


ON FUNDAMENTAL FOURIER COEFFICIENTS OF SIEGEL MODULAR FORMS

SIEGFRIED BÖCHERER¹ AND SOUMYA DAS ^{2,3}

¹*Institut für Mathematik, Universität Mannheim, 68131 Mannheim, Germany*

(boecherer@math.uni-mannheim.de)

²*Department of Mathematics, Indian Institute of Science, Bangalore – 560012, India*

(soumya@iisc.ac.in)

³*Alexander von Humboldt Fellow, Universität Mannheim, 68131 Mannheim, Germany*

(sdas@mail.uni-mannheim.de)

(Received 10 May 2020; revised 3 February 2021; accepted 3 February 2021; first
published online 3 March 2021)

Abstract We prove that if F is a nonzero (possibly noncuspidal) vector-valued Siegel modular form of any degree, then it has infinitely many nonzero Fourier coefficients which are indexed by half-integral matrices having odd, square-free (and thus fundamental) discriminant. The proof uses an induction argument in the setting of vector-valued modular forms. Further, as an application of a variant of our result and complementing the work of A. Pollack, we show how to obtain an unconditional proof of the functional equation of the spinor L -function of a holomorphic cuspidal Siegel eigenform of degree 3 and level 1.

Keywords and phrases: Fourier coefficients, Siegel modular forms, fundamental discriminant, vector-valued, nonvanishing

2020 Mathematics Subject Classification: Primary 11F30
Secondary 11F50

1. Introduction

Fourier coefficients of Siegel modular forms have been objects of continued interest over the years. It is a very useful fact – not only theoretically, but also in computation – that such a form $F \in M_k^n$, where M_k^n (resp., S_k^n) is the space of (resp., cuspidal) Siegel modular forms on $\mathrm{Sp}(n, \mathbf{Z})$ of scalar weight k , is determined by finitely many of its Fourier coefficients $a_F(T)$, described uniformly in terms of the weight k and n . Here and henceforth, for $F \in M_k^n$ we write the Fourier expansion of F as

$$F(Z) = \sum_{T \in \Lambda_n} a_F(T) e(TZ),$$

where Λ_n is the set of all half-integral positive semi-definite matrices (see Section 2) of size n and $e(TZ) := \exp(2\pi i \mathrm{tr} TZ)$. Such a result is known in the literature as ‘Sturm’-bound, see [20, 26], even though this result was known from the works of Hecke or Maaß. For a

more recent version of this in the context of the determination of elliptic cusp forms by ‘square-free’ Fourier coefficients, see [3].

Equally important are results concerning the determination of a Siegel modular form F by Fourier coefficients $a_F(T)$ supported on $T \in \Lambda_n^+$ (which are positive definite members of Λ_n) with the discriminant of $2T$ (which we denote by $\text{disc}(2T)$) varying in an arithmetically interesting subset \mathcal{S} of natural numbers – for example, square-free numbers or fundamental discriminants. For the many works along this line of research, we refer the reader to the introductions in [3, 33]. Let us just note here that in the case of newforms of half-integral weights (via Waldspurger’s formula), such a result is equivalent to the determination of these forms by the twisted L -functions of their Shimura lifts.

When the set \mathcal{S} consists of all fundamental discriminants, Saha [33] proved an affirmative result on S_k^2 ; this result has applications to the representation theory of automorphic forms (see the discussion in [33, Introduction]). It is of course desirable to generalise the results of [33] to higher-degree Siegel cusp forms (including vector-valued modular forms), and to include the space of Eisenstein series. In fact, these aspects were mentioned as ‘difficult open’ problems in [33, remark 2.6, 2.7]. While we addressed the latter (in degree 2) in [10, Prop. 7.7], in this paper we settle both of these questions in the most general case (for full level), in particular including vector-valued modular forms. We note that one of the most natural settings for the problem at hand (also noted in [33, remark 2.7]) is to consider the set \mathcal{S} to be all those lattices $2T$ which are maximal in the set of even integral lattices of a given rank. It is this viewpoint that we consider in this paper.

Let us now state our main result. Let ρ be a polynomial, not necessarily irreducible representation of $\text{GL}(n, \mathbf{C})$. Denote by M_ρ^n the vector space of holomorphic vector-valued Siegel modular forms on $\text{Sp}(n, \mathbf{Z})$ with automorphy factor ρ (see Section 2 for more details) with ‘determinantal’ weight $k(\rho)$ (see Section 2 for the definition). We need one more piece of notation. Let $M \in \Lambda_n^+$ and denote by d_M its ‘absolute discriminant’ (i.e., ignoring the usual sign), defined by

$$d_M := |\text{disc}(2M)| = \begin{cases} \det(2M) & \text{if } n \text{ is even,} \\ \frac{1}{2} \det(2M) & \text{if } n \text{ is odd.} \end{cases}$$

Further, for $X \geq 1$, define

$$\mathfrak{S}_F(X) := \{d \leq X, d \text{ odd, square-free} \mid d_T = d \text{ for some } T \text{ and } a_F(T) \neq 0\}.$$

Define the function $\varrho(n)$ by

$$\varrho(n) = \begin{cases} 3/2 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd.} \end{cases} \tag{1.1}$$

Theorem 1.1. *Let $F \in M_\rho^n$ be nonzero and set $k(\rho) - \frac{n}{2} \geq \varrho(n)$. When n is even, assume that F is cuspidal. Then there exist infinitely many $\text{GL}(n, \mathbf{Z})$ -inequivalent $T \in \Lambda_n^+$ such that d_T is odd and square-free, and $a_F(T) \neq 0$. Moreover, the following stronger*

quantitative result holds: for any given $\epsilon > 0$,

$$\#\mathfrak{S}_F(X) \gg \begin{cases} X \cdot (\log X)^{-1/2} & \text{if } n \text{ is odd,} \\ X^{5/8-\epsilon} & \text{if } n \text{ is even and } F \text{ is cuspidal,} \\ X & \text{if } F \text{ is a scalar-valued, noncuspidal of weight } k, \\ & k \text{ is even, } k > n+1 \text{ and } n \text{ is odd.} \end{cases} \quad (1.2)$$

Here the implied constant depends only on F and ϵ .

Here we call $S, T \in \Lambda_n^+$ equivalent under $GL(n, \mathbf{Z})$ if there exists $U \in GL(n, \mathbf{Z})$ such that $T = U^t S U$. For a version of Theorem 1.1 where we count prime discriminants, see Theorem 5.1.

In particular, taking $\rho = \det^k$ (for $k - \frac{n}{2} \geq \varrho(n)$), Theorem 1.1 applies to scalar-valued Siegel modular forms of weight k . For more information about the lower bound on the quantity $k(\rho)$, see Remark 4.7. Since the T appearing in Theorem 1.1 arise from maximal (even) lattices, the statement of Theorem 1.1 also holds a fortiori for maximal lattices.

We add here that the different lower bounds for the quantity $\#\mathfrak{S}_F(X)$ in Theorem 1.1 depending on the parity of the degree n occur due to our different treatment of these cases. The first and last lower bounds emanate from an argument involving multiplicity 1 for integral weights, whereas the second relies on the existence of unconditional bounds on Fourier coefficients of half-integral elliptic cusp forms. Let us mention that if n is odd, we encounter integral weights, and half-integral weights otherwise. The reader may note that when n is even, we do not have a result on noncusp forms. This is due to some complications arising from half-integral weights.

Our proof uses induction on the degree n , with the Fourier–Jacobi expansion as a main tool. The proof clearly decomposes into a preparatory part (called *Part A*), of algebraic and number-theoretic considerations, and an analytic part (called *Part B*), where nonvanishing properties of Fourier coefficients for elliptic modular forms of half-integral or integral weights via some version of the Rankin–Selberg method play an essential role.

The steps of Part A

The main aim of this part is to reduce the question to a problem on certain elliptic modular forms. The results in this part should hold more generally over the classical tube domains I–IV (as in, e.g., [44]), but we do not pursue that here, mainly because such a treatment might obscure the technical points of this paper. We may return to this point in a future work.

Step 1. This step assures the existence of a nonvanishing Jacobi coefficient $\varphi_T = \varphi_T(\tau, \mathfrak{z})$ of F , $T \in \Lambda_{n-1}^+, \tau \in \mathbf{H}, \mathfrak{z} \in \mathbf{C}^{(1, n-1)}$, with discriminant of T odd and square-free. To prove this, we consider the Taylor expansion of F with respect to \mathfrak{z} around the origin. Then the nonvanishing Taylor coefficients of the lowest homogeneous degree give rise to a possibly vector-valued modular form of degree $n - 1$. By induction, this modular form of degree $n - 1$ has a nonvanishing Fourier coefficient indexed by $T \in \Lambda_{n-1}^+$, with d_T odd and

square-free. For the original modular form F of degree n , this assures the existence of a nonvanishing φ_T , with T as before. We mention here that in order to make the induction work, we have to deal with vector-valued modular forms from the beginning, even if we started from scalar-valued modular forms (see Proposition 3.1).

Step 2. This step ensures that the (possibly vector-valued) nonvanishing Fourier–Jacobi coefficient φ_T from Step 1 has a nonvanishing component $\varphi_T^{(r)}$ which is actually a scalar-valued Jacobi form. It is this Jacobi form which we will focus on throughout the rest of our proof.

Step 3. This step is concerned with the theta expansion of $\varphi_T^{(r)}$: we show that there exists at least one nonzero theta-component $h_\mu = h_\mu(\tau)$ of $\varphi_T^{(r)}$, $\mu \in \mathbf{Z}^{n-1}/(2T)\mathbf{Z}^{n-1}$, such that $T^{-1}[\mu/2]$ has the highest possible denominator (essentially equal to d_T); we call such μ ‘primitive’. This result (Proposition 3.5) is of independent interest, and is an intrinsic result in the theory of Jacobi forms. We finish Step 3 by setting up the desired nonvanishing properties for the Fourier coefficients of such h_μ , which follows from Part B, to prove our theorem. Here we encounter both integral and half-integral weights according to the parity of $n \bmod 2$. We also have to take special care of the prime $p = 2$ during the induction step (while passing from odd to even degrees), and ensure we do not get unnecessary high powers of 2 (see Section 3.4).

The steps of Part B

Our induction steps in Part A do not ‘see’ whether the modular form is cuspidal or not; however, in this part such a distinction becomes prominent. Moreover, let us note that *Part B* actually serves two purposes:

- (i) It covers the base case $n = 1$ of the induction procedure – that is, it proves Theorem 1.1 when $f \in M_k^1$ with $k \in \mathbf{N}$. If $f \in S_k^1$, then such a result is already known from [3, Thm. 6 and Prop. 5.8]. The extension to M_k^1 is trivial.
- (ii) More importantly, it helps to glue the nonzero ‘square-free’ Fourier coefficients of h_μ (see Section 3.4) with $T \in \Lambda_{n-1}^+$ to obtain some $\mathcal{T} \in \Lambda_n^+$ which is also ‘square-free’. The treatment of these h_μ , however, leads us to both integral and half-integral weights over the principal congruence subgroups.

Thus, in the following discussion, we focus only on (ii). We give two approaches for the analytic part: Method 1 (see Sections 4.1.1 and 4.1.2) and Method 2 (see Section 4.2). We feel each method has its own advantages and limitations, which we discuss later.

Step 1. The analytic part first analyses the Fourier expansion of the degree 1 cusp forms h_μ for primitive μ from Step 3 of Part A. Here some analytic number theory of modular forms comes in, and we essentially adapt an argument from [3, 33], using either a classical Rankin–Selberg method or a ‘smoothed’ version (Method 1) in the case of half-integral weights, to the groups $\Gamma_1(N)$. The primitiveness of μ is crucial here. This method has the advantage that it holds uniformly for cusp forms with either integral or

half-integral weights, and does not depend on any multiplicity 1 result (see Theorem 4.5). In the sequel, we use only the result for cusp forms with half-integral weights, as for those with integral weights we prove a *better* quantitative result (see Theorem 4.6), which relies on multiplicity 1 (Method 2). The details can be found in Sections 4.1.1 and 4.1.2.

Step 2. The remaining step is to treat noncusp forms. When $n = 1$, we are reduced to usual elliptic modular forms, for which we present a new method (Method 2) which is actually robust enough to deal with both cusp and noncusp forms! (See Section 4.2 for the details.) Let us only mention here that this method crucially relies on multiplicity 1 for the newspaces and applies only for integral weights. We therefore assume here that n is odd so that the weight of h_μ is integral. But the quality of the quantitative result is better than what can be obtained by Method 1 (see Theorem 4.6).

We finally combine the main results from these steps and finish the proof in Section 4.3.

To put things into perspective, let us mention that in degree 2 our proof looks somewhat similar to that in [33] (here the setting is that of scalar-valued cusp forms), in that we also reduce the question to a suitable Jacobi cusp form, say ϕ . However, there are quite a few interesting differences:

- (i) Instead of using the Eichler–Zagier map to reduce the question further to half-integral-weight elliptic modular forms, we work directly with any of the ‘primitive’ theta-components of ϕ (i.e., those theta-components h_μ for which $(\mu, 4m) = 1$, where $m = \text{index of } \phi$). These h_μ automatically have Fourier expansion supported away from the level, so the analytic treatment becomes easier (compare [3, Prop. 5.1] and [33, Prop. 3.7]). More importantly, these primitive theta-components are crucial for us, since we are led to deal with levels which are squares, and these levels do not satisfy the conditions of [3, Thm. 2] or [33, Thm. 2].
- (ii) Our induction procedure only allows for the index m to be square-free, whereas in [33] one could take m to be an odd prime. This is not serious when $n = 2$, but for higher degrees it is a nontrivial point; it may not be possible to choose a nonzero Fourier–Jacobi coefficient φ_T with $T \in \Lambda_{n-1}^+$ and d_T a prime (see Remark 3.2). However, we show in Corollary 3.3 that one can always choose such a $\varphi_T \neq 0$ with d_T odd and square-free.
- (iii) By choosing m (sticking to $n = 2$ for illustration) odd and square-free and invoking Proposition 3.5, we avoid the subtlety of the injectiveness of the Eichler–Zagier map (this was a nontrivial difficulty in [3]). This injectiveness property ensures smooth passage from Jacobi forms to elliptic modular forms, and is known only when m is a prime and $n = 2$. For degrees $n \geq 3$, the theory of Eichler–Zagier maps is not well developed. We circumvent this by using what we call the ‘primitive’ theta-components of φ_T . Moreover, we believe that this method should work with suitable modifications for other kinds of modular forms, such as Hermitian modular forms.

Concerning an important application of our main result, let us recall some recent work of Pollack [31], where a meromorphic continuation of the degree 8 spinor L -function $Z_F(s)$ attached to a Siegel Hecke eigenform F on $\text{Sp}(3, \mathbf{Z})$ was proved. Further, it was shown

that the functional equation follows under the assumption of nonvanishing of some Fourier coefficient $a_F(T)$, with T corresponding to a maximal order in a quaternion algebra over \mathbf{Q} ramified at ∞ . In the last section of this paper, we show, as an application of a variant of our main result, how to remove the aforementioned assumption to get an unconditional result, which may be stated as follows.

Let $\Lambda_F(s)$ denote the completed spinor L -function attached to the Hecke eigenform F (see, e.g., [31, p. 2] for a description of the gamma factors).

Theorem 1.2. *Let F be a nonzero Siegel cuspidal eigenform form on the group $\mathrm{Sp}(3, \mathbf{Z})$ of weight $k \geq 3$. Then the spinor L -function $Z_F(s)$ attached to F has a meromorphic continuation to \mathbf{C} , is bounded in vertical strips and satisfies the functional equation $\Lambda_F(s) = \Lambda_F(1-s)$.*

As indicated before, note that the proof of Theorem 1.2 does not use Theorem 1.1 directly; instead, it uses a variant of it, which is proved in a self-contained manner in Section 5. To put things into perspective, let us note here that the analytic properties of the spinor L -function for eigenforms on $\mathrm{Sp}(n, \mathbf{Z})$ were conjectured by Andrianov (they are of course special cases of Langlands conjectures; note that we are dealing with nongeneric automorphic forms here), and proved by him when $n = 2$ (compare [4]). The meromorphic continuation of the spinor L -function for eigenforms on $\mathrm{Sp}(3, \mathbf{Z})$ is known from the work of Asgari and Schmidt (compare [7]), but obtaining the functional equation of these objects is a delicate matter, and was not known for $n \geq 3$. Thus, Pollack's work combined with our results from this paper shows the functional equation unconditionally for the first time when $n = 3$.

As another application, let us mention that if one studies the standard L -function via the Andrianov identity [5], one has to use a Rankin convolution involving a theta series attached to a quadratic form T . It is quite convenient to know from the beginning (from Theorem 1.1) that one may choose T to have square-free discriminant (and hence the nebentypus character of the theta series is a primitive quadratic character; see [5, 6, 15] for details).

As a last remark, let us mention that we have not considered the case of higher levels, as the content of the paper is already quite technical; but it definitely is an interesting problem to consider. Let us just mention that our methods should also work in this more general setting, but we expect more complicated answers (compare [3] for $n = 1$). One may have to take into account the Fourier expansions at all cusps simultaneously, and one may expect new difficulties concerning primes dividing the level.

2. Notation and preliminaries

2.1. General notation

- (1) Let $\rho: \mathrm{GL}(n, \mathbf{C}) \rightarrow \mathrm{GL}(V)$ be a finite-dimensional (not necessarily irreducible) rational representation (a morphism in the sense of algebraic groups) with $m = \dim(V)$. We call ρ a *polynomial representation* if it can be realised as a map $\rho: \mathrm{GL}(n, \mathbf{C}) \rightarrow \mathrm{GL}(m, \mathbf{C})$ where all the coordinate functions $g \mapsto \rho_{ij}(g)$ are given by polynomial functions of the entries of g .

For any such ρ , there exists a largest integer k such that $\det^{-k} \otimes \rho$ is polynomial, and we call this k the (*determinantal*) *weight* $k(\rho)$. We tacitly use the fact that this weight does not decrease if we tensor ρ with another polynomial representation or restrict it to some $\mathrm{GL}(n', \mathbf{C})$ sitting inside $\mathrm{GL}(n, \mathbf{C})$ as an algebraic subgroup. This follows easily by looking at the entries of $\rho(g)$ in any matrix realisation of ρ .

- (2) For a commutative ring R with 1, we denote by $M_{m,n}(R)$ the set of $m \times n$ matrices with coefficients in R . If $m = n$, we set $M_{n,n}(R) = M_n(R)$. We denote the transpose of a matrix M by M^t . Further, for matrices A, B of appropriate sizes, $A[B] := B^t AB$. We denote the $n \times n$ identity matrix over a subring of \mathbf{C} by 1_n . For quantities $a_1, \dots, a_n \in R$, we denote by $\mathrm{diag}(a_1, \dots, a_n)$ the matrix consisting of the diagonal entries as a_1, \dots, a_n .

Further, \mathbf{Z}_q denotes the ring of q -adic integers for a prime q , and ν_q the q -adic valuation.

- (3) We define the set of half-integral, symmetric, positive semi-definite matrices by

$$\Lambda_n := \{S = (s_{i,j}) \in M(n, \mathbf{Q}) \mid S = S^t, s_{i,i} \in \mathbf{Z}, s_{i,j} \in \frac{1}{2}\mathbf{Z}, \text{ and } S \text{ is positive semi-definite}\},$$

and denote the subset of positive definite matrices in Λ_n by Λ_n^+ .

- (4) For T real and $Z \in M_n(\mathbf{C})$, we define $e(TZ) := \exp(2\pi i \mathrm{tr}(TZ))$, where $\mathrm{tr}(M)$ is the trace of the matrix M .
- (5) Throughout the paper, ε denotes a small positive number which may vary at different places. Moreover, the symbols $A \ll_c B$ and $O_S(T)$ have their standard meaning, implying that the constants involved depend on c or the set S .

2.2. Siegel modular forms

We denote by

$$\mathbf{H}_n := \{Z \in M_n(\mathbf{C}) \mid Z = Z^t, \Im(Z) > 0\}$$

the Siegel upper half space of degree n . The symplectic group $\mathrm{Sp}(n, \mathbf{R})$ acts on \mathbf{H}_n by $Z \mapsto g\langle Z \rangle = (AZ + B)(CZ + D)^{-1}$; for a polynomial representation ρ with values in $\mathrm{GL}(V)$, we define the stroke operator action on V -valued functions F on \mathbf{H}_n by

$$(F |_\rho g)(Z) := \rho(CZ + D)^{-1} F(g\langle Z \rangle).$$

A Siegel modular form of degree n and automorphy factor ρ is then a V -valued holomorphic function F on \mathbf{H}_n satisfying $F |_\rho \gamma = F$ for all $\gamma \in \mathrm{Sp}(n, \mathbf{Z})$ with the standard additional condition in degree 1. We denote by M_ρ^n the vector space of all such functions and by S_ρ^n the subspace of cusp forms; if ρ is scalar-valued, we write as usual M_k^n and S_k^n if $\rho = \det^k$. An element $F \in M_\rho^n$ has a Fourier expansion

$$F(Z) = \sum_{S \in \Lambda_n} a_F(S) e(SZ), \quad a_F(S) \in V.$$

If F is cuspidal, then this summation is supported on Λ_n^+ . Sometimes we also write $a(F, S)$ for $a_F(S)$.

This definition makes sense for arbitrary rational ρ , but by a theorem of Freitag [21] (if ρ is irreducible), M_ρ^n can be nonzero only if ρ is polynomial; therefore, we only have to take care of polynomial representations.

For $g \in \text{Sp}(n-1, \mathbf{R})$ we denote by g^\downarrow the image of g under the diagonal embedding

$$\text{Sp}(n-1, \mathbf{R}) \hookrightarrow \text{Sp}(n, \mathbf{R}), \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & d \end{pmatrix}. \tag{2.1}$$

We also use the embedding of $\text{GL}(n-1, \mathbf{R}) \hookrightarrow \text{GL}(n, \mathbf{R})$ given by $g \mapsto \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix}$.

Definition 2.1. Let $f \in M_k^n$ and $Z \in \mathbf{H}_{n-1}$. The Siegel Φ -operator is then defined by

$$\Phi(F)(Z) := \lim_{t \rightarrow \infty} F \begin{pmatrix} Z & 0 \\ 0 & it \end{pmatrix} = \sum_{T \in \Lambda_{n-1}} a_F \left(\begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix} \right) e(TZ). \tag{2.2}$$

Then it is well known [20] that $\Phi(F) \in M_k^{n-1}$. Moreover, $F \in M_k^n$ is a cusp form if and only if F is in the kernel of the Φ operator.

2.3. Jacobi forms

Throughout this paper, we use a decomposition for $\mathcal{Z} \in \mathbf{H}_n$ into blocks as follows:

$$\mathcal{Z} = \begin{pmatrix} \tau & \mathfrak{z} \\ \mathfrak{z}^t & Z \end{pmatrix}, \quad \mathfrak{z} \in \mathbf{C}^{(1, n-1)}, Z \in \mathbf{H}_{n-1}. \tag{2.3}$$

Clearly, every $F \in S_\rho^n$ has a Fourier–Jacobi expansion with respect to this decomposition:

$$F(\mathcal{Z}) = \sum_{T \in \Lambda_{n-1}} \varphi_T(\tau, \mathfrak{z}) e(TZ). \tag{2.4}$$

The φ_T are then ‘Jacobi forms’ of automorphy factor ρ and index T – that is, the functions $\psi(\mathcal{Z}) := \varphi_T(\tau, \mathfrak{z}) e(TZ)$ on \mathbf{H}_n are holomorphic, satisfy $\psi|_\rho g = \psi$ for all $g \in C_{n, n-1}(\mathbf{Z}) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid (C, D) = \begin{pmatrix} * & 0 & * & D_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}$, where $*$ denotes some scalar entries, and satisfy the boundedness condition (Fourier expansion at ∞) that

$$\psi(\mathcal{Z}) = \sum_{S = \begin{pmatrix} n & r/2 \\ r^t/2 & T \end{pmatrix} \in \Lambda_n} a_\psi(S) e(S\mathcal{Z}).$$

Note that this definition of vector-valued Jacobi forms does not agree with the one in [45]; for degree 2 our definition is the same as in [22]. In our setup, where we work with Fourier–Jacobi coefficients, this definition is obtained from the corresponding automorphy for Siegel modular forms, and we must use it. In [45], ρ acts only on the Siegel upper half space variable τ . Unless the automorphy factor is $\rho = \det^k$, the definition given here does not match with that in [45] (compare [22, p. 785, eq. (1)]).

The case $\rho = \det^k$ is well known (see, e.g., [26, 45]); in particular, scalar-valued Jacobi forms ϕ_T admit a ‘theta expansion’

$$\varphi_T(\tau, \mathfrak{z}) = \sum_{\mu} h_{\mu}(\tau) \cdot \Theta_T[\mu](\tau, \mathfrak{z}) \tag{2.5}$$

with summation over $\mu \in \mathbf{Z}^{n-1}/2T \cdot \mathbf{Z}^{n-1}$ and

$$\Theta_T[\mu](\tau, \mathfrak{z}) = \sum_{R \in \mathbf{Z}^{n-1}} e^{2\pi i(T[R+\tilde{\mu}]\tau + 2\mathfrak{z}T(R+\tilde{\mu}))}.$$

Here we use $\tilde{\mu} := (2T)^{-1} \cdot \mu$. We note here that the h_{μ} are then modular forms of weight $k - \frac{n-1}{2}$ on some congruence subgroup; the Fourier expansion of h_{μ} is of shape

$$h_{\mu}(\tau) = \sum_n a_{\mu}(n - T^{-1}[\mu/2]) e^{2\pi i(n - T^{-1}[\mu/2]) \cdot \tau} \tag{2.6}$$

and its Fourier coefficients are given by

$$a_{\mu}(n - T^{-1}[\mu/2]) = a_F \left(\begin{pmatrix} n & \mu/2 \\ \mu/2^t & T \end{pmatrix} \right), \tag{2.7}$$

provided that φ_T is the Fourier–Jacobi coefficient of some scalar-valued Siegel modular form $F \in M_k^n$. We denote the space of (scalar-valued) Jacobi forms of weight k and index T by $J_{k,T}$.

2.4. Maximal lattices and primitivity

When we talk about the ‘lattice M ’ for some $M \in \Lambda_n^+$, of course we are tacitly identifying $2M$ with the ‘even’ lattice $\mathcal{L}_M := (\mathbf{Z}^n, \mu \mapsto (2M)[\mu])$ inside \mathbf{Q}^n . We omit the accompanying quadratic form from the notation when there is no danger of confusion. *Henceforth, throughout this paper we assume that d_M is odd and square-free*; in particular, this implies that $2M$ corresponds to a maximal lattice – in other words, there exists no even integral lattice properly containing \mathcal{L}_M . This can be seen easily: if $\mathcal{L}_M \subsetneq \mathcal{L}$ for another even lattice \mathcal{L} with gram matrix A , then $\mathcal{L}_M = H \cdot \mathcal{L}$ for some $H \in M_n(\mathbf{Z})$, and $d_M = \det(H)^2 d_A$. Clearly H can not be unimodular, and thus d_M could not have been square-free.

Let us recall that the level ℓ_M of $2M$ is the smallest $\ell \geq 1$ such that $\ell \cdot (2M)^{-1}$ is even integral. We next compute the level in terms of the (absolute) discriminant.

Lemma 2.2. *Let d_M be odd and square-free. Then ℓ_M is equal to d_M if n is even and equal to $4d_M$ if n is odd.*

Proof. When n is even, this follows from the facts that $\ell_M \mid d_M$ and $d_M \mid \ell_M^n$. When n is odd, we have $\ell \mid 4d_M, 4d_M \mid \ell^n$, which imply $2d_M \mid \ell \mid 4d_M$. To get the exact power of 2 dividing ℓ , we appeal to the local theory of quadratic forms. Namely, we know that $(2M)[U_2] = \mathbb{H} \perp \dots \perp \mathbb{H} \perp 2$ for some $U_2 \in \text{GL}_n(\mathbf{Z}_2)$, with \mathbb{H} being the hyperbolic plane (see Section 3.3.1 for more discussion). Our claim then follows from the following facts:

- (i) The levels of $2M$ and $(2M)[U_2]$ are the same.
- (ii) For the even quadratic form $\mathbb{H} \perp \cdots \perp \mathbb{H} \perp 2$ over \mathbb{Z}_2 , the level clearly equals 4.
- (iii) The level of $2M$ over \mathbf{Z}_2 divides that over \mathbf{Z} . □

Proposition 2.3. *Let d_M be odd and square-free. There is a $\mu \in \mathbf{Z}^n$ such that $\frac{1}{4}M^{-1}[\mu]$ has exact denominator $d = d_M$ if n is even and $4d$ if n is odd.*

This proposition is crucial for us, and we will call such μ *primitive* in the sequel.

Proof. To show this, we first note that maximality is a local property (see, e.g., [25]) and work locally. For any prime p , let us denote the (maximal) lattice $\mathcal{L}_M \otimes \mathbf{Z}_p$ by $\mathcal{L}_{M,p}$ and use the identification $\mathcal{L}_{M,p} := (\mathbf{Z}_p^n, \mu \mapsto (2M)[\mu])$ – or simply $(\mathbf{Z}_p^n, 2M)$ – inside \mathbf{Q}_p^n . We now prove the crucial property in the following claim:

Claim 1. *For any prime p and $\mu \in \mathbf{Z}_p^n$, we have the property*

$$(2M)^{-1}[\mu] \in 2\mathbf{Z}_p \quad \text{if and only if} \quad \mu \in 2M \cdot \mathbf{Z}_p^n \quad (n \geq 1). \tag{2.8}$$

To prove the claim, let us note that the ‘if’ statement is trivial. In the other direction, for any prime p , write $\mu = (2M) \cdot \tilde{\mu}$ and assume that $\tilde{\mu} \notin \mathbf{Z}_p^n$. Consider the lattice $\tilde{\mathcal{L}} := \mathcal{L}_{M,p} + \langle \tilde{\mu} \rangle$, where we have defined $\langle \tilde{\mu} \rangle := \mathbf{Z}_p \cdot \tilde{\mu}$. Then as lattices in \mathbf{Q}_p^n carrying the even integral quadratic form $2M$ (now viewed over \mathbf{Z}_p^n), clearly $\mathbf{Z}_p^n \subsetneq \tilde{\mathcal{L}}$. We will be done if we can show that $(\tilde{\mathcal{L}}, 2M)$ is even integral, as this will contradict the maximality of $(\mathcal{L}_{M,p}, 2M)$.

Let $\nu \in \tilde{\mathcal{L}}$. Writing $\nu := \beta + c\tilde{\mu}$, $\beta \in \mathcal{L}_{M,p}, c \in \mathbf{Z}_p$, we see that $(2M)[\nu] \in 2\mathbf{Z}_p$ if and only if $(2M)[\tilde{\mu}] = (2M)^{-1}[\mu] \in 2\mathbf{Z}_p$. Therefore we are done with Claim 1.

Let us now proceed to prove Proposition 2.3. We choose $\mu \in \mathbf{Z}^n$ such that for all $p \mid \det(2M)$ we have $\mu \notin 2M \cdot \mathbf{Z}_p^n$. This can certainly be done locally. Note that $2M$ is equivalent over \mathbf{Z}_p to the quadratic form $(*\perp*\perp\cdots\perp*\perp*p)$ for odd p (see, e.g., [14, Thm. 3.1]), where the $*$ are units; and similarly for $p = 2$, see the proof of Claim 2 later. Then by strong approximation we get $\mu \in \mathbf{Z}^n$ with the requested properties.

We claim that such a μ is primitive. This follows from first showing that d and $4d$ are indeed the largest possible denominators (because of the level of M). We therefore can write for some $\alpha = \alpha_p \in \mathbf{Z}_p$ (we remind the reader that d is odd),

$$\frac{1}{4}M^{-1}[\mu] = \frac{1}{2}(2M)^{-1}[\mu] = \begin{cases} \frac{\alpha}{d} & \text{if } p \text{ is odd,} \\ \frac{\alpha}{4d} & \text{if } p = 2. \end{cases} \tag{2.9}$$

We have to check that $(\alpha_p, p) = 1$. When p is odd, things are smooth, and the lemma follows from property!(2.8) in Claim 1, along with our choice of μ .

When $p = 2$ and n is odd, then we need to take some care. Namely, we observe the following, whose proof is deferred to the end of the proof of this proposition:

Claim 2.

$$(2M)^{-1}[\mu] \in \mathbf{Z}_2 \quad \text{if and only if} \quad (2M)^{-1}[\mu] \in 2\mathbf{Z}_2. \tag{2.10}$$

Granting this claim, we can now finish the proof of Proposition 2.3 when $p = 2$. If in equation (2.9) $\alpha = \alpha_2$ is odd, we are done. Otherwise, setting $\alpha' = \alpha/2 \in \mathbf{Z}_2$, we get $(2M)^{-1}[\mu] = \alpha'/d$. Since d is odd, this implies that $(2M)^{-1}[\mu] \in \mathbf{Z}_2$, which means, by Claim 2, that $(2M)^{-1}[\mu] \in 2\mathbf{Z}_2$, contradicting Claim 1.

It remains to prove Claim 2. To do that, we appeal to the dyadic theory of quadratic forms (see, e.g., [14, Lem. 4.1]) to recall that since n is odd and $\nu_2(\det(2M)) = 1$, $2M$ is equivalent to the quadratic form $\mathbb{H} \perp \cdots \perp \mathbb{H} \perp 2$ over \mathbf{Z}_2 . Here $\mathbb{H} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is a hyperbolic plane and \perp means the orthogonal direct sum. This shows that with $\mu = (\mu_1, \dots, \mu_n)^t$,

$$(2M)^{-1}[\mu] = \frac{1}{2}\mu_n^2 + 2\mathbf{Z}_2,$$

from which our claim follows. □

3. Part A: Algebraic aspects

3.1. Step 1: A nonvanishing property for Fourier–Jacobi coefficients

The statement given in Proposition 3.1 seems to be new only when we start from a vector-valued function. In the scalar-valued case, variants have appeared in works of Eichler and Zagier [18], Yamana [44], Ibukiyama and Kyomura [22] and others.

We consider the following situation. Let $\rho: \text{GL}(n, \mathbf{C}) \rightarrow \text{GL}(V)$ be a polynomial representation and $F: \mathbf{H}_n \rightarrow V$ be a holomorphic function, not identically zero. We decompose $\mathcal{Z} \in \mathbf{H}_n$ into blocks as in equation (2.3). We then consider the Taylor expansion of F as a function of $\mathfrak{z} = (z_2, \dots, z_n) \in \mathbf{C}^{(1, n-1)}$ as follows. We can write

$$F(\mathcal{Z}) = \sum_{\lambda} F_{\lambda}(\tau, Z) \mathfrak{z}^{\lambda}, \tag{3.1}$$

where $\lambda = (\lambda_2, \dots, \lambda_n) \in \mathbf{N}^{n-1}$ is a polyindex and $\mathfrak{z}^{\lambda} := z_2^{\lambda_2} \cdots z_n^{\lambda_n}$.

We set $\nu = \nu(\lambda) = \sum_{i=2}^n \lambda_i$ and define

$$\nu_0 := \min\{\nu(\lambda) \mid F_{\lambda} \neq 0\}.$$

Then we look at all the Taylor coefficients of homogeneous degree ν_0 and study a polynomial in variables X_2, \dots, X_n of homogeneous degree ν_0 :

$$F^o(\tau, Z) := \sum_{\lambda: \nu(\lambda) = \nu_0} F_{\lambda}(\tau, Z) X_2^{\lambda_2}, \dots, X_n^{\lambda_n}. \tag{3.2}$$

We may view F^o as a function on $\mathbf{H} \times \mathbf{H}_{n-1}$ with values in $V \otimes \mathbf{C}[X_2, \dots, X_n]_{\nu_0}$, where we denote by $\mathbf{C}[X_2, \dots, X_n]_{\nu_0}$ the \mathbf{C} -vector space of homogeneous polynomials of degree ν_0 .

For an integer $m \geq 1$, we denote by Sym^m the symmetric m th power representation of $\text{GL}(n, \mathbf{C})$ realised in the vector space of homogeneous polynomials over \mathbf{C} of degree m :

$$\begin{aligned} \text{Sym}^m: \text{GL}(n, \mathbf{C}) &\rightarrow \text{GL}(\mathbf{C}[Z_1, \dots, Z_n]_m), & g &\mapsto g \cdot f, & f &\in \mathbf{C}[Z_1, \dots, Z_n]_m, \\ & & (g \cdot f)(Z_1, \dots, Z_n) &:= f((Z_1, \dots, Z_n) \cdot g). \end{aligned}$$

Recall the embedding $\text{Sp}(n-1, \mathbf{R}) \hookrightarrow \text{Sp}(n, \mathbf{R})$ given by $g \mapsto g^{\downarrow}$, from formula (2.1).

Proposition 3.1. *Let the setting be as before. Then for any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(n-1, \mathbf{R})$,*

$$(F \mid_{\rho} g^{\downarrow})^{\circ}(\tau, Z) = \rho\left(\begin{pmatrix} 1 & 0 \\ 0 & cZ+d \end{pmatrix}\right)^{-1} \otimes \mathrm{Sym}^{\nu_0}(cZ+d)^{-1} F^{\circ}(\tau, g\langle Z \rangle). \tag{3.3}$$

In particular, if $F \in M_{\rho}^n$, then F° , viewed as a function of Z , is in $M_{\rho' \otimes \mathrm{Sym}^{\nu_0}}^{n-1}$, where ρ' is the restriction of ρ to $GL(n-1) \hookrightarrow GL(n)$ (compare Section 2.2). Moreover, if F is cuspidal, then F° is also cuspidal. Further, if $F \neq 0$, then for some $\tau = \tau_0$, $F^{\circ}(\tau_0, Z)$ is nonzero as a function of Z .

Proof. We recall [20] that

$$g^{\downarrow}\langle Z \rangle = \begin{pmatrix} \tau - \mathfrak{z}(cZ+d)^{-1}c_3^t & \mathfrak{z}(cZ+d)^{-1} \\ (cZ+d)^{-t}c_3^t & g\langle Z \rangle \end{pmatrix}.$$

Then we compute from equation (3.1) that

$$F \mid_{\rho} g^{\downarrow} = \rho\left(\begin{pmatrix} 1 & 0 \\ c_3^t & cZ+d \end{pmatrix}\right)^{-1} \sum_{\lambda} F_{\lambda}(\tau - \mathfrak{z}(cZ+d)^{-1}c_3^t, g\langle Z \rangle) (\mathfrak{z}(cZ+d)^{-1})^{\lambda}. \tag{3.4}$$

We pick out the contributions to \mathfrak{z}^{λ} with $\nu(\lambda) = \nu_0$: due to the minimality of ν_0 , all summands on the right-hand side have degree $\geq \nu_0$ as polynomials in \mathfrak{z} .

Now setting $\mathbf{h} = \mathbf{h}(\mathfrak{z}) := -\mathfrak{z}(cZ+d)^{-1}c_3^t$, and Taylor expanding around τ (with \mathfrak{z} in a sufficiently small neighbourhood of 0), we get

$$F_{\lambda}(\tau + \mathbf{h}, Z) = F_{\lambda}(\tau, Z) + O(\mathbf{h});$$

here $O(\mathbf{h})$ means a multiple of \mathbf{h} . Thus by minimality (see definition (3.2)), only $F_{\lambda}(\tau, g\langle Z \rangle)$ may contribute.

Moreover, only $\rho\left(\begin{pmatrix} 1 & 0 \\ 0 & cZ+d \end{pmatrix}\right)^{-1}$ has to be considered. To see this, note that

$$\rho\left(\begin{pmatrix} 1 & 0 \\ c_3^t & cZ+d \end{pmatrix}\right)^{-1} = \rho\left(\begin{pmatrix} 1 & 0 \\ -(cZ+d)^{-1}c_3^t & 1_{n-1} \end{pmatrix}\right) \cdot \rho\left(\begin{pmatrix} 1 & 0 \\ 0 & cZ+d \end{pmatrix}\right)^{-1}. \tag{3.5}$$

Let us observe here that considering the homogeneous decomposition of the quantity $\rho\left(\begin{pmatrix} 1 & 0 \\ -(cZ+d)^{-1}c_3^t & 1_{n-1} \end{pmatrix}\right)$ as a polynomial in \mathfrak{z} , we can write

$$\rho\left(\begin{pmatrix} 1 & 0 \\ -(cZ+d)^{-1}c_3^t & 1_{n-1} \end{pmatrix}\right) = 1_n + P(\mathfrak{z}),$$

where $P(\mathfrak{z}) \in M(n, \mathbf{C}) \otimes \mathbf{C}[\mathfrak{z}]$ has entries which are polynomial in \mathfrak{z} without a constant term. Since multiplication of the polynomial expression $\rho\left(\begin{pmatrix} 1 & 0 \\ 0 & cZ+d \end{pmatrix}\right)^{-1} F_{\lambda}(\tau, Z) (\mathfrak{z}(cZ+d)^{-1})^{\lambda}$ in \mathfrak{z} by $P(\mathfrak{z})$ can only increase its degree in \mathfrak{z} – which, however, already has degree ν_0 – it follows that only $\rho\left(\begin{pmatrix} 1 & 0 \\ 0 & cZ+d \end{pmatrix}\right)^{-1}$ has to be considered. This proves the automorphy of F° .

The assertion about the cuspidality of F° follows easily by looking at its Fourier expansion as a function of Z , and that about $F^{\circ}(\tau_0, Z)$ being not identically zero for some τ_0 is trivial. This proves the proposition. \square

Remark 3.2. The situation in [33] was very special: a result of Yamana [44] states that a nonvanishing Siegel cusp form of level 1 always has a nonvanishing Fourier coefficient

supported on a primitive (binary) quadratic form. From this, one could immediately get a nonvanishing Fourier–Jacobi coefficient of prime index. The argument here relies on the fact that a primitive binary quadratic form always represents infinitely many primes.

In order to pursue this procedure in our situation, say for degree 3, we would need to prove that every primitive ternary quadratic form represents a binary quadratic form whose determinant is square-free. Unfortunately this is not true in general; we give a counterexample.

From the local theory of ternary quadratic forms, we can find a ternary quadratic form T which for an odd prime p is equivalent over \mathbf{Z}_p to a form

$$\text{diag}(\epsilon, p^2, \mu \cdot p^2)$$

with $\epsilon, \mu \in \mathbf{Z}_p^\times$. Clearly, all binary quadratic forms integrally represented by such T have determinant divisible by p^2 . □

Let us now look at the Fourier–Jacobi expansion of F from equation (2.4). Let $T \in \Lambda_{n-1}^+$ and $\varphi_T(\tau, \mathfrak{z})$ be the Fourier–Jacobi coefficients of F . From the definition of $F^o(\tau_0, \mathfrak{Z})$ (compare definition (3.2)), we see that

$$a_{F^o}(T) = c \cdot \sum_{\nu(\lambda)=\nu_0} \frac{\partial^\lambda}{\partial \mathfrak{z}^\lambda} \varphi_T(\tau_0, \mathfrak{z})|_{\mathfrak{z}=0}, \tag{3.6}$$

where c is a nonzero constant independent of F or T .

Clearly if $a_{F^o}(T) \neq 0$, then $\varphi_T \neq 0$. Thus equation (3.6) provides us our avenue for carrying out an induction argument.

Corollary 3.3. *Let $C \in \mathbf{N}$ be given. Assume that Theorem 1.1 holds for all nonzero forms in M_θ^{n-1} for all polynomial representations θ of $\text{GL}(n-1, \mathbf{C})$ with $k(\theta) \geq C$. Then for any polynomial representation ρ with $k(\rho) \geq C$, all nonzero $F \in M_\rho^n$ have (infinitely many) nonvanishing Fourier–Jacobi coefficients φ_T with T of size $n-1$ and square-free odd discriminant.*

3.2. Step 2: Reduction to the case of scalar-valued Jacobi forms

The Fourier–Jacobi coefficients of (the vector-valued) F do have a theta expansion, which is more complicated than in the scalar-valued case. We do not pursue writing this down, but our aim here is to show that we may choose a nonzero (vector) component of φ_T , which behaves as a Jacobi form in the scalar-valued case. Throughout this section, we use the block decomposition (2.3).

We consider the two transformation laws responsible for the theta expansion. The first one is

$$\varphi_T(\tau, \mathfrak{z} + r) = \varphi_T(t, \mathfrak{z})$$

for any $r \in \mathbf{Z}^{(1, n-1)}$; this comes from the transformation law of F for the matrix $\begin{pmatrix} 1_n & S \\ 0 & 1_n \end{pmatrix}$ with $S = \begin{pmatrix} 0 & r \\ r^t & 0_{n-1} \end{pmatrix}$. The second transformation law is obtained from $M = \begin{pmatrix} U^t & S \\ 0 & U^{-1} \end{pmatrix} \in \text{Sp}(n, \mathbf{Z})$ with $U = \begin{pmatrix} 1 & \ell \\ 0 & 1_{n-1} \end{pmatrix}$; here $\ell \in \mathbf{Z}^{(1, n-1)}$, and it gives

$$\varphi_T(\tau, \tau\ell + \mathfrak{z}) = \rho(U^{-1}) \varphi_T(\tau, \mathfrak{z}) e\left(- (T[\ell^t] \tau + 2 \cdot T\ell^t \mathfrak{z})\right). \tag{3.7}$$

Let $\Delta_n \subset \text{GL}(n, \mathbf{C})$ be the subgroup of all upper triangular matrices. Since Δ_n is a connected, solvable algebraic group, by the Lie–Kolchin theorem (compare [12, Thm. 10.5]), there exists a basis of V such that the set $\rho(\gamma)$ is upper triangular for all $\gamma \in \Delta_n$. Thus without loss, from now on we assume that $V = \mathbf{C}^m$ and that all elements of $\rho(\Delta_n)$ are upper triangular.

We view φ_T as a \mathbf{C}^m -valued function $\varphi_T = (\varphi_T^{(1)}, \dots, \varphi_T^{(m)})^t$. We define

$$r := \max \left\{ i \mid 1 \leq i \leq m, \varphi_T^{(i)} \neq 0 \right\}.$$

For this r , equation (3.7) reads

$$\varphi_T^{(r)}(\tau, \tau\ell + \mathfrak{z}) = \varphi_T^{(r)}(\tau, \mathfrak{z}) e \left(- (T[\ell^t] \tau + 2 \cdot T\ell^t \mathfrak{z}) \right).$$

We must finally check the transformation law for $\text{SL}(2, \mathbf{Z})$. Namely,

$$e \left(\frac{-c}{c\tau + d} \mathfrak{z} T \mathfrak{z}^t \right) \rho \left(\begin{pmatrix} c\tau + d & c\mathfrak{z} \\ 0 & 1_{n-1} \end{pmatrix} \right)^{-1} \varphi_T \left(\frac{a\tau + b}{c\tau + d}, \frac{\mathfrak{z}}{c\tau + d} \right) = \varphi_T(\tau, \mathfrak{z})$$

gives (when applied to $\varphi_T^{(r)}$) the requested transformation property, with weight k' given by

$$\rho(\text{diag}(\lambda, 1, \dots, 1)) = (g_{ij}(\lambda)),$$

where $\lambda \mapsto g_{rr}(\lambda)$ is a (polynomial) character of $\text{GL}(1, \mathbf{C})$ – that is, $g_{rr}(\lambda) = \lambda^{k'}$ for some integer $k' \geq 0$. This follows by looking at the r th components on both sides of the equation in the previous display and here we crucially use the property that $\rho(\Delta_n)$ is upper triangular.

This means that $\varphi_T^{(r)}$ is a nonvanishing scalar-valued Jacobi form of weight $k' \geq k(\rho)$. Summarising, we have shown the following:

Proposition 3.4. *If $\varphi_T \neq 0$ is a vector-valued Jacobi form of index T with respect to ρ in the sense of Section 2.3, then there exists a component $\varphi_T^{(r)}$ of φ_T which is a scalar-valued Jacobi form (of integral weight $k' \geq k(\rho)$) in the sense of [18].*

3.3. Step 3: On primitive components of theta expansions

We now work with the scalar-valued Jacobi form $\varphi_T^{(r)}$ from the previous section. More generally, we prove the existence of ‘primitive’ theta-components of any such form whose index has absolute discriminant odd and square-free. Note the switch from $n - 1$ to n in this subsection, for convenience. We will later apply the result of this subsection to a $T \in \Lambda_{n-1}^+$. The following result is independent of the previous considerations:

Proposition 3.5. *Let $M \in \Lambda_n^+$ be such that $d = d_M$ is odd and square-free. Let*

$$\phi_M(\tau, z) = \sum_{\mu} h_{\mu}(\tau) \Theta_M[\mu](\tau, z) \in J_{k, M}$$

be the theta decomposition of ϕ_M (see, e.g., [45, p. 210]). Assume that $h_{\mu} = 0$ for all primitive μ . Then $\phi_M = 0$.

Remark 3.6.

- (1) This property is weaker than irreducibility of the theta representation; note that in the case of scalar index m , Skoruppa [39] showed that irreducibility holds only for index $m = 1$ or a prime p .
- (2) This result is expected to hold only when M is a maximal lattice; for example, when $n = 1$ and d_M is not square-free, there are nonzero Jacobi forms with vanishing primitive h_μ which ‘come’ from index-old forms (see [40, Lem. 3.1]).

Proof of Proposition 3.5. The proof is rather long, and has been divided into several parts for convenience. Let ϕ_M be a Jacobi form satisfying the assumption of the proposition. Then for all primitive μ , by considering $h_\mu | \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, we obtain the relation

$$\sum_{\nu \in \mathbf{Z}^n / 2M \cdot \mathbf{Z}^n} e\left(\frac{1}{2}\langle \nu, \mu \rangle\right) h_\nu = 0,$$

where for vectors ν, μ we have set $\langle \nu, \mu \rangle = \nu^t M^{-1} \mu$.

By applying translations $\tau \mapsto \tau + t$ with $t \in \mathbf{Z}$, this equation becomes

$$\sum_{\nu \in \mathbf{Z}^n / 2M \mathbf{Z}^n} e\left(\frac{1}{2}\langle \nu, \mu \rangle + \frac{1}{4}\langle \nu, \nu \rangle t\right) \cdot h_\nu = 0.$$

We observe that ν and ν' define the same character $t \mapsto e\left(\frac{1}{4}\langle \nu, \nu \rangle t\right)$ of $(\mathbf{Z}, +)$ if and only if

$$\frac{1}{4}\langle \nu, \nu \rangle - \frac{1}{4}\langle \nu', \nu' \rangle = \frac{1}{4}M^{-1}[\nu] - \frac{1}{4}M^{-1}[\nu'] \in \mathbf{Z}.$$

Using the linear independence of pairwise different characters we get a refined system of equations, we fix some $\nu^o \in \mathbf{Z}^n$ and are led to consider

$$\sum_{\substack{\nu \in \mathbf{Z}^n / 2M \mathbf{Z}^n \\ \frac{1}{4}M^{-1}[\nu] - \frac{1}{4}M^{-1}[\nu^o] \in \mathbf{Z}}} e\left(\frac{1}{2}\langle \nu, \mu \rangle\right) \cdot h_\nu = 0. \tag{3.8}$$

Only the case of imprimitive ν_0 is of interest here, since for primitive ν_0 , all ν appearing in the equation (3.8) will also be primitive, and for these the h_ν are zero anyway. \square

Claim 1. For all fixed ν^o , the matrix $(e\left(\frac{1}{2}\langle \nu, \mu \rangle\right))_{\nu, \mu}$, with μ varying over primitive vectors and ν varying over all vectors as in equation (3.8), is of maximal rank (equal to the cardinality of the set of ν occurring here).

Let us now grant this claim and show how to finish the proof of Proposition 3.5. Indeed, as already noted, we only need to show that all imprimitive h_μ are zero. The condition $\nu \sim \nu^o$ if and only if $\frac{1}{4}M^{-1}[\nu] - \frac{1}{4}M^{-1}[\nu^o] \in \mathbf{Z}$ defines an equivalence relation on the set of imprimitive indices, and equation (3.8) along with Claim 1 just says that all the $h_\nu = 0$ for all ν in the equivalence class of ν^o . But since ν^o can be any arbitrary imprimitive index, we are done.

3.3.1. Reduction to degree 1. Our aim is to show that we can reduce everything to the case of degree 1 – in other words, we show next that it is enough to prove Claim 1 when M, μ, ν are scalars. The idea is to choose representatives of $\mathbf{Z}^n/2M \cdot \mathbf{Z}^n$ which are similar to those for $n = 1$. We give all details when n is odd and indicate the main points for the other case.

When n is odd. First of all, we may find $U \in \text{SL}(n, \mathbf{Z})$ such that $\widetilde{M} := (2M)[U]$ satisfies for, $f \geq 2$,

$$\widetilde{M} \equiv \text{diag}(*, *, \dots, *, \zeta d) \pmod{d^f}, \tag{3.9}$$

where $*, \zeta$ are units mod d , and

$$\widetilde{M} \equiv \mathbb{H} \perp \dots \perp \mathbb{H} \perp 2 \pmod{2^f}, \tag{3.10}$$

where \mathbb{H} denotes the hyperbolic plane $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Indeed, from the local theory of quadratic forms (see, e.g., [14, Chap. 8, Prop. 3.1, Lem. 4.1]), we can find for every $q \mid 2d$ matrices $U_q \in \text{SL}(n, \mathbf{Z}_q)$ such that, since n is odd and $2d$ is square-free,

$$(2M)[U_q] = \begin{cases} \text{diag}(*, *, \dots, *, q) & \text{if } q \neq 2, \\ \mathbb{H} \perp \dots \perp \mathbb{H} \perp 2 & \text{if } q = 2. \end{cases} \tag{3.11}$$

The reader may note that a priori (with the convention in [14]) we can only get a $V_q \in \text{GL}(n, \mathbf{Z}_q)$ with this property; but we can assume $V_q \in \text{SL}(n, \mathbf{Z}_q)$ by multiplying V_q on the left with the matrix $\text{diag}(\det(V_q)^{-1}, 1, \dots, 1)$ without loss.

Then by strong approximation for $\text{SL}(n)$, we may find $U \in \text{SL}(n, \mathbf{Z})$ such that $U \equiv U_q \pmod{q^f}$ for any $f \geq 1$ for any prime $q \mid 2d$. This U works. To get statement (3.9), we use the Chinese remainder theorem for the moduli q^f with $f \geq 2$.

As representatives of $\mathbf{Z}^n/\widetilde{M}\mathbf{Z}^n$ we may choose

$$\widetilde{\nu} := \left\{ (0, 0, \dots, \widetilde{\nu}_n)^t \mid \widetilde{\nu}_n \pmod{2d} \right\}. \tag{3.12}$$

Using formulas (3.9) and (3.10), we see that indeed these are pairwise inequivalent by checking locally and noting that the cardinality of this set is the right one.

Let now $\mathbb{M} = (m_{i,j})$ be the adjoint of \widetilde{M} , so that $\widetilde{M}\mathbb{M} = 2d \cdot I_n$. Then for $\widetilde{\nu}, \widetilde{\mu} \in \mathbf{Z}^n/\widetilde{M}\mathbf{Z}^n$, (assuming they are in the nice form as in definition (3.12), we see that

$$\widetilde{\nu}^t \widetilde{M}^{-1} \widetilde{\mu} = \frac{1}{2d} \widetilde{\nu}_n m_{n,n} \widetilde{\mu}_n. \tag{3.13}$$

We claim that $(m_{n,n}, 2d) = 1$. To see this, we multiply formulas (3.9) and (3.10) by \mathbb{M} on both sides and compare the resulting congruences to obtain

$$\zeta d m_{n,n} \equiv 2d \pmod{d^f}, \quad 2m_{n,n} \equiv 2d \pmod{2^f},$$

which clearly implies our claim, since $f \geq 2$.

Furthermore, for $\widetilde{\nu}, \widetilde{\nu}^o \in \mathbf{Z}^n/\widetilde{M}\mathbf{Z}^n$, we see that

$$\frac{1}{2} \widetilde{M}^{-1} [\widetilde{\nu}] - \frac{1}{2} \widetilde{M}^{-1} [\widetilde{\nu}^o] \in \mathbf{Z} \quad \text{if and only if} \quad \widetilde{\nu}_n^2 \equiv (\widetilde{\nu}_n^o)^2 \pmod{4d}. \tag{3.14}$$

Also, it is clear from equation (3.13) along with $(m_{n,n}, 2d) = 1$ that $\tilde{\nu}$ is primitive (with respect to \widetilde{M}) if and only if $(\tilde{\nu}_n, 2d) = 1$.

These considerations allow us to reduce the case of odd n to $n = 1$ with $d' = 2d$ as follows. We note that

$$\nu^t(2M)^{-1}\mu = \nu^t U \widetilde{M}^{-1} U^t \mu = \tilde{\nu}^t \widetilde{M}^{-1} \tilde{\mu}, \tag{3.15}$$

whence we make a change of variables $\tilde{\nu} = U^t \nu, \tilde{\mu} = U^t \mu$; and we observe that as ν, μ vary over $\mathbf{Z}^n / 2M \cdot \mathbf{Z}^n$, so do $\tilde{\nu}, \tilde{\mu}$ vary over $\mathbf{Z}^n / U^t(2M) \cdot \mathbf{Z}^n = \mathbf{Z}^n / \widetilde{M} \mathbf{Z}^n$. Clearly ν is primitive for M if and only if $\tilde{\nu}$ is primitive for \widetilde{M} . Moreover, the condition on ν in equation (3.8) can be seen to be exactly the one in formula (3.14) upon using equation (3.15).

The reduction to $n = 1$ is now clear from equation (3.13) and formula (3.14), where in equation (3.13) we make a change of variable $\tilde{\mu} \mapsto m_{n,n}^{-1} \tilde{\mu} \pmod{2d}$. We will spell this out explicitly at the end of this subsection after taking care of the analogous case where n is even.

When n is even. We may find, arguing as in the previous case, a $U \in \text{SL}(n, \mathbf{Z})$ such that $\widetilde{M} := (2M)[U]$ satisfies

$$\widetilde{M} \equiv \text{diag}(*, \dots, *, d) \pmod{d^f},$$

where $*$ are units mod d , and

$$\widetilde{M} \equiv \mathbb{H} \perp \dots \perp \mathbb{H} \pmod{4^f} \quad \text{or} \quad \widetilde{M} \equiv \mathbb{H} \perp \dots \perp \mathbb{H} \perp \mathbb{F} \pmod{4^f},$$

where $\mathbb{F} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

As representatives of $\mathbf{Z}^n / 2M \mathbf{Z}^n$ we may choose

$$\nu := \{(0, 0, \dots, \nu_n)^t \mid \nu_n \pmod{d}\},$$

and ν is primitive if and only if $(\nu_n, d) = 1$. Furthermore,

$$e\left(\frac{1}{2} \nu^t M^{-1} \mu\right) = e\left(\frac{1}{d} \nu_n m_{n,n} \mu_n\right),$$

where the matrix adjoint of \widetilde{M} is $(m_{i,j})$ and

$$\frac{1}{4} M^{-1}[\nu] - \frac{1}{4} M^{-1}[\nu^o] \in \mathbf{Z} \iff \nu_n^2 \equiv (\nu_n^o)^2 \pmod{d}.$$

These considerations allow us to reduce the case of even n to $n = 1$ with $d' = d$.

Summarising, we now have to prove the following:

Claim 2. *Let d be a square-free odd positive integer, $d = p_1 \cdots p_t$. To cover even numbers also, we define*

$$d' := \begin{cases} d & \text{if } n \text{ is even,} \\ 2d & \text{if } n \text{ is odd,} \end{cases}$$

with the convention that $d' = 2$ if $t = 0$. We assume that $d' > 1$. We fix $\nu_0 \pmod{d'}$. Then the following matrix has maximal rank:

$$\left(e\left(\frac{\mu\nu}{d'}\right) \right)_{\substack{\mu \bmod d', (\mu, d') = 1 \\ \nu \bmod d', \nu^2 \equiv \nu_0^2 \bmod d'}} \tag{3.16}$$

3.3.2. Proof of Claim 2. By the Chinese remainder theorem, the set

$$\{\nu \bmod d' \mid \nu^2 \equiv \nu_0^2 \bmod d'\}$$

has cardinality $2^{t'}$, where t' is the number of odd primes dividing d and not dividing ν_0 ; note that in the case of even d' , we might as well describe the congruence by $\nu^2 \equiv \nu_0^2 \bmod 4d$.

Lemma 3.7. *With all the conditions from before, we claim that such a matrix always has maximal rank (equal to the number of columns $2^{t'}$).*

To prove this lemma, we argue by induction on t . For $t = 0$ and $d' = 2$, the matrix in question is just a nonzero scalar. For $t = 1$ we consider two cases.

Case I: $t = 1, d' = p$. If $\nu_0 \equiv 0 \bmod p$, the matrix in question is a nonzero column.

Now we look at $p \nmid \nu_0$. We have to consider $\nu = \pm\nu_0$, with a μ still to be determined so that the matrix

$$\begin{pmatrix} e\left(\frac{\nu_0}{p}\right) & e\left(-\frac{\nu_0}{p}\right) \\ e\left(\frac{\mu\nu_0}{p}\right) & e\left(-\frac{\mu\nu_0}{p}\right) \end{pmatrix}$$

of size 2 whose determinant is equal to

$$e\left(\frac{(1-\mu)\nu_0}{p}\right) \left(1 - e\left(-\frac{2(1-\mu)\nu_0}{p}\right)\right)$$

should have the determinant nonzero. This clearly implies the maximality of the rank of the original matrix.

We may choose any μ coprime to p and different from 1. This settles case I.

Case II: $t = 1, d' = 2p$. This works similarly: if $\nu_0 \equiv 0 \bmod p$, the matrix is again just a nonzero column.

Now we assume $p \nmid \nu_0$: we have to consider $\nu = \pm\nu_0$, and with a μ still to be determined, we look at the matrix of size 2:

$$\begin{pmatrix} e\left(\frac{\nu_0}{2p}\right) & e\left(-\frac{\nu_0}{2p}\right) \\ e\left(\frac{\mu\nu_0}{2p}\right) & e\left(-\frac{\mu\nu_0}{2p}\right) \end{pmatrix}.$$

The determinant is equal to

$$e\left(\frac{(1-\mu)\nu_0}{2p}\right) \left(1 - e\left(-\frac{(1-\mu)\nu_0}{p}\right)\right).$$

We may choose any μ coprime to $2p$ and different from 1. Thus the lemma follows in this case as well.

Induction step: $t \mapsto t + 1$, with $t \geq 1$. We write q for the prime p_{t+1} . We decompose $\nu_0 \pmod{d'q}$ as

$$\nu_0 = d' \nu'_0 + q \nu''_0,$$

with $\nu'_0 \pmod{q}$ and $\nu''_0 \pmod{d'}$, and similarly for ν and μ . Then

$$\frac{\mu\nu}{d'q} = \frac{(d'\mu' + q\mu'')(d'\nu' + q\nu'')}{d'q} = \frac{d'\mu'\nu'}{q} + \frac{q\mu''\nu''}{d'} \pmod{\mathbf{Z}}.$$

The matrix attached to $d'q$ and ν_0 is then the tensor product ('Kronecker product') of the matrices attached to q and $d' \cdot \nu'_0$ and attached to d' and $q \cdot \nu''_0$. The induction step follows from the well-known property of Kronecker products that the rank of $A \otimes B$ is the product of the ranks of A and B .

This finishes the proof of Claim 2 and hence also of Proposition 3.5. □

The following lemma, which will be used later, implies that when k is even, all the theta-components of a noncuspidal scalar-valued Jacobi form of index M with d_M odd and square-free are also noncuspidal.

Lemma 3.8. *Let $M \in \Lambda_n^+$ be such that $d = d_M$ is odd and square-free. Suppose that $\phi \in J_{k,M}$ is noncuspidal and that $k > n + 2$ is even. Then all the theta-components $h\mu$ of ϕ are also noncuspidal.*

In particular for a nonzero ϕ as before, there always exists a nonzero, noncuspidal primitive theta-component. Also, we think that the lemma probably holds for all discriminants, even in the vector-valued case (proving either, however, seems nontrivial).

Proof. Let $J_{k,M}^{\mathcal{E}}$ and $J_{k,M}^{cusp}$ be the spaces of Eisenstein series and cusp forms, respectively. We write $\phi = \phi_{\mathcal{E}} + \phi_c$, where $\phi_{\mathcal{E}} \in J_{k,M}^{\mathcal{E}}$ and $\phi_c \in J_{k,M}^{cusp}$. Since ϕ is not a cusp form, $\phi_{\mathcal{E}} \neq 0$.

Let us now recall from [2, Lem. 3.3.14] that

$$\dim J_{k,M}^{\mathcal{E}} = \frac{1}{2} \left(\#\text{Iso} \left(\mathcal{L}_M^{\#} / \mathcal{L}_M \right) + (-1)^k \#\left\{ \gamma \in \text{Iso} \left(\mathcal{L}_M^{\#} / \mathcal{L}_M \right) \mid 2\gamma \in \mathcal{L}_M \right\} \right),$$

where \mathcal{L}_M is the lattice associated with M (compare Section 2.4) and $\text{Iso} \left(\mathcal{L}_M^{\#} / \mathcal{L}_M \right)$ is the set of isotropic elements in the discriminant form $\mathcal{L}_M^{\#} / \mathcal{L}_M$ associated to M . It is now easy to check (note the normalisation of the quadratic form on [2, p. 12]), for M as in the lemma, that $\text{Iso} \left(\mathcal{L}_M^{\#} / \mathcal{L}_M \right)$ is just trivial, and this follows precisely from formula (2.10) if we note that $\mathcal{L}_M^{\#} = ((2M)^{-1} \mathbf{Z}^n, \mu \mapsto (2M)[\mu])$.

Thus $J_{k,M}^{\mathcal{E}} = \mathbf{C}\{E_{k,M,0}\}$ in the notation of [2], as k is even. For the absolute convergence of $E_{k,M,0}$, we need here that $k > n/2 + 2$. Now from the results of [9, Theorem 10], $E_{k,M,0}$ appears as the M th Fourier–Jacobi coefficient of the Siegel Eisenstein series, say E_{n+1}^k , of degree $n + 1$. Here we need to assume that $k > n + 2$. Indeed, [9] describes explicitly the Fourier–Jacobi expansion of any Siegel Eisenstein series. If M is maximal, the summation over w_1 there is trivial, which implies our claim. It is well known that all the Fourier coefficients of E_{n+1}^k are nonzero (see, e.g., [24, Theorem, p. 115]). Thus all the Fourier coefficients of $E_{k,M,0}$ are nonzero – and thus so are all its theta-components. □

3.4. Formulation of the desired properties of h_μ

We consider a nonzero $F \in M_\rho^n$ and choose (by induction hypothesis) a $T \in \Lambda_{n-1}^+$ with d_T odd and square-free such that φ_T is nonzero (compare Corollary 3.3). Then we choose by Proposition 3.4 a suitable nonzero component $\varphi_T^{(r)}$ of φ_T , which is a scalar-valued Jacobi form.

Let $(h_0, \dots, h_\mu, \dots)$ denote the components of the theta expansion of $\varphi_T^{(r)}$ (see Equation (2.5)). By Proposition 3.5, we get hold of a primitive μ such that $h_\mu \neq 0$. We work with this h_μ (of weight $k' - \frac{n-1}{2}$) for the rest of the paper.

The basic starting point is an equation of type

$$\det_n \mathcal{T} = \det_n \left(\begin{pmatrix} \ell & \frac{\mu}{2} \\ \frac{\mu^t}{2} & T \end{pmatrix} \right) = \left(\ell - \frac{1}{4} T^{-1} [\mu^t] \right) \cdot \det_{n-1}(T) \tag{3.17}$$

for the determinant $\det_n(\mathcal{T})$ of a half-integral matrix \mathcal{T} of size n occurring on the left-hand side.

Recall that $d_{\mathcal{T}}$ is the (absolute) discriminant of $2\mathcal{T}$, and similarly for T .

3.4.1. When n is odd. We should multiply equation (3.17) by 2^{n-1} :

$$d_{\mathcal{T}} = \left(\ell - \frac{1}{4} T^{-1} [\mu] \right) \cdot d_T.$$

The first factor has exact denominator d_T , since μ is primitive (see Proposition 2.3). In Part B we will show that there are (infinitely many) nonvanishing Fourier coefficients of h_μ for some primitive μ with summation index

$$\ell - \frac{1}{4} T^{-1} [\mu] = \frac{\alpha}{d_T}$$

(see equation (2.6)) such that α is coprime to d (this is satisfied automatically by the primitiveness of μ) and is square-free and odd.

3.4.2. When n is even. Here we must multiply equation (3.17) by 2^n to get

$$d_{\mathcal{T}} = 4 \left(\ell - \frac{1}{4} T^{-1} [\mu] \right) \cdot d_T = (4\ell - T^{-1} [\mu]) \cdot d_T.$$

The middle factor has exact denominator $4d_T$, since μ is primitive (see Proposition 2.3). In Part B we will show that there are (infinitely many) nonvanishing Fourier coefficients of h_μ for some primitive μ with summation index

$$\ell - \frac{1}{4} T^{-1} [\mu] = \frac{\alpha}{4d_T},$$

(see equation (2.6)) such that α is coprime to $4d_T$ (this is satisfied automatically) and is odd and square-free.

The next proposition summarises the findings from Part A and makes clear the role of Part B in the remainder of the proof. Let us set

$$H_\mu(\tau) = \begin{cases} h_\mu(d_T\tau) & \text{if } n \text{ is odd,} \\ h_\mu(4d_T\tau) & \text{if } n \text{ is even.} \end{cases} \tag{3.18}$$

We now apply the results of this section to $\varphi_T^{(r)}$ from Proposition 3.4 and keep in mind the Fourier expansion of h_μ from equation (2.6).

Proposition 3.9. *Let $n \geq 2$ and $C \in \mathbf{N}$ be given. Assume that Theorem 1.1 holds for all nonzero forms in M_θ^{n-1} for all polynomial representations θ of $\mathrm{GL}(n-1, \mathbf{C})$ with $k(\theta) \geq C$. Consider a nonzero $F \in M_\rho^n$, where ρ is any polynomial representation of $\mathrm{GL}(n, \mathbf{C})$ with $k(\rho) \geq C$.*

Then there exist $T \in \Lambda_{n-1}^+$ with d_T running over infinitely many odd, square-free numbers; and for each such T , there exist a primitive index $\mu \in \mathbf{Z}^{n-1}/(2T)\mathbf{Z}^{n-1}$ and an elliptic modular form $H_\mu(\tau) = \sum_{\ell \geq 1} a_\mu(\ell)q^{d'_T(\ell - \frac{1}{4}T^{-1}[\mu])}$, where $d'_T = d_T$ or $4d_T$ according to whether n is odd or even, of weight at least $k(\rho) - (n-1)/2$ as defined in equation (3.18) with the following property:

$$a_F(\mathcal{T}) \neq 0 \text{ for } \mathcal{T} \in \Lambda_n^+ \text{ of the form } \mathcal{T} = \begin{pmatrix} \ell & \mu \\ \mu & T \end{pmatrix}, \ell \geq 1, \text{ if } a_\mu(\ell) \neq 0.$$

Such a \mathcal{T} satisfies the property that $d_{\mathcal{T}}$ is odd and square-free, provided that $d_T(\ell - \frac{1}{4}T^{-1}[\mu])$ is odd and square-free.

Therefore, in Part B we must investigate the nonvanishing property of the Fourier coefficients $a_\mu(\ell)$. The reader will find the results summarised in Theorem 4.5 and Theorem 4.6. Actually, in our application of Proposition 3.9 to prove Theorem 1.1 by induction on n (see Section 4.3), we would only need one such T as in Proposition 3.9 for each of the induction steps. Say, for example, we are at the r th step – that is, passing from $\mathrm{Sp}(r)$ to $\mathrm{Sp}(r+1)$, where $1 \leq r \leq n-1$; then we only need the nonvanishing of ϕ_T ($T \in \Lambda_r^+$) for one T . The statement about infinitely many such d_T follows from the corresponding property of the $a_\mu(\ell)$, where $\mu \in \mathbf{Z}^{r-1}/(2T)\mathbf{Z}^{r-1}$, $\ell \geq 1$.

4. Part B: The analytic part

We start with a nonzero h_μ , with $\mu \in \mathbf{Z}^{n-1}/2T\mathbf{Z}^{n-1}$ primitive (see Proposition 3.9). The notion that μ is primitive with respect to T can be found in Proposition 2.3. For the convenience of the reader, we reiterate that h_μ is a nonzero theta-component of the scalar-valued Jacobi form $\varphi_T^{(r)}$ from Section 3.2, which in turn arises as a function component of the vector-valued Jacobi form φ_T . Moreover, this φ_T is a Fourier–Jacobi coefficient of F (see Section 3.1, especially Corollary 3.3). Note that d_T is odd and square-free.

The arguments in this section are a little different depending on whether n is even or odd, but we try to treat them simultaneously. Set

$$\kappa = k - \frac{n-1}{2}, \quad d = d_T, \tag{4.1}$$

and for $N \geq 1$ and $\kappa \in \frac{1}{2}\mathbf{Z}$, we set (for the remainder of the paper)

$$N' = \begin{cases} N & \text{if } \kappa \in \mathbf{N}, \\ 4N & \text{otherwise.} \end{cases} \tag{4.2}$$

In any case, we note that $h_\mu \in M_\kappa(\Gamma(d'))$ with $\kappa \in \mathbf{Z}$. Indeed, the transformation properties of the h_μ are inherited from those of the Jacobi form $\varphi_T^{(r)}$ and the Jacobi theta functions of matrix index (compare equation (2.3)). We can read these off from [45, p. 210, eqn. (1) and (2)]. It is clear that the Weil representation defined in, for example, [41, Definition 5.1] is the same as that defined on the theta module $\mathbf{C}\{\Theta_T[\mu]: \mu \in \mathbf{Z}^{n-1}/2T \cdot \mathbf{Z}^{n-1}\}$ (compare section 2.3 and [45, Lem. 3.2]). Now the fact that $\phi_T^{(r)} = \sum_\mu h_\mu \cdot \Theta_T[\mu]$ implies that the tuple $\{h_\mu\}_\mu$ transforms via the dual of the Weil representation just alluded to. Thus our claim about the level of h_μ follows by noting that the kernel of the Weil representation attached to $2T$ factors through (and thus is trivial on) $\Gamma(d)$ when n is odd (i.e., when $\kappa \in \mathbf{Z}$), and in general is trivial on $\Gamma(\widetilde{\text{level}(2T)}) = \widetilde{\Gamma}(d')$ (where $\widetilde{\Gamma}$ denotes the metaplectic cover of Γ ; see, e.g., [41, Lem. 5.5] for this well-known result). For uniformity of notation, we suppress the tilde in $\widetilde{\Gamma}$ even when $\kappa \notin \mathbf{Z}$.

Let us set $f = h_\mu(d'\tau)$. Therefore, in Proposition 3.9 we take $H_\mu := f$.

Then it is clear that $f \in M_\kappa(\Gamma_1(d'^2))$. Crucial for us is the fact that the Fourier expansion of f is supported away from its level (this follows from Section 3.4, precisely because of the primitiveness of μ):

$$f(\tau) = \sum_{n \geq 1, (n, d')=1} a_f(n)e(n\tau). \tag{4.3}$$

For later purposes, we have to consider the modified cusp form

$$g(\tau) := \sum_{(n, \mathbf{M})=1} a_f(n)e(n\tau), \tag{4.4}$$

where \mathbf{M} (to be chosen later) is an odd, square-free integer containing all the prime factors of d' if κ is integral. We note that $g \in S_\kappa(\Gamma_1(d'^2\mathbf{M}^2))$, where M' is the largest divisor of \mathbf{M} coprime to d' .

Lemma 4.1. *The modular form g in definition (4.4) is nonzero.*

Proof. In the case of integral weights, this essentially follows, since f satisfies the property in equation (4.3) and follows easily from classical oldform theory. Let us define $\mathbf{M}' := \mathbf{M}/d'$. Then $g = 0$ implies that $a(f, n) = 0$ for all n such that $(n, \mathbf{M}') = 1$. Since $(\mathbf{M}', d'^2) = 1$, from [29, Theorem 4.6.8 (1)] it then follows that $f = 0$, a contradiction. \square

Lemma 4.1 is true for half-integral weights as well, even though we will not use this fact. Since this is a bit subtle and may have use elsewhere, we give a proof. In this case let us set \mathbf{M} to be the square-free number appearing in [33, Thm. 3] (denoted N_f there). Recall that we need to show that g (as in definition (4.4)) is nonzero.

We refer the reader to [33, Prop. 3.7] for the spaces $S_\kappa(N, \chi)$ and follow its proof. Inspecting the argument in that proof, it is clear that one needs to prove that if $0 \neq \mathbf{f} \in$

$M_\kappa(\Gamma_1(N'))$ for some $N \geq 1$ with $a_f(n) = 0$ for all n such that $(n, p) = 1$, then $p \mid N'$ – and hence $p \mid N$, since we can and will assume that p is odd, since N' and N_f (compare [33, proof of Prop. 3.4]) are both even, and here we are concerned with those primes p which divide N_f but not N' . It is not immediately clear how to adapt the proof in [37, Lem. 7] in our setting. The following lemma, however, completes the proof:

Lemma 4.2. *Set $\kappa \in \frac{1}{2}\mathbf{Z}$ and $\mathbf{f} \in M_\kappa(\Gamma_1(N'))$. If $a_f(n) = 0$ for all n coprime to an odd prime $p \nmid N'$, then $\mathbf{f} = 0$.*

Proof. The trick is to reduce to integral weights. Suppose that $\mathbf{f} \neq 0$ and define $\mathbf{g}(\tau) := \mathbf{f}^2(\tau)$. Then from the formula

$$a_g(n) = \sum_{r+s=n} a_f(r)a_f(s),$$

we see that $\mathbf{g} \in M_{2\kappa}(\Gamma_1(N'))$ is such that $a_g(n) = 0$ for all n with $(n, p) = 1$. Indeed, each summand is zero unless both r and s are divisible by p . This means that the Fourier expansion of \mathbf{g} is supported on multiples of p , and hence we can write $g(q) = \Psi(q^p)$ for some power series Ψ , where $q = e(\tau)$.

However, taking $f_\infty(q) = \Psi(q)$, $j = 2\kappa$ and $L = N'$, this forces \mathbf{g} and hence \mathbf{f} to be zero upon invocation of [27, Chapter VIII, Thm. 4.1], which states:

- Let $L \in \mathbf{N}$ and p be a prime such that $p \nmid L$. If $f_\infty(q)$ is a power series such that $f_\infty(q^p)$ belongs to $M_j(\Gamma_1(L))$ for some integer $j \geq 1$, then $f_\infty = 0$.

This finishes the proof. □

4.1. Method 1: Proof for all cuspidal f without using multiplicity 1 for $\kappa \in \frac{1}{2}\mathbf{Z}$

In this section we want to prove that f has infinitely many nonzero, odd and square-free Fourier coefficients, possibly in a quantitative fashion. However, let us note that we can not just quote the corresponding results from, say, [3, Theorem 2] or [33, Theorem 2], since the results therein are only for cusp forms on $\Gamma_0(N)$ with nebentypus, whereas our setting is on $\Gamma_1(N)$, and the problem of finding nonzero square-free Fourier coefficients does not behave in a desirable way under decomposition by characters.

We now pursue the cases of integral and half-integral weights in separate subsections, by closely following [3, 33]. We will henceforth work with the modular form g from definition (4.4).

4.1.1. f cuspidal and κ integral. Let us set $D = d^2\mathbf{M}^2$ and, for a square-free r such that $(r, D) = 1$, define

$$G := U(r^2)g = \sum_{n \geq 1} a_g(r^2n) e(n\tau),$$

so that $g \in S_\kappa(\Gamma_1(D))$, $G \in S_\kappa(\Gamma_1(Dr^2))$. (Actually the level would be Dr , but we will not need this.)

We apply the Rankin–Selberg method to g . For any $\mathbf{g} \in S_\kappa(\Gamma_1(\mathbf{D}))$, $\mathbf{D} \geq 1$, with Fourier expansion $\mathbf{g} = \sum_{n \geq 1} a'_g(n)n^{\frac{\kappa-1}{2}} e(n\tau)$ – noting that $a_g(n) = a'_g(n)n^{\frac{\kappa-1}{2}}$ – applying this

method (see [32, p. 357, Theorem 1] and [34, eq. (1.14)]) yields

$$\sum_{n \leq X} |a'_g(n)|^2 = A_g X + O_g \left(X^{\frac{3}{5}} \right), \tag{4.5}$$

where the constant A_g is given by

$$A_g := \frac{3}{\pi} \frac{(4\pi)^\kappa}{\Gamma(\kappa)} [\text{SL}(2, \mathbf{Z}) : \Gamma_1(\mathbf{D})]^{-1} \langle \mathbf{g}, \mathbf{g} \rangle_{\mathbf{D}}. \tag{4.6}$$

Here and henceforth, $\langle \mathbf{g}, \mathbf{g} \rangle_{\mathbf{D}}$ denotes the Petersson norm of \mathbf{g} with respect to $\Gamma_1(\mathbf{D})$, defined by

$$\langle \mathbf{g}, \mathbf{g} \rangle_{\mathbf{D}} := \int_{\Gamma_1(\mathbf{D}) \backslash \mathbf{H}} |\mathbf{g}(\tau)|^2 v^k dudv/v^2, \quad \tau = u + iv.$$

It is possible to rework all the calculations done for this in [3] on the spaces $S_\kappa(N, \chi)$ in our situation, but since a major portion of the work requires only upper bounds on the sum of square of Fourier coefficients, we may reduce to [3] via decomposition by characters.

For $Y > 0$ and a modular form \mathbf{g} , let us define $S_g(Y) := \sum_{n \leq Y} |a'_g(n)|^2$. We now prove some results which give upper and lower bounds for the quantity $S_g(Y)$. In particular, they provide suitable bounds on $S_{U(r^2)\mathbf{g}}(Y)$ which are uniform in r .

Proposition 4.3. *For $f \in S_\kappa(\Gamma_1(d^2))$ as before, the following statements hold for some positive constants $c_{f, \mathbf{M}}$ depending on f and \mathbf{M} , and for B_f, C_f depending only on f .*

- (i) *For all $Y \geq c_{f, \mathbf{M}}$, we have $S_g(Y) \geq B_f \prod_{p|\mathbf{M}} \left(1 - \frac{2}{p}\right)^2 \cdot Y$.*
- (ii) *For all $Y > 0$, we have $S_g(Y) \leq C_f \cdot Y$.*

Proof. For the proof of (i), let us write $\mathbf{M}_0 := \mathbf{M}/d$. Since the Fourier expansion of f is supported away from d , we see first of all

$$S_g(Y) = \sum_{n \leq Y, (n, \mathbf{M})=1} |a'_f(n)|^2 = \sum_{n \leq Y, (n, \mathbf{M}_0)=1} |a'_f(n)|^2. \tag{4.7}$$

\mathbf{M}_0 is odd and square-free, since both \mathbf{M} and d are so. Next, we rewrite equation (4.7) as

$$S_g(Y) = \sum_{\beta|\mathbf{M}_0} \mu(\beta) \sum_{n \leq Y/\beta} |a'_f(n\beta)|^2 = \sum_{\beta|\mathbf{M}_0} \mu(\beta) S_{U(\beta)f}(Y/\beta).$$

Therefore, equation (4.5) applied to $U(\beta)f \in S_\kappa(\Gamma_1(d^2\beta))$, with β as before, gives

$$S_g(Y) = \sum_{\beta|\mathbf{M}_0} \frac{\mu(\beta)}{\beta} A_{U(\beta)f} Y + O_{f, \mathbf{M}} \left(Y^{3/5} \right). \tag{4.8}$$

For $\mathbf{D} \geq 1$, let us define $\nu_{\mathbf{D}} := [\text{SL}(2, \mathbf{Z}) : \Gamma_1(\mathbf{D})]^{-1}$. Consider the orthogonal basis $\{f_j\}_j$ away from d^2 that we are considering in this section. Let us further recall from

[3, Theorem 8] and the argument in [3, Proof of Cor. 5.4] that for β and f_j ,

$$\langle U(\beta)f_j, U(\beta)f_j \rangle_{d^2\beta} = Q_\beta \langle f_j, f_j \rangle_{d^2\beta}, \tag{4.9}$$

where Q_β is a multiplicative function given by

$$Q_\beta(f_j) = \prod_{p|\beta} Q_p(f_j), \quad Q_p(f_j) = \left(p^{k-2} + \frac{(p-1)|\lambda_{f_j}(p)|^2}{p+1} \right). \tag{4.10}$$

Here $\lambda_{f_j}(p)$ is the eigenvalue for f_j under T_p . Now from [3, Cor. 5.2] and equation (4.9), we find, with $a_\kappa = \frac{3}{\pi} \frac{(4\pi)^\kappa}{\Gamma(\kappa)}$, that

$$\begin{aligned} A_{U(\beta)f_j} &= a_\kappa \nu_{d^2\beta} \beta^{1-\kappa} Q_\beta(f_j) \langle f_j, f_j \rangle_{d^2\beta} = a_\kappa \nu_{d^2\beta} \beta^{1-\kappa} Q_\beta(f_j) \langle f_j, f_j \rangle_{d^2} \\ &= \beta^{1-\kappa} Q_\beta(f_j) \cdot A_{f_j}. \end{aligned}$$

Now we write $f = \sum_j c_j f_j$ and note that by the orthogonality of the f_j s,

$$\langle f, f \rangle = \sum_j |c_j|^2 \langle f_j, f_j \rangle, \quad A_f = \sum_j |c_j|^2 A_{f_j};$$

and by the orthogonality of the $U_\beta(f_j)$ s (invoking [3, Theorem 8]),

$$\langle U(\beta)f, U(\beta)f \rangle_{d^2\beta} = \sum_j |c_j|^2 \langle U(\beta)f_j, U(\beta)f_j \rangle_{d^2\beta}.$$

Therefore we get

$$A_{U(\beta)f} = \sum_j |c_j|^2 \beta^{1-\kappa} Q_\beta(f_j) \cdot A_{f_j}. \tag{4.11}$$

Putting all these together, we now derive from equation (4.8) that

$$S_g(Y) = \sum_j |c_j|^2 A_{f_j} \sum_{\beta|\mathbf{M}_0} \frac{\mu(\beta)Q_\beta(f_j)}{\beta^\kappa} \cdot Y + O_{f, \mathbf{M}} \left(Y^{3/5} \right). \tag{4.12}$$

Let Q_β (resp., $\lambda(p)$) denote any of the quantities $Q_\beta(f_j)$ (resp., $\lambda_{f_j}(p)$). We define $\mathcal{Q}(\mathbf{M}_0) := \sum_{\beta|\mathbf{M}_0} \frac{\mu(\beta)Q_\beta}{\beta^\kappa}$. By multiplicativity we can write

$$\mathcal{Q}(\mathbf{M}_0) = \prod_{p|\mathbf{M}_0} \left(1 - \frac{Q_p}{p^\kappa} \right) = \prod_{p|\mathbf{M}_0} \left(1 - \frac{1}{p^2} - \frac{(p-1)|\lambda(p)|^2}{p^\kappa(p+1)} \right).$$

Using the Deligne bound for $\lambda(p)$, we see that

$$\begin{aligned} \mathcal{Q}(\mathbf{M}_0) &\geq \prod_{p|\mathbf{M}_0} \left(1 - \frac{4}{p} - \frac{1}{p^2} + \frac{8}{p(p+1)} \right) \\ &\geq \prod_{p|\mathbf{M}_0} \left(\left(1 - \frac{2}{p} \right)^2 + \frac{8}{p(p+1)} - \frac{5}{p^2} \right) \geq \prod_{p|\mathbf{M}_0} \left(1 - \frac{2}{p} \right)^2 \geq a_d \prod_{p|\mathbf{M}} \left(1 - \frac{2}{p} \right)^2, \end{aligned} \tag{4.13}$$

for some constant a_d depending only on d .

We now use the lower bound (4.13) in equation (4.12) to obtain

$$\begin{aligned}
 S_g(Y) &\geq a_d \prod_{p|\mathbf{M}} \left(1 - \frac{2}{p}\right)^2 \sum_j |c_j|^2 A_{f_j} \cdot Y + O_{f,\mathbf{M}}(Y^{3/5}) \\
 &\geq A_f a_d \prod_{p|\mathbf{M}} \left(1 - \frac{2}{p}\right)^2 \cdot Y + O_{f,\mathbf{M}}(Y^{3/5}).
 \end{aligned}
 \tag{4.14}$$

The proof of (i) is therefore complete from formula (4.14), taking for instance $B_f := A_f a_d/2$.

For the proof of (ii) note that for all $Y > 0$,

$$S_g(Y) = \sum_{n \leq Y, (n,\mathbf{M})=1} |a'_f(n)|^2 \leq \sum_{n \leq Y} |a'_f(n)|^2 = S_f(Y) \leq C_f \cdot Y,
 \tag{4.15}$$

for some constant C_f depending only on f , by looking at equation (4.5). □

The space $S_\kappa(\Gamma_1(d^2))$ has an orthogonal basis consisting of eigenforms for all T_n, σ_n with $(n,d) = 1$. Here σ_n are the diamond operators. This set is just the union of eigenforms away from the level d^2 in the spaces $S_\kappa(d^2, \psi)$ with ψ varying mod d^2 . Let this basis be denoted by $\{f_1, f_2, \dots, f_J\}$ where $J = \dim S_\kappa(\Gamma_1(d^2))$.

Lemma 4.4. *For any square-free integer r such that $(r,\mathbf{M}) = 1$ and for all $Y > 0$, we have $S_{U(r^2)_f}(Y) \leq D_f \cdot 11^{\omega(r)} \cdot Y$.*

Proof. We look at the orthogonal decomposition $f = \sum_j c_j f_j$, where the set $\{f_1, \dots, f_J\}$ is an orthogonal basis for $S_\kappa(\Gamma_1(d'^2))$. We first prove the lemma for the eigenforms f_j . Let us define $a'_j(n) := a'_{f_j}(n)$. Let $r = p_1 p_2 \dots p_t$. We proceed by induction on t . Note that by our choice (compare the paragraph preceding Lemma 4.1), $d \mid \mathbf{M}$. Thus $(r,\mathbf{M}) = 1$ implies that $(r,d) = 1$.

From [3, Proof of Prop. 5.3], we recall that for a prime p such that $(p,\mathbf{M}) = 1$,

$$a'_j(p^2 n) = a'_j(n) a'_j(p^2) - \chi(p) a'_j(n) \delta_{p|n} - \chi(p) a'_j(n/p^2),
 \tag{4.16}$$

where $\delta_{p|n} = 1$ if $p \mid n$ and 0 otherwise, and $a'_j(s)$ is zero if s is not an integer. Here we remind the reader that the $|_k$ operator in [3] is normalised so that it defines a group action on $GL(2, \mathbf{R})^+$, the subgroup of $GL(2, \mathbf{R})$ whose elements have positive determinant. Since f_j is an eigenform away from d' , $a'_j(p^2)$ equals the (normalised) eigenvalue of some newform of level dividing D under the Hecke operator T_{p^2} defined as in [3].

By the Cauchy–Schwarz inequality and applying the Deligne bound to $a'_j(p^2)$, we get

$$|a'_j(p^2 n)|^2 \leq 9 |a'_j(n)|^2 + |a'_j(n)|^2 \delta_{p|n} + |a'_j(n/p^2)|^2 \leq 10 |a'_j(n)|^2 + |a'_j(n/p^2)|^2.
 \tag{4.17}$$

Summing formula (4.17) over $n \leq Y$ and using the bound in Proposition 4.3(ii) applied to g_j , we get

$$S_{U(p^2)_f_j}(Y) \leq 11 \cdot S_{f_j}(Y) \leq 11 \cdot C_j \cdot Y.$$

This proves Lemma 4.4 for the eigenform f_j when $t = 1$ with a constant C_j depending only on κ and d .

For $i = 1, \dots, t$, let us set $V_i := U(p_1^2 \cdots p_{i-1}^2) f_j$ and $V_0 := f_j$. By the induction hypothesis, suppose we know the result in the statement of the lemma for V_{i-1} . Since $(p_i, D) = 1$, the Hecke operator $T_{p_i^2}$ commutes with $U(p_1^2 \cdots p_{i-1}^2)$. Therefore, arguing as in the case when $t = 1$, replacing g_j by V_i gives us

$$S_{V_i}(Y) \leq 11S_{V_{i-1}}(Y) \leq C_j \cdot 11^i \cdot Y \tag{4.18}$$

with C_j as before. This proves our result for the elements $\{f_1, \dots, f_J\}$ of the orthogonal basis.

Now again from the equation $f = \sum_j c_j f_j$, we see by the Cauchy–Schwarz inequality that

$$S_{U(r^2)f}(Y) \leq \left(\sum_j |c_j|^2 \right) \cdot \sum_j S_{U(r^2)f_j}(Y) \leq D_f \cdot 11^{\omega(r)} \cdot Y, \tag{4.19}$$

using formula (4.18) for $i = t$. Here $D_f = \left(\sum_j |c_j|^2 \right) \cdot \left(\sum_j C_j \right)$. This completes the proof of the lemma. \square

Let \mathcal{S} be the set of square-free integers and $\mathcal{S}_{\mathbf{M}} = \{n \in \mathcal{S} | (n, \mathbf{M}) = 1\}$. Define

$$S_f(\mathbf{M}, X) := \sum_{n \leq X, n \in \mathcal{S}_{\mathbf{M}}} |a'_f(n)|^2 \left(= \sum_{n \leq X, n \in \mathcal{S}} |a'_g(n)|^2 \right), \tag{4.20}$$

keeping in mind that $a'_g(n)$ is 0 if $(n, \mathbf{M}) > 1$ and equal to $a'_f(n)$ otherwise (see definition (4.4)). Hence¹

$$\begin{aligned} S_f(\mathbf{M}, X) &= \sum_{n \leq X} \sum_{r^2 | n} \mu(r) |a'_g(n)|^2 = \sum_{r^2 \leq X, r \in \mathcal{S}_{\mathbf{M}}} \mu(r) \sum_{m \leq X/r^2} |a'_g(mr^2)|^2 \\ &= \sum_{r^2 \leq X, r \in \mathcal{S}_{\mathbf{M}}} \mu(r) S_{U(r^2)g}(X/r^2). \end{aligned} \tag{4.21}$$

Clearly $S_{U(r^2)g}(X/r^2) \leq S_{U(r^2)f}(X/r^2)$. Therefore for X large enough – that is, $X \geq c_{f, \mathbf{M}}$, where $c_{f, \mathbf{M}}$ is as in Proposition 4.3 – we can use (i) and (ii) in Proposition 4.3 and Lemma 4.4 to write

$$S_f(\mathbf{M}, X) \geq B_f \prod_{p | \mathbf{M}} \left(1 - \frac{2}{p} \right)^2 \cdot X - \sum_{r^2 \leq X, r \in \mathcal{S}_{\mathbf{M}}} S_{U(r^2)f}(X/r^2)$$

¹We take the opportunity to correct an error in [3, Prop. 5.8]. The calculations for the lower bound of the quantity $S_f(M, X)$ there are not correct, and should be replaced by those given here, along with the results supporting them – Proposition 4.3 and Lemma 4.4 of this paper. The rest of the results in [3] still hold. Note also that Proposition 4.3 and Lemma 4.4 in this paper hold for any \mathbf{g} whose Fourier expansion is supported away from its level.

$$\geq \left(B_f \prod_{p|\mathbf{M}} \left(1 - \frac{2}{p} \right)^2 - D_f \cdot \sum_{r^2 \leq X, r \in \mathcal{S}_{\mathbf{M}}} \cdot 11^{\omega(r)} r^{-2} \right) \cdot X. \tag{4.22}$$

Now let us choose $\mathbf{M} = \prod_{2 < p \leq t} p$ with t large enough such that all the primes in d occur in \mathbf{M} (compare equation (4.1) and the few lines after definition (4.4), and recall that d is odd):

$$\sum_{r \geq 2, r \in \mathcal{S}_{\mathbf{M}}} 11^{\omega(r)} / r^2 = -1 + \prod_{p > t} (1 + 11/p^2). \tag{4.23}$$

This sum is bounded as $O(1/t)$. Now we look at the quantity $\prod_{p|\mathbf{M}} \left(1 - \frac{2}{p} \right)^2$. We see that

$$\prod_{p|\mathbf{M}} \left(1 - \frac{2}{p} \right)^{-2} = \exp \left(2 \sum_{2 < p \leq t} \log \left(1 - \frac{2}{p} \right) \right) = (\log t)^4 + O(1). \tag{4.24}$$

We therefore can choose t large enough so that the expression inside the parentheses in formula (4.22) is positive. Hence for $X \geq c_{f, \mathbf{M}}$ (which will depend only on f , since we can choose t large depending only on f),

$$S_f(\mathbf{M}, X) \geq E_f X, \tag{4.25}$$

where $E_f > 0$ depends only on f . Thus we are done in the integral-weight case.

4.1.2. f cuspidal, $\kappa \in \frac{1}{2}\mathbf{Z} \setminus \mathbf{Z}$. Here as well, we try to reduce to the calculations in [33]. In the remainder of this subsection, let us define for convenience

$L := d'^2$, $L_f :=$ the square-free integer divisible by all prime factors of L appearing in [33, Thm. 3] ($=\mathbf{M}$, compare the line preceding Lemma 4.1).

We start with

$$g(\tau) = \sum_{n \leq X, (n, L_f) = 1} a'_f(n) q^n \in S_{\kappa}(\Gamma_1(LL_f^2)),$$

and we do not worry about the precise level of g . Now g is nonzero by Lemma 4.1. Let us decompose g by characters mod LL_f^2 and write

$$g = \sum_{\chi} c_{\chi} g_{\chi},$$

where $c_{\chi} \in \mathbf{C}$ and $g_{\chi} \in S_{\kappa}(LL_f^2, \chi)$. Note that the constants c_{χ} depend only on g (or equivalently only on f). Let M be a square-free integer (to be specified later) divisible by L_f . Let us put $M_0 = M/L_f$, so that M_0 is odd and square-free and $(M_0, L_f) = 1$. We then consider the quantity

$$T_f(Y, M) := \sum_{n \geq 1, (n, M) = 1} |a'(f, n)|^2 e^{-n/Y},$$

which also equals $T_g(Y, M_0)$, whose definition is obvious. Then

$$T_f(Y, M) = T_g(Y, M_0) = \sum_{s|M_0} \mu(s) \sum_{n \geq 1} |a'(g, sn)|^2 e^{-sn/Y}$$

$$= \sum_{\chi, \chi'} c_\chi \overline{c_{\chi'}} \sum_{s|M_0} \mu(s) \sum_{n \geq 1} a'(g_\chi, sn) \overline{a'(g_{\chi'}, sn)} e^{-sn/Y}. \tag{4.26}$$

If we denote the innermost sum in equation (4.26) by $T_s(\chi, \chi'; Y)$, then from the results of [17] we infer that for any square-free s such that $(s, L_f) = 1$ (in particular for $s | M_0$), and for some constant $C_f > 0$ (proportional to $\langle f, f \rangle$) depending only on f ,

$$T_s(\chi, \chi; Y) = C_f \frac{Y}{s} + O_{f, M}(Y^{1/2}) \tag{4.27}$$

$$T_s(\chi, \chi'; Y) = O_{f, M}(Y^{1/2}), \quad \chi \neq \chi'. \tag{4.28}$$

In equation (4.28), we have used the fact that g_χ and $g_{\chi'}$ are orthogonal to each other if $\chi \neq \chi'$. Indeed, equation (4.28) follows from [17, Theorem 5], noting that the first term on the right-hand side of [17, eqn (48)] is proportional to the inner product of the cusp forms in question. Therefore,

$$T_g(Y, M_0) = \sum_{\chi} |c_\chi|^2 \sum_{s|M_0} \frac{\mu(s)}{s} \cdot C_f Y + O_{f, M}(Y^{1/2})$$

$$\geq D_f \frac{\phi(M)}{M} \cdot Y, \tag{4.29}$$

for all $Y > d_{f, M}$ for some constant depending only on f, M .

After this, we follow the argument in the previous subsection – that is, we choose an orthogonal basis of $S_\kappa(\Gamma_1(L))$ and write $f = \sum_j f_j$, where the set $\{f_j\}_j$ consists of a basis of pairwise orthogonal eigenforms on the spaces $S_\kappa(L, \chi)$, $\chi \bmod L$, which are away from L . Analogous to [33, Lem. 3.8], we find that for any square-free r (including 1) coprime with L ,

$$\sum_{n \geq 1} \left| a'_{U(r^2)_f}(n) \right|^2 e^{-n/X} \leq 19^{\omega(r)} B_f X, \tag{4.30}$$

where B_f depends only on f .

Let us now finish the proof. Recall that S_M is the set of square-free integers coprime to M . Using formulas (4.29) and (4.30), we write

$$\sum_{n \geq 1, n \in S_M} |a'_f(n)|^2 e^{-n/X} \geq \frac{\phi(M)}{M} D_f \cdot X - B_f X \cdot \sum_{r \geq 2, (r, M)=1} |\mu(r)| 19^{\omega(r)} r^{-2}. \tag{4.31}$$

The right-hand side is shown to be $\gg X$ in [33, p. 377] by choosing M appropriately (depending only on f ; see also Section 4.1.1), and thus we are done.

4.1.3. Quantitative bounds. To obtain quantitative versions of the nonvanishing results, we use the Deligne bound $-a'_f(n) \ll_f n^\epsilon$, $n \geq 1$ square-free – when $\kappa \in \mathbf{Z}$ and the

Bykovskii bound otherwise: $a'_f(n) \ll_f n^{3/16+\epsilon}$, $n \geq 1$ [13]. More precisely, when the weight κ is integral, referring to the sum in definition (4.20) and looking at its lower bound from formula (4.25), we see (with a given $\epsilon > 0$) that

$$\#\{n \leq X, n \in \mathcal{S}_M | a'_f(n) \neq 0\} \cdot X^\epsilon \gg \sum_{n \leq X, n \in \mathcal{S}_M} |a'_f(n)|^2 \gg_f X, \tag{4.32}$$

which gives the lower bound on $\#\{n \leq X | a'_f(n) \neq 0\}$. When κ is half-integral, we start from the lower bound in formula (4.31), writing it in the form

$$\sum_{n \leq X^{1+\epsilon}, (n, M)=1} |a'_f(n)|^2 e^{-n/X} + \sum_{n > X^{1+\epsilon}, (n, M)=1} |a'_f(n)|^2 e^{-n/X} \gg_f X. \tag{4.33}$$

The second sum can be estimated as

$$\begin{aligned} \sum_{n > X^{1+\epsilon}, (n, M)=1} |a'_f(n)|^2 e^{-n/X} &\leq \int_{X^{1+\epsilon}}^\infty y^{3/8+\epsilon} e^{-y/X} dy \\ &\leq X^{11/8+\epsilon} \Gamma(3/8+\epsilon, X^\epsilon), \end{aligned} \tag{4.34}$$

where $\Gamma(s, x)$ denotes the upper incomplete gamma function defined for $\Re(s) > 0$ as

$$\Gamma(s, x) = \int_x^\infty y^{s-1} e^{-y} dy.$$

From the standard asymptotic properties of $\Gamma(s, x)$ (see, eg., [1]), for $s > 0$ and $x \rightarrow \infty$ we know that $\Gamma(s, x) \sim x^{s-1} e^{-x}$. Therefore for X large enough, we see that the left-hand side of formula (4.34) is

$$\ll_\epsilon X^{11/8+\epsilon} e^{-X^\epsilon} \ll_{\epsilon, A} X^{-A}$$

for any $A > 0$. Therefore the second sum in formula (4.33) contributes only negligibly to it.

Thus for large enough X and the obvious inequality

$$\sum_{n \leq X^{1+\epsilon}, (n, M)=1} |a'_f(n)|^2 \geq \sum_{n \leq X^{1+\epsilon}, (n, M)=1} |a'_f(n)|^2 e^{-n/X} \gg_{f, \epsilon} X,$$

using Bykovskii's bound we get

$$\#\{n \leq X^{1+\epsilon}, n \in \mathcal{S}_M | a'_f(n) \neq 0\} \cdot X^{3/8+\epsilon} \gg \sum_{n \leq X^{1+\epsilon}, (n, M)=1} |a'_f(n)|^2 \gg_f X.$$

Finally, by replacing $X^{1+\epsilon}$ with X and again changing ϵ if necessary, we obtain

$$\#\{n \leq X, n \in \mathcal{S}_M | a'_f(n) \neq 0\} \gg_{f, \epsilon} X^{5/8-\epsilon}. \tag{4.35}$$

We summarise in a theorem.

Theorem 4.5. *Let $\kappa \geq 2, N \geq 1$ and $f \in S_\kappa(\Gamma_1(N'))$, with N' as defined in equation (4.2), be such that $a_f(n) = 0$ for all n such that $(n, N') > 1$. Then there exist infinitely many*

odd and square-free n such that $a_f(n) \neq 0$. More precisely, for any $\epsilon > 0$,

$$\#\{n \leq X, n \text{ square-free} \mid a_f(n) \neq 0\} \gg_{f,\epsilon} \begin{cases} X^{1-\epsilon} & \text{if } \kappa \in \mathbf{Z}, \\ X^{5/8-\epsilon} & \text{otherwise.} \end{cases}$$

4.2. Method 2: Proof for all $f \in M_\kappa(\Gamma(N'))$ using multiplicity 1 where $\kappa \in \mathbf{Z}$

We will show that when $\kappa \in \mathbf{Z}$, one can prove an optimal result about the nonvanishing of odd, square-free Fourier coefficients of any nonzero $h \in M_\kappa(\Gamma(N'))$ whose Fourier expansion is supported away from the level. This result can then be applied to f from equation (4.3) with $N = d^2$. The crucial input is the multiplicity 1 on the space of newforms. The method is based on a useful argument involving sieving of eigenforms, originally due to Balog and Ono [8]. Variants of this method have already appeared in [3]. For convenience, we recall the details of the method below.

We start with an $h \in M_\kappa(\Gamma(N'))$ such that $a_h(n) = 0$ for all n such that $(n, N) > 1$. We consider, as before, $H(\tau) = h(N\tau) \in M_\kappa(\Gamma_1(D))$, where we put $D = N^2$. Note that $a_H(n) = 0$ for all $(n, D) > 1$.

Let us denote the set of normalised newforms on $\Gamma_1(D)$ of weight k by \mathcal{F}_D^1 , and those on $\Gamma_0(D)$ with nebentypus χ by $\mathcal{F}_{D,\chi}$. In this section we allow $\kappa \geq 2$.

Since $\mathcal{F}_D^1 = \cup_\chi \mathcal{F}_{D,\chi}$, where χ runs over Dirichlet characters mod D , we can write

$$H(\tau) = \sum_i \sum_\delta \alpha_{i,\delta} f_i(\delta\tau). \tag{4.36}$$

Here f_i runs over $\cup_\chi \mathcal{F}_{D,\chi}$, δ runs over the divisors of D (such that $\delta \cdot (\text{level of } f_i) \mid D$) and $c_{j,\delta}$ are scalars. For this result, see [29, Lem. 4.6.9] for the space of cusp forms and [43, Prop. 5] for the space of Eisenstein series.

Now by classical (new-/oldform theory it follows that H can not entirely lie in the oldspace (see [29, Theorem 4.6.8]), and hence must have a nonzero ‘newform’ component of some level. In terms of equation (4.36), this means that there exists an i' such that $c_{i',1} \neq 0$. For ease of notation, by renumbering if necessary, we can assume that $i' = 1$ and omit the nebentypus χ' of the corresponding newform f_1 (of some level $M \mid D$) from display.

We now move on to the argument mentioned at the beginning of this subsection. This involves sieving out the entire old-classes corresponding to all f_i with $i \neq 1$ by means of Hecke operators T_p with $p \nmid D$ (for $\Gamma_1(D)$) and using multiplicity 1. Let the Fourier expansion of f_i be written as

$$f_i(\tau) = \sum_{n \geq 1} b_i(n)q^n.$$

By multiplicity 1, we can choose an odd prime $q = q_{1,2} \nmid D$ such that $b_1(q) \neq b_2(q)$. In fact, we can bound such a q solely in terms of κ and D (see, e.g., [11, Prop. 3.2]). Then consider the form $g_1(\tau) = \sum_{n=1}^\infty a_1(n)q^n := T(q)f(\tau) - b_2(q)f(\tau)$, so that

$$g_1(\tau) = \sum_{i=1}^s (b_i(q) - b_2(q)) \sum_{\delta \mid N} \alpha_{i,\delta} f_i(\delta\tau), \quad s = \dim M_\kappa(\Gamma_1(D)).$$

The modular forms $f_2(\delta\tau)$ for any $\delta|N$ do not appear in the decomposition of $g_1(\tau)$, but f_1 does. Proceeding inductively in this way, we can remove all the nonzero newform components and their translates $f_i(\delta\tau)$ for all $i = 2, \dots, s$ one by one, to obtain a modular form F in $M_\kappa(\Gamma_1(D))$ such that on the one hand we have

$$F(\tau) = \sum_{n=1}^{\infty} A(n)q^n := \prod_{2 \leq j \leq s} (b_1(q_{1,j}) - b_j(q_{1,j})) \sum_{\delta|N} \alpha_{1,\delta} f_1(\delta\tau). \tag{4.37}$$

To be precise, we set $g_0 := H$, and having determined g_{i-1} , we define $g_i := T(q_{1,i})g_{i-1} - b_{q_{1,i}}g_i$, where $q_{1,i}$ is any prime coprime to D such that $b_1(q_{1,i}) \neq b_i(q_{1,i})$. At each stage we choose the smallest odd prime with these properties. As mentioned before, such primes are bounded solely in terms of κ and D (compare [11, Prop. 3.2]). Then $F = g_{s-1}$.

By the construction, the product appearing on the right-hand side of the equality in formula (4.37) is nonzero. Therefore, rescaling F and calling the resulting function again F , we note on the other hand that the inductive procedure gives us finitely many ($\leq 3^{s-1}$) algebraic numbers β_t (polynomials in the $p_{1,i}$ s and certain Dirichlet characters) and positive, square-free rational numbers γ_t (which are quotients of the $p_{1,i}$ s) such that for every $n \geq 1$,

$$A(n) = \sum_{\delta|N} \alpha_{1,\delta} b_1(n/\delta) \equiv \sum_t \beta_t a_H(\gamma_t n). \tag{4.38}$$

In equation (4.38), $\alpha_{1,\delta} \neq 0$. We define $Q := \prod_{i=2}^s q_{1,i}$. Note that $\max_t \gamma_t \leq Q$.

Therefore, if we choose $n \geq 1$ such that $(n, QN) = 1$, we get

$$\alpha_{1,1} b_1(n) = \sum_t \beta_t a_H(\gamma_t n). \tag{4.39}$$

Then by our choice, all the γ_t s appearing in equation (4.39) are odd, square-free integers. This is because the primes $q_{1,i}$ are pairwise distinct.

Let us define for $L \geq 1$ and $\mathbf{g} \in M_\kappa(\Gamma_1(D))$ the counting functions

$$\begin{aligned} \Pi_{\mathbf{g}}(L; X) &:= \{n \leq X \mid (n, L) = 1, a_{\mathbf{g}}(n) \neq 0\}, \\ \Pi_{\mathbf{g}}^*(L; X) &:= \#\{n \leq X \mid (n, L) = 1, n \text{ odd, square-free, } a_{\mathbf{g}}(n) \neq 0\}. \end{aligned}$$

Now from equation (4.39) we see that for any $n \leq X$, $(n, QN) = 1$ such that $b_1(n) \neq 0$, there is a *unique* t such that $\gamma_t n$ satisfies $a_H(\gamma_t n) \neq 0$. Clearly $\gamma_t n \leq QX$. This implies

$$\Pi_{f_1}^*(QN; X) \leq \Pi_H^*(QN, QX). \tag{4.40}$$

The size of the set $\Pi_{\mathbf{g}}(L; X)$ has been studied in [35, 37] when \mathbf{g} is a newform. In particular when \mathbf{g} is cuspidal, we quote [35, Théorème 16, Prop. 18] for $\kappa \geq 2$ and [36, Théorème 4.2(ii)] for $\kappa = 1$. For our purposes we need analogous statements about the quantity $\Pi_{\mathbf{g}}^*(L; X)$. Namely, we want to show that for some constants $\rho_j, \alpha > 0$ depending

only on \mathfrak{g} and L , the following hold:

$$\#\Pi_{\mathfrak{g}}^*(L; X) \sim \begin{cases} \rho_1 X & \text{if } \mathfrak{g} \text{ is cuspidal, not of CM type, } \kappa \geq 2, \\ \rho_2 \frac{X}{(\log X)^{1/2}} & \text{if } \mathfrak{g} \text{ is cuspidal, of CM type, } \kappa \geq 2, \\ \rho_3 \frac{X}{(\log X)^\alpha} & \text{if } \mathfrak{g} \text{ is cuspidal, } \kappa = 1, \\ \rho_4 X & \text{if } \mathfrak{g} \text{ is an Eisenstein newform, } \kappa \geq 2. \end{cases} \tag{4.41}$$

These bounds have been proved in [35, 36] for $\Pi_{\mathfrak{g}}(L; X)$. We need to address two additional points in our case:

- (a) Serre states his results for $L = 1$, but we require them to hold for any fixed $L \geq 1$.
- (b) We have to count odd, square-free integers – that is, consider the quantity $\Pi_{\mathfrak{g}}^*(L; X)$.

We show in the following how to adapt Serre’s arguments in a simple manner to deal with (a) and (b). For this we refer the reader to [36, §1, §2], and follow the arguments presented therein.

Define $P_{\mathfrak{g}}(L) := \{p \nmid L \mid a_{\mathfrak{g}}(p) = 0\}$ and $\Pi_{\mathfrak{g}}^*(L) := \{(n, L) = 1, n \text{ square-free} \mid a_{\mathfrak{g}}(n) = 0\}$. Then the generating function (Dirichlet series) $\mathcal{F}(s)$ of $\Pi_{\mathfrak{g}}^*(L)$ is just, say,

$$\mathcal{F}(s) = \sum_{n \notin \Pi_{\mathfrak{g}}^*(L)} n^{-s} = \prod_{p \notin P_{\mathfrak{g}}(L)} (1 + p^{-s}) = \sum_{n \geq 1} b_n n^{-s}.$$

Clearly $\Pi_{\mathfrak{g}}^*(L; X) = \sum_{n \leq X} b_n$. Further, the set $P_{\mathfrak{g}}(L)$ is ‘Frobenian’ and has natural (and hence analytic or Dirichlet) density $\alpha := \alpha(\mathfrak{g})$ such that $0 \leq \alpha < 1$. Moreover, if \mathfrak{g} is of CM type, then $\alpha = 1/2$; and if $\kappa \geq 2$ and \mathfrak{g} is not of CM type, then $\alpha = 0$. For these facts, see, for example, [35, §7.4, p. 178] and [35, §7.5, p. 180], respectively.

$\mathcal{F}(s)$ is holomorphic in the region $\Re(s) > 1$ and is nonzero there. We can then write

$$\log \mathcal{F}(s) = \sum_{p \notin P_{\mathfrak{g}}(L)} p^{-s} + \theta_1(s) = (1 - \alpha) \cdot \frac{1}{s - 1} + \theta_2(s),$$

where $\theta_1(s), \theta_2(s)$ are holomorphic in $\Re(s) \geq 1$. This can be seen, for example, from [36, equation (1.6)]. This gives us

$$\mathcal{F}(s) = \frac{1}{(s - 1)^{1 - \alpha}} \cdot \exp(\theta_2(s)), \quad \Re(s) \geq 1.$$

Since $\mathcal{F}(s)$ has nonnegative Dirichlet coefficients, our desired results follow from the (generalised) Ikehara–Weiner theorem (see [36, equation (2.7)] and [16]), and we conclude that

$$\Pi_{\mathfrak{g}}^*(L; X) = \sum_{n \leq X} b_n \sim c \cdot \frac{X}{(\log X)^\alpha}, \quad c > 0.$$

The assertions in formula (4.41) now follow from the different values of α .

From the foregoing discussion and formula (4.40), we now arrive at the following theorem:

Theorem 4.6. *Let $\kappa \in \mathbf{Z}$, $N \geq 1$ and $h \in M_\kappa(\Gamma(N))$ be such that $a_f(n) = 0$ for all n such that $(n, N) > 1$. Then there exist infinitely many odd and square-free n such that*

$a_f(n) \neq 0$. More precisely, for some $0 < \alpha < 1$,

$$\#\{n \leq X, n \text{ odd, square-free} \mid a_h(n) \neq 0\} \gg_h \begin{cases} \frac{X}{(\log X)^{1/2}} & \text{if } h \text{ is cuspidal, } \kappa \geq 2, \\ \frac{X}{(\log X)^\alpha} & \text{if } h \text{ is cuspidal, } \kappa = 1, \\ X & \text{if } h \text{ is not cuspidal, } \kappa \geq 2. \end{cases}$$

4.3. Conclusion of the proof of Theorem 1.1

As mentioned in the introduction, we prove Theorem 1.1 by induction on n . When $n = 1$ and ρ is as in the statement of the theorem, we can write $F = (f_1, \dots, f_m)$, where $m = \dim \rho$ and each f_j is a modular form of some weight $k_j \geq 0$. Moreover, at least one $f_j \neq 0$. To prove Theorem 1.1 for $n = 1$, it is therefore enough to prove it for any of the nonzero f_j . This in turn follows directly from Theorem 4.6.

We next move to treat higher degrees and assume that $n > 1$.

We first demonstrate the proof of the first two lower bounds of $\#\mathfrak{S}_F(X)$ in Theorem 1.1. We start with a vector-valued $F \neq 0$ of degree n with the given condition on the weight that $k(\rho) - n/2 \geq \varrho(n)$, and consider the nonzero modular form $F^o \in M_{\rho''}^{n-1}$ (with ρ'' denoting the representation appearing in Proposition 3.1) of weight $k(\rho'') \geq k(\rho)$. To apply the induction hypothesis to F^o , we need to verify that $k(\rho'') - (n - 1)/2 \geq \varrho(n - 1)$. Indeed, this follows from the $k(\rho'') \geq k(\rho)$ inequality and the hypothesis on F . Now using Corollary 3.3, we obtain a scalar-valued Jacobi form $\varphi_T^{(r)} \neq 0$, $T \in \Lambda_{n-1}^+$, which is a Jacobi form of weight $k' \geq k(\rho)$ and is a certain vector component of a (vector-valued) Fourier-Jacobi coefficient φ_T of F . Here T has odd, square-free discriminant.

Now we again invoke the fact that $k(\rho) - \frac{n}{2} \geq \varrho(n)$, so that $\kappa := k' - (n - 1)/2 \geq k(\rho) - (n - 1)/2 \geq \varrho(n) + 1/2$, and hence $\kappa \geq 5/2$, since κ is half-integral when n is even. This allows us to use the results of Section 4. If n is odd, we get $\kappa \geq 2$ by a similar reasoning, which puts us into the setting of Section 4.2. See Remark 4.7 for more on the restriction on the weights.

Then we move on to the realm of elliptic modular forms by using Proposition 3.9 to obtain a primitive theta-component h_μ of $\varphi_T^{(r)}$ of weight κ . In fact, we work with its close relative H_μ (compare equation (3.18)) and then use Theorem 4.5 and Theorem 4.6 to get suitable nonvanishing properties of its Fourier coefficients. In the vector-valued setting that we are in, let us point out that even when we start with a noncuspidal form F , it is not clear to us how ensure that the scalar-valued $\varphi_T^{(r)}$ is also noncuspidal. This is highly probable, but we cannot prove it. It can be proved, however, if we start from a noncuspidal scalar-valued modular form (see later). Therefore, the first lower bound in Theorem 1.1 is actually the infimum of the lower bounds appearing in Theorem 4.6.

Thus we can demonstrate (via Proposition 3.9) the requisite nonvanishing properties of the Fourier coefficients of F . This finishes the proof of the first two lower bounds of $\#\mathfrak{S}_F(X)$ in Theorem 1.1.

To demonstrate the proof of the last lower bound in Theorem 1.1, we start with a scalar-valued $F \neq 0$. We then consider $G := \Phi(F) \in M_k^{n-1}$, where Φ is the Siegel Φ -operator (see definition (2.2)). Now the first lower bound in Theorem 1.1 gives us at least one $M \in \Lambda_{n-1}^+$

such that d_M is odd and square-free and

$$a_{\Phi(F)}(M) = a_F\left(\begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix}\right) \neq 0. \tag{4.42}$$

If the Fourier–Jacobi coefficients of F are denoted by φ_T , $T \in \Lambda_{n-1}$, then equation (4.42) clearly implies that $\varphi_M \neq 0$ and φ_M is noncuspidal. Therefore, by Lemma 3.8 and Proposition 3.5, φ_M has a nonzero *noncuspidal* primitive theta-component h_μ . From this point on, the demonstration proceeds exactly as described in the first part of this proof. We apply Theorem 4.6 here. The restriction on the weight comes from Lemma 3.8. \square

Remark 4.7 (Condition on weights). Let us remark here that the condition on the weight in Theorem 1.1 – namely, $k(\rho) - \frac{n}{2} \geq \varrho(n)$ – is technical, and is used in many places in Section 4. Let us set $\ell := k(\rho) - \frac{n-1}{2}$. When n is odd, so that ℓ is an integer, the bound $\ell \geq 2$ is enough. When n is even, so that ℓ is a half-integer, the bound $\ell \geq 5/2$ is needed to invoke results from [33].

To be more precise, let us point out that the results of Sections 4.1.1 and 4.1.2 are valid for κ (= weight of the cusp form considered there) at least 1 and 5/2, respectively. The lower bound 5/2 can be improved to 3/2 if we use the results of [28], but we have to ensure that we do not encounter unary theta series of weight 3/2. This is an interesting thing to consider. In section 4.2 we request $\kappa \geq 1$ in the cuspidal case and $\kappa \geq 2$ otherwise. (The second condition perhaps can be relaxed.)

We note here that in the scalar-valued *cuspidal* case it is necessary to have $k - \frac{n}{2} \geq 0$; otherwise F is singular. Thus our condition on k implies that F is not singular. Moreover, our result is false for small weights, such as when $k = n/2 + 1$. Counterexamples are furnished by the theta series $\vartheta \in S_{n/2+1}^n$, given as

$$\vartheta(Z) = \sum_{X \in M_n(\mathbf{Z})} \det(X) e(S[X]Z),$$

where S is even unimodular and does not have an automorphism with $\det = -1$. In particular, it is not clear whether our theorem would hold for $k = (n + 1)/2$. Similar remarks as before apply to the vector-valued case as well.

In the *noncuspidal* case as well, our theorems may not hold for small weights; the classical theta function of weight $n/2$ already defined as ϑ , but with the $\det(X)$ removed, is a counterexample.

5. Refinement for prime discriminants and an application to the spinor L -function

In this section we first show how a variant of our main result (Theorem 1.1) can be used to refine it to a statement about ‘*prime discriminants*’. This is explicitly stated in the following. Let \mathbf{P} denote the set of all primes.

Theorem 5.1. *Let n be odd. Let $F \in S_\rho^n$ be nonzero and $k(\rho) - \frac{n}{2} \geq 3/2$. Then there exist $T \in \Lambda_n^+$ with d_T assuming infinitely many odd prime values, such that $a_F(T) \neq 0$.*

Moreover, the following stronger quantitative result holds:

$$\#(\mathfrak{S}_F(X) \cap \mathbf{P}) \gg X/\log X,$$

where the implied constant depends only on F .

We now show how to use this theorem (for $n = 3$) along with the work of Pollack [31] to obtain the standard analytic properties (meromorphic continuation, functional equation, etc.) of the spinor L -function $Z_F(s)$ of a holomorphic Siegel cuspidal eigenform F on $\mathrm{Sp}(3, \mathbf{Z})$ unconditionally (compare Theorem 1.2). We briefly discuss the background behind this result. Pollack used the correspondence between ternary quadratic forms and quaternion algebras to study a certain Rankin–Selberg integral (with respect to a suitable Eisenstein series) indexed by orders in quaternion algebras (or equivalently by some $T \in \Lambda_3^+$). This integral could be evaluated by unfolding using an expression for the spinor L -function as a Dirichlet series (essentially due to Evdokimov [19]), with a factor $a_F(T)$ in the front, where T corresponds to a maximal order in the quaternion algebra in question. The moment we know that $a_F(T) \neq 0$, we can read off the analytic properties of $Z_F(s)$ from those of the Eisenstein series in question. This is what we are going to do in this section. But first, let us postpone the proof of Theorem 5.1 and show how Theorem 1.2 can be obtained from it.

5.1. Proof of Theorem 1.2

In view of [31, Theorem 1.2] and Theorem 5.1, it is enough to check that $T \in \Lambda_3^+$ with $d_T = p$ (p odd prime) defines a maximal order in a quaternion algebra (necessarily) ramified at ∞ , since T is positive definite.

Since the correspondence between Λ_3^+ and orders in quaternion algebras (see, e.g., [31, Proposition 3.3] or [42, Chapter 22]) preserves discriminants (compare [31, Corollary 3.4]), if T corresponds to an order \mathcal{O}_T in some quaternion algebra Q over \mathbb{Q} ramified at ∞ , then $d_T = p = |rd(\mathcal{O}_T)|$. Here $rd(\mathcal{O}_T)$ denotes the reduced discriminant of \mathcal{O}_T . This implies that \mathcal{O}_T is a maximal order. Indeed, if $\mathcal{O}_T \subset \mathcal{O}$ for some maximal order \mathcal{O} of Q , then $|rd(\mathcal{O})| \mid |rd(\mathcal{O}_T)| = p$. However, $rd(\mathcal{O})$ cannot be 1, as it must be ramified at another finite place (p), since the number of ramified places is even. Thus $|rd(\mathcal{O})| = |rd(\mathcal{O}_T)|$ – that is, $\mathcal{O} = \mathcal{O}_T$ and \mathcal{O}_T is maximal. □

In order to prove Theorem 5.1, we need a lemma.

Let us denote the set of normalised newforms on $\Gamma_1(M)$ of weight k by \mathcal{F}_M^1 and those on $\Gamma_0(M)$ with nebentypus χ (with $m_\chi \mid M$) by $\mathcal{F}_{M,\chi}$. We begin with a lemma on integral-weight cusp forms. For $\ell \geq 1$, $f \in S_k(\Gamma_1(N))$, let $f \mid_k V_\ell(\tau) := f(\ell\tau)$. In this section we allow $k \geq 1$.

Lemma 5.2. *Let $f \in S_k(\Gamma_1(N))$ be nonzero. Suppose that f does not belong to the \mathbb{C} -span of $\mathcal{F}_M^1 \mid_k V_\ell$ (where $M \mid N$ and $\ell \mid N/M$) with $\ell > 1$. Then there exist infinitely many primes p such that $a_f(p) \neq 0$. More precisely,*

$$\#\{p \leq x, (p, N) = 1 \mid a_f(p) \neq 0\} \gg_f x/\log x.$$

The proof of this lemma is based on the celebrated Ikehara–Wiener theorem on Dirichlet series with nonnegative coefficients, which we recall next.

Theorem 5.3 (Ikehara and Wiener [30]). *Let $A(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series. Suppose there exists another Dirichlet series $B(s) = \sum_{n=1}^{\infty} b_n n^{-s}$ with $b_n \geq 0$ such that*

- (a) $|a_n| \leq b_n$,
- (b) $B(s)$ converges in $\Re(s) > 1$,
- (c) $B(s)$ (resp., $A(s)$) can be extended meromorphically to $\Re(s) \geq 1$ having no poles except (resp., except possibly) a simple pole at $s = 1$ with residue $R \geq 0$ (resp., $r \geq 0$). Then

$$(i) \sum_{n \leq x} a_n = rx + o(x), \quad (ii) \sum_{n \leq x} b_n = Rx + o(x).$$

Proof of Lemma 5.2. Since $\mathcal{F}_M^1 = \cup_{\chi} \mathcal{F}_{M,\chi}$, where χ runs over Dirichlet characters mod M such that $m_{\chi} | M$, we can write

$$f(z) = \sum_{\chi} \sum_M \sum_{\ell} c_{\chi,M,\ell} f_M(\ell z), \tag{5.1}$$

where χ runs over Dirichlet characters mod N , M runs over divisors of N such that $m_{\chi} | M$, f_M runs over $\mathcal{F}_{M,\chi}$, ℓ runs over the divisors of N/M and $c_{\chi,M,\ell}$ are scalars. By our assumption, there exist a χ and an M such that $c_{\chi,M,1} \neq 0$.

From equation (5.1), it follows immediately that for primes $(p, N) = 1$ we can write

$$a_f(p) = \sum_{\mathfrak{f}} c_{\mathfrak{f}} \lambda_{\mathfrak{f}}(p), \tag{5.2}$$

where \mathfrak{f} runs over the set $\cup_{\chi,M} \mathcal{F}_{M,\chi}$, with χ and M varying as before, and not all the $c_{\mathfrak{f}}$ are zero. The point to note here is that all the \mathfrak{f} are newforms of level dividing N .

By a theorem of Shahidi [38], we know that the Rankin–Selberg convolution (in the sense of Langlands) $L(\mathfrak{f} \otimes \bar{\mathfrak{g}}, s) \neq 0$ on the line $\Re(s) = 1$, if $\mathfrak{f} \neq \mathfrak{g}$. Moreover, it is classical (see, e.g., [23]) that $L(\mathfrak{f} \otimes \bar{\mathfrak{g}}, s)$ is analytic in \mathbf{C} except for a simple pole with positive residue at $s = 1$ if and only if $\mathfrak{f} = \mathfrak{g}$.

Then we compute

$$\begin{aligned} & \sum_{p \leq x, p \nmid N} |a_{\mathfrak{f}}(p)|^2 \log p \\ &= \sum_{\mathfrak{f}} |c_{\mathfrak{f}}|^2 \sum_{p \leq x, p \nmid N} |\lambda_{\mathfrak{f}}(p)|^2 \log p + \sum_{\mathfrak{f} \neq \mathfrak{g}} c_{\mathfrak{f}} \bar{c}_{\mathfrak{g}} \sum_{p \leq x, p \nmid N} \lambda_{\mathfrak{f}}(p) \overline{\lambda_{\mathfrak{g}}(p)} \log p \\ &= \sum_{\mathfrak{f}} |c_{\mathfrak{f}}|^2 \sum_{p \leq x, p \nmid N} \left| \lambda_{\mathfrak{f} \otimes \bar{\mathfrak{f}}}(p) \right| \log p + \sum_{\mathfrak{f} \neq \mathfrak{g}} c_{\mathfrak{f}} \bar{c}_{\mathfrak{g}} \sum_{p \leq x, p \nmid N} \lambda_{\mathfrak{f} \otimes \bar{\mathfrak{g}}}(p) \log p. \end{aligned}$$

We now appeal to (a relative version of) the Ikehara–Wiener theorem (see Theorem 5.3) applied to the Dirichlet series defined by the logarithmic derivatives $\frac{L'}{L}(\mathfrak{f} \otimes \bar{\mathfrak{f}}, s)$ and then to $\frac{L'}{L}(\mathfrak{f} \otimes \bar{\mathfrak{g}}, s)$. Let us denote the Satake parameters of \mathfrak{f} and \mathfrak{g} at a prime p (which we suppress mostly) by $\{a_{i,p}\}$ and $\{b_{j,p}\}$, respectively. We drop the suffix p from $\{a_{i,p}\}$, $\{b_{i,p}\}$ for convenience.

Further, let us set

$$\frac{L'}{L}(\mathfrak{f} \otimes \bar{\mathfrak{g}}, s) = \sum_{n=1}^{\infty} \Lambda_{\mathfrak{f} \otimes \bar{\mathfrak{g}}}(n) n^{-s}, \tag{5.3}$$

where

$$\Lambda_{\mathfrak{f} \otimes \bar{\mathfrak{g}}}(n) = \begin{cases} \left(\sum_{i,j} a_i^m \bar{b}_j^{-m}\right) \log p & \text{if } n = p^m, \\ 0 & \text{otherwise.} \end{cases}$$

Thus by the Cauchy-Schwarz inequality, we see that

$$|\Lambda_{\mathfrak{f} \otimes \bar{\mathfrak{g}}}(n)| \leq \frac{1}{2} \left(\Lambda_{\mathfrak{f} \otimes \bar{\mathfrak{f}}}(n) + \Lambda_{\mathfrak{g} \otimes \bar{\mathfrak{g}}}(n) \right), \tag{5.4}$$

and moreover, $\Lambda_{\mathfrak{f} \otimes \bar{\mathfrak{f}}}(n) = |\sum_i a_i^m|^2 \geq 0$ for all n .

Thus all the conditions of the Ikehara–Wiener theorem (Theorem 5.3) for $\frac{L'}{L}(\mathfrak{f} \otimes \bar{\mathfrak{f}}, s)$ and $\frac{L'}{L}(\mathfrak{f} \otimes \bar{\mathfrak{g}}, s)$ are satisfied, and we have

$$(i) \sum_{n \leq x} \Lambda_{\mathfrak{f} \otimes \bar{\mathfrak{f}}}(n) = x + o(x), \quad (ii) \sum_{n \leq x} \Lambda_{\mathfrak{f} \otimes \bar{\mathfrak{g}}}(n) = o(x) \tag{5.5}$$

as $x \rightarrow \infty$.

It is then easy to finish the proof by noting that

$$\sum_{n \leq x} \Lambda_{\mathfrak{f} \otimes \bar{\mathfrak{g}}}(n) = \sum_{p \leq x} \Lambda_{\mathfrak{f} \otimes \bar{\mathfrak{g}}}(p) + O\left(x^{1/2} \log x\right) \tag{5.6}$$

and $\Lambda_{\mathfrak{f} \otimes \bar{\mathfrak{g}}}(p) = \lambda_{\mathfrak{f} \otimes \bar{\mathfrak{g}}}(p) \log p$. Indeed, combining equations (5.3), (5.5) and (5.6), we get

$$\sum_{p \leq x, (p, N)=1} |a_{\mathfrak{f}}(p)|^2 \log p = \sum_{\mathfrak{f}} |c_{\mathfrak{f}}|^2 x + o_{\mathfrak{f}}(x) \tag{5.7}$$

as $x \rightarrow \infty$, where $\sum_{\mathfrak{f}} |c_{\mathfrak{f}}|^2 > 0$. This immediately implies the assertion of Lemma 5.2. \square

Remark 5.4. One may look for a better error term in equation (5.7) from the point of view of analytic number theory. This may be obtained (using the same arguments) by using the prime-number theorem for the Rankin–Selberg L -functions $L(\mathfrak{f} \otimes \bar{\mathfrak{g}}, s)$. However, one has to be careful about Siegel zeros in case $L(\mathfrak{f} \otimes \bar{\mathfrak{g}}, s)$ has a quadratic Dirichlet L -function as a factor.

5.2. Proof of Theorem 5.1

First we note that in the statement of Proposition 3.9, we can replace the condition ‘*odd and square-free*’ in the second part of the proposition with ‘*odd and prime*’.

We appeal to Corollary 3.3 (which is now unconditional, as we have proved Theorem 1.1) to conclude that our $F \in S_{\rho}^n$ has infinitely many nonzero Fourier–Jacobi coefficients ϕ_T , $T \in \Lambda_{n-1}^+$, with d_T odd and square-free. Then Proposition 3.9 gives us the nonzero cusp form H_{μ} , μ primitive, as defined just before Proposition 3.9. Let us keep the notation used there.

From now on we assume that n is odd. Now let us observe that Lemma 5.2 can be applied to $H_\mu \in S_{k' - \frac{n-1}{2}}(\Gamma_1(d'^2))$ – recall that $d' = d_T$ in the present case and $k' - \frac{n-1}{2} \in \mathbf{N}$ – because its Fourier expansion is supported on indices which are coprime to d_T by the primitiveness of μ . This in turn implies by oldform theory that it cannot entirely lie in the oldspace (see [29, Theorem 4.6.8]), and hence must have a nonzero new component.

Thus by the first paragraph of this section, we conclude that when n is odd, there exist infinitely many odd primes p such that for each such p , F has at least one nonzero Fourier coefficient $a_F(\mathcal{T})$ with $d_{\mathcal{T}} = p$. The quantitative version follows immediately from the corresponding statement of Lemma 5.2. \square

Remark 5.5. It is desirable to prove an analogue of Theorem 5.1 for n even; however, the necessary properties of elliptic modular forms of half-integral weight (i.e., a suitable version of Lemma 5.2) seem not to be available.

Acknowledgments

We would like to thank P. Anamby, A. Pollack and R. Schulze-Pillot for useful conversations. The first author thanks IISc. Bangalore and the second author thanks Universität Mannheim and Alfréd Rényi Institute for Mathematics, where parts of this work were done, for generous hospitality and for providing excellent working conditions. The second author also thanks IISc. Bangalore, DST (India) and the UGC Centre for Advanced Studies for financial support. During a substantial part of this work, the second author held a Humboldt Fellowship from the Alexander von Humboldt Foundation and was supported by MATRICS grant MTR/2017/000496 from DST-SERB, India. The authors are grateful to the referees for their detailed comments and suggestions and for pointing out some errors in an earlier version, which led to improved results.

References

- [1] N. ABRAMOWITZ AND I. STEGUN, *Handbook of Mathematical Functions* (Dover, New York, 1965).
- [2] A. AJOUZ, *Hecke Operators on Jacobi Forms of Lattice Index and the Relation to Elliptic Modular Forms*, Ph.D. thesis, University of Siegen, 2015, https://dspace.uni-siegen.de/bitstream/ubsi/938/1/Dissertation_Ali_Ajouz.pdf.
- [3] P. ANAMBY AND S. DAS, Distinguishing Hermitian cusp forms of degree 2 by a certain subset of all Fourier coefficients, *Publ. Mat.* **63** (2019), 307–341.
- [4] A. N. ANDRIANOV, Euler products corresponding to Siegel modular forms of genus 2, *Russian Math. Surveys* **29**(3) (1974), 45–116.
- [5] A. N. ANDRIANOV, Euler decompositions of theta-transformations of Siegel modular forms, *Math. USSR-Sb.* **34**(1978), 291–341.
- [6] A. N. ANDRIANOV, The multiplicative arithmetic of Siegel modular forms, *Russian Math. Surveys* **34** (1979), 75–148.
- [7] M. ASGARI AND R. SCHMIDT, Siegel modular forms and representations, *Manuscripta math.* **104** (2001), 173–200.
- [8] A. BALOG AND K. ONO, The Chebotarev density theorem in short intervals and some questions of Serre, *J. Number Theory*, **91**(2) (2001), 356–371.

- [9] S. BÖCHERER, Über die Fourier-Jacobi-Entwicklung Siegelscher Eisensteinreihen, *Math. Z.* **183** (1983), 21–46.
- [10] S. BÖCHERER AND S. DAS, Petersson norms of not necessarily cuspidal Jacobi modular forms and applications, *Adv. Math.* **336** (2018), 336–375.
- [11] S. BÖCHERER AND S. DAS, On Fourier coefficients of elliptic modular forms mod ℓ with applications to Siegel modular forms, *Manuscripta math.* (to appear), <https://arxiv.org/abs/2001.11700>.
- [12] A. BOREL, *Linear Algebraic Groups*, 2nd ed., Graduate Texts in Mathematics, **126** (Springer-Verlag, New York, 1991).
- [13] V. A. BYKOVSKIĬ, A trace formula for the scalar product of Hecke series and its applications, *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* **226** (1996), 235–236.
- [14] J. W. S. CASSELS, *Rational Quadratic Forms* (Academic Press, London-New York, 1978).
- [15] M. COURTIEU AND A. PANCHISHKIN, *Non-Archimedean L-Functions and Arithmetical Modular Forms*, Lecture Notes in Mathematics, 1471 (Springer, Berlin, 2004).
- [16] H. DELANGE, Généralisation du théorème de Ikehara, *Ann. Sci. Éc. Norm. Supér. (9)* **3**(71) (1954), 213–242.
- [17] W. DUKE AND H. IWANIEC, Bilinear forms in the Fourier coefficients of half-integral weight cusp forms and sums over primes, *Math. Ann.* **286** (1990), 783–802.
- [18] M. EICHLER AND D. ZAGIER, *The Theory of Jacobi Forms*, Progress in Mathematics, **55** (Birkhäuser, Boston, 1985).
- [19] S. A. EVDOKIMOV, Dirichlet series, multiple Andrianov zeta-functions in the theory of Siegel modular forms of genus 3, *Dokl. Akad. Nauk SSSR* **277** (1984), 25–29.
- [20] E. FREITAG, *Siegelsche Modulfunktionen*, Grundlehren der Mathematischen Wissenschaften, **254** (Springer-Verlag, Berlin, 1983).
- [21] E. FREITAG, Ein Verschwindungssatz für automorphe Formen zur Siegelschen Modulgruppe, *Math. Z.* **165** (1979), 11–18.
- [22] T. IBUKIYAMA AND R. KYOMURA, A generalization of vector-valued Jacobi forms, *Osaka J. Math.* **48** (2011), 783–808.
- [23] H. IWANIEC, *Topics in Classical Automorphic Forms*, Graduate Studies in Mathematics, **17** (American Mathematical Society, Providence, RI, 1997).
- [24] Y. KITAOKA, Modular forms of degree n and representations by quadratic forms, *Nagoya Math. J.* **74** (1979), 95–122.
- [25] Y. KITAOKA, *Arithmetic of Quadratic Forms*, Cambridge Tracts in Mathematics, **106** (Cambridge University Press, Cambridge, UK, 1993).
- [26] H. KLINGEN, *Introductory Lectures on Siegel Modular Forms*, Cambridge Studies in Advanced Mathematics, **20** (Cambridge University Press, Cambridge, UK, 1990).
- [27] S. LANG, *Introduction to Modular Forms*, Grundlehren der Mathematischen Wissenschaften, **222** (Springer-Verlag, Berlin, 1995). Corrected reprint of the 1976 original, with appendixes by D. ZAGIER AND W. FEIT.
- [28] Y. LI, Restriction of Coherent Hilbert Eisenstein series, *Math. Ann.* **368**(1-2) (2017), 317–338.
- [29] T. MIYAKE, *Modular Forms*, Springer Monographs in Mathematics, (Springer-Verlag, Berlin, 2006). Reprint of the first 1989 English edition, translated from the 1976 Japanese original by Y. Maeda.
- [30] M. RAM. MURTY AND V. KUMAR MURTY, *Non-Vanishing of L-Functions and Applications*, Progress in Mathematics, **157** (Birkhäuser Verlag, Basel, 1997).
- [31] A. POLLACK, The spin L -function on GSp_6 for Siegel modular forms, *Compos. Math.* **153**(7) (2017), 1391–1432.

- [32] R. A. RANKIN, Contributions to the theory of Ramanujan's function $\tau(n)$ and similar arithmetic functions, II: Order of the Fourier coefficients of integral modular forms, *Proc. Cambridge Philos. Soc.* **35** (1939), 357–372.
- [33] A. SAHA, Siegel cusp forms of degree 2 are determined by their fundamental Fourier coefficients, *Math. Ann.* **355**(1) (2013), 363–380.
- [34] A. SELBERG, On the estimation of Fourier coefficients of modular forms, *Proc. Sympos. Pure Math.* **VIII** (1965), 1–15.
- [35] J.-P. SERRE, Quelques applications du théorème de densité de Chebotarev, *Publ. Math. Inst. Hautes Études Sci.* **54** (1981), 323–401.
- [36] J.-P. SERRE, Divisibilité de certaines fonctions arithmétiques, Séminaire Delange-Pisot-Poitou, *Théor. Nombres* **16**(1) (1974-1975), 1–28.
- [37] J.-P. SERRE AND H. STARK, Modular forms of weight 1/2: Modular functions of one variable, VI, in *Proceedings of the Second International Conference, University of Bonn, Bonn, 1976*, pp. 27–67, Lecture Notes in Mathematics, 627, (Springer, Berlin, 1977).
- [38] F. SHAHIDI, On nonvanishing of L -functions, *Bull. Amer. Math. Soc. (N.S.)* **2**(3) (1980), 462–464.
- [39] N.-P. SKORUPPA, *Über den Zusammenhang zwischen Jacobiformen und Modulformen halbganzen Gewichts*, Ph.D. thesis, Rheinische Friedrich-Wilhelms-Universität, Bonn, 1985.
- [40] N.-P. SKORUPPA AND D. ZAGIER, Jacobi forms and a certain space of modular forms, *Invent. Math.* **94** (1988), 113–146.
- [41] F. STROMBERG, Weil representations associated with finite quadratic modules, *Math. Z.* **275** (2013), 509–527.
- [42] J. VOIGHT, *Quaternion Algebras*, <https://math.dartmouth.edu/~jvoight/quat-book.pdf>.
- [43] J. WEISINGER, *Some Results on Classical Eisenstein Series and Modular Forms over Function Fields*, Ph.D. thesis, Harvard University, 1977.
- [44] S. YAMANA, Determination of holomorphic modular forms by primitive Fourier coefficients, *Math. Ann.* **344** (2009), 853–863.
- [45] C. ZIEGLER, Jacobi forms of higher degree, *Abh. Math. Semin. Univ. Hambg.* **59** (1989), 191–224.