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A quasi-linear model of particle, momentum and heat transport in a stochastic magnetic field

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We present a quasi-linear treatment of the drift-kinetic equation in the presence of a stochastic magnetic field, which provides a self-contained description of particle, parallel momentum and heat transport. Explicit analytical expressions, which satisfy the Onsager reciprocal relations, are obtained by approximating the distribution function by a local shifted Maxwellian. This theory completes previous formulations (Harvey *et al.*, *Phys. Rev. Lett.*, vol. 47, 1981, p. 102) by including the momentum transport and by generalizing the derivation from the cylindrical tokamak configuration to an arbitrary cylindrical pinch. Application to the reversed field pinch provides satisfactory results.

Key words: plasma nonlinear phenomena, plasma instabilities, plasma dynamics

1. Introduction

The purpose of this paper is to present an integrated analytical model of particle, parallel momentum and heat transport in magnetized plasmas, in the presence of a stochastic magnetic field produced by resonant perturbations. Particle and energy transport have been already studied in such conditions by adopting the quasi-linear approximation of the drift-kinetic equation (Harvey *et al.* 1981; Rice, Molvig & Helava 1982; Predebon & Paccagnella 2007). In particular Harvey *et al.* (1981) and Rice *et al.* (1982) derive the ambipolar electric field, which develops in order to equalize the electrons' and ions' particle fluxes. Later work (Predebon & Paccagnella 2007) completes the analysis by including the collisional effects. All these studies, made in slab or cylindrical geometry, implicitly adopt the tokamak ordering of the magnetic field components, which allows some simplifications in the derivation of the transport equations. Following the same line of the previous work, the present paper develops some new elements. First, no ordering assumption is made for the magnetic field components: this gives rise to extra terms in the transport equations, whose relevance is assessed. Second, we also consider the parallel momentum transport, an issue not addressed in the above-mentioned works, showing that it is described in terms of diffusion and convection, like the particle and heat fluxes. The transport equations are obtained by approximating the distribution function by a local shifted Maxwellian. We find that the collisional effects addressed in Predebon & Paccagnella (2007) must be included to avoid singularities in some expressions related to the momentum transport. The coefficient regulating the momentum flux is similar to that obtained in Finn, Guzdar & Chernikov (1992) with a different approach. The inclusion of

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momentum transport enriches the particle and heat equations by further terms (for example the viscous heating) not considered in the above-mentioned works. The expressions for the particle, momentum and heat fluxes satisfy the Onsager reciprocal relations (Onsager 1931), which are a general property of the transport phenomena.

Cylindrical geometry is adopted, with periodicity $2\pi R$ in the z direction, and simulated toroidal angle $\phi = z/R$. Hence, the coordinates are (r, θ, ϕ). The minor radius of the device, location of the material wall, is at $r = a$.

The paper is organized as follows. In § 2 we present the basis of our transport model: the kinetic diffusion equation obtained from the quasi-linear approximation of the drift-kinetic equation. In § 3 we take velocity moments of the kinetic diffusion equation. Then, by assuming a local shifted Maxwellian for the distribution function, we derive explicit analytical expressions for particle, momentum and heat transport in § 4. In § 5 we discuss a stationary solution of the above equation for the reversed field pinch (RFP) configuration. In § 6 we compare the present model with previous works on the subject. Conclusions are drawn in § 7. Appendix A presents the details of the derivation of the kinetic diffusion equation.

2. Kinetic diffusion equation

The starting point is the drift-kinetic equation for the gyro-averaged distribution function of particles of species j (Hinton & Hazeltine 1976):

$$\partial f_j/\partial t + v_\parallel \boldsymbol{b}\cdot\nabla f_j + Z_j e v_\parallel \boldsymbol{b}\cdot\boldsymbol{E}_{\rm in}\partial f_j/\partial\varepsilon = C_j, \tag{2.1}$$

with $f_j = f_j(r, \theta, \phi, \varepsilon, \mu)$, $\varepsilon = \frac{1}{2}m_j v^2 + Z_j e\Phi$ the total energy, m_j the mass, $v^2 = {v_\parallel}^2 + v_\perp^2$, $v_\parallel$, $v_\perp$ the velocities of the parallel and gyration motion respectively, $Z_j e$ the electric charge, $e > 0$ the elementary charge, Φ the electrostatic potential, $\mu = m_j v_\perp^2/(2B)$ the magnetic moment, $\boldsymbol{b} = \boldsymbol{B}/B$ the magnetic unit vector, $\boldsymbol{E}_{\rm in}$ the externally applied inductive electric field and C_j the collision term. From now on we drop the subscript j, if not strictly necessary. As in Harvey *et al.* (1981) and Rice *et al.* (1982), we neglect the drift velocity perpendicular to the magnetic field and we assume both the magnetic field (including its perturbation) and the electrostatic potential constant in time ($\partial B/\partial t = \partial\Phi/\partial t = 0$). The plasma here considered has destroyed magnetic surfaces due to the overlapping of several resonant magnetic perturbations. The total magnetic field is written as the sum of the equilibrium $\boldsymbol{B}_0$ and the perturbation $\boldsymbol{B}_1$:

$$\boldsymbol{B} = \boldsymbol{B}_0(r) + \boldsymbol{B}_1(r, \theta, \phi), \quad B_{0,r} = 0, \quad \nabla\cdot\boldsymbol{B} = \nabla\cdot\boldsymbol{B}_0 = \nabla\cdot\boldsymbol{B}_1 = 0. \tag{2.2a–c}$$

We take the ratio $\delta = B_1/B_0$ to be a small parameter. It is also useful to split the magnetic field unit vector $\boldsymbol{b}$ into the equilibrium $\boldsymbol{b}_0 = \boldsymbol{B}_0/B_0$ and perturbation components:

$$\boldsymbol{b} = \boldsymbol{b}_0 + \tilde{\boldsymbol{b}}, \quad \tilde{\boldsymbol{b}} = \boldsymbol{B}_1/B_0 - (\boldsymbol{B}_0\cdot\boldsymbol{B}_1/B_0^2)\boldsymbol{b}_0 + o(\delta^2). \tag{2.3a,b}$$

In the presence of a stochastic magnetic field, the parallel gradient $v_\parallel\boldsymbol{b}\cdot\nabla f$ of (2.1) gives rise to a diffusive term for the distribution function, stemming from quasi-linear approximation (Harvey *et al.* 1981; Rice *et al.* 1982). To prove it, let us write the distribution function as

$$f = \langle f\rangle(r) + \tilde{f}(r, \theta, \phi), \tag{2.4}$$

where $\langle\rangle$ denotes the average over the angles θ, ϕ and $\tilde{f} \sim o(\delta)\langle f\rangle$ is the perturbation component. Then, we approximate (2.1) by the following two equations:

$$\partial\tilde{f}/\partial t + v_{\parallel}\boldsymbol{b}_0\cdot\nabla\tilde{f} + v_{\parallel}\tilde{\boldsymbol{b}}\cdot\nabla\langle f\rangle \cong 0, \tag{2.5}$$

$$\partial\langle f\rangle/\partial t + v_{\parallel}\langle\tilde{\boldsymbol{b}}\cdot\nabla\tilde{f}\rangle + Zev_{\parallel}\boldsymbol{b}_0\cdot\boldsymbol{E}_{\rm in}\partial\langle f\rangle/\partial\varepsilon \cong C(\langle f\rangle). \tag{2.6}$$

All the terms in (2.5) are of order $(v_{\parallel}/a)o(\delta)\langle f\rangle$. All the terms in (2.6) are assumed to be of higher order: $(v_{\parallel}/a)o(\delta^2)\langle f\rangle$. As far as $\partial\langle f\rangle/\partial t$ and $C(\langle f\rangle)$ are concerned, this is justified in conditions close to an equilibrium described by a Maxwellian distribution. As far as the $\boldsymbol{E}_{\rm in}$ term is concerned, the underlying idea is that the electric field balances the diffusion $v_{\parallel}\langle\tilde{\boldsymbol{b}}\cdot\nabla\tilde{f}\rangle$ and the collisions. Equation (2.5) is time-integrated along the characteristics, given by the field line of the equilibrium magnetic field $\boldsymbol{b}_0$. Since collisions are not considered in (2.5), this case corresponds to the collision-less solution of $\tilde{f}$. The collisional correction is introduced heuristically in a second step. By inserting the $\tilde{f}$ solution into the term $v_{\parallel}\langle\tilde{\boldsymbol{b}}\cdot\nabla\tilde{f}\rangle$ of (2.6), one gets the following kinetic diffusion equation for $\langle f\rangle$:

$$\left.\begin{aligned}\partial\langle f\rangle/\partial t \cong |v_{\parallel}|\frac{1}{r}\mathcal{L}[rD_0\mathcal{L}\langle f\rangle] + |v_{\parallel}|L_0^{-1}D_0\mathcal{L}\langle f\rangle - Zev_{\parallel}\boldsymbol{b}_0\cdot\boldsymbol{E}_{\rm in}\partial\langle f\rangle/\partial\varepsilon + C(\langle f\rangle),\\ \mathcal{L} = \left(\frac{\partial}{\partial r}\right)_{\varepsilon,\mu}, \quad L_0 = B_0/(\mathrm{d}B_0/\mathrm{d}r).\end{aligned}\right\} \tag{2.7}$$

Details of the derivation are given in Appendix A. The first two terms on the right-hand side stem from $v_{\parallel}\langle\tilde{\boldsymbol{b}}\cdot\nabla\tilde{f}\rangle$, and they describe radial diffusion processes due to the stochastic magnetic field. The coefficient D_0 is the magnetic diffusivity, which relates the average squared radial displacement $\langle(\Delta r)^2\rangle$ of the field lines to the distance l covered along the lines: $\langle(\Delta r)^2\rangle = 2D_0 l$ (Rechester & Rosenbluth 1978; D'Angelo & Paccagnella 1996). By defining the Fourier expansion $\tilde{\boldsymbol{b}} = \sum_{m,n\in\mathbb{Z}}\tilde{\boldsymbol{b}}^{m,n}\,\mathrm{e}^{\mathrm{i}(m\theta-n\phi)}$, D_0 is expressed in terms of the harmonics $\tilde{b}_r^{m,n}$:

$$D_0(r) = \sum_{\substack{m\,\in,\,\mathbb{Z}\\ n>0}} 2\pi\delta(m-nq)|\tilde{b}_r^{m,n}|^2\left|\frac{Rq}{b_{0,\phi}}\right|, \quad q(r) = rB_{0,\phi}/(RB_{0,\theta}). \tag{2.8a,b}$$

Only the resonant harmonics, i.e. those satisfying the condition $q(r^{m,n}) = m/n$ inside the plasma ($r^{m,n} < a$), give a contribution to D_0. Note that the sum runs over half of the harmonics: as shown in Appendix A, the delta-function includes the contribution of the complex-conjugate harmonics (i.e. the complementary half $m\in\mathbb{Z}$, $n<0$). The δ function in (2.8) derives from integrating (2.5) along the unperturbed magnetic field line, and it is a typical feature of the quasi-linear theory. Such a singular function can be smoothed by exploiting nonlinear resonance broadening techniques (Ishihara, Xia & Hirose 1992), which instead consider the effect of the perturbed orbits. The development of such a method for our specific case is beyond the scope of the present paper, and it is left as future refinement of the work.

The term $|v_{\parallel}|L_0^{-1}D_0\mathcal{L}\langle f\rangle$ of (2.7) is not present in previous derivations (Harvey *et al.* 1981; Rice *et al.* 1982), where the tokamak ordering there assumed makes it negligible: $L_0^{-1} = a^{-1}o(\epsilon^2)$, $\epsilon = a/R$, with ϵ small. Nonetheless, this term cannot be discarded *a priori*, since $L_0^{-1}\sim a^{-1}$ for a general pinch.

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Expression (2.8) corresponds to the collision-less case, since it derives from the $\tilde{f}$ obtained in this limit. Collisions hamper the free motion of the particles along the field lines, thereby reducing the stochastic transport. Following Predebon & Paccagnella (2007), we include their effect by an *ad hoc* factor in front of D_0:

$$D_j = g_j(v_\parallel)D_0, \quad g_j(v_\parallel) = \frac{1}{1 + L_K/\lambda_j(v_\parallel)}, \quad \lambda_j(v_\parallel) = \frac{v_\parallel}{\nu_{0j}}, \quad \nu_{0j} = \frac{Z_j^2 e^4 \ln(\Lambda) n_i}{4\pi \varepsilon_0^2 m_j^2 v_\parallel^3}, \tag{2.9a–d}$$

where $\lambda_j(v_\parallel)$ is the mean free path of a test particle j in a hydrogen background Maxwellian plasma whose main ion density is n_i, computed by the fundamental collision frequency ν_{0j} (Trubnikov 1965) (the actual frequencies of the various relaxation processes involve extra coefficients); $\ln(\Lambda)$ is the Coulomb logarithm; and L_K is the Kolmogorov–Lyapunov length (Rechester, Rosenbluth & White 1979; D'Angelo & Paccagnella 1996), which is the characteristic length for the exponential divergence of close magnetic field lines. The collision-less limit $L_K/\lambda_j \to 0$ corresponds to (2.8). In the opposite limit $L_K/\lambda_j \to +\infty$ (as occurs for particles with $v_\parallel \to 0$), collisions are so frequent that particles do not experience the stochastic nature of the magnetic field: the vanishing of expression (2.9) for D in this limit means that other transport mechanisms take over the stochastic one. Actually, L_K should be corrected by a logarithmic factor involving further characteristic lengths (Rechester & Rosenbluth 1978; Krommes, Oberman & Kleva 1983; Predebon & Paccagnella 2007): these details, not crucial for our purposes, are here omitted. Hereafter, we replace D_0 with D in (2.7).

A solution of (2.7) for $\langle f \rangle$ is not attempted here. Instead, we take velocity moments of (2.7), and then we convert them into transport equations for macroscopic plasma quantities by approximating $\langle f \rangle$ with a local shifted Maxwellian.

3. Velocity moments of the kinetic diffusion equation

In order to take the velocity moments of (2.7), we perform the variable change $(\varepsilon, \mu) \to (v_\parallel, v_\perp)$ by

$$v_\parallel = \pm\sqrt{\frac{2}{m}(\varepsilon - \mu B - Ze\Phi)}, \quad v_\perp = \sqrt{\frac{2}{m}\mu B}. \tag{3.1a,b}$$

Moreover, we add a source term of particles coming from the wall with zero velocity. Equation (2.7) for $\langle f \rangle(r, v_\parallel, v_\perp)$ becomes

$$\begin{aligned}\partial\langle f\rangle/\partial t = |v_\parallel|\frac{1}{r}\mathcal{L}[r D\mathcal{L}\langle f\rangle] + |v_\parallel| L_0^{-1} D\mathcal{L}\langle f\rangle - Ze/m\boldsymbol{b}_0 \cdot \boldsymbol{E}_{\text{in}}\partial\langle f\rangle/\partial v_\parallel + C \\ + \delta(v_\parallel)\delta(v_\perp)/(2\pi v_\perp)\, S(r),\end{aligned} \tag{3.2}$$

with

$$\mathcal{L} = \left(\frac{\partial}{\partial r}\right)_{\varepsilon,\mu} = \left(\frac{\partial}{\partial r}\right)_{v_\parallel, v_\perp} - \frac{\frac{1}{2}m v_\perp^2 L_0^{-1} - ZeE_a}{m v_\parallel}\left(\frac{\partial}{\partial v_\parallel}\right)_{r, v_\perp} + \frac{1}{2}v_\perp L_0^{-1}\left(\frac{\partial}{\partial v_\perp}\right)_{r, v_\parallel}, \tag{3.3}$$

where $S(r)$ is the radial dependence of the particle source. The form of the delta-function factors derives from the cylindrical symmetry in the velocity space introduced by the magnetic field, which gives $\mathrm{d}^3v = 2\pi v_\perp\, \mathrm{d}v_\perp\, \mathrm{d}v_\parallel$, with $v_\parallel \in [-\infty, +\infty]$ in the role of axial coordinate and $v_\perp \in [0, +\infty]$ in the role of radial coordinate. Since $D \sim o(\delta^2)$, in (3.3) we approximate the radial derivative of B by means of the equilibrium component L_0^{-1}, and the radial derivative of the potential by its angular average $E_a = -\langle \partial\Phi/\partial r\rangle$. This radial

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electric field E_a is named ambipolar, because it equalizes the particle fluxes of electrons and ions, as we will see later.

Let $F(v_\parallel, v_\perp)$ be a generic polynomial function of the velocity components. After multiplying (3.2) by F, we integrate it over the velocity space by assuming $\langle f\rangle, \mathcal{L}\langle f\rangle$ to vanish faster than any power of the velocity when $|v_{\parallel,\perp}| \to +\infty$. The integration over $v_\parallel$ can be limited to the interval $[0, +\infty]$ by splitting any function $X(v_\parallel)$ of the parallel velocity into its even $X_+(v_\parallel) = [X(v_\parallel) + X(-v_\parallel)]/2$ and odd $X_-(v_\parallel) = [X(v_\parallel) - X(-v_\parallel)]/2$ components: $X(v_\parallel) = X_+(v_\parallel) + X_-(v_\parallel)$. Since both the operator (3.3) and the collisional factor $g(v_\parallel)$ defined in (2.9) are even in $v_\parallel$, and the latter also vanishes for $v_\parallel \to 0$, the velocity integration can be cast into the following form:

$$\frac{\partial}{\partial t}(\mathcal{F}_+ + \mathcal{F}_-) + \frac{\partial}{r\partial r}[r(\Gamma_+ + \Gamma_-)] = -\frac{ZeE_a}{m}(A_+ + A_-) + \frac{1}{2}L_0^{-1}(B_+ + B_-) + \frac{Ze}{m}\boldsymbol{b}_0 \cdot \boldsymbol{E}_{\text{in}}\text{E} + \bar{C} + F(0,0)S(r), \tag{3.4}$$

$$\mathcal{F}_\pm = 4\pi \int_0^{+\infty} dv_\parallel \int_0^{+\infty} \mathrm{d}v_\perp v_\perp F_\pm \langle f\rangle_\pm, \tag{3.5}$$

$$\Gamma_\pm = -4\pi \int_0^{+\infty} \mathrm{d}v_\parallel v_\parallel \int_0^{+\infty} \mathrm{d}v_\perp v_\perp F_\pm D\mathcal{L}\langle f\rangle_\pm, \tag{3.6}$$

$$A_\pm = 4\pi \int_0^{+\infty} \mathrm{d}v_\parallel \int_0^{+\infty} \mathrm{d}v_\perp v_\perp \left[\frac{\partial}{\partial v_\parallel}F_\pm\right] D\mathcal{L}\langle f\rangle_\pm, \tag{3.7}$$

$$B_\pm = 4\pi \int_0^{+\infty} \mathrm{d}v_\parallel \int_0^{+\infty} \mathrm{d}v_\perp \left[v_\perp^3 \frac{\partial}{\partial v_\parallel}F_\pm - v_\parallel {v_\perp}^2 \frac{\partial}{\partial v_\perp}F_\pm\right] D\mathcal{L}\langle f\rangle_\pm, \tag{3.8}$$

$$E = 4\pi \int_0^{+\infty} \mathrm{d}v_\parallel \int_0^{+\infty} \mathrm{d}v_\perp v_\perp \left[\langle f\rangle_+ \frac{\partial}{\partial v_\parallel}F_- + \langle f\rangle_- \frac{\partial}{\partial v_\parallel}F_+\right], \tag{3.9}$$

$$\bar{C} = 2\pi \int_{+\infty}^{+\infty} \mathrm{d}v_\parallel \int_0^{+\infty} \mathrm{d}v_\perp v_\perp F\, C(\langle f\rangle). \tag{3.10}$$

Note that the original integration interval of $v_\parallel$ is retained in the collision term (3.10).

4. Transport equations

As written above, we do not seek for a solution of (3.2). Instead, according to the ordering assumption previously discussed for (2.6), we approximate $\langle f\rangle$ by a local shifted Maxwellian:

$$\langle f\rangle \cong n\left(\frac{m}{2\pi T}\right)^{3/2} \mathrm{e}^{-m[(v_\parallel - u)^2 + v_\perp^2]/2T}, \tag{4.1}$$

with $n(r), T(r)$ the particle density and temperature (in energy units) and $u(r)$ the parallel drift velocity. All these quantities depend on the particle species. In such a way, the velocity integrals (3.5)–(3.9) can be carried out analytically, as done in previous publications (Harvey *et al.* 1981; Rice *et al.* 1982; Predebon & Paccagnella 2007), and (3.4) gives rise to transport equations for $n(r), T(r), u(r)$. Approximation (4.1) implies that collisions can establish a local thermal equilibrium. Accordingly, their effect has been included in (2.9).

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Denoting $v_{th} = (T/m)^{1/2}$ the thermal velocity, and assuming $\zeta = (u/v_{th})^2 = mu^2/T \ll 1$, we approximate the even/odd components in $v_\parallel$ of $\langle f\rangle$ with the following expansions:

$$\langle f\rangle_+ \cong \left(1 - \frac{1}{2}\zeta + \frac{m\,v_\parallel^2}{2T}\zeta\right) f_0, \quad f_0 = n\left(\frac{m}{2\pi T}\right)^{3/2} \mathrm{e}^{-m\,v^2/2T}, \tag{4.2}$$

$$\langle f\rangle_- \approx \frac{m}{T} u\,v_\parallel f_0 = \zeta^{1/2} v_\parallel / v_{th} f_0. \tag{4.3}$$

Accordingly, in (3.6)–(3.8) we make use of the following relations:

$$\mathcal{L}\langle f\rangle_+ = \left[\mathcal{A} + \zeta\mathcal{B} + \frac{mv^2}{2T}\frac{T'}{T}\left(1 - \frac{1}{2}\zeta\right) + \zeta\left(\frac{m\,v_\parallel^2}{2T}\mathcal{C} + \frac{m\,v_\parallel^2}{2T}\frac{mv^2}{2T}\frac{T'}{T} - \frac{m\,v_\perp^2}{2T}L_0^{-1}\right)\right] f_0, \tag{4.4}$$

$$\mathcal{A} = \frac{n'}{n} - \frac{3}{2}\frac{T'}{T} - \frac{ZeE_a}{T}, \quad \mathcal{B} = -\frac{n'}{2n} + \frac{5}{4}\frac{T'}{T} - \frac{u'}{u} + \frac{3}{2}\frac{ZeE_a}{T},$$
$$\mathcal{C} = \frac{n'}{n} - \frac{7}{2}\frac{T'}{T} - \frac{ZeE_a}{T} + 2\frac{u'}{u}, \tag{4.5a–c}$$

$$\mathcal{L}\langle f\rangle_- = \left(\frac{n'}{n} - \frac{5}{2}\frac{T'}{T} + \frac{mv^2}{2T}\frac{T'}{T} - \frac{ZeE_a}{T} + \frac{u'}{u} - \frac{\frac{1}{2}mv_\perp^2 L_0^{-1} - ZeE_a}{m{v_\parallel}^2}\right)\frac{m}{T}u\,v_\parallel f_0. \tag{4.6}$$

Only terms up to $o(\zeta^{1/2})$ have been retained in Predebon & Paccagnella (2007). Instead, here we push the approximation to $o(\zeta)$ to disclose velocity-related contributions in the particle and heat transport equations. Equation (3.4) is specialized to specific functions $F(v_\parallel, v_\perp)$ in the following subsections.

4.1. *Particle transport*

By taking $F_+ = 1$, $F_- = 0$, we have $A_\pm = B_\pm = E = 0$ in (3.4). Since we are considering collisions that do not change the number of particles, $\bar{C} = 0$ as well. In this case (3.4) provides the following continuity equation:

$$\frac{\partial n}{\partial t} + \frac{\partial}{r\partial r}(r\Gamma_n) = S(r), \tag{4.7}$$

with the particle flux Γ_n given by

$$\Gamma_n = -D_n n'\left[1 + \zeta\left(\frac{H_3}{H_2} - \frac{1}{2}\right)\right] + n\,V_n, \quad ' = \mathrm{d}/\mathrm{d}r, \tag{4.8}$$

where the stochastic particle diffusivity is

$$D_n = \left(\frac{2}{\pi}\right)^{1/2} v_{th} D_0 H_2, \tag{4.9}$$

the convective velocity is

$$V_n = -D_n\left\{\frac{T'}{T}\left(\frac{H_3}{H_2} - \frac{1}{2}\right) - \frac{ZeE_a}{T} + \zeta\left[\frac{T'}{T}\left(\frac{H_4}{H_2} - \frac{3H_3}{H_2} + \frac{3}{4}\right) + \frac{u'}{u}\left(\frac{2H_3}{H_2} - 1\right)\right.\right.$$
$$\left.\left. + \frac{ZeE_a}{T}\left(\frac{3}{2} - \frac{H_3}{H_2}\right) - L_0^{-1}\right]\right\} \tag{4.10}$$

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and H_k are the following family of integral functions:

$$H_k(\alpha) = \int_0^{+\infty} \mathrm{d}x \frac{x^k}{\alpha + x^2}\, \mathrm{e}^{-x}, \tag{4.11}$$

stemming from the collision term $g(v_\parallel)$ defined in (2.9). Therefore, their argument depends on the particle species:

$$\alpha_j = \frac{1}{4} L_K/\lambda_j(v_{\mathrm{th},j}) = \frac{Z_j^2 e^4 \ln(\varLambda) n_i L_K}{4\pi \varepsilon_0^2 (2T_j)^2}. \tag{4.12}$$

We note that the combination of (4.8), (4.10) becomes (16) of Predebon & Paccagnella (2007) in the limit $\zeta \to 0$. The particle radial flux (4.8) is given by a diffusion term, proportional to n', plus a convective term. The integral functions (4.11) are decreasing functions of α; hence collisions tend to decrease D_n. The collision-less case corresponds to $\alpha = 0$. Note that $\alpha_i \sim Z_i^2 \alpha_e$ for the ions and $\alpha_i \sim \alpha_e$ for the common case of hydrogen isotope species. Therefore, the electron diffusivity is much larger than the ion diffusivity: $D_{n,i} \sim v_{\mathrm{th},i}/v_{\mathrm{th},e} D_{n,e} \sim (m_e/m_i)^{1/2} D_{n,e}$. Nonetheless, the particle fluxes (4.8) of ions and electrons are equalized, condition required by the plasma quasi-neutrality, by the ambipolar electric field E_a. The constraint $\varGamma_{n,i} = \varGamma_{n,e}$ implies the vanishing, apart from terms $o((m_e/m_i)^{1/2})$, of $\varGamma_{n,e}/D_{n,e}$. One gets

$$\begin{aligned} \frac{eE_a}{T_e} \cong -\Bigg\{ \frac{n'_e}{n_e} + \frac{T'_e}{T_e}\left(\frac{H_3}{H_2} - \frac{1}{2}\right) + \zeta_e \Bigg[\frac{n'_e}{n_e} + \frac{T'_e}{T_e}\left(\frac{H_4}{H_2} - \frac{H_3}{H_2} - \left(\frac{H_3}{H_2}\right)^2\right) \\ + \frac{u'_e}{u_e}\left(\frac{2H_3}{H_2} - 1\right) - L_0^{-1} \Bigg]\Bigg\}. \end{aligned} \tag{4.13}$$

In (4.13) the functions (4.11) are evaluated on the electron population: $H_k = H_k(\alpha_e)$. Note that the L_0^{-1} term enters only as higher-order correction ($o(\zeta)$) in (4.10) and (4.13).

4.2. *Parallel momentum transport*

By taking $F_+ = 0$, $F_- = m_j v_\parallel$, we get from (3.4) the following equation for the parallel momentum:

$$\frac{\partial}{\partial t}(nmu) + \frac{\partial}{r\partial r}(r\varGamma_u) = -\frac{ZeE_a}{T}\varTheta_u + L_0^{-1}\varLambda_u + Zen\boldsymbol{b}_0 \boldsymbol{\cdot} \boldsymbol{E}_{\mathrm{in}} + R_\parallel. \tag{4.14}$$

The radial flux of parallel momentum $\varGamma_u$ on the left-hand side of (4.14) is the sum of a diffusion term and a convection term:

$$\varGamma_u = nm(-D_u u' + u V_u), \tag{4.15}$$

$$D_u = 2\left(\frac{2}{\pi}\right)^{1/2} v_{\mathrm{th}} D_0 H_3 = 2\frac{H_3}{H_2} D_n, \tag{4.16}$$

$$V_u = -D_u \left[\frac{n'}{n} + \frac{T'}{T}\left(\frac{H_4}{H_3} - \frac{3}{2}\right) + \frac{ZeE_a}{T}\left(\frac{H_2}{2H_3} - 1\right) - \frac{1}{2} L_0^{-1} \frac{H_2}{H_3} \right]. \tag{4.17}$$

The coefficient (4.16) represents the stochastic kinematic viscosity. On the right-hand side of (4.14), the first term is a parallel force generated by the ambipolar electric field, with

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Θ_u similar to a momentum flux:

$$\Theta_u = n\,m\frac{H_2}{2H_3}(D_u u' + u\,V_{u,\Theta}), \tag{4.18}$$

$$V_{u,\Theta} = D_u\left[\frac{n'}{n} + \frac{T'}{T}\left(\frac{H_3}{H_2} - \frac{3}{2}\right) + \frac{ZeE_a}{T}\left(\frac{H_1}{2H_2} - 1\right) - \frac{1}{2}L_0^{-1}\frac{H_1}{H_2}\right]. \tag{4.19}$$

The second term in the right-hand side of (4.14) is a parallel force generated by the gradient of B_0, with Λ_u similar to a momentum flux:

$$\Lambda_u = nm\frac{H_2}{2H_3}(D_u u' + u\,V_{u,\Lambda}), \tag{4.20}$$

$$V_{u,\Lambda} = D_u\left[\frac{n'}{n} + \frac{T'}{T}\left(\frac{H_3}{H_2} - \frac{1}{2}\right) + \frac{ZeE_a}{T}\left(\frac{H_1}{2H_2} - 1\right) - L_0^{-1}\frac{H_1}{H_2}\right]. \tag{4.21}$$

The function H_1, which enters (4.19), (4.21), diverges in the collision-less case $\alpha = 0$. This divergence stems from the singularity of (4.6) for $v_\parallel \to 0$, and it is symptomatic of the inconsistency of taking a collision-less limit in transport expressions derived by a local Maxwellian approximation of $\langle f\rangle$ (a distribution indeed established by collisions). Therefore, we have to keep $\alpha \neq 0$ to be consistent with our assumptions, and in this case H_1 is finite. Nonetheless, the divergence at $\alpha = 0$ is logarithmic, hence very weak: for instance, $H_1/H_2 \sim 1$ for $\alpha \sim 1$, and $H_1/H_2 \sim 10$ only at $\alpha \sim 10^{-9}$. Moreover, the explicit computation discussed in the next paragraph estimates the two forces under discussion as almost negligible with $\alpha \sim 1$. Therefore, even the case of $\alpha \neq 0$ but very small is not problematic.

The third term on the right-hand side of (4.14) is the drive due to the applied electric field. Finally, the fourth term is the collisional transfer of parallel momentum:

$$R_\parallel = 2\pi\int_{-\infty}^{+\infty} \mathrm{d}v_\parallel \int_0^{+\infty} \mathrm{d}v_\perp v_\perp m v_\parallel C. \tag{4.22}$$

Following the procedure adopted in Bragiinkii (1965), we replace the time derivative of the density in (4.14) by the continuity equation (4.7):

$$nm\frac{\partial}{\partial t}u + m\Gamma_n\frac{\partial}{\partial r}u + \frac{\partial}{r\partial r}(r\hat{\Gamma}_u) = -\frac{ZeE_a}{T}\Theta_u + L_0^{-1}\Lambda_u + Zen\boldsymbol{b}_0\cdot\boldsymbol{E}_{\mathrm{in}} + R_\parallel - muS, \tag{4.23}$$

$$\hat{\Gamma}_u = \Gamma_u - mu\Gamma_n = nm(-D_u u' + u\hat{V}_u), \tag{4.24}$$

$$\hat{V}_u = -D_u\left[\frac{n'}{n}\left(1 - \frac{H_2}{2H_3}\right) + \frac{T'}{T}\left(\frac{H_4}{H_3} - 2 + \frac{H_2}{4H_3}\right) + \frac{ZeE_a}{T}\left(\frac{H_2}{H_3} - 1\right) - \frac{1}{2}L_0^{-1}\frac{H_2}{H_3}\right]. \tag{4.25}$$

Since $u = \zeta^{1/2}v_{\mathrm{th}}$, terms of order ζ are neglected in (4.25). Equation (4.23) is formally akin to the Braginskii motion equation ((2.2) of Bragiinkii 1965), with an extra term related to the particle source S. In particular, two correspondences can be identified:

$$m\Gamma_n\frac{\partial}{\partial r}u \leftrightarrow nm\boldsymbol{V}\cdot\nabla\boldsymbol{V}, \quad \frac{\partial}{r\partial r}(r\hat{\Gamma}_u) \leftrightarrow \nabla\cdot\overline{\overline{\boldsymbol{\Pi}}}, \tag{4.26}$$

with $\boldsymbol{V}$, $\overline{\overline{\boldsymbol{\Pi}}}$ the Braginskii mean velocity and stress tensor, respectively.

4.3. *Heat transport*

Since we do not distinguish between parallel and perpendicular temperatures in the distribution function (4.1), the meaningful second-order moment is $F_+ = \frac{1}{2}mv^2 = \frac{1}{2}m(v_\parallel^2 + v_\perp^2)$, $F_- = 0$. Note an exact cancellation of the terms in the square bracket of expression (3.8) for $B_\pm$: therefore, L_0^{-1} gives no contribution in (3.4). From (3.4), one gets the following energy transport equation:

$$\frac{\partial}{\partial t}\left(\frac{3}{2}nT + \frac{1}{2}nmu^2\right) + \frac{\partial}{r\partial r}(r\Gamma_E) = ZeE_a\Gamma_n + Zenu\boldsymbol{b}_0\cdot\boldsymbol{E}_{\text{in}} + R_\parallel u + Q, \tag{4.27}$$

where Γ_E is the energy flux, not written here for the sake of brevity, and

$$Q = 2\pi\int_{--\infty}^{+\infty}\mathrm{d}v_\parallel\int_0^{+\infty}\mathrm{d}v_\perp v_\perp\frac{1}{2}m[(v_\parallel - u)^2 + v_\perp^2]C \tag{4.28}$$

is the heat produced by collisions. Following again Bragiinkii (1965), we express the time derivative of $\frac{1}{2}nmu^2$ by (4.7) and (4.23) to obtain the heat transport equation:

$$\frac{\partial}{\partial t}\left(\frac{3}{2}nT\right) + \frac{\partial}{r\partial r}(r\Gamma_Q) = -\hat{\Gamma}_u\frac{\partial}{\partial r}u + ZeE_a\Gamma_n + \frac{Ze\,E_a}{T}\Theta_u u - L_0^{-1}\Lambda_u u + Q + \frac{1}{2}mu^2 S. \tag{4.29}$$

The heat flux is

$$\Gamma_Q = \Gamma_E - \frac{1}{2}mu^2\Gamma_n - \hat{\Gamma}_u u = -D_T nT'\left[H_4 + \frac{H_3}{2} + \frac{H_2}{2} + \zeta\left(\frac{9}{4}H_3 - 4H_4 + H_5\right)\right] + nTV_T, \tag{4.30}$$

$$D_T = \left(\frac{2}{\pi}\right)^{1/2} v_{\text{th}}D_0 = D_n H_2^{-1}, \tag{4.31}$$

$$V_T = -D_T\left\{\left(\frac{n'}{n} - \frac{ZeE_a}{T}\right)(H_3 + H_2) + \zeta\left[\frac{n'}{n}\left(H_4 - \frac{3}{2}H_3\right) + \frac{u'}{u}(2H_4 - H_2 - H_3) + \frac{ZeE_a}{T}\left(\frac{5}{2}H_3 - H_4\right) - L_0^{-1}(H_3 + H_2)\right]\right\}, \tag{4.32}$$

with D_T the stochastic thermal diffusivity. Since $H_4(\alpha) = 2 - \alpha H_2(\alpha)$, we note that the flux (4.30) becomes expression (31) of Predebon & Paccagnella (2007) for $\zeta \to 0$.

The terms on the right-hand side of (4.29) are heating sources. In particular, by comparing (4.29) with the Braginskii heat equation ((1.23) of Bragiinkii 1965), and taking into account the second of (4.26), we can identify the viscous heating as

$$Q_{visc} \to -\hat{\Gamma}_u\frac{\partial}{\partial r}u. \tag{4.33}$$

The presence of the particle source S in (4.29) is explained by the fact that zero-velocity particles from the wall are seen with energy $\frac{1}{2}mu^2$ in the plasma reference. However, this contribution is estimated as negligible (see § 5).

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4.4. *The Onsager reciprocal relations*

The above expressions for particle, momentum and heat flux satisfy the Onsager reciprocal relations (Onsager 1931), a general property of the thermodynamics of the transport processes, stemming from the principle of microscopic reversibility. This result is discussed in full because it represents a verification of the model. Assuming a small deviation from thermal equilibrium due to the 'thermodynamic forces' X_i (such as the gradients), the 'thermodynamic fluxes' Γ_i are linearly related to X_i by kinetic coefficients L_{ik}:

$$\Gamma_i = \sum_k L_{ik} X_k. \tag{4.34}$$

Fluxes and forces are 'conjugate' if the entropy production for unit of volume can be expressed in the form

$$\theta = \sum_i \Gamma_i X_i. \tag{4.35}$$

The reciprocal relations of the kinetic coefficients apply to conjugate fluxes and forces. In the presence of a magnetic field they are (Bragiinkii 1965)

$$L_{ik}(\boldsymbol{B}) = L_{ki}(-\boldsymbol{B}), \tag{4.36}$$

when both forces X_i, X_k are even or odd functions of the particle velocity,

$$L_{ik}(\boldsymbol{B}) = -L_{ki}(-\boldsymbol{B}), \tag{4.37}$$

if one force is odd and the other is even. To show that the present theory satisfies the constraints (4.36), (4.37), we first identify the pairs of conjugate thermodynamic fluxes and forces. Following Bragiinkii (1965), we start from the entropy per particle (j-species label omitted):

$$s = \tfrac{3}{2}\ln(T) - \ln(n) + \text{const.} \tag{4.38}$$

Then we compute the transport equation for the entropy per unit volume, ns, by exploiting the continuity equation for density (4.7), and the heat transport equation (4.29). Discarding the particle source S, one gets

$$\frac{\partial}{\partial t}(ns) + \frac{\partial}{r\partial r}\left[r\left(s\Gamma_n - \frac{5}{2}\Gamma_n + \frac{\Gamma_Q}{T}\right)\right] = \theta = \sum_{i=1,4} \Gamma_i X_i + \frac{Q}{T}. \tag{4.39}$$

The entropy production θ is given by the collisional heating and by the following four pairs of conjugate fluxes and thermodynamic forces:

$$\left.\begin{aligned} &\Gamma_1 = \Gamma_n, \quad X_1 = \frac{3}{2}\frac{T'}{T} - \frac{n'}{n} + \frac{ZeE_a}{T}; \quad \Gamma_2 = \hat{\Gamma}_u, \quad X_2 = -\frac{u'}{T}; \\ &\Gamma_3 = \Gamma_Q, \quad X_3 = -\frac{T'}{T^2}; \quad \Gamma_4 = \frac{ZeE_a}{T}\Theta_u - L_0^{-1}\Lambda_u, \quad X_4 = \frac{u}{T}. \end{aligned}\right\} \tag{4.40}$$

Note that Γ_4 is a force density (which enters (4.23)), but we keep the term 'thermodynamic flux' in this context (as done in Bragiinkii (1965) for the collisional momentum transfer $\boldsymbol{R}$, for instance). The forces X_2, X_3 are expressed by gradients of u, T respectively, and they are conjugate of the parallel momentum and heat fluxes Γ_2, Γ_3, as expected. Instead, X_1, the conjugate of the particle flux Γ_1, depends on T' and E_a, in addition to n'. The

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ambipolar electric field enters X_1 because $\Gamma_1 ZeE_a$ is a heating source in (4.29). Moreover, we speculate that T' enters X_1 because particles carry energy. As written above, Γ_4 is actually a parallel force density, and it is conjugate of X_4 because $\Gamma_4 X_4 T$ is a heating source in (4.29).

By exploiting equations (4.8)–(4.10), (4.16), (4.18)–(4.21), (4.24), (4.25), (4.30)–(4.32), it is possible to establish the linear relationship (4.34) between Γ_i and X_i defined in (4.40), with the following L_{ik}. The diagonal coefficients are

$$\left.\begin{aligned} L_{11} &= nD_n\left[1+\zeta\left(\frac{H_3}{H_2}-\frac{1}{2}\right)\right]; \quad L_{22} = 2nmTD_n\frac{H_3}{H_2};\\ L_{33} &= nT^2D_n\left[\frac{H_4}{H_2}+\frac{2H_3}{H_2}+2+\zeta\left(\frac{H_5}{H_2}-\frac{5}{2}\frac{H_4}{H_2}\right)\right];\\ L_{44} &= nmTD_n\frac{H_1}{H_2}\left[\frac{1}{2}\left(\frac{Ze\,E_a}{T}\right)^2+L_0^{-2}-\frac{Ze\,E_a}{T}L_0^{-1}\right].\end{aligned}\right\} \tag{4.41}$$

The off-diagonal coefficients are symmetric:

$$\left.\begin{aligned} L_{12} = L_{21} &= nmuD_n\left(\frac{2H_3}{H_2}-1\right); \quad L_{13} = L_{31} = nTD_n\left[\frac{H_3}{H_2}+1+\zeta\left(\frac{H_4}{H_2}-\frac{3}{2}\frac{H_3}{H_2}\right)\right];\\ L_{14} = L_{41} &= -nmuD_n\left[\frac{ZeE_a}{T}-L_0^{-1}\right]; \quad L_{23} = L_{32} = nmTuD_n\left(\frac{2H_4}{H_2}-1-\frac{H_3}{H_2}\right);\\ L_{24} = L_{42} &= -nmTD_n\left[\frac{ZeE_a}{T}-L_0^{-1}\right]; \quad L_{34} = L_{43} = -nmTuD_n\left[\frac{ZeE_a}{T}\frac{H_3}{H_2}-L_0^{-1}\left(\frac{H_3}{H_2}+1\right)\right].\end{aligned}\right\} \tag{4.42}$$

The equalities $L_{13} = L_{31}, L_{24} = L_{42}$ correspond to (4.36), because both the forces involved are even or odd in the velocity. Instead, the other coefficients relate even and odd forces: the equalities $L_{12} = L_{21}$, $L_{14} = L_{41}$, $L_{23} = L_{32}$, $L_{34} = L_{43}$ indeed correspond to (4.37), because the average parallel velocity u, contained in these coefficients, is odd with $\boldsymbol{B}$. In conclusion, this theory fulfils the Onsager reciprocal relations.

The off-diagonal coefficients formalize the interdependence of the transport processes in a thermodynamic system (Onsager 1931). Indeed, the fluxes derived in the previous paragraph depend on all the quantities $n'/n, T'/T, u'/u, ZeE_a/T, L_0^{-1}$, so that they are represented by convective velocities, in addition to diffusive components. By inspection of (4.41), (4.42), we note that the terms $L_{12}X_2, L_{14}X_4$ are $o(\zeta)$ with respect to $L_{11}X_1, L_{13}X_3$, so that the particle flux Γ_1 mostly depends on X_1, X_3. Instead, the terms $L_{2k}X_k$ share the same order, so that the parallel momentum flux Γ_2 depends on all the X_k in the same way. The terms $L_{32}X_2, L_{34}X_4$ are $o(\zeta)$ with respect to $L_{33}X_3, L_{31}X_1$, so that the heat flux Γ_3 mostly depends on X_1, X_3. The terms $L_{4k}X_k$ share the same order, so that Γ_4 depends on all the X_k in the same way. Of course, the actual magnitude of the various terms $L_{ik}X_k$ can be inferred only by an explicit computation, and we comment on that in the next paragraph.

As far as the collisional heating Q is concerned, we can adopt the Braginskii expressions (Bragiinkii 1965) in the present context without violating (4.36), (4.37). Braginskii's formulas are simplified here, because: (i) the mean velocity is taken only in the direction parallel to $\boldsymbol{B}$ and (ii) due to the cylindrical geometry, the gradients parallel to $\boldsymbol{B}$ can be discarded as an effect produced by the radial magnetic field ($\nabla_\parallel \sim a^{-1}o(\delta)$). For

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electrons and main ions one gets

$$Q_e = R_{\|,e}(u_i - u_e) - Q_{ei}, \quad Q_i = Q_{ei}, \tag{4.43a,b}$$

$$R_{\|,e} = (en_e)^2\eta_\|(u_i - u_e), \quad Q_{ei} \cong 1.37 \times 10^{-30}\frac{m_e}{m_i}Z_i \ln \Lambda n_e^2(T_e - T_i)/T_e^{3/2}, \tag{4.44a,b}$$

with $\eta_\|$ the parallel resistivity and Q_{ei} the heat exchanged between the two species (the Q_{ei} expression is in SI units, except T in eV). For the electrons, (4.39), (4.43) introduce the thermodynamic force $(u_i - u_e)/T_e$ and the conjugate flux $R_{\|,e}$. The same equations, summed for electrons and ions, introduce the force $(T_e - T_i)/(T_eT_i)$ and the conjugate flux Q_{ei}. These two pairs of fluxes and forces are completely decoupled by the previous ones (4.40). Therefore, (4.43), (4.44) can be included without violating the Onsager reciprocal relations.

5. Example of solution

Within the cylindrical geometry approximation, the present theory can be applied to the stochastic region of any magnetic configuration. The above equations for particle and heat transport, the latter without the terms related to the velocity u, have been already integrated into a transport model for the RFP (Predebon *et al.* 2009), which is currently applied to the RFX-mod device ($R = 2m$, $a = 0.459m$) (Zuin *et al.* 2017). Indeed, this theory is suitable for the RFP in the so-called multiple-helicity regime (MH), which is characterized by many $m = 1$ tearing modes with comparable amplitudes, whose overlap gives rise to a wide stochastic region. This is not true in the quasi-single-helicity regime (QSH), where the innermost resonant mode grows at the expense of the others, thus healing the magnetic chaos (Lorenzini et al 2009). The inclusion of parallel momentum transport, the novelty of the present derivation, still allows finding plausible solutions for RFX-mod in MH conditions. The unavoidable uncertainty about several input quantities, as well as the lack of the transport terms due to the drift velocity perpendicular to the magnetic field, prevent a detailed comparison of this model with the experiment. Nonetheless, the solutions are in the expected range of values. We consider a case at plasma current $I_p \sim 1$ MA (shot 29509 as reference). We adopt SI units, but the temperature is given in eV.

First of all, at least an *indicative* estimate of the collision-less magnetic diffusion coefficient D_0, given by (2.8), is required. The RFX-mod sensors' layout allows determination of the harmonics $\tilde{b}_r^{m,n}$ for the core-resonant $m = 1$ modes and for the reversal-resonant $m = 0$ modes ($r^{0,n} \approx 0.4$ in this shot), in the interval $n = 1$–24. We focus on $m = 1$, whose amplitude is much larger than that of $m = 0$. The harmonics $\tilde{b}_r^{1,n}$ are estimated at the resonant radii $r^{1,n}$ by a reconstruction, from the edge measurements, of both the equilibrium (assumed force-free, $\boldsymbol{J}_0 \propto \boldsymbol{B}_0$) and the perturbation radial profile (Zanca & Terranova 2004). Their amplitudes are plotted in figure 1(*a*): even in a MH case, the innermost resonant $n = 7$ is twice as large as the following $n = 8$ (for a QSH case it would be 20 times larger). The same figure plots the Chirikov parameter $s(n, n+1) = \frac{1}{2}(w^{1,n} + w^{1,n+1})/(r^{1,n+1} - r^{1,n})$ (Rechester & Rosenbluth 1978; D'Angelo & Paccagnella 1996), with $w^{1,n}$ the island width (the increase of s with n is due to the resonances getting denser). The overlapping condition $s > 1$, which defines a stochastic magnetic field, is largely satisfied. As written before, we do not apply any resonance broadening method to regularize the singular functions $\delta(m - nq) = \delta(r - r^{1,n})/|nq'|$. We simply average them by radial integration of (2.8) over suitable intervals centred about the resonant radii $r^{1,n}$. Since magnetic islands overlap ($s > 1$), we cannot take $w^{1,n}$ as amplitudes of these intervals. To avoid overlapping, we adopt neighbouring intervals $\Delta r^{1,n} = (r^{1,n+1} - r^{1,n-1})/2$, with extrema halfway to $r^{1,n+1}$ and $r^{1,n-1}$. For the innermost

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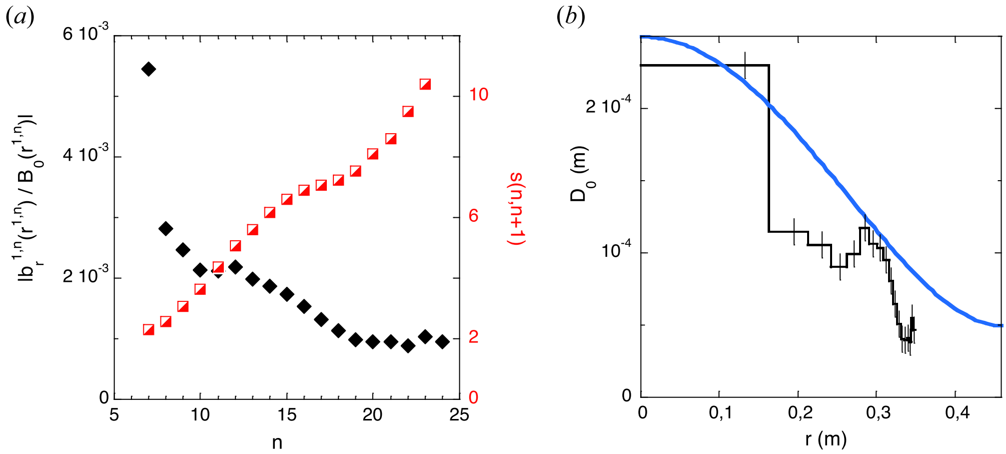

FIGURE 1. (*a*) Values of $|\tilde{b}_r^{1,n}(r^{1,n})|$ for modes $m=1$, $n=7$–24, averaged over a significant time interval for shot 29509 (black diamonds), and stochasticity parameter from these amplitudes (red squares). (*b*) Stepwise profile of $D_0(r)$ obtained from averaging equation (2.8) (black; the vertical segments mark the resonant radii), alongside the continuous profile taken to solve the equations (blue).

resonant $n=7$ the left extremum is set at $r=0$. The contribution of the $m=0$ modes cannot be computed in this way, due to the impossibility of defining a radial interval to average: in fact, both resonances of $m=1, n>0$ and $m=1, n<0$ (a secondary branch resonating between the reversal and the plasma edge, not detectable by the RFX-mod sensors) become dense approaching the reversal. The average values of D_0 on these intervals give the stepwise profile plotted in figure 1(*b*) (black). We have verified that a more regular profile makes solving the equations easier. Therefore, we choose the profile plotted in blue ($D_0(0)=2.5\times10^{-4}$ m). We also keep a finite value at the edge ($D_0(a)=5\times10^{-5}$ m), since numerical analyses of magnetic diffusion in MH conditions give a stochastic region in the whole plasma: see figure 9(*b*) of Innocente *et al.* (2017).

The Kolmogorov–Lyapunov length L_K, which enters α_j (equation (4.12)), is here fixed by a scaling found numerically in D'Angelo & Paccagnella (1996): $L_K(r)=(0.127D_0^{0.275})^{-1}$. An increasing radial profile with $L_K(0)\approx 80$ m is obtained. This scaling is close to the analytical result derived in Rechester *et al.* (1979).

Stochastic diffusion is expected to be the dominant transport mechanism in the plasma core, but electrostatic turbulence can emerge in the edge region where D_0 becomes smaller. Indeed, the edge RFX-mod particle transport is compatible with the Bohm diffusivity $D_{Bohm}=\frac{1}{16}(T/B)$ (Antoni *et al.* 1998; Spolaore *et al.* 2004); hence this additional transport mechanism should be included when solving the equations. Diffusivity D_{Bohm} is typically negligible compared with the electron stochastic diffusivity, but, in the case under examination, it is of the same order of magnitude ($\sim$10 m^2 s^{-1}) as the ion stochastic diffusivity (see figure 2*d*). One option is to take a background of Bohm diffusion in the whole plasma, besides the stochastic transport, letting the equations to weigh the two different mechanisms. We now present the solution obtained by this assumption. Another option is to restrict the Bohm diffusion only to the edge region: this second case is briefly discussed later on.

Taking stationary conditions ($\partial/\partial t=0$), the transport equations become second-order ordinary differential equations in the radial coordinate, for density, velocity and temperature. Regularity at $r=0$ requires the vanishing of the on-axis radial derivative of

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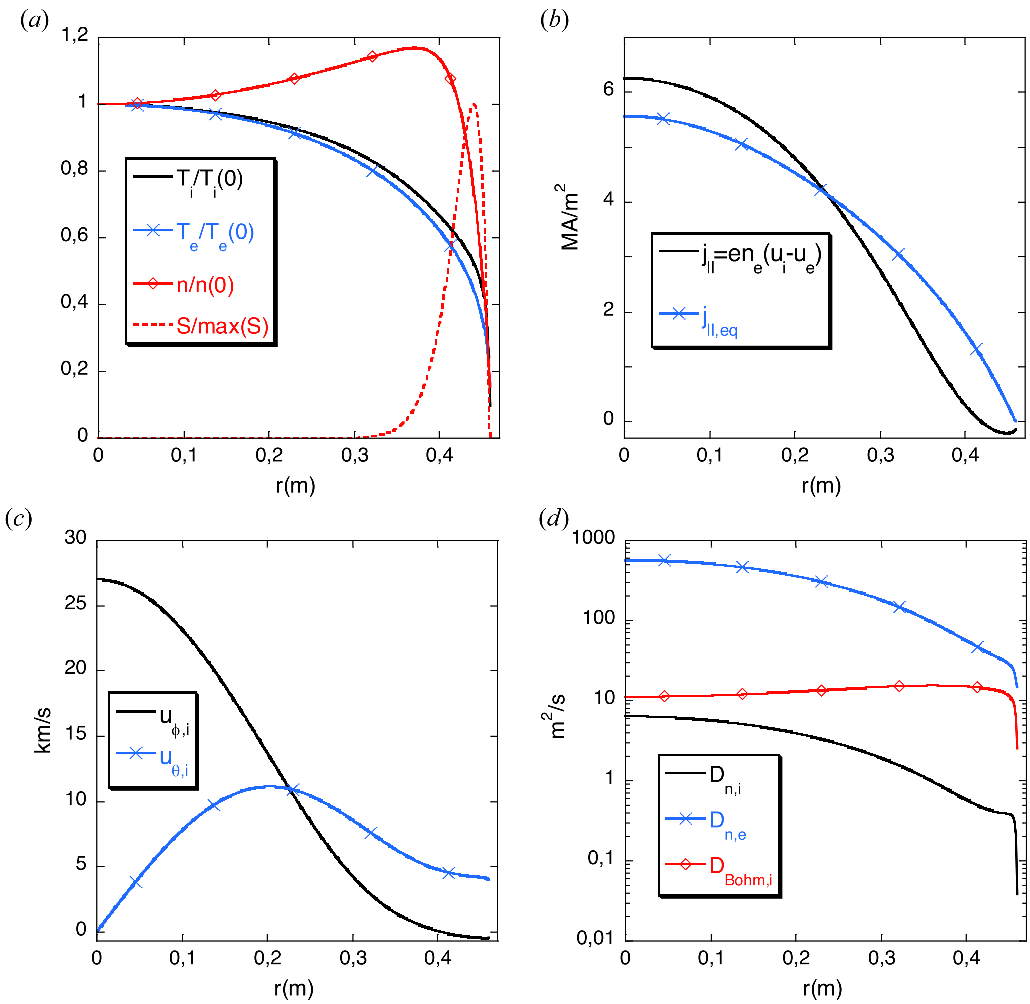

FIGURE 2. (*a*) Normalized profiles obtained for density (red with diamonds), T_e (blue with crosses) and T_i (black). The normalized particle source is also reported (red dashed). (*b*) Parallel current from velocity and density solution (black) and from equilibrium reconstruction (blue with crosses). (*c*) Toroidal (black) and poloidal (blue with crosses) components of u_i. (*d*) Electron (blue with crosses) and ion (black) stochastic particle diffusivity, alongside the ion Bohm coefficient (red with diamonds). These solutions are obtained by superimposing the Bohm diffusion on the stochastic transport, in the whole plasma.

these quantities. Therefore, one free parameter is left in each equation to make the profiles decreasing to small values towards $r = a$. The equations solved are described below.

(i) Equations (4.7)–(4.13) for the normalized density $n_i/n_i(0)$ of the hydrogen ion ($Z_i = 1$). As explained before, the diffusion term $-D_{\mathrm{Bohm},i}n_i'$ is added to the flux (4.8). Without a model for the source, we fix a plausible $S(r)/n_i(0)$ profile localized at the edge, but we leave a global coefficient S_0 as free parameter. We take the same normalized density profile for all the particle species, with on-axis electron density $n_e(0) = 2.5 \times 10^{19}\ \mathrm{m}^{-3}$.

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(ii) Equations (4.14)–(4.22) for the parallel velocities of electrons u_e and ions u_i, with $\boldsymbol{E}_{\text{in}} = E_\phi \boldsymbol{e}_\phi$, $E_\phi = 25/(2\pi R)$ V m^{-1}. The electron collisional momentum transfer $R_{\parallel,e}$ is expressed by (4.44) with Spitzer parallel resistivity: $\eta_\parallel \cong 5.2 \times 10^{-5} \ln \varLambda Z_{\text{eff}} N(Z_{\text{eff}}) T_e^{-3/2}$ (Wesson 2004). Due to the significant dependence of $R_{\parallel,e}$ on the plasma effective charge Z_{eff}, with $N(Z_{\text{eff}})$ a corrective function, it is important to include impurity species, which, though present only in trace, can raise Z_{eff}. We consider the Carbon impurity (the material of the RFX-mod first wall), with mean charge $Z_C = 5$, and we take $Z_{\text{eff}} = 2.5$, a suitable value for this density level (Carraro *et al.* 2005). Equation (4.14) is considered separately for the electrons, whereas those for the main ion and the Carbon impurity are summed together, with assumption of same velocity and temperature for the two species, and exploiting the momentum conservation $R_{\parallel,i} + R_{\parallel,C} = -R_{\parallel,e}$. The on-axis values $u_e(0)$, $u_i(0)$ are free parameters. The background Bohm diffusion terms $-n_j m_j D_{\text{Bohm},j} u_j'$ are added to the momentum flux (4.15) for the different species.

(iii) Equations (4.29)–(4.32) for the electron T_e and ion T_i temperatures. The collisional heating is given by (4.43), (4.44). The background diffusion terms $-n_j D_{\text{Bohm},j} T_j'$ are added to the heat flux (4.30). The on-axis values $T_e(0)$, $T_i(0)$ are free parameters.

All the above equations are coupled together.

To summarize, several inputs of the equations are taken from the experiments: the equilibrium magnetic field and the perturbed radial magnetic field, used to estimate D_0 and L_K, the on-axis electron density, the applied electric field and Z_{eff}. Instead, S_0, $u_e(0)$, $u_i(0)$, $T_e(0)$, $T_i(0)$ are left as free parameters, tuned to have density, velocity and temperature profiles with values at $r = a$ below 15 % of the on-axis inputs. Therefore, they turn out to be an output of the solution.

The normalized solution profiles for density and temperature are shown in figure 2(*a*). The very steep gradient at $r \approx a$ is an artefact due to transport coefficients getting very small there (they depend on T). The density has the mildly hollow profile typical of RFX-mod, and the particle source $S(r)$ (red dashed line in the same figure) is very similar to the edge-localized profiles reported in figures 5 and 6 of Lorenzini *et al.* (2006), with a maximum about 2.4×10^{23} m^{-3} s^{-1}. As far as the on-axis temperature is concerned, we get $T_e(0) \approx 335$ eV, $T_i(0) \approx 195$ eV. The T_e value is compatible with typical measurements at $I_p \sim 1$ MA in MH conditions (see figure 6 of Innocente *et al.* 2009), whereas for T_i we cannot say, lacking reliable diagnostics of this quantity in RFX-mod. The collisional heating $Q_i = Q_{ei}$ dominates the other source terms on the right-hand side of (4.29) for the ions. Hence, this model does not contain any relevant anomalous heating source for the ions. Apart from the high-density regime, Q_i is not strong enough to equalize the ion and electron temperatures, hence our result $T_i(0) \approx 0.58\ T_e(0)$. During the magnetic reconnection events, characteristic of the RFP, it is often observed that $T_i > T_e$ (Scime *et al.* 1992; Gobbin *et al.* 2022), supporting the existence of an important transient anomalous ion heating, outside the quasi-linear description of the present work. The identification of the process responsible for that is still a matter of investigation: see McChesney, Stern & Bellan (1987), Guo, Paccagnella & Romanelli (1994) and Fiksel *et al.* (2009) as examples.

In the core region the parallel velocity profiles are steeper than those of the temperature, because the ratio between the source and the stochastic diffusion coefficient is larger for the momentum than for the temperature. Moreover, the two terms with $\boldsymbol{E}_{\text{in}}$, $R_\parallel$ dominate the other sources on the right-hand side of (4.14). We get $u_e(0) \approx 1540$ km s^{-1}, $u_i(0) \approx 27$ km s^{-1}, corresponding to Mach numbers $\approx$5.17, 0.09 respectively (use is made of the sound speed defined in formula (6.1) below). The density and velocity solutions allow

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computing the parallel current density profile, $j_{\|} = en_e(u_i - u_e)$, which is compared with the magnetic equilibrium reconstruction $j_{\|,\mathrm{eq}}$ in figure 2(*b*). The good deal of uncertainty about several quantities of the model (for instance D_0, L_k) would leave space for an adjustment of the same in order to get a better agreement between $j_{\|}$ and $j_{\|,\mathrm{eq}}$. Nonetheless, the result is already satisfactory. Since $u_i \ll u_e$, $j_{\|}$ involves mostly u_e. As far as u_i is concerned, a comparison with the RFX-mod spectroscopic measurement of the impurity flow (Zaniol *et al.* 2015) is more problematic, because the present model misses the perpendicular drift velocity, particularly relevant in the edge region (Spolaore *et al.* 2004; Vianello *et al.* 2005). However, the toroidal and poloidal components of u_i plotted in figure 2(*c*) are in the correct experimental range (< 20 km s^{-1} for impurities which emit in the region $r > a/2$). In conclusion, the solution is compatible with the experiment.

The solution so obtained weakly depends on the profile shape chosen for $D_0(r)$. For instance, a profile with the same on-axis and edge values as that considered so far (blue continuous line in figure 1*b*), but much flatter in the central region $r < a/2$, and steeper for $r > a/2$, leaves the density profile and the source $S(r)$ almost unchanged and gives about 10 % variations for the other quantities: $T_e(0) \approx 377$ eV, $T_i(0) \approx 205$ eV, $u_e(0) \approx 1750$ km s^{-1}, $u_i(0) \approx 30$ km s^{-1}.

As far as the stochastic transport coefficients are concerned, both the kinematic viscosity (4.16) and the thermal diffusivity (4.31) are proportional to the particle diffusivity (4.9). Therefore, in figure 2(*d*) we plot $D_{n,e}$, $D_{n,i}$ alongside $D_{\mathrm{Bohm},i}$ as terms of comparison: the electron stochastic diffusivity dominates the ion stochastic diffusivity, which instead is of the same order of magnitude as the Bohm diffusivity in the central region ($r < a/2$).

The stochastic fluxes derived in § 4 are all of the form $\varGamma_x \propto D_x x' + xV_x$, with V_x mainly depending on n'/n and T'/T (taking into account expression (4.13) for the ambipolar electric field), apart from the L_0^{-1} dependence of $\hat{V}_u$, and the $o(\zeta)$ corrections. Figures 3(*a*) and 3(*b*) plot the ratio between the convective and diffusive components of the particle, heat and momentum flux, for ions and electrons, respectively, in the core region (the edge region is not plotted, due the unrealistically large n'/n and T'/T obtained there). This ratio is everywhere close to unity for the electron particle flux, due to the ambipolar electric field (4.13), which makes $\varGamma_{n,e}/D_{n,e} = o((m_e/m_i)^{1/2})$. Therefore, it is not plotted. The ratio is small for the momentum, because the density and temperature profiles (which determine V_x) are rather flat in the core, while the velocity is not. Recalling the discussion below (4.42), this means that the momentum flux is mainly driven by the diagonal term $L_{22}X_2$, though all $L_{2k}X_k$ are of the same order *a priori*. Nonetheless, convection is not entirely negligible for the ion momentum flux. For heat flux, diffusion exceeds convection: indeed, the diagonal term $L_{33}X_3$ is larger than $L_{31}X_1$ (see figure 3*c*) whereas $L_{32}X_2$, $L_{34}X_4$ (not plotted) are $o(\zeta)$ corrections (as stated below (4.42)). For ion particle flux, convection exceeds diffusion: indeed $L_{13}X_3$ is larger than $L_{11}X_1$ (see figure 3*c*), whereas $L_{12}X_2$, $L_{14}X_4$ (not plotted) are $o(\zeta)$ corrections. This large convection produces the hollow density profile typically observed in RFX-mod. The parallel force density $\varGamma_4$ (not plotted) is estimated as negligible; hence the cumulative effects of the terms $L_{4k}X_k$ are very small. We cannot say whether all these properties are specific to the stochastic RFP core, or more general.

A comment is made about the terms depending on the magnetic field radial variation L_0^{-1}, which is one of the new elements of this analysis. The one entering $\hat{V}_u$ (equation (4.25)) is not negligible, whereas those in V_n, V_T are higher-order ($o(\zeta)$) corrections. As written above, the related force and heat source, on the right-hand side of (4.23), (4.29) respectively, are estimated as negligible, as those produced by E_a. Therefore, the L_0^{-1} effect is relevant only in the convective component of the momentum flux.

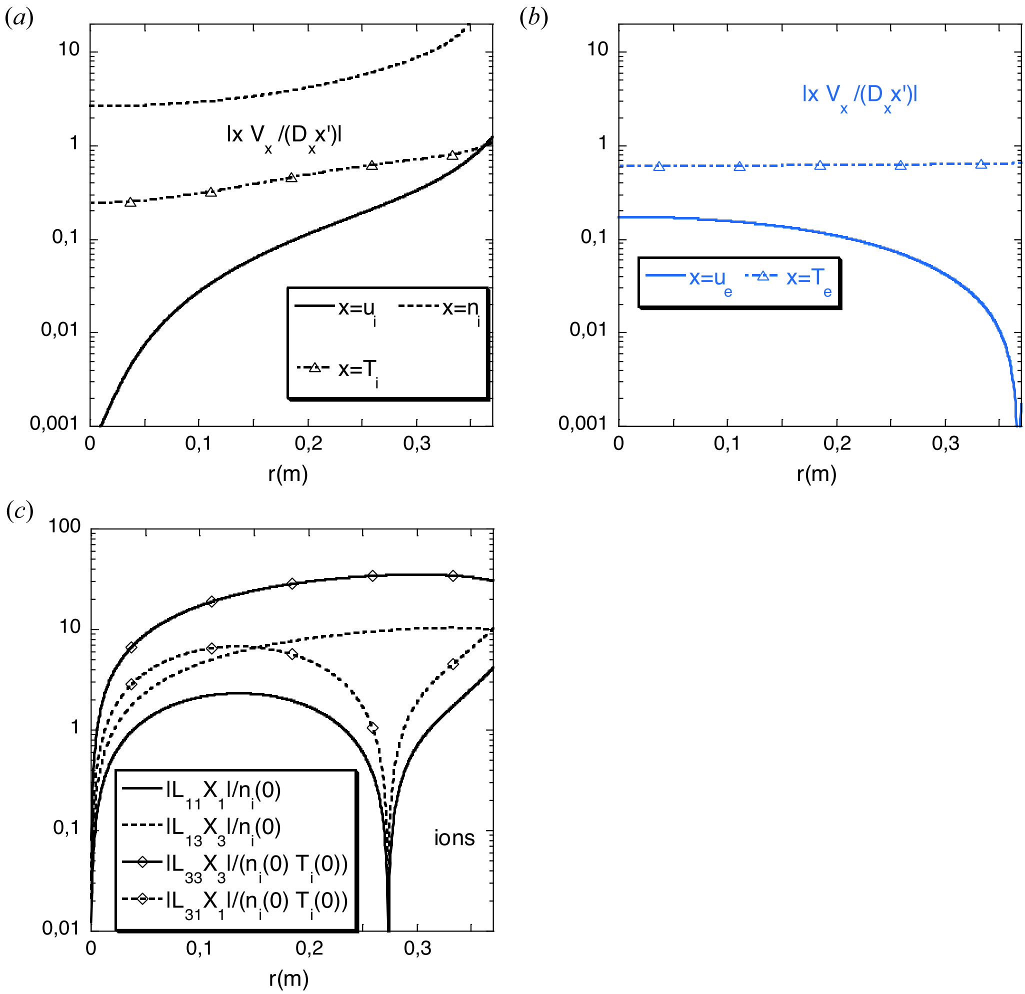

FIGURE 3. (*a*) Ratio between the convective and diffusive components of the stochastic fluxes in the core region ($r/a < 0.8$) for the ion quantities. (*b*) Ratio between the convective and diffusive components of the stochastic fluxes in the core region ($r/a < 0.8$) for the electron quantities, but for the density. (*c*) Some of the terms $L_{ik}X_k$ for the main ion, suitably normalized to make them comparable. All these quantities refer to the solution displayed in figure 2.

Finally, we summarize what changes in the solution when restricting the Bohm diffusion to the very edge region ($r \geq 0.9a$). As expected, T_e, u_e are almost unaffected, whereas T_i, u_i increase: $T_i(0) \approx 0.8T_e(0)$, $u_i(0) \approx 54$ km s^{-1} (nonetheless, in the region $r > a/2$ we still get the correct experimental range < 20 km s^{-1} for the toroidal and poloidal components of u_i). The hollowness of the density profile increases (but it is still plausible for RFX-mod), due to the removal of the additional Bohm diffusion in the core. In conclusion, the ion solution is sensitive to the inclusion of anomalous transport processes different from the stochastic one.

6. Discussion

A previous work (Finn *et al.* 1992) applied the quasilinear theory to the propagation of electrostatic sound wave perturbations along a stochastic magnetic field. The ensuing

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transport is purely diffusive, and it is written for two quantities defined by linear combinations of parallel velocity and density. The diffusion coefficient is similar to those here derived, except for the sound speed c_s in the place of the thermal velocity. Borrowing from Finn *et al.* (1992) the c_s dependence, the following expression for the perpendicular dynamic viscosity (assumed spatially constant):

$$\mu_\perp = n_i m_i c_s \sum_{\substack{m \in \mathbb{Z} \\ n > 0}} |\tilde{b}_r^{m,n}|^2 L_c, \quad c_s = \left(\frac{T_e + \gamma T_i}{m_i} \right)^{1/2}, \quad L_c \approx 1m, \quad \gamma = 3, \tag{6.1}$$

gave a good description of Madison Symmetric Torus (MST) momentum transport experiments (Frindström *et al.* 2018). For the solutions presented in the previous paragraph, and taking the sum over the $m = 1$, $n = 7$–24 modes, (6.1) gives $\mu_\perp/(n_i m_i) \approx 23\ \text{m}^2\ \text{s}^{-1}$, a value almost identical to that obtained from the on-axis ion coefficient (4.16): $D_{u,i}(0) \approx 26\ \text{m}^2\ \text{s}^{-1}$. This is not surprising, since (6.1) shares the same structure of our transport coefficients, and c_s is not far from the ion thermal speed. Instead, the perpendicular kinematic viscosity $\eta_2^i/(n_i m_i)$ of Braginskii's theory (Bragiinkii 1965) is estimated as much smaller, with a maximum value $\approx 0.08\ \text{m}^2\ \text{s}^{-1}$. We remark on some differences between Finn *et al.* (1992) and the present derivation. Our model is based on the drift-kinetic equation only, and it does not require electrostatic instabilities. Our fluxes include diffusive and convective terms, whereas the latter are absent in Finn *et al.* (1992). Finally, the present model integrates the momentum transport with those for particles and heat, within a formalism that satisfies the Onsager reciprocal relations.

According to (2.8) and other similar expressions of previous works (Rechester & Rosenbluth 1978; Harvey *et al.* 1981; Rice *et al.* 1982), the quasi-linear theory predicts a quadratic dependence of the magnetic diffusion coefficient on the normalized perturbation amplitudes: $(\tilde{b})^\alpha, \alpha = 2$. Nonetheless, a numerical analysis of magnetic diffusion (D'Angelo & Paccagnella 1996) showed that the exponent α actually depends on the shape of the magnetic spectrum: $\alpha = 2$ for a flat spectrum of $m = 1$ modes; $\alpha = 1.5$ for mode amplitudes decreasing as $1/n$. Another numerical analysis of particle diffusion, with a background magnetic spectrum from the output of a magnetohydrodynamic code run in RFP configuration, found a sub-diffusive regime, and a phenomenological coefficient described by $\alpha = 1.6$ (Spizzo, White & Cappello 2007). Therefore, it is not surprising that the experimental indications are not unanimous. In agreement with the quasi-linear theory, the above MST analysis (Frindström *et al.* 2018) found $\alpha \approx 2$ for the experimental perpendicular viscosity divided by the sound speed, $(\mu_\perp)_{\text{exp}}/c_s$. Instead, a study of the global energy confinement in RFX obtained the best fit with $\alpha \approx 1.5$ alongside the correction (2.9) for collisions (with the replacement $v_\parallel \rightarrow v_{\text{th}}$) (Sattin *et al.* 2005). In conclusion, the quasi-linear approximation is a straightforward method to obtain general analytical transport expressions, but it cannot describe effects due to the details of the magnetic topology, for which specific numerical studies are required.

7. Conclusions

We have presented a self-contained theory of particle, parallel momentum and heat transport in a stochastic magnetic field. Transport equations are derived from the drift-kinetic equation, treated within the quasi-linear approximation, with velocity perpendicular to the magnetic field neglected, and from approximating the distribution function as a local shifted Maxwellian. The theory refers to a general pinch in cylindrical geometry, and extends previous formulations (Harvey *et al.* 1981; Rice *et al.* 1982;

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Predebon & Paccagnella 2007) specific of the tokamak. The fluxes are all made by diffusive and convective components, and they satisfy the Onsager reciprocal relations. This theory, applied to an RFX-mod case in MH condition, and completed by a background of Bohm diffusion suitable for the edge region, provides density, parallel current and temperature profiles compatible with the experiment. This is in agreement with previous application to RFX-mod of a subset of the equations here derived (Predebon *et al.* 2009). As far as the momentum flux is concerned, the convective component is estimated as smaller than the diffusive one, at least in the examined case, but not negligible for ions. Convection turns out to be particularly important for particle flux. The radial variation of the equilibrium magnetic field magnitude, negligible for the tokamak, gives rise to several new terms in the equations. However, an explicit evaluation shows that they are relevant only in the convective velocity of the momentum flux. The predicted anomalous heating source terms are not strong enough to raise the ion temperature above the electron one, as transiently observed in the RFP during the reconnection events. Therefore, other mechanisms, out of this anyway self-consistent theory, need to be included for a complete description of the RFP transport. Future applications of this theory will concern the edge stochastic region of a tokamak, produced by application of resonant magnetic perturbations.

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Declaration of interests

The authors report no conflict of interest.

Appendix A. Derivation of (2.7)

The solution method is similar to that reported in Rice *et al.* (1982), but no ordering assumption of the equilibrium magnetic field components is made here. First, (2.5) is integrated along the characteristics, which are given by the magnetic field line of the equilibrium field (orbit-0):

$$\tilde{f}-\tilde{f}_0=-v_\parallel\int_{t_0}^{t}\mathrm{d}t'\tilde{\boldsymbol{b}}\boldsymbol{\cdot}\nabla\langle f\rangle\bigg|_{orbit-0}=-v_\parallel\frac{\mathrm{d}\langle f\rangle}{\mathrm{d}r}\int_{t_0}^{t}\mathrm{d}t'\tilde{b}_r\bigg|_{orbit-0}. \tag{A1}$$

The orbit-0 are identified by the following equations:

$$\theta(t')=\theta(t)+\frac{v_\parallel}{r}b_{0,\theta}(t'-t),\quad \phi(t')=\phi(t)+\frac{v_\parallel}{R}b_{0,\phi}(t'-t). \tag{A2a,b}$$

The distinction between the velocity parallel to $\boldsymbol{B}$ and that parallel to $\boldsymbol{B}_0$ gives a higher-order correction in (A1), and hence it is neglected. The term $\mathrm{d}\langle f\rangle/\mathrm{d}r$ is put outside the integral for two reasons: (i) the orbit-0 develop at constant r and (ii) the time dependence of $\langle f\rangle$ can be neglected as a higher-order effect, because $\partial\langle f\rangle/\partial t\sim(v_\parallel/a)\,o(\delta^2)\langle f\rangle$ according to (2.6). In (A1) we Fourier-expand $\tilde{b}_r$ (see (2.8)),

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exploiting (A2):

$$\tilde{f}-\tilde{f}_0=-v_\parallel \mathrm{d}\langle f\rangle/\mathrm{d}r\sum_{m,n}\left\{\int_{t_0}^{t}\mathrm{d}t'\,\mathrm{e}^{\mathrm{i}(v_\parallel/r)b_{0,\theta}(t'-t)[m-nq]}\right\}\tilde{b}_r^{m,n}(r)\,\mathrm{e}^{\mathrm{i}[m\,\theta(t)-n\,\phi(t)]}. \tag{A3}$$

By the variable change $(v_\parallel/r)b_{0,\theta}(t'-t)=x'$, and letting $t_0\to-\infty, \tilde{f}_0=0$ one gets

$$\tilde{f}=-\mathrm{sgn}(v_\parallel)\left|\frac{Rq}{b_{0,\phi}}\right|\frac{\mathrm{d}\langle f\rangle}{\mathrm{d}r}\sum_{m,n}\mathcal{R}(m-nq)\tilde{b}_r^{m,n}\,\mathrm{e}^{\mathrm{i}[m\theta-n\phi]}, \tag{A4}$$

with $\mathcal{R}$ the following resonant function:

$$\mathcal{R}(m-nq)=\begin{cases}\int_{-\infty}^{0}\mathrm{d}x'\,\mathrm{e}^{\mathrm{i}x'[m-nq]} & v_\parallel b_{0,\theta}>0,\\ \int_{0}^{+\infty}\mathrm{d}x'\,\mathrm{e}^{\mathrm{i}x'[m-nq]} & v_\parallel b_{0,\theta}<0.\end{cases} \tag{A5}$$

Then, solution (A4) is replaced into the term $v_\parallel\langle\tilde{\boldsymbol{b}}\cdot\nabla\tilde{f}\rangle$ of (2.6), written as

$$v_\parallel\langle\tilde{\boldsymbol{b}}\cdot\nabla\tilde{f}\rangle=v_\parallel\langle\nabla\cdot(\tilde{\boldsymbol{b}}\tilde{f})\rangle-v_\parallel\langle\tilde{f}\nabla\cdot\tilde{\boldsymbol{b}}\rangle=v_\parallel\frac{1}{r}\frac{\partial}{\partial r}r\langle\tilde{b}_r\tilde{f}\rangle-v_\parallel\langle\tilde{f}\nabla\cdot\tilde{\boldsymbol{b}}\rangle. \tag{A6}$$

From (2.3), discarding $o(\delta^2)$ terms, we have

$$\begin{aligned}\nabla\cdot\tilde{\boldsymbol{b}}&\cong\boldsymbol{B}_1\cdot\nabla(1/B_0)-\boldsymbol{b}_0\cdot\nabla(\boldsymbol{B}_0\cdot\boldsymbol{B}_1/B_0^2)\\&=-B_{1,r}/B_0L_0^{-1}-ib_{0,\theta}/r\sum_{m,n}(m-nq)(b_{0,\theta}B_{1,\theta}^{m,n}+b_{0,\phi}B_{1,\phi}^{m,n})/B_0\,\mathrm{e}^{\mathrm{i}(m\theta-n\phi)}.\end{aligned} \tag{A7}$$

In a tokamak, (A7) gives $\nabla\cdot\tilde{\boldsymbol{b}}=o(\delta\epsilon^2)$, $\epsilon=a/R$, according to the ordering of the field components in this device (Wesson 2004): in this case the second term on the right-hand side of (A6) can be discarded, as done in Rice *et al.* (1982). But, this is not true in general, and hence we retain it.

By inserting (A4), (A7) into (A6) and performing the angular averages $\langle\rangle$, one gets

$$\begin{aligned}v_\parallel\langle\tilde{\boldsymbol{b}}\cdot\nabla\tilde{f}\rangle=&-|v_\parallel|\frac{1}{r}\frac{\partial}{\partial r}\left\{r\frac{\mathrm{d}\langle f\rangle}{\mathrm{d}r}\left|\frac{Rq}{b_{0,\phi}}\right|\sum_{m,n}\mathcal{R}(m-nq)|\tilde{b}_r^{m,n}|^2\right\}\\&-|v_\parallel|L_0^{-1}\frac{\mathrm{d}\langle f\rangle}{\mathrm{d}r}\left|\frac{Rq}{b_{0,\phi}}\right|\sum_{m,n}\mathcal{R}(m-nq)|\tilde{b}_r^{m,n}|^2\\&+|v_\parallel|\frac{b_{0,\theta}}{r}\frac{\mathrm{d}\langle f\rangle}{\mathrm{d}r}\left|\frac{Rq}{b_{0,\phi}}\right|\sum_{m,n}\mathcal{R}(m-nq)(m-nq)\tilde{b}_r^{m,n}i(\boldsymbol{b}_0\cdot\boldsymbol{B}_1^{m,.n})^*/B_0.\end{aligned} \tag{A8}$$

Then, we limit the sum to the modes $m\in\mathbb{Z}, n>0$, by including the contribution of the complex-conjugate harmonic $(-m,-n)$. Since $\mathcal{R}(m-nq)+\mathcal{R}(nq-m)=2\pi\delta(m-nq)$,

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we get

$$v_\parallel \langle \tilde{\boldsymbol{b}} \boldsymbol{\cdot} \nabla \tilde{f} \rangle = -|v_\parallel| \frac{1}{r}\frac{\partial}{\partial r}\left\{ r \frac{\mathrm{d}\langle f\rangle}{\mathrm{d}r} \left|\frac{Rq}{b_{0,\phi}}\right| \sum_{\substack{m\in\mathbb{Z}\\ n>0}} 2\pi\,\delta(m-nq)|\tilde{b}_r^{m,n}|^2 \right\} - |v_\parallel| L_0^{-1} \frac{\mathrm{d}\langle f\rangle}{\mathrm{d}r} \left|\frac{Rq}{b_{0,\phi}}\right| \sum_{\substack{m\in\mathbb{Z}\\ n>0}} 2\pi\,\delta(m-nq)|\tilde{b}_r^{m,n}|^2. \tag{A9}$$

The last summation on the right-hand side of (A8) is discarded, by assuming $\tilde{b}_r^{m,n} i(\boldsymbol{b}_0 \boldsymbol{\cdot} \boldsymbol{B}_1^{\boldsymbol{m},\boldsymbol{n}})^*$ to be a real quantity, as is the case for tearing modes in the presence of an ideal shell (Fitzpatrick 1999). In fact, with this assumption the sum is zero:

$$\begin{aligned}
&\sum_{m\in\mathbb{Z} n>0} \{\mathcal{R}(m-nq)(m-n\,q)\tilde{b}_r^{m,n} i(\boldsymbol{b}_0 \boldsymbol{\cdot} \boldsymbol{B}_1^{\boldsymbol{m},\boldsymbol{n}})^*/B_0 \\
&\quad + \mathcal{R}(nq-m)(m-nq)(\tilde{b}_r^{m,n} i)^* \boldsymbol{b}_0 \boldsymbol{\cdot} \boldsymbol{B}_1^{\boldsymbol{m},\boldsymbol{n}}/B_0\} \\
&= \sum_{m\in\mathbb{Z} n>0} 2\pi\,\delta(m-nq)(m-nq)\tilde{b}_r^{m,n} i(\boldsymbol{b}_0 \boldsymbol{\cdot} \boldsymbol{B}_1^{\boldsymbol{m},\boldsymbol{n}})^*/B_0 = 0.
\end{aligned} \tag{A10}$$

Replacing (A9) into (2.6) we obtain (2.7) with definition (2.8).

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