ_n and will be determined explicitly in what follows. Note that this choice is compatible with the boundary conditions (1.7)–(1.9). Then, taking into account the definition of λ_n given by (3.9) and δ, the system (3.10)–(3.12) is equivalent to finding A, B and C such that

$$K\delta\lambda_1 B + K\delta\lambda_2 C = \rho_1, \tag{3.13}$$

$$K\delta\lambda_1 A + \left[-\lambda_n^2\left(\rho_2 - D\frac{\rho_1}{K}\right) - D\left(\frac{1+\mu}{2}\right)\delta^2\lambda_2^2 + K + i\lambda_n d_1\right]B + D\left(\frac{1+\mu}{2}\right)\delta^2\lambda_1\lambda_2 C = 0, \tag{3.14}$$

$$K\delta\lambda_2 A + D\left(\frac{1+\mu}{2}\right)\delta^2\lambda_1\lambda_2 B + \left[-\lambda_n^2\left(\rho_2 - D\frac{\rho_1}{K}\right) + - D\left(\frac{1+\mu}{2}\right)\delta^2\lambda_1^2 + K + i\lambda_n d_2\right]C = 0. \tag{3.15}$$

From (3.13) we have

$$C := C_n = \frac{\delta}{\lambda_2} - \frac{\lambda_1}{\lambda_2}B. \tag{3.16}$$

Substituting (3.16) into (3.14) and (3.15), respectively, we get

$$K\delta\lambda_1 A + \left[-\lambda_n^2\left(\rho_2 - D\frac{\rho_1}{K}\right) - D\left(\frac{1+\mu}{2}\right)\delta^2\lambda_n^2 + K + i\lambda_n d_1\right]B = -D\left(\frac{1+\mu}{2}\right)\delta^3\lambda_1, \tag{3.17}$$

https://doi.org/10.1017/S0956792517000092 Published online by Cambridge University Press

and

$$K\delta\lambda_2 A - \left[-\lambda_n^2\left(\rho_2 - D\frac{\rho_1}{K}\right) - D\left(\frac{1+\mu}{2}\right)\delta^2\lambda_n^2 + K + i\lambda_n d_2\right]\frac{\lambda_1}{\lambda_2}B$$
$$= -\left[-\lambda_n^2\left(\rho_2 - D\frac{\rho_1}{K}\right) - D\left(\frac{1+\mu}{2}\right)\delta^2\lambda_1^2 + K + i\lambda_n d_2\right]\frac{\delta}{\lambda_2}. \tag{3.18}$$

Multiplying, respectively, equations (3.17) and (3.18) by λ_2 and $-\lambda_1$ and summing up the product result we obtain

$$\left[-\lambda_n^2\left(\rho_2 - D\frac{\rho_1}{K}\right) - D\left(\frac{1+\mu}{2}\right)\delta^2\lambda_n^2 + K + i\lambda_n d_1\right]\lambda_2 B$$
$$+ \left[-\lambda_n^2\left(\rho_2 - D\frac{\rho_1}{K}\right) - D\left(\frac{1+\mu}{2}\right)\delta^2\lambda_n^2 + K + i\lambda_n d_2\right]\frac{\lambda_1^2}{\lambda_2}B$$
$$= \left[-\lambda_n^2\left(\rho_2 - D\frac{\rho_1}{K}\right) - D\left(\frac{1+\mu}{2}\right)\delta^2\lambda_1^2 + K + i\lambda_n d_2\right]\delta\frac{\lambda_1}{\lambda_2}$$
$$- D\left(\frac{1+\mu}{2}\right)\delta^3\lambda_1\lambda_2,$$

from where we obtain

$$B := B_n = \frac{\left[-\lambda_n^2 D\left(\frac{\rho_2}{D} - \frac{\rho_1}{K}\right) - D\left(\frac{1+\mu}{2}\right)\delta^2\lambda_n^2 + K + i\lambda_n d_2\right]\delta\lambda_1}{-\lambda_n^4 D\left(\frac{\rho_2}{D} - \frac{\rho_1}{K}\right) - D\left(\frac{1+\mu}{2}\right)\delta^2\lambda_n^4 + K\lambda_n^2 + i\lambda_n(d_1\lambda_2^2 + d_2\lambda_1^2)}. \tag{3.19}$$

Substituting B given by (3.19) into (3.17), we get

$$A := A_n = -\frac{D}{K}\left(\frac{1+\mu}{2}\right)\delta^2 - \frac{Q_1(\lambda_n)Q_2(\lambda_n)}{Q_3(\lambda_n)}, \tag{3.20}$$

where

$$Q_1(\lambda_n) = -\lambda_n^2\left(\rho_2 - D\frac{\rho_1}{K}\right) - D\left(\frac{1+\mu}{2}\right)\delta^2\lambda_n^2 + K + i\lambda_n d_1,$$
$$Q_2(\lambda_n) = -\lambda_n^2\left(\rho_2 - D\frac{\rho_1}{K}\right) - D\left(\frac{1+\mu}{2}\right)\delta^2\lambda_n^2 + K + i\lambda_n d_2,$$
$$Q_3(\lambda_n) = K\left[-\lambda_n^4 D\left(\frac{\rho_2}{D} - \frac{\rho_1}{K}\right) - D\left(\frac{1+\mu}{2}\right)\delta^2\lambda_n^4 + K\lambda_n^2 + i\lambda_n(d_1\lambda_2^2 + d_2\lambda_1^2)\right].$$

From (3.16), (3.19) and (3.20) we can conclude that

$$A_n \to \frac{D}{K}\left(\frac{\rho_2}{D} - \frac{\rho_1}{K}\right), \quad B_n \to 0, \quad C_n \to 0, \tag{3.21}$$

when $n \to \infty$. Then using the definition of $||U_n||_{\mathcal{H}}$ and the hypotheses (3.8), we have

$$||U_n||_{\mathcal{H}}^2 \geqslant \rho_1 \int_\Omega |W_n|^2\, d(x,y) = \rho_1 \int_\Omega |\lambda_n A_n \sin(\delta\lambda_1 x)\sin(\delta\lambda_2 y)|^2\, d(x,y)$$
$$= \rho_1|\lambda_n A_n|^2\frac{L_1L_2}{4} \to \infty,$$

https://doi.org/10.1017/S0956792517000092 Published online by Cambridge University Press

as $n \to \infty$. Therefore, from Theorem 1.1 we conclude that the semigroup $S(t)$ associated with the system (1.1)–(1.9) does not have exponential decay. □

4 Exponential stability

In this section, we establish some properties of the asymptotic behaviour of the energy associated with the Reissner–Mindlin–Timoshenko system (1.1)–(1.9) on a rectangular domain. In particular, we show that the system is exponentially stable when the speeds of wave propagation are equal, that is, $v_1^2 = v_2^2$. To do this, first we show, following Theorem 1.1, that the resolvent is uniformly bounded over the imaginary axis. Then, let us consider the product in $\mathcal{H}$ of $U = (\omega, W, \psi, \Psi, \varphi, \Phi)' \in \mathcal{D}(\mathcal{A})$ with the resolvent equation of $\mathcal{A}$, that is

$$i\lambda ||U||_{\mathcal{H}}^2 - (\mathcal{A}U, U)_{\mathcal{H}} = (F, U)_{\mathcal{H}}.$$

Then taking the real part and using the inequality (2.6), we obtain

$$d_1 \int_\Omega \Psi^2 \, d(x,y) + d_2 \int_\Omega \Phi^2 \, d(x,y) \leqslant ||U||_{\mathcal{H}} ||F||_{\mathcal{H}}, \tag{4.1}$$

where $F = (f^1, f^2, f^3, f^4, f^5, f^6)' \in \mathcal{H}$. Now, taking account of the resolvent system in terms of the coefficients, we obtain

$$i\lambda\omega - W = f^1, \tag{4.2}$$
$$i\lambda\rho_1 W - K(\psi + \omega_x)_x - K(\varphi + \omega_y)_y = f^2, \tag{4.3}$$
$$i\lambda\psi - \Psi = f^3, \tag{4.4}$$
$$i\lambda\rho_2\Psi - D\left(\psi_{xx} + \frac{1-\mu}{2}\psi_{yy} + \frac{1+\mu}{2}\varphi_{xy}\right) + K(\psi + \omega_x) + d_1\Psi = f^4, \tag{4.5}$$
$$i\lambda\varphi - \Phi = f^5, \tag{4.6}$$
$$i\lambda\rho_2\Phi - D\left(\frac{1-\mu}{2}\varphi_{xx} + \varphi_{yy} + \frac{1+\mu}{2}\psi_{xy}\right) + K(\varphi + \omega_y) + d_2\Phi = f^6, \tag{4.7}$$

where $\lambda \in \mathbb{R}$. Our starting point is to show that $i\mathbb{R} \cap \sigma(\mathcal{A}) = \emptyset$, where $\sigma(\mathcal{A})$ is the spectrum of $\mathcal{A}$. Using the Lax–Milgram Theorem (see [7]) we have that $0 \in \rho(\mathcal{A})$ therefore $\mathcal{A}^{-1}$ is bounded and it is a bijection between $\mathcal{H}$ and the domain $\mathcal{D}(\mathcal{A})$. Since $\mathcal{D}(\mathcal{A})$ has compact embedding into $\mathcal{H}$ it follows that $\mathcal{A}^{-1}$ is a compact operator, which implies that the spectrum of $\mathcal{A}$ is discrete.

Lemma 4.1 *With the above notation we have*

$$i\mathbb{R} \subset \rho(\mathcal{A}). \tag{4.8}$$

Proof To prove (4.8), it suffices to show that $\mathcal{A}$ has no imaginary eigenvalue. Let us reason by contradiction. Let us suppose that there exists an imaginary eigenvalue $i\lambda$ with eigenvector $U = (\omega, W, \psi, \Psi, \varphi, \Phi)' \in \mathcal{D}(\mathcal{A})$ with $||U||_{\mathcal{H}} = 1$ such that $\mathcal{A}U - i\lambda U = 0$.

From (2.6), we have $\Psi = \Phi = 0$. Then, from (4.4) and (4.6), with $f^3 = f^5 = 0$, we get $\psi = \varphi = 0$. Now, from (4.5) and (4.7), with $f^4 = f^6 = 0$, and using Poincaré's inequality, we conclude that $\omega = 0$. Consequently, from (4.2), with $f^1 = 0$, it follows that $W = 0$. Therefore, $U = 0$, but this is a contradiction. □

Remark. In particular this result implies that the semigroup is strongly stable, that is $S(t)U_0 \to 0$ as $t \to \infty$, where $S(t) := e^{\mathcal{A}t}$ is the C_0-semigroup of contractions on Hilbert space $\mathcal{H}$ and U_0 is the initial data.

Now, we will prove that the Reissner–Mindlin–Timoshenko system is exponentially stable for the condition $v_1^2 = v_2^2$. This proof involves some auxiliary lemmas.

Lemma 4.2 *There exists a positive constant M, such that*

$$
\begin{aligned}
&D\int_\Omega |\psi_x|^2\, d(x,y) + D\int_\Omega |\varphi_y|^2\, d(x,y) + D\left(\frac{1-\mu}{2}\right)\int_\Omega |\psi_y+\varphi_x|^2\, d(x,y)\\
&\quad + D\mu\int_\Omega \psi_x\overline{\varphi}_y\, d(x,y) + D\mu\int_\Omega \varphi_y\overline{\psi}_x\, d(x,y)\\
&\leqslant \frac{K}{|\lambda|^2}\int_\Omega |\psi+\omega_x|^2\, d(x,y) + \frac{K}{|\lambda|^2}\int_\Omega |\varphi+\omega_y|^2\, d(x,y) + M||U||_{\mathcal{H}}||F||_{\mathcal{H}}, \qquad (4.9)
\end{aligned}
$$

for $|\lambda| > 1$ large enough.

Proof Multiplying equation (4.5) by $\overline{\psi}$ and integrating on Ω, we get

$$
\begin{aligned}
&\underbrace{i\lambda\rho_2\int_\Omega \Psi\overline{\psi}\, d(x,y)}_{:=I_1} - D\int_\Omega \psi_{xx}\overline{\psi}\, d(x,y) - D\left(\frac{1-\mu}{2}\right)\int_\Omega \psi_{yy}\overline{\psi}\, d(x,y)\\
&\quad - D\left(\frac{1-\mu}{2}\right)\int_\Omega \varphi_{xy}\overline{\psi}\, d(x,y) - D\mu\int_\Omega \varphi_{xy}\overline{\psi}\, d(x,y)\\
&\quad + K\int_\Omega (\psi+\omega_x)\overline{\psi}\, d(x,y) + d_1\int_\Omega \Psi\overline{\psi}\, d(x,y) = \int_\Omega f^4\overline{\psi}\, d(x,y).
\end{aligned}
$$

Substituting ψ given by (4.4) into I_1, performing a integration by parts and using the boundary conditions (1.7)–(1.9), one has that

$$
\begin{aligned}
&D\int_\Omega |\psi_x|^2\, d(x,y) + D\left(\frac{1-\mu}{2}\right)\int_\Omega |\psi_y|^2\, d(x,y) + D\left(\frac{1-\mu}{2}\right)\int_\Omega \varphi_x\overline{\psi}_y\, d(x,y)\\
&\quad + D\mu\int_\Omega \varphi_y\overline{\psi}_x\, d(x,y) = \rho_2\int_\Omega |\Psi|^2\, d(x,y) - K\int_\Omega (\psi+\omega_x)\overline{\psi}\, d(x,y)\\
&\quad - d_1\int_\Omega \Psi\overline{\psi}\, d(x,y) + \int_\Omega f^4\overline{\psi}\, d(x,y) + \rho_2\int_\Omega \Psi\overline{f^3}\, d(x,y). \qquad (4.10)
\end{aligned}
$$

On the other hand, multiplying equation (4.7) by $\overline{\varphi}$, integrating by parts on Ω and

https://doi.org/10.1017/S0956792517000092 Published online by Cambridge University Press

using (4.6), we get

$$D\left(\frac{1-\mu}{2}\right)\int_\Omega |\varphi_x|^2\,d(x,y)+D\int_\Omega |\varphi_y|^2\,d(x,y)+D\left(\frac{1-\mu}{2}\right)\int_\Omega \psi_y\overline{\varphi}_x\,d(x,y)$$
$$+\,D\mu\int_\Omega \psi_x\overline{\varphi}_y\,d(x,y)=\rho_2\int_\Omega |\Phi|^2\,d(x,y)-K\int_\Omega(\varphi+\omega_y)\overline{\varphi}\,d(x,y)$$
$$-\,d_2\int_\Omega \Phi\overline{\varphi}\,d(x,y)+\int_\Omega f^6\overline{\varphi}\,d(x,y)+\rho_2\int_\Omega \Phi\overline{f^5}\,d(x,y). \tag{4.11}$$

Summing equations (4.10) and (4.11), and using Young's inequality, we get

$$D\int_\Omega |\psi_x|^2\,d(x,y)+D\int_\Omega |\varphi_y|^2\,d(x,y)+D\left(\frac{1-\mu}{2}\right)\int_\Omega |\psi_y+\varphi_x|^2\,d(x,y)$$
$$+\,D\mu\int_\Omega \psi_x\overline{\varphi}_y\,d(x,y)+D\mu\int_\Omega \varphi_y\overline{\psi}_x\,d(x,y)$$
$$\leqslant\left(\rho_2+\frac{d_2}{|\lambda|}+\frac{K}{4}\right)\int_\Omega |\Psi|^2\,d(x,y)+\left(\rho_2+\frac{d_2}{|\lambda|}+\frac{K}{4}\right)\int_\Omega |\Phi|^2 d(x,y)$$
$$+\,\frac{K}{|\lambda|^2}\int_\Omega |\psi+\omega_x|^2\,d(x,y)+\frac{K}{|\lambda|^2}\int_\Omega |\varphi+\omega_y|^2\,d(x,y)$$
$$+\,\frac{K}{|\lambda|}\int_\Omega |f^3||\psi+\omega_x|\,d(x,y)+\frac{K}{|\lambda|}\int_\Omega |f^5||\varphi+\omega_y|\,d(x,y)$$
$$+\,\frac{d_1}{|\lambda|}\int_\Omega |\Psi||f^3|\,d(x,y)+\frac{d_2}{|\lambda|}\int_\Omega |\Phi||f^5|\,d(x,y)+\int_\Omega |f^4||\psi|\,d(x,y)$$
$$+\int_\Omega |f^6||\varphi|\,d(x,y)+\rho_2\int_\Omega |\Psi||f^3|\,d(x,y)+\rho_2\int_\Omega |\Phi||f^5|\,d(x,y), \tag{4.12}$$

where ε is a small positive constant. Then, from the above inequality and from (4.1) we conclude the proof of the lemma. □

The next lemma gives the important relation between the coefficients for obtaining the necessary and sufficient condition for exponential stability.

Lemma 4.3 *There exists a positive constant M such that any solution of system* (1.1)–(1.9) *satisfies*

$$\frac{K}{2}\int_\Omega |\psi+\omega_x|^2\,d(x,y)+\frac{K}{2}\int_\Omega |\varphi+\omega_y|^2\,d(x,y)$$
$$\leqslant\lambda\left|\frac{D\rho_1}{K}-\rho_2\right|\int_\Omega |W||\psi_x+\varphi_y|\,d(x,y)+M||U||_{\mathcal{H}}||F||_{\mathcal{H}}, \tag{4.13}$$

for $|\lambda|>1$ large enough.

https://doi.org/10.1017/S0956792517000092 Published online by Cambridge University Press

Proof Multiplying equation (4.5) by $(\overline{\psi + \omega_x})$, integrating by parts on Ω and using (4.4), we have

$$\begin{aligned}
&i\lambda\rho_2 \int_\Omega \Psi \overline{\omega}_x \, d(x,y) + D\int_\Omega \psi_x (\overline{\psi + \omega_x})_x \, d(x,y) \\
&+ D\left(\frac{1-\mu}{2}\right)\int_\Omega \psi_y (\overline{\psi + \omega_x})_y \, d(x,y) \\
&+ D\left(\frac{1-\mu}{2}\right)\int_\Omega \varphi_x (\overline{\psi + \omega_x})_y \, d(x,y) + D\mu \int_\Omega \varphi_y (\overline{\psi + \omega_x})_x \, d(x,y) \\
&- D\int_{\Gamma_2}\left(\psi_x + \mu\varphi_y, \frac{1-\mu}{2}(\varphi_x + \psi_y)\right)\cdot \nu(\overline{\psi + \omega_x})\, d\Gamma_2 \\
&+ K\int_\Omega |\psi + \omega_x|^2 \, d(x,y) = \rho_2 \int_\Omega |\Psi|^2 \, d(x,y) + \rho_2 \int_\Omega \Psi \overline{f^3}\, d(x,y) \\
&- d_1 \int_\Omega \Psi(\overline{\psi + \omega_x})\, d(x,y) + \int_\Omega f^4 (\overline{\psi + \omega_x})\, d(x,y).
\end{aligned} \tag{4.14}$$

On the other hand, multiplying equation (4.7) by $(\overline{\varphi + \omega_y})$ integrating by parts on Ω and using (4.6), we have

$$\begin{aligned}
&i\lambda\rho_2 \int_\Omega \Phi \overline{\omega}_y \, d(x,y) + D\left(\frac{1-\mu}{2}\right)\int_\Omega \varphi_x (\overline{\varphi + \omega_y})_x \, d(x,y) \\
&+ D\int_\Omega \varphi_y (\overline{\varphi + \omega_y})_y \, d(x,y) + D\left(\frac{1-\mu}{2}\right)\int_\Omega \psi_y (\overline{\varphi + \omega_y})_x \, d(x,y) \\
&+ D\mu\int_\Omega \psi_x (\overline{\varphi + \omega_y})_y d(x,y) + K\int_\Omega |\varphi + \omega_y|^2 d(x,y) \\
&- D\int_{\Gamma_1}\left(\frac{1-\mu}{2}(\varphi_x + \psi_y), \varphi_y + \mu\psi_x\right)\cdot \nu(\overline{\varphi + \omega_y})\, d\Gamma_1 \\
&= \rho_2 \int_\Omega |\Phi|^2 \, d(x,y) + \rho_2 \int_\Omega \Phi \overline{f^5}\, d(x,y) \\
&- d_2 \int_\Omega \Phi(\overline{\varphi + \omega_y})\, d(x,y) + \int_\Omega f^6 (\overline{\varphi + \omega_y})\, d(x,y).
\end{aligned} \tag{4.15}$$

Summing up the results obtained and taking into account the boundary conditions (1.7)–(1.9), we arrive at

$$\begin{aligned}
&i\lambda\rho_2 \int_\Omega \Psi \overline{\omega}_x \, d(x,y) + i\lambda\rho_2 \int_\Omega \Phi \overline{\omega}_y \, d(x,y) + K\int_\Omega |\psi + \omega_x|^2 \, d(x,y) \\
&+ K\int_\Omega |\varphi + \omega_y|^2 \, d(x,y) + \underbrace{D\int_\Omega \psi_x (\overline{\psi + \omega_x})_x \, d(x,y) + D\int_\Omega \varphi_y (\overline{\varphi + \omega_y})_y \, d(x,y)}_{:=I_2} \\
&+ D\left(\frac{1-\mu}{2}\right)\int_\Omega \psi_y (\overline{\psi + \omega_x})_y \, d(x,y) + D\left(\frac{1-\mu}{2}\right)\int_\Omega \varphi_x (\overline{\varphi + \omega_y})_x \, d(x,y) \\
&+ D\left(\frac{1-\mu}{2}\right)\int_\Omega \varphi_x (\overline{\psi + \omega_x})_y \, d(x,y) + D\left(\frac{1-\mu}{2}\right)\int_\Omega \psi_y (\overline{\varphi + \omega_y})_x \, d(x,y)
\end{aligned}$$

https://doi.org/10.1017/S0956792517000092 Published online by Cambridge University Press

$$+ D\mu \int_\Omega \varphi_y(\overline{\psi + \omega_x})_x \, d(x,y) + D\mu \int_\Omega \psi_x(\overline{\varphi + \omega_y})_y \, d(x,y) = \rho_2 \int_\Omega |\Psi|^2 \, d(x,y)$$
$$+ \rho_2 \int_\Omega |\Phi|^2 \, d(x,y) + \rho_2 \int_\Omega \Psi \overline{f^3} \, d(x,y) + \rho_2 \int_\Omega \Phi \overline{f^5} \, d(x,y) - d_1 \int_\Omega \Psi(\overline{\psi + \omega_x}) \, d(x,y)$$
$$- d_2 \int_\Omega \Phi(\overline{\varphi + \omega_y}) \, d(x,y) + \int_\Omega f^4(\overline{\psi + \omega_x}) \, d(x,y) + \int_\Omega f^6(\overline{\varphi + \omega_y}) \, d(x,y).$$

Now, from (4.3), we have

$$\begin{aligned} K \int_\Omega (\overline{\psi + \omega_x})_x \psi_x \, d(x,y) &+ K \int_\Omega (\overline{\varphi + \omega_y})_y \varphi_y \, d(x,y) \\ &= -i\lambda\rho_1 \int_\Omega \overline{W}(\psi_x + \varphi_y) \, d(x,y) - K \int_\Omega (\overline{\varphi + \omega_y})_y \psi_x \, d(x,y) \\ &\quad - K \int_\Omega (\overline{\psi + \omega_x})_x \varphi_y \, d(x,y) - \int_\Omega \overline{f^2}(\psi_x + \varphi_y) \, d(x,y). \end{aligned} \tag{4.16}$$

Substituting (4.16) into I_2, we get

$$\begin{aligned} &\underbrace{i\lambda\rho_2 \int_\Omega \Psi \overline{\omega}_x \, d(x,y)}_{:=I_3} + \underbrace{i\lambda\rho_2 \int_\Omega \Phi \overline{\omega}_y \, d(x,y)}_{:=I_4} - i\lambda \frac{D\rho_1}{K} \int_\Omega \overline{W}(\psi_x + \varphi_y) \, d(x,y) \\ &\quad + K \int_\Omega |\psi + \omega_x|^2 \, d(x,y) + K \int_\Omega |\varphi + \omega_y|^2 \, d(x,y) \\ &\quad - D(1-\mu) \int_\Omega \varphi_y(\overline{\psi + \omega_x})_x \, d(x,y) - D(1-\mu) \int_\Omega \psi_x(\overline{\varphi + \omega_y})_y \, d(x,y) \\ &\quad + D\left(\frac{1-\mu}{2}\right) \int_\Omega \psi_y(\overline{\psi + \omega_x})_y \, d(x,y) + D\left(\frac{1-\mu}{2}\right) \int_\Omega \varphi_x(\overline{\varphi + \omega_y})_x \, d(x,y) \\ &\quad + D\left(\frac{1-\mu}{2}\right) \int_\Omega \varphi_x(\overline{\psi + \omega_x})_y \, d(x,y) + D\left(\frac{1-\mu}{2}\right) \int_\Omega \psi_y(\overline{\varphi + \omega_y})_x \, d(x,y) \\ &= \rho_2 \int_\Omega |\Psi|^2 \, d(x,y) + \rho_2 \int_\Omega |\Phi|^2 \, d(x,y) + \rho_2 \int_\Omega \Psi \overline{f^3} \, d(x,y) + \rho_2 \int_\Omega \Phi \overline{f^5} \, d(x,y) \\ &\quad - d_1 \int_\Omega \Psi(\overline{\psi + \omega_x}) \, d(x,y) - d_2 \int_\Omega \Phi(\overline{\varphi + \omega_y}) \, d(x,y) + \frac{D}{K} \int_\Omega \overline{f^2}(\psi_x + \varphi_y) \, d(x,y) \\ &\quad + \int_\Omega f^4(\overline{\psi + \omega_x}) \, d(x,y) + \int_\Omega f^6(\overline{\varphi + \omega_y}) \, d(x,y). \end{aligned} \tag{4.17}$$

Now, substituting ω given by (4.2) into I_3 and I_4, and using (4.4) and (4.6), we have

$$\begin{aligned} I_3 + I_4 = \; & i\lambda\rho_2 \int_\Omega \overline{W}(\psi_x + \varphi_y) \, d(x,y) - \rho_2 \int_\Omega \overline{W}(f_x^3 + f_y^5) \, d(x,y) \\ & - \rho_2 \int_\Omega \overline{f_x^1} \Psi \, d(x,y) - \rho_2 \int_\Omega \overline{f_y^1} \Phi \, d(x,y). \end{aligned} \tag{4.18}$$

https://doi.org/10.1017/S0956792517000092 Published online by Cambridge University Press

Substituting (4.18) into (4.17) and after simplifications, we obtain

$$
\begin{aligned}
&K\int_\Omega |\psi+\omega_x|^2\,d(x,y)+K\int_\Omega |\varphi+\omega_y|^2\,d(x,y)+D\left(\frac{1-\mu}{2}\right)\int_\Omega |\psi_y-\varphi_x|^2\,d(x,y)\\
&\quad = i\lambda\left(\frac{D\rho_1}{K}-\rho_2\right)\int_\Omega \overline{W}(\psi_x+\varphi_y)\,d(x,y)+\rho_2\int_\Omega |\Psi|^2\,d(x,y)+\rho_2\int_\Omega |\Phi|^2\,d(x,y)\\
&\qquad +\rho_2\int_\Omega \Psi\overline{f^3}\,d(x,y)+\rho_2\int_\Omega \Phi\overline{f^5}\,d(x,y)+\rho_2\int_\Omega \overline{f^1}_x\Psi\,d(x,y)+\rho_2\int_\Omega \overline{f^1}_y\Phi\,d(x,y)\\
&\qquad +\rho_2\int_\Omega \overline{W}\left(f^3_x+f^5_y\right)\,d(x,y)-d_1\int_\Omega \Psi(\overline{\psi+\omega_x})\,d(x,y)-d_2\int_\Omega \Phi(\overline{\varphi+\omega_y})\,d(x,y)\\
&\qquad +\frac{D}{K}\int_\Omega \overline{f^2}\psi_x\,d(x,y)+\frac{D}{K}\int_\Omega \overline{f^2}\varphi_y\,d(x,y)+\int_\Omega f^4(\overline{\psi+\omega_x})\,d(x,y)\\
&\qquad +\int_\Omega f^6(\overline{\varphi+\omega_y})\,d(x,y).
\end{aligned}
$$

Then, using Young's inequality, it follows that

$$
\begin{aligned}
&\frac{K}{2}\int_\Omega |\psi+\omega_x|^2\,d(x,y)+\frac{K}{2}\int_\Omega |\varphi+\omega_y|^2\,d(x,y)\\
&\quad\leqslant |\lambda|\left|\frac{D\rho_1}{K}-\rho_2\right|\int_\Omega |W||\psi_x+\varphi_y|\,d(x,y)\\
&\qquad +\left(\rho_2+\frac{d_1^2}{2K}\right)\int_\Omega |\Psi|^2\,d(x,y)+\left(\rho_2+\frac{d_2^2}{2K}\right)\int_\Omega |\Phi|^2\,d(x,y)\\
&\qquad +\rho_2\int_\Omega |\Psi||f^3|\,d(x,y)+\rho_2\int_\Omega |\Phi||f^5|\,d(x,y)+\rho_2\int_\Omega |f^1_x||\Psi|\,d(x,y)\\
&\qquad +\rho_2\int_\Omega |f^1_y||\Phi|\,d(x,y)+\rho_2\int_\Omega |W||f^3_x+f^5_y|\,d(x,y)+\frac{D}{K}\int_\Omega |f^2||\psi_x|d(x,y)\\
&\qquad +\frac{D}{K}\int_\Omega |f^2||\varphi_y|\,d(x,y)+\int_\Omega |f^4||\overline{\psi+\omega_x})|\,d(x,y)+\int_\Omega |f^6||\varphi+\omega_y|\,d(x,y). \qquad (4.19)
\end{aligned}
$$

Now, using the inequality (4.1) we have the conclusion of the lemma. □

Lemma 4.4 *There exists a positive constant M, such that*

$$
\begin{aligned}
\rho_1\int_\Omega |W|^2\,d(x,y)\leqslant{}& \frac{1}{4}\int_\Omega |\Psi|^2\,d(x,y)+\frac{1}{4}\int_\Omega |\Phi|^2\,d(x,y)+\left(K+\frac{1}{|\lambda|^2}\right)\int_\Omega |\psi+\omega_x|^2 d(x,y)\\
&+\left(K+\frac{1}{|\lambda|^2}\right)\int_\Omega |\varphi+\omega_y|^2 d(x,y)+M||U||_{\mathcal{H}}||F||_{\mathcal{H}}, \qquad (4.20)
\end{aligned}
$$

for $|\lambda|>1$ large enough.

Proof Multiplying equation (4.3) by $\overline{\varpi}$ and integrating on Ω, we get

$$
\underbrace{i\lambda\rho_1\int_\Omega W\overline{\varpi}\,d(x,y)}_{:=I_7}-K\int_\Omega (\psi+\omega_x)_x\overline{\varpi}\,d(x,y)-K\int_\Omega (\varphi+\omega_y)_y\overline{\varpi}\,d(x,y)=\int_\Omega f^2\overline{\varpi}\,d(x,y).
$$

https://doi.org/10.1017/S0956792517000092 Published online by Cambridge University Press

Substituting ω given by (4.2) into I_7 and integrating by parts, we have

$$\rho_1 \int_\Omega |W|^2\, d(x,y) - K \int_\Omega |\psi + \omega_x|^2\, d(x,y) - K \int_\Omega |\varphi + \omega_y|^2\, d(x,y)$$
$$+ K \int_\Omega (\psi + \omega_x)\overline{\psi}\, d(x,y) + K \int_\Omega (\varphi + \omega_y)\overline{\varphi}\, d(x,y) = -\rho_1 \int_\Omega W\overline{f^1}\, d(x,y)$$
$$- \int_\Omega f^2 \overline{\omega}\, d(x,y).$$

Then, using Young's inequality, we get

$$\rho_1 \int_\Omega |W|^2 d(x,y) \leqslant \frac{1}{4}\int_\Omega |\Psi|^2\, d(x,y) + \frac{1}{4}\int_\Omega |\Phi|^2\, d(x,y)$$
$$+ \left(K + \frac{1}{|\lambda|^2}\right)\int_\Omega |\psi + \omega_x|^2\, d(x,y) + \left(K + \frac{1}{|\lambda|^2}\right)\int_\Omega |\varphi + \omega_y|^2\, d(x,y)$$
$$+ \frac{1}{|\lambda|}\int_\Omega |f^3||\psi + \omega_x|\, d(x,y) + \frac{1}{|\lambda|}\int_\Omega |f^5||\varphi + \omega_y|\, d(x,y)$$
$$+ \rho_1 \int_\Omega |W||f^1|\, d(x,y) + \int_\Omega |f^2||\omega|\, d(x,y).$$

Therefore, using (4.1) we have the conclusion of the lemma. □

Now, we are in the position to prove the main result of this article.

Theorem 4.5 *The semigroup associated with the Reissner–Mindlin–Timoshenko system* (1.1)–(1.9) *is exponentially stable if and only if* $v_1^2 = v_2^2$.

Proof From Lemmas 4.2–4.4, we can conclude that

$$||U||^2_{\mathcal{H}} \leqslant M||U||_{\mathcal{H}}||F||_{\mathcal{H}}, \quad \forall U \in \mathcal{D}(\mathcal{A}),$$

from where it follows that

$$||U||_{\mathcal{H}} \leqslant M||F||_{\mathcal{H}}, \ \forall U \in \mathcal{D}(\mathcal{A}).$$

Using Prüss's result [26], one has the conclusion of the theorem. □

5 Polynomial decay and its optimal estimate

In the Section 3, we have seen that the Reissner–indlin–Timoshenko system is not exponentially stable when $v_1^2 \neq v_2^2$. Then, taking into account this assumption, we will show that in general the Reissner–Mindlin–Timoshenko system goes to zero polynomially as $1/\sqrt{t}$.

Theorem 5.1 *Let us suppose that* $v_1^2 \neq v_2^2$. *Then the semigroup associated with the Reissner–Mindlin–Timoshenko system* (1.1)–(1.9) *is polynomially stable, that is, there exists a positive*

https://doi.org/10.1017/S0956792517000092 Published online by Cambridge University Press

constant C such that

$$||S(t)U_0||_{\mathcal{H}} \leqslant \frac{C}{\sqrt{t}}||U_0||_{\mathcal{D}(\mathcal{A})}, \ \forall U_0 \in \mathcal{D}(\mathcal{A}).$$

Moreover, this rate of decay is optimal, in the sense that decay must be slower than $t^{-\frac{1}{2-\epsilon}}$ for any $\epsilon > 0$.

Proof From (4.9), we have

$$D\int_\Omega \left(|\psi_x|^2 + |\varphi_y|^2\right)\ d(x,y) \leqslant \frac{K\varepsilon}{|\lambda|^2}\int_\Omega |\psi + \omega_x|^2\ d(x,y) + \frac{K\varepsilon}{|\lambda|^2}\int_\Omega |\varphi + \omega_y|^2\ d(x,y) + C_1||U||_{\mathcal{H}}||F||_{\mathcal{H}}, \tag{5.1}$$

where C_1 is a positive constant. Now, from (4.13), we obtain

$$\frac{K}{2}\int_\Omega \left(|\psi + \omega_x|^2 + |\varphi + \omega_y|^2\right)\ d(x,y) \leqslant |\lambda|\left|\frac{D\rho_1}{k} - \rho_2\right|\int_\Omega |W||\psi_x + \varphi_y|\ d(x,y) + M||U||_{\mathcal{H}}||F||_{\mathcal{H}}. \tag{5.2}$$

Using Young's inequality it follows that

$$\frac{K}{2}\int_\Omega \left(|\psi + \omega_x|^2 + |\varphi + \omega_y|^2\right)\ d(x,y) \leqslant C_2|\lambda|\int_\Omega |W|^2\ d(x,y) + C_2|\lambda|\int_\Omega \left(|\psi_x|^2 + |\varphi_y|^2\right) d(x,y) + M||U||_{\mathcal{H}}||F||_{\mathcal{H}}, \tag{5.3}$$

where C_2 is a positive constant. Substituting (5.3) into (5.1), we obtain

$$D\int_\Omega \left(|\psi_x|^2 + |\varphi_y|^2\right)\ d(x,y) \leqslant \frac{2C_2\varepsilon}{|\lambda|}\int_\Omega |W|^2\ d(x,y) + \frac{2C_2\varepsilon}{|\lambda|}\int_\Omega \left(|\psi_x|^2 + |\varphi_y|^2\right)\ d(x,y) + M||U||_{\mathcal{H}}||F||_{\mathcal{H}}. \tag{5.4}$$

Combining Lemmas 4.2–4.4 with inequality (5.4), then choosing $\varepsilon > 0$ small enough it follows that there exists a positive constant C, such that

$$||U||^2_{\mathcal{H}} \leqslant C|\lambda|^2||U||_{\mathcal{H}}||F||_{\mathcal{H}},$$

for $|\lambda| > 1$ large enough and, consequently, we have

$$\frac{1}{|\lambda|^2}||U||_{\mathcal{H}} \leqslant C||F||_{\mathcal{H}},$$

which is equivalent to

$$||(\lambda I - \mathcal{A})^{-1}||_{\mathcal{L}(\mathcal{H})} \leqslant C|\lambda|^2.$$

Then using Theorem 1.2 one gets the first conclusion of the theorem. To prove that the rate is optimal we use the same ideas as in the proof of Theorem 3.1 and the details can be omitted here. The proof is now complete. □

6 Numerical approach

In this section, we consider a numerical scheme using finite differences and we reproduce numerically the analytical results established on exponential decay for the Reissner–Mindlin–Timoshenko system. We are concerned mainly with the lack of exponential decay reached in Section 3. That is to say, if (3.8) holds, then the dissipative system of Reissner–Mindlin–Timoshenko system treated here is not exponentially stable.

6.1 A fully discrete finite-difference scheme and its properties

Given $I, J, N \in \mathbb{N}$, we set $\Delta x = \dfrac{L_1}{I+1}, \Delta y = \dfrac{L_2}{J+1}$ and $\Delta t = \dfrac{T}{N+1}$, and we introduce the grids

$$x_0 = 0 < x_1 = \Delta x < \cdots < x_I = I\Delta x < x_{I+1} = (I+1)\Delta x = L_1, \tag{6.1}$$

$$y_0 = 0 < y_1 = \Delta y < \cdots < y_J = J\Delta y < y_{J+1} = (J+1)\Delta y = L_2, \tag{6.2}$$

$$t_0 = 0 < t_1 = \Delta t < \cdots < t_N = N\Delta t < t_{N+1} = (N+1)\Delta t = T, \tag{6.3}$$

with $x_i = i\Delta x,\ y_j = j\Delta y$ and $t_n = n\Delta t$ for $i = 0, 1, 2, \dots, I+1,\ j = 0, 1, 2, \dots, J+1$ and $n = 0, 1, 2, \dots, N+1$.

In numerical setting our problem consists of finding $(\omega^n_{i,j}, \psi^n_{i,j}, \varphi^n_{i,j})$ satisfying the following numerical scheme:

$$\rho_1 \bar{\partial}_t \partial_t \omega^n_{i,j} = K \bar{\partial}_x \partial_x \omega^n_{i,j} + K \frac{\partial_x + \bar{\partial}_x}{2} \psi^n_{i,j} + K \bar{\partial}_y \partial_y \omega^n_{i,j} + K \frac{\partial_y + \bar{\partial}_y}{2} \varphi^n_{i,j}, \tag{6.4}$$

$$\begin{aligned} \rho_2 \bar{\partial}_t \partial_t \psi^n_{i,j} = {} & D \bar{\partial}_x \partial_x \psi^n_{i,j} + D \frac{1-\mu}{2} \bar{\partial}_y \partial_y \psi^n_{i,j} + D \frac{1+\mu}{2} \left(\frac{\partial_y + \bar{\partial}_y}{2} \frac{\partial_x + \bar{\partial}_x}{2} \right) \varphi^n_{i,j} \\ & - \frac{K}{2} (\psi^n_{i+1/2,j} + \psi^n_{i-1/2,j} + \psi^n_{i,j+1/2} + \psi^n_{i,j-1/2}) \\ & - K \frac{\partial_x + \bar{\partial}_x}{2} \omega^n_{i,j} - d_1 \frac{\partial_t + \bar{\partial}_t}{2} \psi^n_{i,j}, \end{aligned} \tag{6.5}$$

$$\begin{aligned} \rho_2 \bar{\partial}_t \partial_t \varphi^n_{i,j} = {} & D \bar{\partial}_y \partial_y \varphi^n_{i,j} + D \frac{1-\mu}{2} \bar{\partial}_x \partial_x \varphi^n_{i,j} + D \frac{1+\mu}{2} \left(\frac{\partial_x + \bar{\partial}_x}{2} \frac{\partial_y + \bar{\partial}_y}{2} \right) \psi^n_{i,j} \\ & - \frac{K}{2} (\varphi^n_{i+1/2,j} + \varphi^n_{i-1/2,j} + \varphi^n_{i,j+1/2} + \varphi^n_{i,j-1/2}) \\ & - K \frac{\partial_y + \bar{\partial}_y}{2} \omega^n_{i,j} - d_2 \frac{\partial_t + \bar{\partial}_t}{2} \varphi^n_{i,j}, \end{aligned} \tag{6.6}$$

for all $i = 1, 2, \dots, I$ $j = 1, 2, \dots, J$ and $n = 1, 2, \dots, N$. To simplify our numerical

https://doi.org/10.1017/S0956792517000092 Published online by Cambridge University Press

calculations, we consider the homogeneous boundary conditions given by

$$\omega_{0,j}^{n} = \omega_{I+1,j}^{n} = u_{i,0}^{n} = \omega_{i,J+1}^{n} = 0, \quad \forall n = 1, 2, \ldots, N, \tag{6.7}$$
$$\psi_{0,j}^{n} = \psi_{I+1,j}^{n} = \psi_{i,0}^{n} = \psi_{i,J+1}^{n} = 0, \quad \forall n = 1, 2, \ldots, N, \tag{6.8}$$
$$\varphi_{0,j}^{n} = \varphi_{I+1,j}^{n} = \varphi_{i,0}^{n} = \varphi_{i,J+1}^{n} = 0, \quad \forall n = 1, 2, \ldots, N, \tag{6.9}$$

and initial conditions given by

$$\omega_{i,j}^{0} = \omega(x_i, y_j, 0), \ \omega_{i,j}^{1} = \omega_{i,j}^{0} + \Delta t \omega_t(x_i, y_j, 0), \quad \forall i = 1, \ldots, I, j = 1, \ldots, J, \tag{6.10}$$
$$\psi_{i,j}^{0} = \psi(x_i, y_j, 0), \ \psi_{i,j}^{1} = \psi_{i,j}^{0} + \Delta t \psi_t(x_i, y_j, 0), \quad \forall i = 1, \ldots, I, j = 1, \ldots, J, \tag{6.11}$$
$$\varphi_{i,j}^{0} = \varphi(x_i, y_j, 0), \ \varphi_{i,j}^{1} = \varphi_{i,j}^{0} + \Delta t \varphi_t(x_i, y_j, 0), \quad \forall i = 1, \ldots, I, j = 1, \ldots, J. \tag{6.12}$$

The numerical operators used in (6.4)–(6.6) are given by

$$\partial_x \omega_{i,j}^{n} := \frac{\omega_{i+1,j}^{n} - \omega_{i,j}^{n}}{\Delta x}, \ \overline{\partial}_x \omega_{i,j}^{n} := \frac{\omega_{i,j}^{n} - \omega_{i-1,j}^{n}}{\Delta x}, \ \partial_y \omega_{i,j}^{n} := \frac{\omega_{i,j+1}^{n} - \omega_{i,j}^{n}}{\Delta y},$$

$$\overline{\partial}_y \omega_{i,j}^{n} := \frac{\omega_{i,j}^{n} - \omega_{i,j-1}^{n}}{\Delta y}, \ \partial_t \omega_{i,j}^{n} := \frac{\omega_{i,j}^{n+1} - \omega_{i,j}^{n}}{\Delta t}, \ \overline{\partial}_t \omega_{i,j}^{n} := \frac{\omega_{i,j}^{n-1} - \omega_{i,j}^{n}}{\Delta t},$$

$$\frac{\partial_x + \overline{\partial}_x}{2} \omega_{i,j}^{n} := \frac{\omega_{i+1,j}^{n} - \omega_{i-1,j}^{n}}{2\Delta x}, \quad \frac{\partial_y + \overline{\partial}_y}{2} \omega_{i,j}^{n} := \frac{\omega_{i,j+1}^{n} - \omega_{i,j-1}^{n}}{2\Delta y},$$

$$\frac{\partial_t + \overline{\partial}_t}{2} \omega_{i,j}^{n} := \frac{\omega_{i,j}^{n+1} - \omega_{i,j}^{n-1}}{2\Delta t}, \quad \overline{\partial}_x \partial_x \omega_{i,j}^{n} := \frac{\omega_{i+1,j}^{n} - 2\omega_{i,j}^{n} + \omega_{i-1,j}^{n}}{\Delta x^2},$$

$$\overline{\partial}_y \partial_y \omega_{i,j}^{n} := \frac{\omega_{i,j+1}^{n} - 2\omega_{i,j}^{n} + \omega_{i,j-1}^{n}}{\Delta y^2}, \quad \overline{\partial}_t \partial_t \omega_{i,j}^{n} := \frac{\omega_{i,j}^{n+1} - 2\omega_{i,j}^{n} + \omega_{i,j}^{n-1}}{\Delta t^2},$$

with the same approximations to the functions ψ and φ on the mesh. Here, we are denoting by $\omega_{i,j}^{n}, \varphi_{i,j}^{n}$ and $\psi_{i,j}^{n}$ the numerical approximations to the exact solutions ω, ψ and φ respectively, evaluated on the mesh. More precisely, we have $\omega_{i,j}^{n} \approx \omega(x_i, y_j, t_n)$, $\psi_{i,j}^{n} \approx \psi(x_i, y_j, t_n)$ and $\varphi_{i,j}^{n} \approx \varphi(x_i, y_j, t_n)$. Also $\psi_{i-1/2,j}^{n}$ and $\psi_{i+1/2,j}^{n}$ denote the average of $\psi_{i,j}^{n}$ at the points $(x_{i-1}, y_j, t_n), (x_i, y_j, t_n)$ and $(x_{i+1}, y_j, t_n), (x_i, y_j, t_n)$, respectively. Similar meanings hold for $\psi_{i,j-1/2}^{n}$ and $\psi_{i,j+1/2}^{n}$. Using this discretization, we obtain the approximation given by

$$\psi(x_i, y_j, t_n) \approx \frac{\psi_{i+1,j}^{n} + 2\psi_{i,j}^{n} + \psi_{i-1,j}^{n}}{4} + \frac{\psi_{i,j+1}^{n} + 2\psi_{i,j}^{n} + \psi_{i,j-1}^{n}}{4}. \tag{6.13}$$

Numerical discretization like (6.13) avoids a numerical anomaly known as the locking phenomenon of shear force (see [33] and references contained therein). More precisely, it avoids an over-estimation on rigidity coefficient $b = EI$. Moreover, the numerical scheme presented here is consistent and explicit in sense of the definitions given by

Jovanović and Süli [15] and also by Süli and Mayers [31] (see also Wright [32,33]) and its computational implementation requires knowledge of the approximations at time levels t_n and t_{n-1} in order to approximate the numerical solutions at time level t_{n+1}. It is expected that the stability criterion obeys the restriction given by $\Delta t \leqslant 2/\omega_{\max}$, where $\omega_{\max}$ is the high frequency of the Reissner–Mindlin–Timoshenko system (for references on this issue see [2,32,33]), but the proof is still to be done. However, for the purposes of numerical convergence, we fix the thickness h and we choose $\Delta t \ll \Delta x$ for $\Delta x = \Delta y$.

6.2 Discrete energy

In this section, we prove that the numerical scheme (6.4)–(6.12) has a property of consistency that makes it a useful method in the study of asymptotic behaviour of dissipative systems. With this aim in mind, we present a first property concerning the energy of our method.

The total energy to the numerical equations (6.4)–(6.12) at the time step t_n will be computed using the expression

$$
\begin{aligned}
E^n := \frac{\Delta x \Delta y}{2} \sum_{i=0}^{I} \sum_{j=0}^{J} \Bigg[& \rho_1 \left(\frac{\omega_{i,j}^{n+1} - \omega_{i,j}^{n}}{\Delta t} \right)^2 + \rho_2 \left(\frac{\psi_{i,j}^{n+1} - \psi_{i,j}^{n}}{\Delta t} \right)^2 + \rho_2 \left(\frac{\varphi_{i,j}^{n+1} - \varphi_{i,j}^{n}}{\Delta t} \right)^2 \\
& + D \frac{\psi_{i+1,j}^{n+1} - \psi_{i,j}^{n+1}}{\Delta x} \frac{\psi_{i+1,j}^{n} - \psi_{i,j}^{n}}{\Delta x} + D \left(\frac{1-\mu}{2} \right) \frac{\psi_{i,j+1}^{n+1} - \psi_{i,j}^{n+1}}{\Delta y} \frac{\psi_{i,j+1}^{n} - \psi_{i,j}^{n}}{\Delta y} \\
& + D \left(\frac{1-\mu}{2} \right) \frac{\varphi_{i+1,j}^{n+1} - \varphi_{i,j}^{n+1}}{\Delta x} \frac{\varphi_{i+1,j}^{n} - \varphi_{i,j}^{n}}{\Delta x} + D \frac{\varphi_{i,j+1}^{n+1} - \varphi_{i,j}^{n+1}}{\Delta y} \frac{\varphi_{i,j+1}^{n} - \varphi_{i,j}^{n}}{\Delta y} \\
& + K \left(\frac{\omega_{i+1,j}^{n+1} - \omega_{i,j}^{n+1}}{\Delta x} + \frac{\psi_{i+1,j}^{n+1} + \psi_{i,j}^{n+1}}{2} \right) \left(\frac{\omega_{i+1,j}^{n} - \omega_{i,j}^{n}}{\Delta x} + \frac{\psi_{i+1,j}^{n} + \psi_{i,j}^{n}}{2} \right) \\
& + K \frac{\psi_{i,j+1}^{n+1} + \psi_{i,j}^{n+1}}{2} \frac{\psi_{i,j+1}^{n} + \psi_{i,j}^{n}}{2} \\
& + K \left(\frac{\omega_{i,j+1}^{n+1} - \omega_{i,j}^{n+1}}{\Delta y} + \frac{\varphi_{i,j+1}^{n+1} + \varphi_{i,j}^{n+1}}{2} \right) \left(\frac{\omega_{i,j+1}^{n} - \omega_{i,j}^{n}}{\Delta y} + \frac{\varphi_{i,j+1}^{n} + \varphi_{i,j}^{n}}{2} \right) \\
& + K \frac{\varphi_{i+1,j}^{n+1} + \varphi_{i,j}^{n+1}}{2} \frac{\varphi_{i+1,j}^{n} + \varphi_{i,j}^{n}}{2} \\
& + D \left(\frac{1+\mu}{2} \right) \Bigg(\frac{\psi_{i+1,j+1}^{n+1} - \psi_{i,j}^{n+1}}{2\Delta x} \frac{\varphi_{i+1,j+1}^{n} - \varphi_{i,j}^{n}}{2\Delta y} + \frac{\psi_{i,j+1}^{n+1} - \psi_{i+1,j}^{n+1}}{2\Delta x} \frac{\varphi_{i+1,j}^{n} - \varphi_{i,j+1}^{n}}{2\Delta y} \\
& + \frac{\varphi_{i+1,j+1}^{n+1} - \varphi_{i,j}^{n+1}}{2\Delta x} \frac{\psi_{i+1,j+1}^{n} - \psi_{i,j}^{n}}{2\Delta y} + \frac{\varphi_{i,j+1}^{n+1} - \varphi_{i+1,j}^{n+1}}{2\Delta x} \frac{\psi_{i+1,j}^{n} - \psi_{i,j+1}^{n}}{2\Delta y} \Bigg) \Bigg]. \qquad (6.14)
\end{aligned}
$$

We note that E^n is the discrete version of the continuous energy (2.7). Moreover, one can show that E^n decreases for any $d_i > 0,\ i = 1, 2$ and that it is constant for $d_i = 0, i = 1, 2$. Instead of computing the time derivative of the energy we can use summation by parts. The discrete energy E^n is an important numerical instrument to certify our analytical results concerning the stabilization of dissipative Reissner–Mindlin–Timoshenko system established in previous sections.

https://doi.org/10.1017/S0956792517000092 Published online by Cambridge University Press

Next, we establish the discrete counterpart of the Proposition 2.2. In that direction, we say that the numerical scheme (6.4)–(6.8) is qualitatively stable with respect to discrete rate of change of the energy E^n for all step sizes $\Delta x, \Delta y$ and Δt (see Definition 1.1 on page 43 of [4]).

Theorem 6.1 (Discrete energy) *Let* $(\omega^n_{i,j}, \varphi^n_{i,j}, \psi^n_{i,j})$ *be a solution of the finite difference scheme* (6.4)–(6.8) *with* $d_i \geqslant 0$ $i = 1, 2$. *Then for all* $\Delta t, \Delta x$ *and* Δy, *the discrete rate of change of energy of the numerical scheme* (6.4)–(6.8) *at the instant of time* t_n *is given by*

$$\frac{E^n - E^{n-1}}{\Delta t} = -d_1 \Delta x \sum_{j=1}^{J} \left(\frac{\varphi^{n+1}_{i,j} - \varphi^{n-1}_{i,j}}{2\Delta t} \right)^2 - d_2 \Delta x \sum_{j=1}^{J} \left(\frac{\psi^{n+1}_{j} - \psi^{n-1}_{j}}{2\Delta t} \right)^2 \leqslant 0, \tag{6.15}$$

for all $n = 1, \ldots, N, N+1$.

Proof The proof is too long and we omit it here. Analogously to continuous case, we use the multipliers at discrete level given by $(\frac{\partial_t + \bar{\partial}_t}{2} \omega^n_{i,j})$, $(\frac{\partial_t + \bar{\partial}_t}{2} \varphi^n_{i,j})$ and $(\frac{\partial_t + \bar{\partial}_t}{2} \psi^n_{i,j})$ and we organize the results in order to make up the difference $E^n - E^{n-1}$. □

The above theorem says that E^n is non-increasing regardless of any relation between mesh parameters. That is to say, the CFL condition is not necessary to get the decreasing of E^n. On the other hand, it is not assured that E^n preserves the positivity property and then the discrete energy E^n is not signed in general. In order to prove the positivity property, it is necessary to have the stability criterion as necessary condition. See for example Negreanu and Zuazua [23].

6.3 Numerical simulations

In this section, we focus on the numerical scheme (6.4)–(6.12) and its energy E^n to illustrate by means of the numerical experiments the analytical results established in previous sections.

We emphasize that we are not concerned with issues of numerical convergence between exact solution and numerical solution and the respective rate of convergence. The accuracy of the numerical scheme (6.4)–(6.12) can be seen through of the energy conservation law. Indeed, taking $d_i = 0, i = 1, 2$ in (6.15) we obtain that $E^n = E^0$, $n = 1, \ldots, N+1$.

For simulations in Figure 1, we use the following main data: $L_1 = L_2 = 1$, $T = 4$, thickness $h = 0.015$, $E = 21 \times 10^4$ N/m^2, $\rho = 7{,}850$ kg/m^3, $k' = 5/6$ and $\mu = 0.29$. The initial conditions are given by

$$\omega(x_i, y_j, 0) = \psi(x_i, y_j, 0) = \varphi(x_i, y_j, 0) = 0, \tag{6.16}$$

$$\omega_t(x_i, y_j, 0) = \sin\left(\nu \frac{\pi x_i}{L_1} \right) \sin\left(\nu \frac{\pi y_j}{L_2} \right), \quad \forall \nu \in \mathbb{N}, \tag{6.17}$$

https://doi.org/10.1017/S0956792517000092 Published online by Cambridge University Press

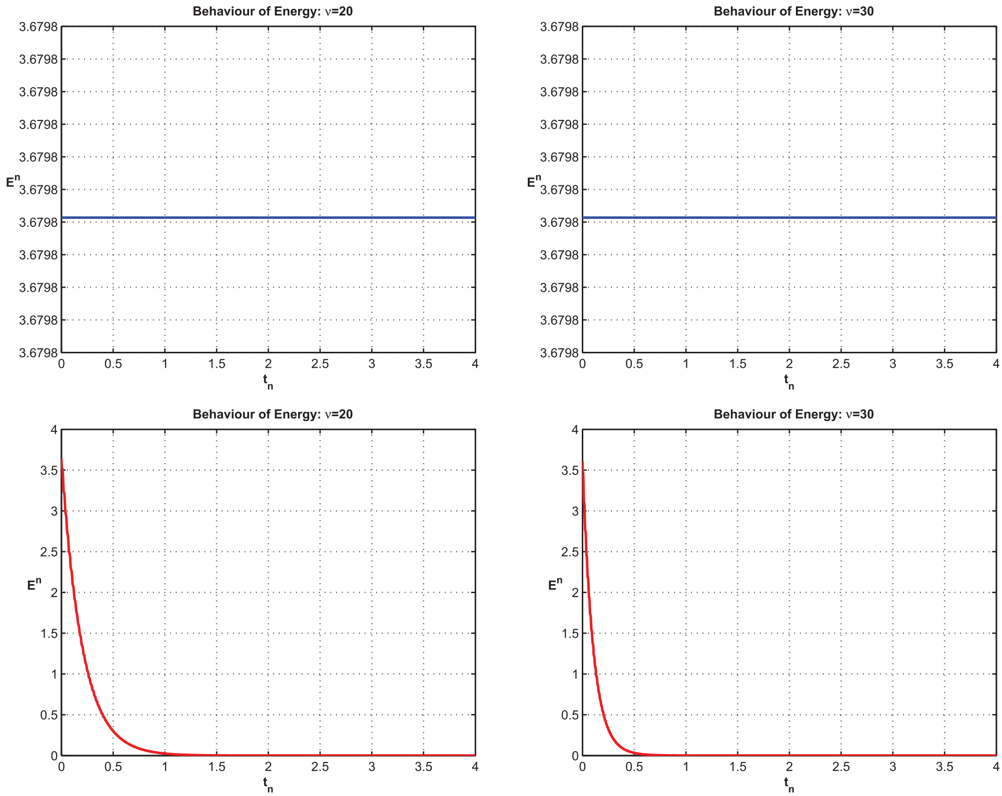

FIGURE 1. Plots of the numerical energy E^n. The energy conservation law (2.10) and its discrete counterpart are compatible according to two first graphs. The last two graphs show that the energy E^n is like an exponential function $e^{-\omega t_n}$ for $\omega > 0$.

$$\psi_t(x_i, y_j, 0) = \cos\left(v\frac{\pi x_i}{L_1}\right)\sin\left(v\frac{\pi y_j}{L_2}\right), \quad \forall v \in \mathbb{N}, \tag{6.18}$$

$$\varphi_t(x_i, y_j, 0) = \sin\left(v\frac{\pi x_i}{L_1}\right)\cos\left(v\frac{\pi y_j}{L_2}\right), \quad \forall v \in \mathbb{N}. \tag{6.19}$$

In the computational mesh, we use $\Delta x = \Delta y = 0.03125$ and $\Delta t = 0.00195$ such that $\Delta t/\Delta x = 0.0624$.

6.3.1 *Undamped and fully damped cases*

The Figure 1 shows that the energy behaviour in time t_n in two situations: undamped ($d_i = 0,\ i = 1, 2$) and full damping cases. For the fully damped case, we consider three internal frictional dissipations in the system, that is, we consider terms $d_0\omega_t$, $d_1\psi_t$ and $d_2\varphi_t$, for d_0, d_1 and d_2 to be positive constants. In the first one, the energy E^n is constant for all t_n and this property is a measure of the accurate of the numerical scheme (6.4)–(6.12). In the simulations given Figure 1, we use different speeds of wave propagation.

https://doi.org/10.1017/S0956792517000092 Published online by Cambridge University Press

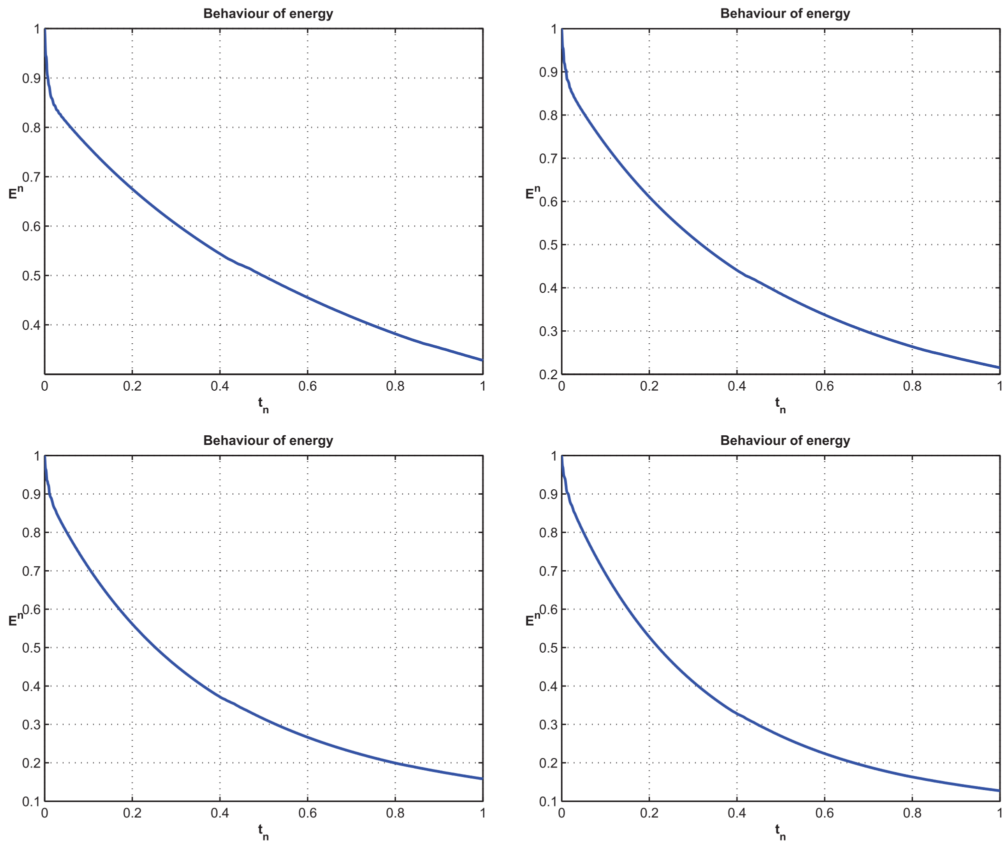

FIGURE 2. Discrete time versus discrete energy for dissipative Reissner–Mindlin-Timoshenko plates when $v_1^2 \neq v_2^2$. The curve decays at time t_n and in Table 1 we assumed the rate equal to 1/2 for polynomial decay. In these cases, in order to get the optimal rate of decay, we refine the mesh size by taking $\Delta = \Delta x = \Delta y \in \{1/40, 1/60, 1/80, 1/100\}$ and $\Delta t = 1/2048$.

6.3.2 *Damping on two rotational angles*

The next numerical experiments show a fast or slow decay according to whether $v_1^2 - v_2^2$ is equal to zero or not, respectively (see Figures 2 and 3). That is to say, if $v_1^2 = v_2^2$ we obtain the exponential decay (see Theorem 4.5). On the other hand, for $v_1^2 \neq v_2^2$, decay is not exponential. (see Theorem 3.1). In both cases we calculate, from a code performed with Matlab, the decay rates in order to give an accurate measure of exponential or polynomial decay for energy. To be precise, we calculate the decay rates from the data given by $(\log E^n, t_n)$ in order to obtain an exponential rate and from the data $(\log E^n, \log t_n)$ in order to obtain a polynomial decay. As we can see in Table 1, for the exponential decay case ($v_1^2 = v_2^2$), we observed the rate decreases and approaches -2 as we refine the step size h. On the other hand, when $v_1^2 \neq v_2^2$, we can see that the rate goes to -0.5, this result corroborates with the estimate obtained in the Section 5. For these cases, the data for discrete energy were normalized, i.e., we take E^n/E^0. Moreover, in this case for initial data we consider $\omega(x_i, y_j, 0) = \psi(x_i, y_j, 0) = \varphi(x_i, y_j, 0) = \omega_t(x_i, y_j, 0) = \psi_t(x_i, y_j, 0) = \varphi_t(x_i, y_j, 0) = 10(x_i + 1)^3 x_i^3 (y_j + 1)^3 y_j^3$ (see [5]).

https://doi.org/10.1017/S0956792517000092 Published online by Cambridge University Press

Table 1. *Decay rates*

	$\Delta = 1/40$	$\Delta = 1/60$	$\Delta = 1/80$	$\Delta = 1/100$
$v_1^2 \neq v_2^2$	-0.2548	-0.3744	-0.4644	-0.5295
$v_1^2 = v_2^2$	-1.0591	-1.5827	-1.9480	-2.1535

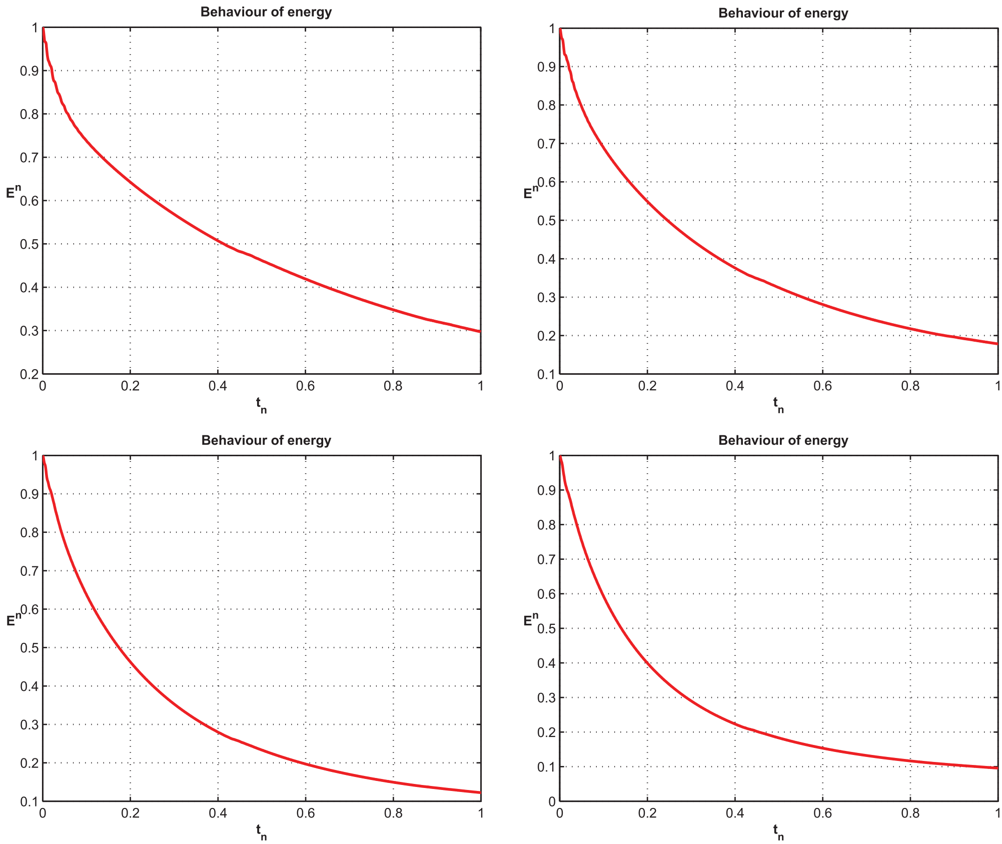

FIGURE 3. Discrete time versus discrete energy for dissipative Reissner–Mindlin–Timoshenko plates when $v_1^2 = v_2^2$. The curve decays over time t_n and the decay rate approaches 2.15. This is an expected result for standard finite difference schemes.

7 Conclusions and outlook

Taking into account the literature on Timoshenko beams, it is well known that the relationship between the speeds of wave propagation plays an important role in the asymptotic behaviour of solutions of weakly dissipative Timoshenko systems. For example, if we consider the Timoshenko system with a dissipative mechanism being present only on the equation for the rotation angle, Soufyane [30] and Muñoz Rivera and Racke [22] proved that the Timoshenko system is exponentially stable if and only if $\rho_1/k = \rho_2/b$. In this paper, we proved a similar result for the Reissner–Mindlin–Timoshenko system for the case where we have frictional dissipations on the equations of the rotational angles. Analogously to the one-dimensional case we conclude that if the speeds of wave propagation are equal, then the energy of solutions of system decreases exponentially. This result has been observed in the numerical experiments, by calculations on decay rates presented in Table 1.

On the other hand, considering a frictional damping working only on the vertical displacement for the model of Timoshenko beams, Almeida Júnior *et al.* [1] proved that the system is exponentially stable if and only if the speeds of wave propagation are equal. As previously mentioned, Campelo *et al.* [8] proved an analogous result to the Reissner–Mindlin–Timoshenko system with frictional damping acting only on the displacement equation.

It is increasingly clear that the two-dimensional model of Reissner–Mindlin–Timoshenko preserves some qualitative properties of the one-dimensional model of Timoshenko. At this point, knowing that research involving problems of Timoshenko beams is better consolidated, we design new questions and perspectives of dissipative models of Reissner–Mindlin–Timoshenko, and that can be studied from the point of view of mathematical analysis, especially regarding the exponential stability and taking into account the speeds of wave propagation. For example, the dynamic model plates can be studied with other dissipative mechanisms such as thermal dissipation, which can be obtained by considering heat conduction Fourier's or Cattaneo's law. Therefore, for these systems we hope that there exists a relationship between the coefficients, which makes them exponentially stable as in the 1-D case.

Acknowledgements

The authors are grateful to the reviewers of this article for constructive suggestions. These were of great help in improvement of this paper in its final form. In particular, we acknowledge the helpful comments and suggestions from Dr Michael Pokojovy. His valuable remarks have contributed to improve this study.

References

[1] Almeida Júnior, D. S., Santos, M. L. & Muñoz Rivera, J. E. (2013) Stability to weakly dissipative Timoshenko systems. *Math. Methods Appl. Sci.* **36**(14), 1965–1976.

[2] Almeida Júnior, D. S. & Muñoz Rivera, J. E. (2015) Stability criterion to explicit finite difference applied to the Bresse system. *Afrika Mathematika* **26**(5), 761–778.

[3] Ammar-Khodja, F., Benabdallah, A., Muñoz Rivera, J. E. & Racke, R. (2003) Energy decay for Timoshenko systems of memory type. *J. Differ Equ.* **194**(1), 82–115.

[4] Anguelov, R., Djoko, J. K. & Lubuma, J. M.-S. (2008) Energy properties preserving schemes for Burgers' equation. *Numer. Methods Partial Differ. Equ.* **24**(1), 41–59.

[5] Alves, M., Muñoz Rivera, J. E., Sepúlveda, M., Villagrán, O. V. & Zegarra Garay, M. (2014) The asymptotic behavior of the linear transmission problem in viscoelasticity. *Math. Nachr.* **287**(5–6), 483–497.

[6] Borichev, A. & Tomilov, Y. (2009) Optimal polynomial decay of functions and operator semigroups. *Math. Ann.* **347** (2), 455–478.

[7] Brezis, H. (1992) *Analyse Fonctionelle, Théorie et Applications*. Masson, Paris.

[8] Campelo, A., Almeida Júnior, D. & Santos, M. L. (2016) Stability to the dissipative Reissner-Mindlin-Timoshenko acting on displacement equation. *Eur. J. Appl. Math.* **27**(2), 157–193.

[9] Fernándes Sare, H. D. (2009) On the stability of Mindlin–Timoshenko plates. *Q. Appl Math* **LXVII**(2), 249–263.
[10] Grobbelaar-Van Dalsen, M. (2011) Strong stabilization of models incorporating the thermoelastic Reissner–Mindlin plate equations with second sound. *Appl. Anal.* **90**(9), 1419–1449.
[11] Grobbelaar-Van Dalsen, M. (2013) Stabilization of a thermoelastic Mindlin-Timoshenko plate model revisited, *Z. Angew. Math. Phys.* **64**(4), 1305–1325.
[12] Grobbelaar-Van Dalsen, M. (2015) Polynomial decay rate of a thermoelastic Mindlin–Timoshenko plate model with Dirichlet boundary conditions. *Z. Angew. Math. Phys.* **66**(1), 113–128.
[13] Haraux, A. (1989) Une remarque sur la stabilisation de certains systèmes du deuxième ordere en tems. *Port. Math.* **46**(3), 245–258.
[14] Huang, F. (1985) Characteristic conditions for exponential stability of linear dynamical systems in Hilbert spaces. *Ann. Differential Equations*, **1**(1), 43–56.
[15] Jovanović, B. S. & Süli, E. (2014) *Analysis of Finite Difference Schemes. For Linear Partial Differential Equations with Generalized Solutions*. Springer Series in Computational Mathematics, Vol. 46, Springer London, 408 pages.
[16] Kim, J. & Renardy, Y. (1987) Boundary control of the Timoshenko beam. *SIAM J. Control Optim.* **25**(6), 1417–1429.
[17] Lagnese, J. & Lions, J. (1988) *Modelling, Analysis and Control of Thin Plates*. Collection RMA, Masson, Paris.
[18] Lagnese, J. (1989) *Boundary Stabilization of Thin Plates*. SIAM, Philadelphia.
[19] Liu, Z. & Zheng, S. (1999) *Semigroups Associated with Dissipative Systems*, In CRC Reseach Notes in Mathematics, Vol. 398, Chapman & Hall, CRC Press. Taylor & Francis Group.
[20] Muñoz Rivera, J. E. & Portillo Oquendo, H. (2003) Asymptotic behavior on a Mindlin–Timoshenko plate with viscoelastic dissipation on the boundary. *Funkcialaj Ekvacioj.* **46**(3), 363–382.
[21] Muñoz Rivera, J. E. & Racke, R. (2002) Mildly dissipative nonlinear Timoshenko systems–Global existence and exponential stability. *J. Math. Anal. Appl.* **276**(1), 248–278.
[22] Muñoz Rivera, J. & Racke, R. (2003) Global stability for damped Timoshenko systems. *Discrete Continuous Dyn. Syst.* **9**(6), 1625–1639.
[23] Negreanu, M. & Zuazua, E. (2003) Uniform boundary controllability of a discrete 1-D wave equations. *Syst. Control Lett.* **48**(3–4), 261–279.
[24] Pazy, A. (1983) *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Vol. 44, Springer-Verlag, New York, 282 pages.
[25] Pokojovy, M. (2015) On stability of hyperbolic thermoelastic Reissner-Mindlin-Timoshenko plates. *Math. Methods Appl. Sci.* **38**(7), 1225–1246.
[26] Prüss, J. (1984) On the spectrum of C_0-semigroups. *Trans. Am. Math. Soc.* **284**, 847–857.
[27] Raposo, C., Ferreira, J., Santos, M. L. & Castro, N. (2005) Exponential stability for the Timoshenko beam with two weak dampings. *Appl. Math. Lett.* **18**(5), 535–541.
[28] Santos, M. (2002) Decay rates for solutions of a Timoshenko system with a memory condition at the boundary. *Abstr. Appl. Anal.* **7**(10), 531–546.
[29] Santos, M., Almeida Júnior, D. S. & Muñoz Rivera, J. (2012) The stability number of the Timoshenko system with second sound. *J. Differ. Equ.* **253**(9), 2715–2733.
[30] Soufyane, A. (1999) Stabilisation de la poutre de Timoshenko. *C. R. Acad. Sci., Paris, Série I - Math.* **328**(8), 731–734.
[31] Süli, E. & Mayers, D. (2003) *An Introduction to Numerical Analysis*. Cambridge University Press, 433 pages.
[32] Wright, J. (1987) A mixed time integration method for Timoshenko and Mindlin type elements. *Commun. Appl. Numer. Methods* **3**(3), 181–185.
[33] Wright, J. (1998) Numerical stability of a variable time step explicit method for Timoshenko and Mindlin type structures. *Commun. Numer. Methods Eng.* **14**(2), 81–86.