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Existence of a global smooth solution to the initial boundary-value problem for the *p*-system with nonlinear damping

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This paper is concerned with the initial boundary-value problem for the p-system with nonlinear damping. We prove the existence of a global smooth solution under the assumption that only the C^0-norm of the derivative of the initial data is sufficiently small, while the C^0-norm of the initial data is not necessarily small. The proof is based on several key *a priori* estimates, the maximum principle and the characteristic method.

1. Introduction

In this paper, we consider the existence of a global smooth solution to the initial boundary-value problem for the so-called p-system with nonlinear damping

$$\left.\begin{aligned} v_t - u_x &= 0, \\ u_t + p(v)_x &= -\alpha u - g(u), \quad x \in \mathbb{R}^+,\ t > 0, \end{aligned}\right\} \tag{1.1}$$

with the initial data

$$(v(x,0), u(x,0)) = (v_0(x), u_0(x)) \tag{1.2}$$

and the null Dirichlet boundary condition

$$u|_{x=0} = 0. \tag{1.3}$$

System (1.1) can be used to model the compressible flow through porous media. Here, $v > 0$ is the specific volume, u is the velocity, α is a positive constant and $g(u)$ is a smooth function satisfying $g(0) = 0$ and $g'(u) \geqslant 0$. The pressure $p(v)$ satisfies the assumptions in (P).

(P) $p(v) \in C^2(0,\infty)$, $p'(v) < 0$, $p''(v) > 0$ for $v \in (0,\infty)$.

The motion of the adiabatic gas flow through porous media can be modelled by the damped hyperbolic system

$$\left.\begin{aligned} v_t - u_x &= 0, \\ u_t + p(v,s)_x &= -\alpha u, \quad t > 0, \\ s_t &= 0. \end{aligned}\right\} \tag{1.4}$$

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For the isentropic flow, namely, $s = \text{const.}$, (1.4) takes the form as in (1.1), that is,

$$\left.\begin{aligned} v_t - u_x &= 0, \\ u_t + p(v)_x &= -\alpha u, \quad x \in \mathbb{R},\ t > 0. \end{aligned}\right\} \tag{1.5}$$

The global existence of smooth solutions for the Cauchy problem of system (1.4) has been studied by many authors (see [5,23]). For the Cauchy problem of (1.5), the global existence of smooth solutions with small or large initial data has been studied by many authors (see [12–16, 22]). For the p-system under other damping and relaxation conditions, we refer the reader to [16, 18]. Yang and Zhu [18] considered the Cauchy problem of the p-system with relaxation:

$$\left.\begin{aligned} v_t - u_x &= 0, \\ u_t + p(v)_x &= \frac{1}{\varepsilon}(f(v) - u), \end{aligned}\right\} \tag{1.6}$$

with the initial data (1.2), where $\varepsilon > 0$ is the relaxation time and $f(v) \in C^1(0, \infty)$ satisfies the subcharacteristic condition

$$-\sqrt{-p'(v)} < f'(v) < \sqrt{-p'(v)}.$$

They proved the global existence of the smooth solution under the assumption that only the C^0-norm of the derivative of the initial data is sufficiently small, while the C^0-norm of the initial data is not necessarily small. Wang and Li [16] studied the damping p-system

$$\left.\begin{aligned} v_t - u_x &= 0, \\ u_t + p(v)_x &= -2\alpha u, \end{aligned}\right\} \tag{1.7}$$

and pointed out that the Cauchy problem (1.7) admits a unique global smooth solution under the assumption that only the C^0-norm of the initial data is suitably small, while the C^1-norm of the initial data is not necessarily small. Consequently, Zhu and Zhao [25] extended the results of [16] to the case of nonlinear dissipation and found the same results. For other related results in this direction, we refer the reader to [1, 3, 7–11, 17, 19, 24, 26].

For the initial boundary-value problem, because of the effect of the boundary value, the characteristic method becomes more difficult. Hsiao and Pan [4] considered system (1.4) with the initial data

$$(v, u, s)(x, t) = (v_0, u_0, s_0)$$

and the boundary value

$$u(0, t) = u(1, t) = 0, \quad t \geqslant 0.$$

When $p(v, s) = (\gamma - 1)v^{-\gamma}\mathrm{e}^s (1 < \gamma < 3)$, they prove the global existence of the smooth solution by using the characteristic method. For the generalized function $p(v)$, the global existence of the solution was proved by Jiang and Ruan [6]. However, the existence of global smooth solutions to the initial boundary-value problem for a p-system with nonlinear damping is still an open problem. The main purpose of this paper is to give a conclusive answer to this problem.

REMARK 1.1. In this paper, we only consider the case of the isentropic flow; for the non-isentropic flow, we can get the same results by using a similar method and the proof will be more complicated.

2. Existence of a global smooth solution

In this section, we consider the existence of global smooth solutions for the initial boundary-value problem (1.1)–(1.3). Since the local existence and uniqueness of a C^1-solution can be proved by standard arguments, such as the Brouwer fixed-point theorem, we need only establish the uniform C^1-estimates for the solution $(v(x,t), u(x,t))$ of (1.1)–(1.3) on the domain where the classical solution exists.

System (1.1) has two eigenvalues:

$$\lambda = -\sqrt{-p'(v)}, \qquad \mu = \sqrt{-p'(v)}, \tag{2.1}$$

and the Riemann invariants are taken as

$$r = u + h(v), \qquad s = u - h(v), \tag{2.2}$$

where

$$h(v) = \int_1^v \mu(\tau)\,\mathrm{d}\tau. \tag{2.3}$$

First, we rewrite (1.1)–(1.3) in the diagonal form

$$\left.\begin{aligned} r_t + \lambda(v) r_x &= -\frac{\alpha}{2}(r+s) - g\left(\frac{r+s}{2}\right), \\ s_t + \mu(v) s_x &= -\frac{\alpha}{2}(r+s) - g\left(\frac{r+s}{2}\right), \\ (r(x,0), s(x,0)) &= (u_0(x) + h(v_0), u_0(x) - h(v_0)), \quad x \in \mathbb{R}^+, \\ (r+s)(0,t) &= 0. \end{aligned}\right\} \tag{2.4}$$

According to the local existence theorem of classical solutions to first-order quasilinear hyperbolic systems (see [2]), for the diagonal form (2.4), there exists a constant $T > 0$ that depends only on the C^1-norm of the initial data such that, in the domain

$$\pi(T) = \{(x,t) \colon x \in \mathbb{R}^+,\ 0 \leqslant t \leqslant T\},$$

(2.4) possesses a unique smooth solution $(r(x,t), s(x,t))$, provided the *a priori* hypothesis (H) holds.

(H) $0 < v < \infty$.

In order to get the global existence of the smooth solutions on $t \geqslant 0$, it is sufficient to prove that the C^1-norm of the solution is bounded on the domain where the classical solution exists. To do this, we first give C^0-norm estimates of the solution.

LEMMA 2.1. *Under the assumptions in (P), if the initial data* $(r_0(x), s_0(x)) \in C^1(\mathbb{R}^+)$ *and there exists a positive constant* M_0 *such that*

$$|r_0(x)| \leqslant M_0, \qquad |s_0(x)| \leqslant M_0, \tag{2.5}$$

then, under the a priori *hypothesis in (H), the solution* $(r(x,t), s(x,t))$ *to the initial boundary-value problem (2.4) satisfies the estimates*

$$|r(x,t)| \leqslant M_0, \qquad |s(x,t)| \leqslant M_0. \tag{2.6}$$

Proof. We prove the lemma by applying the maximum principle (see [18, 20]).

Let

$$\left.\begin{aligned} r(x,t) &= \bar{r}(x,t) + M_0 + \frac{N}{L}(x + C\mathrm{e}^t), \\ -s(x,t) &= \bar{s}(x,t) + M_0 + \frac{N}{L}(x + C\mathrm{e}^t), \end{aligned}\right\} \tag{2.7}$$

where C is a positive constant to be defined below, L is a constant, which can be arbitrary large, N is an upper bound of $|r(x,t)|$, $|s(x,t)|$ on $\pi(T)$ (N can be obtained by the local existence of the smooth solution).

From (2.4) and (2.7), it is easy to deduce that $\bar{r}(x,t)$, $\bar{s}(x,t)$ satisfy

$$\left.\begin{aligned} \bar{r}_t + \lambda(v)\bar{r}_x + \frac{N}{L}(C\mathrm{e}^t + \lambda) &= -\tfrac{1}{2}(\alpha + g'(\xi))(\bar{r} - \bar{s}), \\ \bar{s}_t + \mu(v)\bar{s}_x + \frac{N}{L}(C\mathrm{e}^t + \mu) &= -\tfrac{1}{2}(\alpha + g'(\xi))(\bar{s} - \bar{r}), \end{aligned}\right\} \tag{2.8}$$

where ξ is between 0 and $\frac{1}{2}(\bar{r} - \bar{s})$. We consider system (2.8) in the region $[0,L] \times [0,T]$; then, the initial and boundary conditions are

$$\left.\begin{aligned} \bar{r}(x,0) &= r(x,0) - M_0 - \frac{N}{L}(x + C) < 0, \\ \bar{s}(x,0) &= -s(x,0) - M_0 - \frac{N}{L}(x + C) < 0, \\ \bar{r}(L,t) &= r(L,t) - M_0 - N - \frac{N}{L}C\mathrm{e}^t < 0, \\ \bar{s}(L,t) &= -s(L,t) - M_0 - N - \frac{N}{L}C\mathrm{e}^t < 0, \\ \bar{r}(0,t) &- \bar{s}(0,t) = 0. \end{aligned}\right\} \tag{2.9}$$

From (2.8) and (2.9), we claim that

$$\bar{r}(x,t) < 0, \quad \bar{s}(x,t) < 0, \quad (x,t) \in [0,L] \times [0,T]. \tag{2.10}$$

Otherwise, we let

$$\bar{t} = \sup_t \{t \mid \bar{r}(x,\tau) < 0,\ \bar{s}(x,\tau) < 0,\ \forall x \in [0,L],\ \tau \in (0,t)\}.$$

Then,

$$0 < \bar{t} \leqslant T < +\infty.$$

By the continuity of $\bar{r}(x,t)$ and $\bar{s}(x,t)$, there exists $(\bar{x},\bar{t})$ with $\bar{x} \in [0,L)$ such that one of the following cases holds.

(1) When $\bar{x} \in (0, L)$,

$$\bar{r}(\bar{x}, \bar{t}) = 0, \qquad \bar{s}(\bar{x}, \bar{t}) \leqslant 0, \qquad \frac{\partial \bar{r}(x,t)}{\partial x}\bigg|_{(\bar{x},\bar{t})} = 0, \qquad \frac{\partial \bar{r}(x,t)}{\partial t}\bigg|_{(\bar{x},\bar{t})} \geqslant 0$$

or

$$\bar{s}(\bar{x}, \bar{t}) = 0, \qquad \bar{r}(\bar{x}, \bar{t}) \leqslant 0, \qquad \frac{\partial \bar{s}(x,t)}{\partial x}\bigg|_{(\bar{x},\bar{t})} = 0, \qquad \frac{\partial \bar{s}(x,t)}{\partial t}\bigg|_{(\bar{x},\bar{t})} \geqslant 0.$$

(2) When $\bar{x} = 0$, from $(2.4)_4$ and (2.7), we have that

$$\bar{r}(\bar{x}, \bar{t}) = \bar{s}(\bar{x}, \bar{t}) = 0; \quad \text{then } \frac{\partial \bar{r}(x,t)}{\partial x}\bigg|_{(\bar{x},\bar{t})} \leqslant 0, \qquad \frac{\partial \bar{r}(x,t)}{\partial t}\bigg|_{(\bar{x},\bar{t})} \geqslant 0.$$

For the above cases, using the maximum principle (see [18, 20]), when $C > 2\sup \mu$ for all v under consideration, we will have a contradiction. Therefore, (2.10) holds.

From (2.7) and (2.10), we get that

$$\left.\begin{aligned} r(x,t) &< M_0 + \frac{N}{L}(x + Ce^t), \\ s(x,t) &> -M_0 - \frac{N}{L}(x + Ce^t). \end{aligned}\right\} \tag{2.11}$$

Since L can be arbitrarily large, by letting $L \to \infty$ we have that

$$r(x,t) \leqslant M_0, \qquad s(x,t) \geqslant -M_0.$$

Similarly, if we let

$$\left.\begin{aligned} r(x,t) &= \bar{\bar{r}}(x,t) - M_0 - \frac{N}{L}(x + Ce^t), \\ -s(x,t) &= \bar{\bar{s}}(x,t) - M_0 - \frac{N}{L}(x + Ce^t), \end{aligned}\right\} \tag{2.12}$$

we can show that

$$r(x,t) \geqslant -M_0, \qquad s(x,t) \leqslant M_0.$$

Hence,

$$|r(x,t)| \leqslant M_0, \qquad |s(x,t)| \leqslant M_0.$$

Lemma 2.1 is proved. □

LEMMA 2.2. *Consider the following condition.*

(G) $M_0 < \min\{-h(0),\ h(\infty)\}$.

Under the hypothesis of lemma 2.1, if the initial data $(r(x,0), s(x,0))$ satisfy (G), then the smooth solution of $(v(x,t), u(x,t))$ of (1.1)–(1.3) satisfies

$$|u(x,t)| \leqslant M_0, \qquad 0 < v_* \leqslant v(x,t) \leqslant v^* < \infty, \tag{2.13}$$

where v_* and v^* are constants that depend only on M_0, but are independent of α.

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Proof. From (2.2), we have that

$$u = \frac{r+s}{2}, \qquad 2h(v) = 2\int_1^v \sqrt{-p'(\tau)}\,\mathrm{d}\tau = r - s. \tag{2.14}$$

Hence, we get from (2.6), (G) and (2.14) that

$$|u(x,t)| \leqslant \tfrac{1}{2}(|r| + |s|) \leqslant M_0,$$
$$2h(0) < -2M_0 \leqslant 2h(v) = r - s \leqslant 2M_0 < 2h(\infty).$$

From $h'(v) = \sqrt{-p'(v)} > 0$, there exist $v_* > 0$ and $v^* < \infty$ such that

$$0 < v_* \leqslant v \leqslant v^* < \infty.$$

This completes the proof of lemma 2.2. □

We next estimate the derivatives of $r(x,t)$ and $s(x,t)$. Because the value $u_x(0,t)$ cannot be determined, we cannot use the maximum principle as in [18]. But, noting that $|r(0,t)| = |s(0,t)|$ from $(2.4)_4$, we can prove the following lemma by using the characteristic method as in [4, 6, 21].

Lemma 2.3. *Under the assumptions of lemmas 2.1 and 2.2, and if there exists a small enough constant* M_1 *such that*

$$|r_0'(x)| \leqslant \alpha M_1, \qquad |s_0'(x)| \leqslant \alpha M_1, \tag{2.15}$$

then the solution $(r(x,t), s(x,t))$ *of (2.4) in the domain where the classical solution exists has the following estimates:*

$$|r_x(x,t)| \leqslant \alpha M_2, \qquad |s_x(x,t)| \leqslant \alpha M_2, \tag{2.16}$$

where M_2 *is a positive constant independent of* α*.*

Proof. From (2.2)–(2.4) and (1.1), we have that

$$\frac{\mathrm{d}}{\mathrm{d}_\lambda t}(r - s) = 2\mu s_x, \qquad \frac{\mathrm{d}}{\mathrm{d}_\mu t}(r - s) = 2\mu r_x, \tag{2.17}$$

$$\frac{\mathrm{d}v}{\mathrm{d}_\lambda t} = s_x, \qquad \frac{\mathrm{d}v}{\mathrm{d}_\mu t} = r_x, \qquad v_x = \frac{1}{2\sqrt{-p'(v)}}(r_x - s_x), \tag{2.18}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}_\lambda t} = \frac{\partial}{\partial_t} + \lambda\frac{\partial}{\partial_x}, \qquad \frac{\mathrm{d}}{\mathrm{d}_\mu t} = \frac{\partial}{\partial_t} + \mu\frac{\partial}{\partial_x}.$$

Differentiating $(2.4)_1$ and $(2.4)_2$ with respect to x we have, respectively,

$$\left.\begin{aligned} (r_x)_t + \lambda(v)(r_x)_x &= -\frac{\alpha}{2}(r_x + s_x) + \frac{p''(v)}{4p'(v)}(r_x - s_x)r_x - \frac{1}{2}g'\left(\frac{r+s}{2}\right)(r_x + s_x), \\ (s_x)_t + \mu(v)(s_x)_x &= -\frac{\alpha}{2}(r_x + s_x) + \frac{p''(v)}{4p'(v)}(s_x - r_x)s_x - \frac{1}{2}g'\left(\frac{r+s}{2}\right)(r_x + s_x). \end{aligned}\right\} \tag{2.19}$$

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Let

$$F(x,t) = (-p'(v))^{1/2} r_x, \qquad G(x,t) = (-p'(v))^{1/2} s_x. \tag{2.20}$$

From (2.17)–(2.19), we deduce that $(F(x,t), G(x,t))$ satisfies

$$\left.\begin{aligned} F_t + \lambda(v) F_x &= -A_1(F+G),\\ G_t + \mu(v) G_x &= -A_2(F+G), \end{aligned}\right\} \tag{2.21}$$

with the initial boundary data

$$\left.\begin{aligned} F(x,0) := F_0 &= ((-p'(v))^{1/2} r_x)(x,0),\\ G(x,0) := G_0 &= ((-p'(v))^{1/2} s_x)(x,0),\\ F(0,t) - G(0,t) &= 0, \end{aligned}\right\} \tag{2.22}$$

where

$$\left.\begin{aligned} A_1(v,u,F) &= \frac{\alpha}{2} + \frac{p''(v)}{4(-p'(v))^{3/2}} F + \frac{g'(u)}{2},\\ A_2(v,u,G) &= \frac{\alpha}{2} + \frac{p''(v)}{4(-p'(v))^{3/2}} G + \frac{g'(u)}{2}. \end{aligned}\right\} \tag{2.23}$$

Let

$$B_1 = \sup_{v\in[v_*,v^*]} \frac{p''(v)}{4(-p'(v))^{3/2}}.$$

If we assume that in the domain where the smooth solution exists

$$|F(x,t)| \leqslant \frac{\alpha}{4B_1}, \qquad |G(x,t)| \leqslant \frac{\alpha}{4B_1}, \tag{2.24}$$

noting that $g'(u) \geqslant 0$, we have that

$$A_1(v,u,F) > 0, \qquad A_2(v,u,G) > 0. \tag{2.25}$$

Next, we show that

$$\sup_{0\leqslant\tau\leqslant t} \max\{\|F(\cdot,\tau)\|_{L^\infty}, \|G(\cdot,\tau)\|_{L^\infty}\} \leqslant \max\{\|F_0\|_{L^\infty}, \|G_0\|_{L^\infty}\}. \tag{2.26}$$

Let

$$M(t) = \sup_{0\leqslant\tau\leqslant t} \max\{\|F(\cdot,\tau)\|_{L^\infty}, \|G(\cdot,\tau)\|_{L^\infty}\}.$$

Let $x_\lambda = x_\lambda(a,t)$ and $x_\mu = x_\mu(b,t)$ be the λ-characteristic curve and the μ-characteristic curve passing through the points $(a,0)$ and $(b,0)$, respectively, i.e.

$$\left.\begin{aligned} \frac{\mathrm{d}x_\lambda(a,t)}{\mathrm{d}t} &= \lambda(v(x_\lambda(a,t)),t),\\ x_\lambda(a,0) &= a \end{aligned}\right\} \tag{2.27}$$

and

$$\left.\begin{aligned} \frac{\mathrm{d}x_\mu(b,t)}{\mathrm{d}t} &= \mu(v(x_\mu(b,t)),t),\\ x_\mu(b,0) &= b. \end{aligned}\right\} \tag{2.28}$$

For every fixed $T > 0$, there are only three cases.

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Case 1. $M(T)$ is reached by $F(x,t)$ first at some point $(x,t)\in[0,\infty)\times[0,T]$. Then, integrating $(2.21)_1$ along the λ-characteristic curve that intersects $t=0$ at $(x_1,0)$, we have that

$$F(x,t)=F_0(x_1)\exp\left(-\int_0^t A_1\,\mathrm{d}s\right)+\int_0^t(-A_1G)\exp\left(-\int_\tau^t A_1\,\mathrm{d}s\right)\mathrm{d}\tau. \tag{2.29}$$

Then,

$$\begin{aligned}|F(x,t)|&\leqslant|F_0(x_1)|\exp\left(-\int_0^t A_1\,\mathrm{d}s\right)+\int_0^t|-A_1G|\exp\left(-\int_\tau^t A_1\,\mathrm{d}s\right)\mathrm{d}\tau\\&\leqslant|F_0(x_1)|\exp\left(-\int_0^t A_1\,\mathrm{d}s\right)+M(T)\int_0^t A_1\exp\left(-\int_\tau^t A_1\,\mathrm{d}s\right)\mathrm{d}\tau\\&\leqslant|F_0(x_1)|\exp\left(-\int_0^t A_1\,\mathrm{d}s\right)+M(T)\left(1-\exp\left(-\int_0^t A_1\,\mathrm{d}s\right)\right).\end{aligned} \tag{2.30}$$

Case 2. $M(T)$ is reached by $G(x,t)$ first at some point $(x,t)\in[0,\infty)\times[0,T]$. From (x,t), we draw a forward characteristic that intersects $x=0$ at $(0,t_1)$. Then, integrating $(2.21)_2$ along the μ-characteristic curve, we have that

$$\begin{aligned}|G(x,t)|&\leqslant|G(0,t_1)|\exp\left(-\int_{t_1}^t A_2\,\mathrm{d}s\right)+\int_{t_1}^t|-A_2F|\exp\left(-\int_\tau^t A_2\,\mathrm{d}s\right)\mathrm{d}\tau\\&\leqslant|G(0,t_1)|\exp\left(-\int_{t_1}^t A_2\,\mathrm{d}s\right)+M(T)\left(1-\exp\left(-\int_{t_1}^t A_2\,\mathrm{d}s\right)\right).\end{aligned} \tag{2.31}$$

Then, from $(0,t_1)$, we draw a λ-characteristic curve that intersects $t=0$ at $(x_2,0)$, along this characteristic, similarly to (2.30), and, noting $(2.22)_3$, we have that

$$\begin{aligned}|G(0,t_1)|&=|F(0,t_1)|\\&\leqslant|F_0(x_2)|\exp\left(-\int_0^{t_1}A_1\,\mathrm{d}s\right)+\int_0^{t_1}|-A_1G|\exp\left(-\int_\tau^{t_1}A_1\,\mathrm{d}s\right)\mathrm{d}\tau\\&\leqslant|F_0(x_2)|\exp\left(-\int_0^{t_1}A_1\,\mathrm{d}s\right)+M(T)\int_0^{t_1}A_1\exp\left(-\int_\tau^{t_1}A_1\,\mathrm{d}s\right)\mathrm{d}\tau\\&\leqslant|F_0(x_2)|\exp\left(-\int_0^{t_1}A_1\,\mathrm{d}s\right)+M(T)\left(1-\exp\left(-\int_0^{t_1}A_1\,\mathrm{d}s\right)\right).\end{aligned} \tag{2.32}$$

Substituting (2.32) into (2.31), we have that

$$\begin{aligned}|G(x,t)|&\leqslant\exp\left(-\int_{t_1}^t A_2\,\mathrm{d}s\right)\exp\left(-\int_0^{t_1}A_1\,\mathrm{d}s\right)|F_0(x_2)|\\&\quad+\left(1-\exp\left(-\int_{t_1}^t A_2\,\mathrm{d}s\right)\exp\left(-\int_0^{t_1}A_1\,\mathrm{d}s\right)\right)M(T).\end{aligned} \tag{2.33}$$

Case 3. $M(T)$ is reached by $G(x,t)$ first at some point $(x,t)\in[0,\infty)\times[0,T]$. From (x,t), we draw a forward characteristic that intersects $t=0$ at $(x_3,0)$. Then,

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integrating $(2.21)_2$ along the μ-characteristic curve, we have that

$$G(x,t) = G_0(x_3)\exp\left(-\int_0^t A_2\,\mathrm{d}s\right) + \int_0^t (-A_2F)\exp\left(-\int_\tau^t A_2\,\mathrm{d}s\right)\mathrm{d}\tau. \quad (2.34)$$

Then,

$$\begin{aligned}|G(x,t)| &\leqslant |G_0(x_3)|\exp\left(-\int_0^t A_2\,\mathrm{d}s\right) + \int_0^t |-A_2F|\exp\left(-\int_\tau^t A_2\,\mathrm{d}s\right)\mathrm{d}\tau \\ &\leqslant |G_0(x_3)|\exp\left(-\int_0^t A_2\,\mathrm{d}s\right) + M(T)\int_0^t A_2\exp\left(-\int_\tau^t A_2\,\mathrm{d}s\right)\mathrm{d}\tau \\ &\leqslant |G_0(x_3)|\exp\left(-\int_0^t A_2\,\mathrm{d}s\right) + M(T)\left(1-\exp\left(-\int_0^t A_2\,\mathrm{d}s\right)\right). \quad (2.35)\end{aligned}$$

Noting that $A_1, A_2 > 0$ from (2.25), then

$$\exp\left(-\int_0^t A_1\,\mathrm{d}s\right) < 1, \qquad \exp\left(-\int_0^t A_2\,\mathrm{d}s\right) < 1$$

and

$$\exp\left(-\int_{t_1}^t A_2\,\mathrm{d}s\right)\exp\left(-\int_0^{t_1} A_1\,\mathrm{d}s\right) < 1.$$

From (2.30), (2.33) and (2.35), we can prove (2.26). Finally, we show that the *a priori* assumption (2.24) can be closed. In fact, from (2.22), we have that

$$|F_0| \leqslant B_2\alpha M_1, \qquad |G_0| \leqslant B_2\alpha M_1, \quad (2.36)$$

where

$$B_2 = \sup_{v\in[v_*,v^*]}\sqrt{-p'(v)}.$$

Therefore, (2.24) holds provided that M_5 is sufficiently small, i.e. $M_1 < 1/8B_1B_2$. Then, combining (2.20) and (2.24), we prove (2.16). This completes the proof of lemma 2.3. □

By lemmas 2.1–2.3, we have the following main theorem.

THEOREM 2.4. *Assume that (P) and (G) hold, and that if there exist positive constants v_1, v_2, M', M'' such that*

$$v_1 \leqslant v_0(x) \leqslant v_2, \qquad |u_0(x)| \leqslant M', \quad (2.37)$$

$$|r_0'(x)| \leqslant \alpha M'', \qquad |s_0'(x)| \leqslant \alpha M'', \quad (2.38)$$

where M'' is a sufficiently small constant, then the initial boundary-value problem (1.1)–(1.3) admits a unique global smooth solution $(v(x,t), u(x,t))$ satisfying

$$v_* \leqslant v(x,t) \leqslant v^*, \qquad |u(x,t)| \leqslant M', \quad (2.39)$$

$$|v_x(x,t)| \leqslant CM'', \qquad |u_x(x,t)| \leqslant CM'', \quad (2.40)$$

where v_*, v^* are two positive constants depending only on v_1, v_2, M', M''.

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References

1 F. Bloom. On the damped nonlinear evolution equation $W_{tt} = \sigma(W)_{xx} - \gamma W_t$. *J. Math. Analysis Applic.* **96** (1983), 551–583.

2 A. Douglis. Existence theorems for hyperbolic systems. *Commun. Pure Appl. Math.* **5** (1952), 119–154.

3 L. Hsiao and T. T. Li. Global smooth solutions of Cauchy problems for a class of quasilinear hyperbolic systems. *Chin. Annals Math.* **4** (1983), 109–115.

4 L. Hsiao and R. H. Pan. Initial boundary-value problem for the system of compressible adiabatic flow through porous media. *J. Diff. Eqns* **159** (1999), 280–305.

5 L. Hsiao and D. Serre. Global existence of solutions for the system of compressible adiabatic flow through porous media. *SIAM J. Math. Analysis* **27** (1996), 70–77.

6 M. N. Jiang, L. Z. Ruan and J. Zhang. Existence of global smooth solution to the initial boundary-value problem for *p*-system with damping. *Nonlin. Analysis* **70** (2009), 2471–2479.

7 J. L. Johnson. Global continuous solutions of hyperbolic systems of quasilinear equations. *Bull. Am. Math. Soc.* **73** (1967), 639–641.

8 P. D. Lax. Development of singularities of solutions on nonlinear hyperbolic partial differential equations. *J. Math. Phys.* **5** (1964), 611–613.

9 T. T. Li and T. H. Qin. Global smooth solutions for a class of quasilinear hyperbolic systems with dissipative terms. *Chin. Annals Math.* **6** (1985), 199–210.

10 T. T. Li and W. Z. Yu. Cauchy's problems for quasilinear hyperbolic systems of first-order partial differential equations. *Math. Prog. Sinica* **7** (1964), 152–171.

11 L. W. Lin. On the global existence of the continuous solutions of the reducible quasilinear hyperbolic systems. *Acta Sci. Nat. Jilin Univ.* **4** (1963), 83–96.

12 L. W. Lin and T. Yang. Existence and nonexistence of global smooth solutions for damped *p*-system with 'really large' initial data. *J. PDEs* **4** (1991), 45–51.

13 L. W. Lin and Y. S. Zheng. Existence and nonexistence of global smooth solutions for quasilinear hyperbolic systems. *Chin. Annals Math.* **9** (1988), 372–377.

14 T. Nishida. *Nonlinear hyperbolic equations and related topics in fluid dynamics*, Publications Mathematiques D'Orsay (Paris: Department de Mathematique, Université de Paris-Sud, 1978).

15 M. Slemrod. Instability of steady shearing flows in a nonlinear viscoelastic fluid. *Arch. Ration. Mech. Analysis* **68** (1978), 211–225.

16 J. H. Wang and C. Z. Li. Global regularity and formation of singularities of solutions for quasilinear hyperbolic system with dissipation. *Chin. Annals Math.* **9** (1988), 509–523.

17 M. Yamaguti and T. Nishida. On the global solution for quasilinear hyperbolic equations. *Funkcial. Ekvac.* **11** (1968), 51–57.

18 T. Yang and C. J. Zhu. Existence and non-existence of global smooth solution for *p*-system with relaxation. *J. Diff. Eqns* **161** (2000), 321–336.

19 T. Yang, C. J. Zhu and H. J. Zhao. Global smooth solutions for a class of quasilinear hyperbolic systems with dissipative terms. *Proc. R. Soc. Edinb.* A **127** (1997), 1311–1324.

20 L. A. Ying, T. Yang and C. J. Zhu. Existence of global smooth solutions for Euler equations with symmetry. *Commun. PDEs* **22** (1997), 1361–1387.

21 Y. H. Zhang and Z. Tan. Existence and asymptotic behavior of global smooth solution for *p*-system with damping and boundary effect. *Nonlin. Analysis* **72** (2010), 2499–2513.

22 Y. S. Zheng. Global smooth solutions for systems of gas dynamics with dissipation. *Acta Math. Sci.* **7** (1987), 383–390.

23 Y. S. Zheng. Global smooth solutions for systems of adiabatic gas dynamics with dissipation. *Chin. Annals Math.* **17** (1996), 155–162.

24 C. J. Zhu. Global resolvability for a viscoelastic model with relaxation. *Proc. R. Soc. Edinb.* A **125** (1995), 1277–1285.

25 C. J. Zhu and H. J. Zhao. Existence, uniqueness and stability of solution for a class of quasilinear wave equation. *Chin. Annals Math.* **18** (1997), 223–234.

26 C. J. Zhu, C. Z. Li and H. J. Zhao. Existence of globally continuous solutions to a class of non-strictly hyperbolic conservation laws. *Acta Math. Sci.* **14** (1994), 96–106.

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