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USING EXCURSIONS TO ANALYZE SIMULATION OUTPUT

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We consider the steady-state simulation output analysis problem for a process that satisfies a functional central limit theorem. We construct an estimator for the time-average variance constant that is based on excursions of a process above the minimum. The resulting estimator does not require a fixed run length, and the memory requirement can be dynamically bounded. Standardized time series methods based on excursions are also described.

1. INTRODUCTION

A basic simulation problem is to estimate the steady-state mean and variance of a simulated stochastic process. One of the most commonly used and widely applicable methods is the method of batch means. This method can be applied when the simulated process satisfies a functional central limit theorem, which is the case with many classes of processes that might arise in a steady-state simulation context; see Glynn and Iglehart [7]. The idea is to break up the suitably normalized and centered simulated path into a number of fixed-size batches and compute a statistic based on the increments over the batches; see Goldsman and Schmeiser [8]. Variations include the method of overlapping batch means (Meketon and Schmeiser [11]).

In this article we consider methods that break the simulated path into random length segments. In some settings there can be computational advantages as well as advantages in terms of statistical efficiency.

The methods we describe are partly motivated by the problem of simulation-based optimization. Some optimization algorithms use a large collection of estimates based on different parameter values. It must be possible to halt and resume simulations at different parameter values, as those parameters become more or less important for identifying the optimum. Therefore, it is important for the simulations to use limited memory (since there may be many of them going on in parallel), and it is not known in advance how long the simulation at a particular parameter value will ultimately run. Thus, it is also important to have the flexibility to increase or decrease the resources allocated to a simulation as time advances and to always have an estimate available; that is, we should not need to plan in advance a fixed horizon to simulate until we obtain useful estimates. This issue is discussed, for example, in [10], in the context of sequential methods for ranking and selection; see also [6].

We emphasize that we are concerned with the case in which the simulation run length is not known in advance. The use of batching in this setting was investigated in Yeh and Schmeiser [16]. As stated in Goldsman and Schmeiser [8], the storage requirements should be $O(1)$ or certainly at most $O(n)$.

The methods that we propose are economical in their use of memory compared with the batch-means alternative. The price we pay is that at least two independent simulations must be run in parallel in order to center the output. To the extent that initialization bias is a problem, it could be exacerbated by this requirement.

In the next section we describe the basic setup. Section 3 describes the variance estimators based on excursions above the minimum and also the batch-means alternative. Section 4 outlines how the excursion approach can be used in the context of standardized time series methods to produce asymptotically valid confidence intervals for the steady-state mean. The results of some numerical experiments are presented at the end of Sections 3 and 4.

2. EXCURSIONS ABOVE THE MINIMUM

Suppose that a simulation generates a real-valued sequence $Y_1, Y_2, \ldots$. We assume throughout the article that there exist constants $\mu \in \mathbb{R}$, $\sigma \in (0, \infty)$, such that if we define the process

$$X_n^*(t) = n^{-1/2}\left(\sum_{i=1}^{\lfloor nt\rfloor}(Y_i - \mu) + (nt - \lfloor nt\rfloor)\left(Y_{\lfloor nt\rfloor+1} - \mu\right)\right), \qquad t \geq 0,$$

then $X_n^* \xrightarrow{d} \sigma B$, where B is a Brownian motion and $\xrightarrow{d}$ denotes convergence in distribution. In this case we say that the process satisfies a functional central limit theorem; see Revuz and Yor [12].

In discrete-event simulation, we are often interested in estimating the parameters μ and σ. We now outline such a procedure.

Run two (or more) simulations in parallel, producing independent output sequences $\{Y_1^1, Y_2^1, \ldots\}$ and $\{Y_1^2, Y_2^2, \ldots\}$. For $j = 1, 2$, set

$$X_n^j(t) = n^{-1/2}\left(\sum_{i=1}^{\lfloor nt\rfloor}\left(Y_i^j - \mu\right) + (nt - \lfloor nt\rfloor)(Y_{\lfloor nt\rfloor+1}^j - \mu)\right), \qquad t \geq 0.$$

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Our functional central limit theorem assumption implies that $X_n^j \xrightarrow{d} \sigma B$ for $j = 1, 2$.

Let

$$\mu_n = \frac{1}{2n} \sum_{i=1}^{n} \left(Y_i^1 + Y_i^2 \right),$$

so that

$$\sqrt{n}\,(\mu_n - \mu) = \frac{1}{2} \left(X_n^1(1) + X_n^2(1) \right) \xrightarrow{d} \sigma N(0, 1).$$

Let

$$A(t) = \sum_{i=1}^{\lfloor t \rfloor} \left(Y_i^1 - Y_i^2 \right) + (t - \lfloor t \rfloor) \left(Y_{\lfloor t \rfloor + 1}^1 - Y_{\lfloor t \rfloor + 1}^2 \right), \qquad t \geq 0.$$

This is the process that our output analysis algorithm uses. Note that

$$\frac{A(n\,\cdot)}{\sqrt{n}} \xrightarrow{d} \sigma B.$$

Finally, set

$$X_n = X_n^1 - X_n^2 = n^{-1/2} A(n\cdot) \xrightarrow{d} \sigma B.$$

Although the X_n^j are not observable from the simulation since they depend on the unknown μ, X_n can be constructed from the simulation.

Let $\{[\alpha_i, \beta_i], i \geq 1\}$ denote the intervals of excursion above the minimum for the process A; that is,

$$A(\alpha_i) = A(\beta_i) = \min_{s \leq \alpha_i} A(s)$$

and

$$A(s) > A(\alpha_i)$$

for $\alpha_i < s < \beta_i$. Set

$$h_i = \sup\{A(s) - A(\alpha_i) : \alpha_i \leq s \leq \beta_i\}$$

and

$$l_i = \beta_i - \alpha_i.$$

We call the process A over the interval $[\alpha_i, \beta_i]$ the ith excursion above the minimum. For the ith excursion, consider the pair $(l_i, h_i \cdot l_i^{-1/2})$ (we refer to the second component as the normalized excursion height). Note that if A has an excursion of length l and height h from α to β, then $X_n = n^{-1/2} A(n\,\cdot)$ has an excursion of length l/n and height $h/\sqrt{n}$ from α/n to β/n.

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We summarize some results on Brownian excursions; see, for example, Revuz and Yor [12]. Let B be a standard Brownian motion and suppose that

$$B(\alpha) = B(\beta) = 0$$

and

$$B(s) > 0, \quad \forall s \in (\alpha, \beta).$$

Then the process

$$e(t) = \frac{B(\alpha + t(\beta - \alpha))}{\sqrt{\beta - \alpha}}, \qquad 0 \le t \le 1,$$

is a *standard (positive) Brownian excursion*. Let H denote the maximum of a standard Brownian excursion. Then

$$P(H \le x) = \sqrt{2}\pi^{5/2}x^{-3} \sum_{n=1}^{\infty} n^2 \exp\left(-\frac{1}{2}\pi^2 \frac{n^2}{x^2}\right),$$

$$EH = \sqrt{\frac{\pi}{2}}, \qquad EH^2 = \frac{\pi^2}{6}, \tag{1}$$

and $\mathrm{Var}(H) = (\pi/2)(\pi/3 - 1)$.

Let u be the location of the last minimum before time v. Then

$$M(t) = \frac{B(u + t(v - u))}{\sqrt{v - u}}, \qquad 0 \le t \le 1,$$

is a *standard Brownian meander*. The distribution of $M(1)$ is the same as the square root of an exponential random variable with mean 2. Therefore,

$$EM(1) = \sqrt{\pi/2}, \qquad \mathrm{Var}(M(1)) = 2 - \pi/2.$$

The processes $(B(t) - \min_{0 \le s \le t} B(s))$ and $(|B(t)|)$ have the same law (Revuz and Yor [12, p. 230]). Therefore, the absolute value of an excursion away from 0 of B has the same law as an excursion of B above its running minimum.

Consider a standard Brownian motion B on the unit interval, and for $\epsilon > 0$, let $\nu(\epsilon)$ be the number of excursions of length $\ge \epsilon$. Then by Theorem 2.21 in [9, p. 415], as $\epsilon \to 0$,

$$\nu(\epsilon)\sqrt{\epsilon} \longrightarrow \sqrt{8/\pi}\, L_1(0) \quad \text{a.s.}, \tag{2}$$

where $L_1(0)$ is the local time process at 0 of a Brownian motion, evaluated at time 1. If $\{h_i \cdot l_i^{-1/2},\ i = 1, 2, \ldots, \nu(\epsilon)\}$ are the normalized excursion heights of excursions of length at least ϵ (including perhaps the terminal meander), then as $\epsilon \downarrow 0$,

$$\sqrt{\nu(\epsilon)}\left(\frac{1}{\nu(\epsilon)} \sum_{i=1}^{\nu(\epsilon)} h_i \cdot l_i^{-1/2} - \sqrt{\frac{\pi}{2}}\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{\pi}{2}\left(\frac{\pi}{3} - 1\right)\right). \tag{3}$$

This follows from Theorem 17.2 in Billingsley [1], (2), and the fact that the normalized heights are independent and identically distributed. (The final meander does not have the same distribution, but it does not change the limit.)

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Let F_ϵ denote the distribution of

$$\frac{1}{\nu(\epsilon)} \sum_{i=1}^{\nu(\epsilon)} h_i \cdot l_i^{-1/2}.$$

3. THE SIMULATION ESTIMATOR

After the simulation time has reached n, we keep all excursions of A up to time n that have length at least ϵn, where $\epsilon > 0$. Let L_n denote the time of the last minimum of A before time n. The final meander above the minimum before time n is treated exactly like an excursion, except that we record its final instead of maximum value; if its length is at least ϵn, then we keep its length $n - L_n$ and terminal value $X_n(n) - X_n(L_n)$. Thus, the retained excursions correspond to excursions of length $\geq \epsilon$ for the scaled process that converges to σB; that is, an excursion of length ϵn for A corresponds to an excursion of length ϵ for X_n.

Let $\nu_n(\epsilon)$ be the number of retained excursions of A at time n (including the final meander, if it is kept). Let $h_{i,n}$ and $l_{i,n}$ denote the height and length, respectively, of the ith retained excursion of A above the minimum, for $1 \leq i \leq \nu_n(\epsilon)$ (if the final meander is long enough, then $h_{\nu_n(\epsilon),n}, l_{\nu_n(\epsilon),n}$ is the terminal value and length of the last meander). As n increases, the elements of $\{(h_{i,n}, l_{i,n}), i \leq \nu_n(\epsilon)\}$ can change as some pairs are dropped and others added.

Define the estimator

$$\sigma_n(\epsilon) \triangleq \frac{\sqrt{2/\pi}}{\nu_n(\epsilon)} \sum_{i=1}^{\nu_n(\epsilon)} \frac{h_{i,n}}{\sqrt{l_{i,n}}}. \tag{4}$$

THEOREM 3.1: *For any $\epsilon > 0$, as the simulation run length $n \to \infty$,*

$$\sqrt{\nu_n(\epsilon)}\,(\sigma_n(\epsilon) - \sigma) \xrightarrow{d} \sigma\sqrt{\nu(\epsilon)}\left(\frac{\sqrt{2/\pi}}{\nu(\epsilon)} \sum_{i=1}^{\nu(\epsilon)} \frac{h_i}{\sqrt{l_i}} - 1\right).$$

PROOF: By the Skorokhod representation theorem (Rogers and Williams [13]), there exists a probability space $(\Omega', \mathcal{F}', P')$ on which are defined processes X_n' and X', with $X_n' \stackrel{d}{=} X_n$ and X' a Brownian motion with diffusion coefficient σ and such that $X_n' \to X'$ uniformly on $[0, 1]$. With probability 1, X' does not have the same local minimum occurring more than once (Revuz and Yor [12, p. 108, Ex. 3.26]). Therefore, there exists a set $G \in \mathcal{F}'$ with $P'(G) = 1$, and for all $\omega \in G$, $X_n'(\omega)$ converges uniformly to $X'(\omega)$, and $X'(\omega)$ does not have the same local minimum occurring more than once.

Fix $\omega \in G$ and suppose that $X'(\omega)$ has an excursion above the minimum of length $\geq \epsilon$; say $X'(\omega, l) = X'(\omega, r) = \min_{s \leq r} X'(\omega, s)$ for some $0 < l < r < 1$, $r - l \geq \epsilon$,

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and $\max_{l \le s \le r} X'(\omega, s) = m$. Since $X'_n(\omega)$ converges uniformly to $X'(\omega)$,

$$\lim_{n \to \infty} \max_{l \le s \le r} X'_n(\omega, s) = m$$

and

$$\lim_{n \to \infty} \min_{l \le s \le r} X'_n(\omega, s) = \lim_{n \to \infty} X'_n(\omega, l) = \lim X'_n(\omega, r) = X'(\omega, l).$$

Thus, for large enough n, $X'_n(\omega)$ has an excursion of height m_n with left end point l_n and right end point r_n and $m_n \to m$; the only thing that could prevent the excursion of X'_n from converging to that of X' is if its length in the limit is less than $r - l$. Suppose that $l_n \to \hat{l} > l$. This would imply that $X'(\omega, \hat{l}) = X'(\omega, l)$. However, since $\omega \in G$, $X'(\omega)$ does not have the same local minimum twice, and so $= l$; similarly, $r_n \to r$ and

$$(l_n, r_n, m_n) \longrightarrow (l, r, m).$$

Thus, the normalized heights of the excursions of length $\ge \epsilon$ of X'_n converge to the normalized heights of excursions of length $\ge \epsilon$ of X'. Since $\nu_n(\epsilon) \to \nu(\epsilon)$, where $\nu(\epsilon)$ is defined like $\nu_n(\epsilon)$ but for X' instead of X_n, the proof is complete. ∎

By (3) and Theorem 3.1,

$$\lim_{\epsilon \downarrow 0} \lim_{n \uparrow \infty} P\left(\sqrt{\nu_n(\epsilon)}\,(\sigma_n(\epsilon) - \sigma) \le z\right) = \Phi\left(\frac{z}{\sigma\sqrt{\pi/3 - 1}}\right). \quad \mathbf{(5)}$$

For $\epsilon > 0$, define $\nu'(\epsilon)$, l'_i, h'_i, $1 \le i \le \nu'(\epsilon)$ like the unprimed variables but for excursions below the maximum instead of above the minimum. Let $\hat{F}_\epsilon$ denote the distribution of

$$\frac{1}{2}\left(\frac{1}{\nu(\epsilon)} \sum_{i=1}^{\nu(\epsilon)} h_i \cdot l_i^{-1/2} + \frac{1}{\nu'(\epsilon)} \sum_{i=1}^{\nu'(\epsilon)} h'_i \cdot (l'_i)^{-1/2}\right).$$

As $\epsilon \downarrow 0$, both averages converge in probability to $\sqrt{\pi/2}$, and so $\hat{F}_\epsilon$ converges to a point mass at $\sqrt{\pi/2}$. Define $\hat{\sigma}_n(\epsilon)$ the same way as $\sigma_n(\epsilon)$ was defined but with X_n replaced by $-X_n$; that is, we average the normalized heights of excursions below the maximum. Adapting the proof of Theorem 3.1 to consider excursions below the maximum as well as excursions above the minimum shows that

$$\frac{\sigma_n(\epsilon) + \hat{\sigma}_n(\epsilon)}{2\sigma\sqrt{2/\pi}} \xrightarrow{d} \hat{F}_\epsilon$$

as $n \to \infty$. Then

$$\lim_{\epsilon \downarrow 0} \lim_{n \uparrow \infty} P\left(\left|\frac{1}{2}\left(\sigma_n(\epsilon) + \hat{\sigma}_n(\epsilon)\right) - \sigma\right| > \delta\right) = 0$$

for any $\delta > 0$. This averaged estimator, which we rename $\sigma_n(\epsilon)$, makes use of more of the simulated path and is the estimator that we use in the numerical experiments in Section 3.3.

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A variation on the estimator is to fix the memory allocation m and keep only the m longest excursions. This estimator was presented, without proofs, in Calvin [3].

3.1. Continuity Correction

Our estimator is biased due to the discrete nature of the process we are approximating with a Brownian excursion. In order to examine this source of bias, let us assume that the simulated process is a Brownian motion, so we are working with the linear interpolation of a skeleton of a Brownian motion. The situation is depicted in Figure 1.

One source of bias is that the discrete maximum is less than the continuous maximum. The other source of bias is that in a “discrete” excursion, the continuous path might sometimes go below the discrete minimum; thus, our estimate of excursion length overestimates the actual continuous excursion length. Therefore, in the expression for the normalized excursion height, the discrete estimator underestimates the numerator and overestimates the denominator, leading to underestimation.

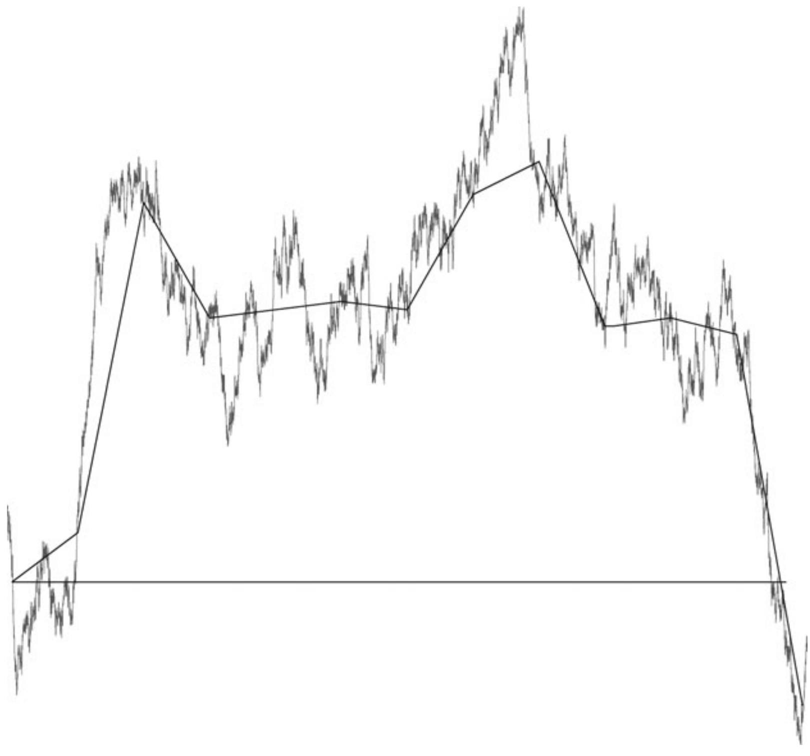

FIGURE 1. Comparison of discrete and continuous excursions.

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In order to reduce the first source of bias, note that for a Brownian motion B,

$$E\left(\min_{i\le k} B(i/k) - \min_{0\le t\le 1} B(t)\right) = \frac{\zeta(1/2)}{\sqrt{2\pi k}} + o(k^{-1/2}) \approx \frac{0.5826}{\sqrt{k}} + o(k^{-1/2}),$$

where ζ is Riemann's zeta function; see Calvin [2]. To account for the underestimation of the height, for each excursion of length k, completed at time n, we add

$$\sigma_n \frac{0.5826}{\sqrt{k}} \quad \textbf{(6)}$$

to the height of the excursion.

To reduce the second source of bias, we proceed as follows. At each simulation step, simulate the continuous minimum of an interpolating Brownian motion with variance σ_n. If the continuous minimum sets a new record low, use it in place of the discrete minimum.

More precisely, if we have simulated to time n, we generate the continuous minimum Z over the interval $[n-1, n]$ according to the distribution

$$P(Z < z) = \exp(-2\sigma_n(A(n-1) - z)(A(n) - z))$$

for $z < \min(A(n-1), A(n))$. If Z is smaller than the current record minimum, we replace the minimum with Z. For computational simplicity, we take the location where Z was attained to be $n - 1/2$.

Combining the two corrections results in a reduction in the bias by a factor of approximately 5 in the numerical experiments. The first correction requires a negligible increase in computation, whereas the second correction increases running time proportional to the length of the simulation.

3.2. Batching

A natural alternative to the method described in the previous subsection is to use the increments over a fixed grid. Suppose that we store $\{X_n(i/m) : i = 0, 1, 2, \ldots, m\}$ for some fixed m and define

$$\sigma_{b,m} \triangleq \left(\sum_{i=1}^{m} (X_n(i/m) - X_n((i-1)/m))^2\right)^{1/2}.$$

Due to the functional central limit theorem assumption and the continuous mapping theorem, for fixed m we have

$$\sigma_{b,m} \xrightarrow{d} \sigma \left(\frac{\chi_m^2}{m}\right)^{1/2}$$

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as $n \to \infty$, where χ^2_m is a chi-square random variable with m degrees of freedom. If $\sigma_{b,m}$ is uniformly integrable, then in the limit we have

$$E\sigma_{b,m} \approx \sigma \left(\frac{2}{m}\right)^{1/2} \frac{\Gamma((m+1)/2)}{\Gamma(m/2)}, \qquad E\sigma^2_{b,m} \approx \sigma^2,$$

and

$$\mathrm{var}\left(\sigma_{b,m}\right) \approx \sigma^2 \left(1 - \frac{2}{m} \frac{\Gamma^2((m+1)/2)}{\Gamma^2(m/2)}\right).$$

Comparing this with the variance of the height of a standard excursion, the variance of the batch estimator with $m = 6$ is about the same as for a single excursion.

As n increases, one could "forget" some of the values (combine adjacent batches) to maintain a memory of fixed size m or to limit the growth of the memory required. Thus, we can construct a method for which the run length would not need to be known in advance, although it would be cumbersome.

3.3. Numerical Experiments

For the numerical experiments we used the following first-order autoregressive model. For $0 < \varphi < 1$, let $Y_0 \sim N(0, 1)$, and for $i \geq 1$, set

$$Y_i = \varphi Y_{i-1} + \epsilon_i,$$

with the $\epsilon_i \sim N(0, 1 - \varphi^2)$ independent and independent of Y_0. We set the parameter $\varphi = 1/2$, which results in $\sigma^2 = 3$; see Chien, Goldsman and Melamed [4]. In order to center the output, we ran two independent simulations for each of the 10^3 independent replications, each of which was stopped after $n = 10^5$ transitions. We chose $\epsilon = 10^{-5/2}$, which resulted in an average of 21.2 excursions per replication.

Figure 2 shows the mean-squared error for the excursion scheme with this choice of ϵ and the batching scheme with the number of batches varied from 2 up to 200. The curves cross at around 115, so to obtain the same mean-squared error with on average 21.2 excursions requires around 115 batches. Figures 3 and 4 break down the mean-squared error into the standard deviation and bias. Figure 5 reveals that the bias in the excursion estimator is greater than that of the batch-means estimator even for 20 batches.

We repeated the experiments using the bias-reduction methods described in Section 3.1. Figure 5 shows that the mean-squared error with the bias correction is improved, so that the curves cross at about 140 batches instead of at 115. Figure 7 shows that with the correction terms the bias of the excursion estimator is comparable to that of the batch estimator with the number of batches equal to the average number of excursions.

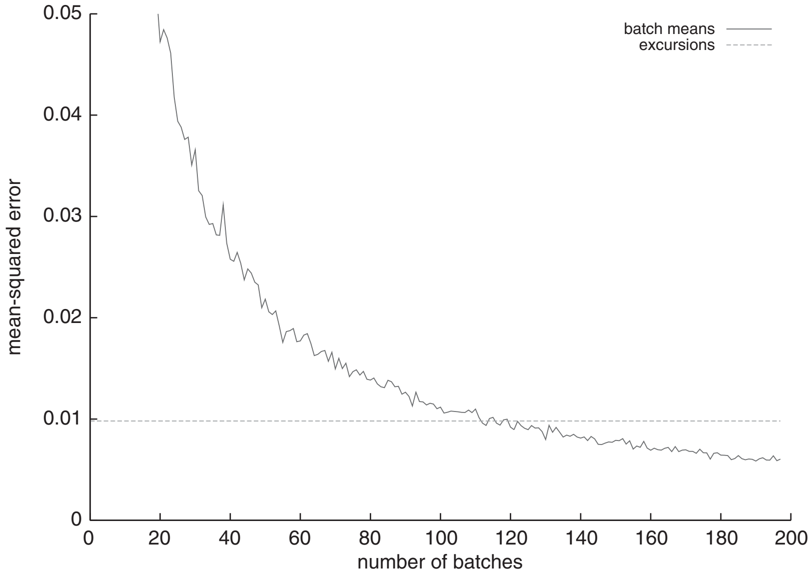

FIGURE 2. Comparison of mean-squared error for batching and excursions.

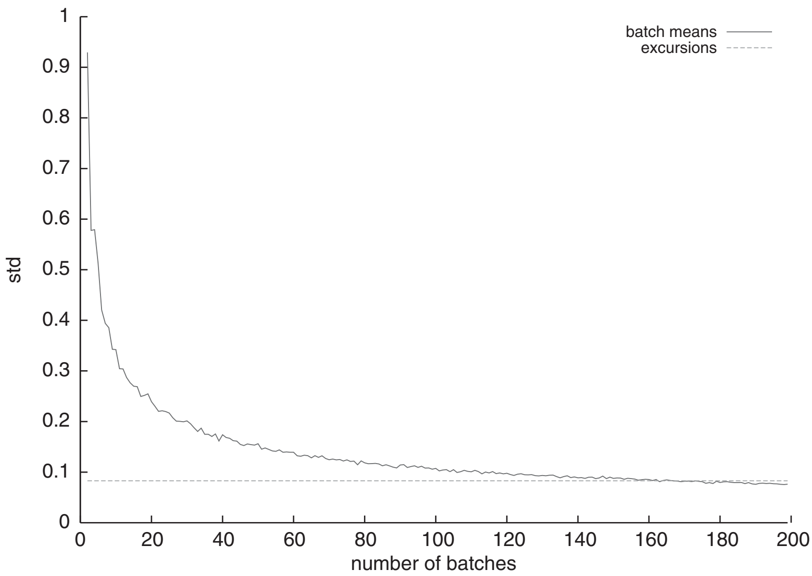

FIGURE 3. Comparison of standard deviation for batching and excursions.

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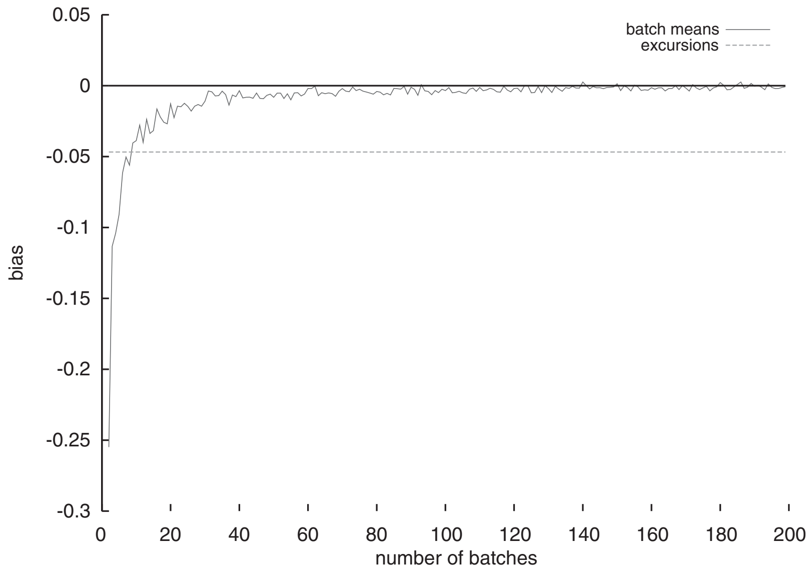

FIGURE 4. Comparison of bias for batching and excursions.

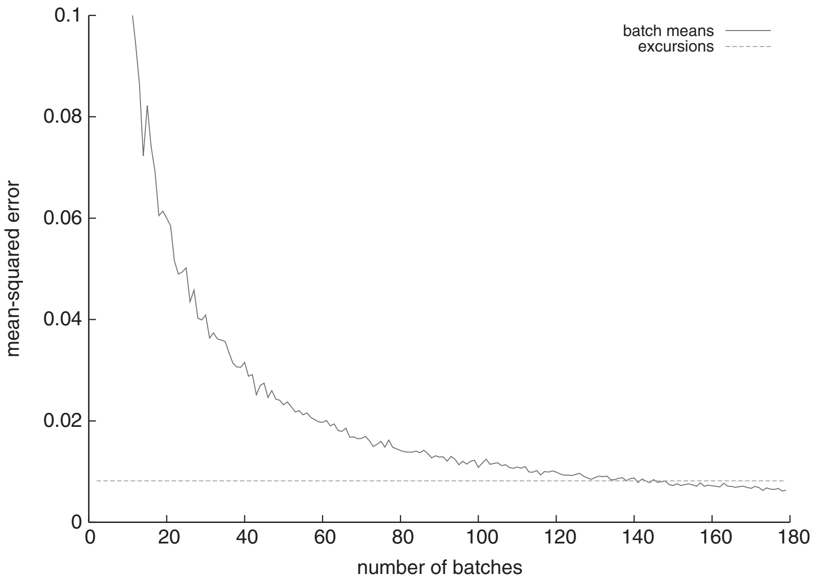

FIGURE 5. Comparison of mean-squared error for batching and excursions with the bias correction.

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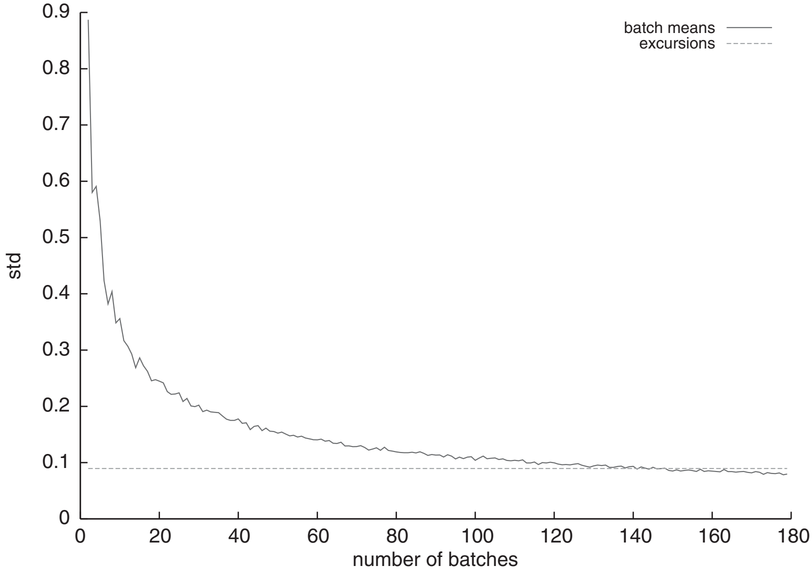

FIGURE 6. Comparison of standard deviation for batching and excursions with the bias correction.

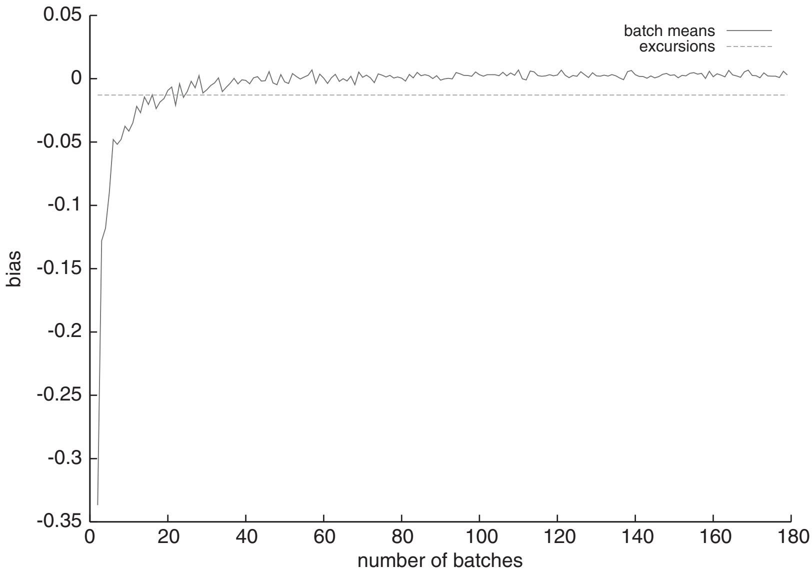

FIGURE 7. Comparison of bias for batching and excursions with the bias correction.

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4. STANDARDIZED TIME SERIES

Estimators based on excursions can be used in the context of standardized time series, or cancellation methods; these methods are described in Schruben [14] and Glynn and Iglehart [7]. In this section we illustrate the approach with one example. A preliminary version of this approach appeared in Calvin [3].

Let $C_0 = \{x \in C([0,1]) : x(0) = 0\}$ and $C_0^* = \{x \in C_0 : \min_{0 \le s \le 1} x(s) < 0 < \max_{0 \le s \le 1} x(s)\}$. For $x \in C_0^*$, define

$$\tau(x) = \sup\{s \le 1 : x(s) = 0 \text{ and } x \text{ changes sign at } s\},$$

$$R(x) = \frac{\max_{0 \le s \le \tau(x)} x(s) - \min_{0 \le s \le \tau(x)} x(s)}{\sqrt{\tau(x)}},$$

$$U(x) = \frac{|x(1)|}{\sqrt{1 - \tau(x)}}.$$

For $x \in C_0 \setminus C_0^*$, set $\tau(x) = R(x) = U(x) = 0$.

THEOREM 4.1: *As $n \to \infty$,*

$$(R(X_n), U(X_n)) \xrightarrow{d} \sigma \cdot (H, M(1)),$$

where H and M are independent, H is the maximum of a standard Brownian excursion, and M is a standard Brownian meander.

PROOF: The process

$$\overline{B}(t) \equiv B(t\tau(B)) / \sqrt{\tau(B)}, \qquad 0 \le t \le 1,$$

is a *Brownian bridge* (roughly, Brownian motion conditioned on the value at time 1 being zero). By a theorem of Vervaat (see Vervaat [15]), the range of a Brownian bridge has the same distribution as the height of a standard Brownian excursion; thus, $R(B) \stackrel{d}{=} H$.

The maps τ, R, and U are continuous on C_0^* and $P(C_0^*) = 1$. Therefore, by the continuous mapping theorem (Billingsley [1]) and the functional central limit theorem assumption,

$$(R(X_n), U(X_n)) \xrightarrow{d} (R(\sigma B), U(\sigma B)) \stackrel{d}{=} \sigma \cdot (H, M(1)).$$

■

Let B^1 and B^2 be independent standard Brownian motions.

LEMMA 4.2: *As $n \to \infty$,*

$$\left(\frac{X_n^1 + X_n^2}{\sqrt{2}\sigma}, \frac{X_n^1 - X_n^2}{\sqrt{2}\sigma} \right) \xrightarrow{d} (B^1, B^2).$$

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PROOF: By the continuous mapping theorem,

$$\left(\frac{X_n^1 + X_n^2}{\sqrt{2}\sigma}, \frac{X_n^1 - X_n^2}{\sqrt{2}\sigma}\right) \xrightarrow{d} \left(\frac{B^1 + B^2}{\sqrt{2}}, \frac{B^1 - B^2}{\sqrt{2}}\right) \stackrel{d}{=} (B^1, B^2).$$

The last equality is due to the rotational invariance of the Gaussian distribution. ■

The following theorem is the basis for constructing asymptotically valid confidence intervals for μ.

THEOREM 4.3: *As $n \to \infty$,*

$$\frac{\sqrt{n}(\mu_n - \mu)}{(R(X_n^1 - X_n^2)^2 + U(X_n^1 - X_n^2)^2/2)^{1/2}} \xrightarrow{d} \frac{N}{(H^2 + \gamma)^{1/2}}, \tag{7}$$

where $N \sim N(0,1)$, H is the height of a standard Brownian excursion, γ is an exponential random variable with mean 1, and N, γ, and H are independent.

PROOF: For $x, y \in C_0^*$, define

$$G(x,y) = \frac{x(1)}{2}(R(y)^2 + U(y)^2/2)^{-1/2}.$$

Since G is continuous on $C_0^* \times C_0^*$ and $P(B^1 \in C_0^*, B^2 \in C_0^*) = 1$,

$$\frac{\sqrt{n}(\mu_n - \mu)}{(R(X_n^1 - X_n^2)^2 + U(X_n^1 - X_n^2)^2/2^{1/2}} = G(X_n^1 + X_n^2, X_n^1 - X_n^2) \xrightarrow{d} G(B^1, B^2)$$

by Lemma 4.2. However,

$$G(B^1, B^2) \stackrel{d}{=} \frac{N}{(H^2 + \gamma)^{1/2}}.$$

■

We derive the cumulative distribution function of the limiting random variable appearing on the right-hand side of (7) by conditioning on N and γ. For $z > 0$,

$$P\left(\frac{N}{\sqrt{H^2 + \gamma}} \le z\right) = \int_{x=-\infty}^{\infty} \int_{y=0}^{\infty} P\left(\frac{x}{\sqrt{H^2 + y}} \le z\right) \frac{e^{-x^2/2}}{\sqrt{2\pi}} e^{-y}\, dy\, dx.$$

The probability in the integrand is 1 for $x \le 0$, and for $x > 0, y \ge (x/z)^2$, and so

$$P\left(\frac{N}{\sqrt{H^2 + \gamma}} \le z\right) = \frac{1}{2} + \frac{1}{2\sqrt{1 + 2/z^2}} + \int_{x=0}^{\infty} \int_{y=0}^{(x/z)^2} P\left(H \ge \sqrt{(x/z)^2 - y}\right) \frac{e^{-x^2/2}}{\sqrt{2\pi}} e^{-y}\, dy\, dx.$$

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By an alternative formula for H (see [5, Thm. 7]),

$$P(H > b) = 2\sum_{k=1}^{\infty}\left(4k^2b^2 - 1\right)\exp(-2k^2b^2).$$

Then

$$\int_{x=0}^{\infty}\int_{y=0}^{(x/z)^2} P\left(H \geq \sqrt{(x/z)^2 - y}\right)\frac{e^{-x^2/2}}{\sqrt{2\pi}}e^{-y}\,dy\,dx$$
$$= \int_{x=0}^{\infty}\int_{y=0}^{(x/z)^2} 2\sum_{k=1}^{\infty}\left(4k^2[(x/z)^2 - y] - 1\right)\exp\left(-2k^2[(x/z)^2 - y]\right)$$
$$\times \frac{e^{-x^2/2}}{\sqrt{2\pi}}e^{-y}\,dy\,dx$$
$$= 2\int_{x=0}^{\infty}\int_{w=0}^{(x/z)^2}\sum_{k=1}^{\infty}\left(4k^2w - 1\right)\exp\left(-2k^2w\right)\frac{e^{-x^2/2}}{\sqrt{2\pi}}e^{-(x/z)^2}e^{w}\,dw\,dx,$$

where we have made the substitution $w = x/z - y$. We can interchange the order of integration and summation, since

$$\int_{x=0}^{\infty}\int_{w=0}^{(x/z)^2}\sum_{k=1}^{\infty}(4k^2w + 1)\exp(-2k^2w)\frac{e^{-x^2/2}}{\sqrt{2\pi}}e^{-(x/z)^2}e^{w}\,dw\,dx < \infty.$$

Therefore,

$$2\int_{x=0}^{\infty}\int_{w=0}^{(x/z)^2}\sum_{k=1}^{\infty}\left(4k^2w - 1\right)\exp\left(-2k^2w\right)\frac{e^{-x^2/2}}{\sqrt{2\pi}}e^{-(x/z)^2}e^{w}\,dw\,dx$$
$$= 2\sum_{k=1}^{\infty}\int_{x=0}^{\infty}\int_{w=0}^{(x/z)^2}\left(4k^2w - 1\right)\exp\left(-2k^2w\right)\frac{e^{-x^2/2}}{\sqrt{2\pi}}e^{-(x/z)^2}e^{w}dw\,dx$$
$$= 2\sum_{k=1}^{\infty}\int_{x=0}^{\infty}\frac{\exp(-(x^2/2)(1 + 2/z^2))}{\sqrt{2\pi}}\int_{w=0}^{(x/z)^2}\left(4k^2w - 1\right)$$
$$\times \exp\left(-w(2k^2 - 1)\right)dw\,dx$$
$$= 2\sum_{k=1}^{\infty}\int_{x=0}^{\infty}\frac{\exp\left(-(x^2/2)(1 + 2/z^2)\right)}{\sqrt{2\pi}}$$
$$\left(\frac{1 + 2k^2}{(1 - 2k^2)^2} - \left(\frac{1 + 2k^2}{(1 - 2k^2)^2} - \frac{4k^2}{1 - 2k^2}(x/z)^2\right)\right.$$
$$\times \exp\left((1 - 2k^2)(x/z)^2\right)\,dx$$
$$= \sum_{k=1}^{\infty}\frac{1 + 2k^2}{(1 - 2k^2)^2}\frac{1}{\sqrt{1 + 2/z^2}} - 2\sum_{k=1}^{\infty}\int_{x=0}^{\infty}\frac{\exp\left(-(x^2/2)(1 + 4k^2/z^2)\right)}{\sqrt{2\pi}}$$

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$$\times \left(\frac{1 + 2k^2 + 4k^2(2k^2 - 1)(x/z)^2}{(1 - 2k^2)^2} \right) dx$$

$$= \sum_{k=1}^{\infty} \frac{1 + 2k^2}{(1 - 2k^2)^2} \frac{1}{\sqrt{1 + 2/z^2}} - \sum_{k=1}^{\infty} \frac{1 + 2k^2}{(2k^2 - 1)^2} \frac{1}{\sqrt{1 + 4k^2/z^2}} - \sum_{k=1}^{\infty} \frac{4k^2}{z^2(2k^2 - 1)} \frac{1}{(1 + 4k^2/z^2)^{3/2}}.$$

Combining these expressions, we obtain

$$P\left(\frac{N}{\sqrt{H^2 + \gamma}} \leq z \right) = \frac{1}{2} + \frac{1}{2\sqrt{1 + 2/z^2}} + \frac{1}{\sqrt{1 + 2/z^2}} \sum_{k=1}^{\infty} \frac{1 + 2k^2}{(2k^2 - 1)^2} - \sum_{k=1}^{\infty} \frac{1 + 2k^2}{(2k^2 - 1)^2} \frac{1}{\sqrt{1 + 4k^2/z^2}} - \sum_{k=1}^{\infty} \frac{4k^2}{z^2(2k^2 - 1)} \frac{1}{(1 + 4k^2/z^2)^{3/2}}. \tag{8}$$

Confidence intervals can be constructed based on quantiles computed numerically from this formula. The FindRoot procedure in Mathematica gave the 95th percentile as approximately 1.11262.

We used the Ehrenfest urn model with nine states for a numerical experiment. The number of transitions was $n = 10^5$, and we constructed 95% confidence intervals based on 10^4 independent replications. The bias correction terms described in Section 3.1 were incorporated into the estimator (with twice the term defined in (6) added to the range component). To avoid numerical instability, in cases where the last zero crossing takes place prior to time $0.05n$, we consider only the meander part of the estimator defined in Theorem 4.3, and, similarly, if the last zero occurs after time $0.95n$, we use only the excursion part of the estimator (as described in [3]). The quantiles for the estimator based on only the meander are computed from the distribution of $N/\sqrt{\gamma}$, which has the distribution

$$P\left(\frac{N}{\sqrt{\gamma}} > z \right) = \frac{1}{2} \left(1 - \frac{1}{\sqrt{1 + 2/z^2}} \right)$$

for $z > 0$, which can be established by a calculation similar to the one used to derive (8). The 95th quantile is $\sqrt{162/19} \approx 2.91998$. Table 1 shows the results of comparing coverage and confidence interval half-width mean and variance for the excursion estimator and batch-means with one, two, four, and five batches. Compared with the batch-means estimator with two batches, the excursion confidence intervals have coverage much closer to the nominal level and have narrower and less variable confidence intervals.

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TABLE 1. 95% Confidence Interval Characteristics for Excursions and Batch Means

	Coverage	Avg. Half-width	Var. Half-width
Excursion	0.9528	0.252745	0.008468
1 batch	0.9309	0.904173	0.474060
2 batches	0.9028	0.326427	0.030561
4 batches	0.7952	0.178619	0.004677
5 batches	0.7382	0.152188	0.002840

An alternative approach explored in [3] is based on the limit theorem that has the same form as (7) but without the U and γ in the denominators; that is, the meander after the last zero crossing is ignored. This estimator suffers from the problem mentioned earlier that the last zero crossing can be near the origin. Additionally, the variability is greater since the estimator is based on less of the path.

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References

1. Billingsley, P. (1968). *Convergence of probability measures*. New York: Wiley.
2. Calvin, J. (1995). Average performance of passive algorithms for global optimization. *Journal of Mathematical Analysis and Applications* 191: 608–617.
3. Calvin, J.M. (2004) Simulation output analysis based on excursions. In R.G. Ingalls, M.D. Rosetti, J.S. Smith, & B.A. Peters (eds.), *Proceedings of the 2004 winter simulation conference*. (Piscataway, NJ), Institute of Electrical and Electronics Engineers, pp. 681–684.
4. Chien, C., Goldsman, D., & Melamed, B. (1997). Large-sample results for batch means. *Management Science* 43: 1288–1295.
5. Chung, K.L. (1976). Excursions in Brownian motion. *Arkiv för Matematik* 14: 155–177.
6. Futschik, A. & Pflug, G. (1997). Optimal allocation of simulation experiments in discrete stochastic optimization and approximative algorithms. *European Journal of Operational Research* 101: 245–260.
7. Glynn, P.W. & Iglehart, D.L. (1990). Simulation output analysis using standardized time series. *Mathematics of Operations Research* 14: 1–16.
8. Goldsman, D. & Schmeiser, B.W. (1997). Computational efficiency of batching methods. In S. Andradóttir, K.J. Healy, D.H. Withers & B.L. Nelson (eds.), *Proceedings of the 1997 winter simulation conference*. Piscataway, NJ: Institute of Electrical and Electronics Engineers, pp. 202–207.
9. Karatzas, I. & Shreve, S.E. (2000). *Brownian motion and stochastic calculus*, 2nd ed. New York: Springer-Verlag.
10. Kim, S.-H. & Nelson, B.L. (2006). On the asymptotic validity of fully sequential selection procedures for steady-state simulation. *Operations Research* 54: 475–488.
11. Meketon, M.S. & Schmeiser, B.W. (1984). Overlapping batch means: Something for nothing? In S. Sheppard, U. Pooch & D. Pegden (eds.), *Proceedings of the 1984 winter simulation conference*. Piscataway, NJ: Institute of Electrical and Electronics Engineers, pp. 227–230.
12. Revuz, D. & Yor, M. (1994). *Continuous martingales and brownian motion*. Berlin: Springer-Verlag.
13. Rogers, L.C.G. & Williams, D. (1994). *Diffusions, Markov processes and martingales: Vol. 1: Foundations*. Cambridge: Cambridge University Press.

https://doi.org/10.1017/S0269964809990234 Published online by Cambridge University Press

14. Schruben, L.W. (1983). Confidence interval estimation using standardized time series. *Operations Research*, Volume 31, pp. 1090–1108.
15. Vervaat, W. (1979). A relation between Brownian bridge and Brownian excursion. *Annals of Probability*, Volume 7, pp. 141–149.
16. Yeh, Y. & Schmeiser, B.W. (2000). Simulation output analysis via dynamic batch means. In J.A. Joines, R.R. Barton, K. Kang & P.A. Fishwick (eds.), *Proceedings of the 2000 winter simulation conference*. Piscataway, NJ: Institute of Electrical and Electronics Engineers, pp. 637–645.