Economics and Philosophy, **25** (2009) 153–159
doi:10.1017/S0266267109990022

DE FINETTI ON RISK AVERSION

Joseph B. Kadane and Gaia Bellone
Carnegie Mellon University

According to Mark Rubinstein (2006) 'In 1952, anticipating Kenneth Arrow and John Pratt by over a decade, he [de Finetti] formulated the notion of absolute risk aversion, used it in connection with risk premia for small bets, and discussed the special case of constant absolute risk aversion.' The purpose of this note is to ascertain the extent to which this is true, and at the same time, to correct certain minor errors that appear in de Finetti's work.

1. INTRODUCTION

Bruno de Finetti (1906–1985) was an Italian philosopher, statistician and mathematician. His work was not so well-known in the English-speaking world until his last few years. Through his work as an actuary, he studied decision-making under uncertainty, and is now regarded, with L. J. Savage and F. Ramsey, as one of the founders of the modern subjective Bayesian school.

In 1952, de Finetti participated in a conference in Paris on utility theory, also attended by G. T. Guilbaud, L. J. Savage, K. J. Arrow, M. Friedman, M. Allais, H. Wold, P. A. Samuelson, P. Massé, G. Morlat, J. Marschack, R. Frisch, M. Boiteux, F. Divisia, J. Ville, and D. van Dantzig (what a group!). De Finetti's 1952 paper apparently contains both the content of his talk at the Paris meeting, and his response to several of the other talks given there.

This paper concerns only a small section of de Finetti's 1952 paper, the section concerning risk aversion. A translation of that section is given here as Appendix A. Appendix B reviews and gives proofs of the results claimed by de Finetti (and corrects some minor errors in them). Section 2 gives comments on the context of de Finetti's work, and section 3 concludes.

2. BACKGROUND AND CONTEXT OF DE FINETTI'S WORK

Bruno de Finetti is writing in the context of the maximization of expected utility. His technique is to expand the utility function in a Taylor series around an arbitrary point. Because he is looking for the change in expected utility as a consequence of a (bounded and small) random prospect, the constant term cancels. The first order term yields the mean, of course. It is the second order term that gives the correction for risk aversion. Not surprisingly, the correction for risk aversion operates on the variance of the random prospect. This is the heart of de Finetti's treatment of risk aversion, and is expressed in equation (B.9).

Why was de Finetti's work on risk aversion not recognized at the time? Perhaps one clue is that other important work of de Finetti was also overlooked, namely his work on mean-variance analysis of portfolios with correlated risks, which de Finetti published in 1940 and Markovitz rediscovered in 1952 (see de Finetti 1940, Markovitz 1952, 2006). Bruno de Finetti wrote in Italian, and in a formal, difficult style. It is, of course, commonplace that when the groundwork is 'ready' for an idea, it can be discovered independently in several places. Both Pratt (1964) and Arrow (1971) agree that their work on risk aversion was independent of the other. Neither acknowledges de Finetti, although Arrow apparently attended the 1952 meeting where de Finetti gave his paper.

Bruno de Finetti's work on risk aversion appears almost as an afterthought in his paper, most of which concerns his insistence on operational definitions. For example, de Finetti rejects interpersonal utility comparisons, for lack of an experiment that would show that person 1 cares more strongly about the preference of A over B than does person 2.

Bruno de Finetti was a powerful mathematician. This makes it a bit surprising that we find errors in his work. Leaving aside sign errors, there is only one error of importance, the result (c), where he says that, for all probabilities p, the price of a gain equal to h with probability p (and to 0 otherwise) is $ph(1 + h/\lambda)$. But this can't be right, because in the special case when $p = 1$ (so the gain h is certain), the price must be h. But de Finetti's expression specializes to $h(1 + h/\lambda) > h$. So de Finetti would pay more than h for a certain gain of h, which is incoherent. The alternative expression I derive, $ph(1 - (1 - p)h/\lambda)$ does simplify to h when $p = 1$. A referee suggests that perhaps there is a typographical error, and that what de Finetti meant was $ph(1 - qh/\lambda)$, where $q = 1 - p$.

3. CONCLUSION

Each of Rubinstein's claims for de Finetti is sustained by this reading of his work. While there are some minor errors in his paper, he deserves credit for having anticipated many of the results subsequently published by others.

Although de Finetti's work on risk aversion was not apparently influential, while Arrow's and Pratt's was, that observation does not diminish the originality of this work.

APPENDIX A. A PASSAGE FROM DE FINETTI (1952) (PP. 700–1)

Note: These authors' comments are [in brackets].

The introduction of the [utility] function $u(x)$ serves to bring the most general coherent behavior in the probabilistic sense to the classical one based on mathematical expectation (*i.e.* on the concept of a fair bet); everything is reduced to performing the calculation in terms of u instead of in terms of x. The mathematical aspect brings to the discussion associative means[15] and their properties, in particular the relationships with variances brought to light by A. Chimenti[16] will help us in the following.

It's easy to see that the probabilistic behavior in a neighborhood of a certain value of x depends on the degree of relative convexity of u, where, by relative convexity I mean the ratio $-u''/2u'$ between the second and first derivatives (half, because it will be convenient). Denote by $\lambda(x)$ the inverse of the relative convexity ($\lambda = -2u'/u''$) (to have a magnitude of the same dimensions as a 'value' instead of the reciprocal of a value). The significance of λ, together with the characteristics of probabilistic behavior for small bets, is brought to light by the following relationships (all asymptotically valid for $h \to 0$).[17]

a) The risk to gain or lose h with the same probability ($\pm h$ with probability $1/2$) is equivalent to a sure loss of h^2/λ;
b) to make this operation indifferent, the probability of winning must exceed that of losing by $d = h/\lambda$ (*i.e.* gain $\pm h$ with probability $(1 \pm d)/2$);
c) the price of a gain h (loss if $h < 0$) with probability p is $ph(1 + h/\lambda)$ [this should be $ph(1 - (1 - p)h/\lambda]$.
d) in general, a bet involving an uncertain gain X (with X certainly lying between $\pm h$) is advantageous, disadvantageous, indifferent if σ^2 is less, greater than, or equal to $-\lambda m$ [this should be λm] (m and σ are respectively the expectation and standard deviation of X).

[15] A. Kolmogorov, Sur la notion de moyenne, 'Rend. Licei,' 1930; O. Chisini, Sul concetto di media, 'Period. di Matematiche,' 1929; B. de Finetti, Sul concetto di media, 'Giorn. Ist. It. Attuari,' 1931.

[16] Chimenti, Disuguaglianze tra medie associative, 'Statistica,' 1947.

[17] For proof, it will be enough to observe that

$$u(x + h) = u + hu' + (1/2)h^2(u'' + \epsilon)$$

(denoting briefly with u, u', and u'' the values at x, and with ϵ a function of h that goes to 0 with h; we will have for example for a:

$$u(x + h) + u(x - h) = 2u(x - d)h^2(u'' + \epsilon) = 2du' + d^2(u'' + \epsilon), \text{ etc.}$$

https://doi.org/10.1017/S0266267109990022 Published online by Cambridge University Press

We note that, for $h \to 0$, the expectation rule holds, to a first approximation; the correction provided by λ (in the general form (d) and in special cases (a)–(c)), serves as a second approximation.

Besides, it is easy to see how, assigning or determining experimentally, according to one of a) – d) the value of λ corresponding to different x, we can construct u integrating the differential equation $u''/u' = 2/\lambda$ [should be $-2/\lambda$], thus $u = \int e^{-2\int 1/\lambda(x)dx}$, (with two integration constants corresponding to the inessential linear transformation $u = A + Bu_0$). In particular, we should have $u = K - e^{-2x/\lambda}$ for $\lambda = $ constant, $u = \log x$ for $\lambda = 2x$ and, in greater generality, $u = x^{1-2c}$ for $\lambda = x/c$ ($c = 1/2$ gives us the previous case).

APPENDIX B. DE FINETTI'S RESULTS

Suppose $u(x) : \mathcal{R} \to \mathcal{R}$ is an increasing (and typically concave) utility function with several derivatives, so that $u' > 0$ and (typically) $u'' < 0$. Then de Finetti defines the function λ as follows: $\lambda(x) = -2u'(x)/u''(x)$.

The principal result of de Finetti may now be stated as:

Theorem 1. *Let X be a random variable such that $P\{-h \leq X \leq h\} = 1$, with mean m and standard deviation σ. Then, asymptotically as $h \to 0$ the uncertain gain X is respectively advantageous, disadvantageous or indifferent if σ^2 is less than, greater than or equal to λm.*

Proof: Let g be the certain amount that would make the decision-maker indifferent between the uncertain gain $X - g$ and the status quo. Then by definition, g satisfies the implicit equation

(B.1) $0 = Eu(x + X - g) - u(x).$

By assumption, $X = O_p(h)$, which means that X/h is a bounded random variable, and $g = O(h)$, which means that g/h is bounded.

Expanding u in a Taylor series to first order around x yields

(B.2) $0 = E[u + (X - g)u' + O_p(X - g)^2] - u$

where u and its derivatives are evaluated at x.

Evaluating the expectation, we have

(B.3) $0 = u + (m - g)u' + O(h^2) - u.$

Solving for g yields

(B.4) $g = m + O(h^2).$

[This is what de Finetti means by his remark that the first approximation is valid for the expectation rule. It amounts to the observation that locally to first order every smooth function is linear.]

Now expanding to second order, we have

$$0 = E\left[u + (X - g)u' + \frac{(X-g)^2}{2}u'' + O_p(h^3)\right] - u. \tag{B.5}$$

The new term in this expansion is

$$\begin{aligned} E\frac{(X-g)^2}{2} &= \frac{1}{2}E(X - m + m - g)^2 \\ &= \frac{1}{2}E(X-m)^2 + E\,[(X-m)(m-g)] + \frac{1}{2}E(m-g)^2 \\ &= \frac{1}{2}\sigma^2 + 0 + \frac{1}{2}(m-g)^2. \end{aligned} \tag{B.6}$$

But (B.4) implies that

$$(m-g)^2 = O(h^4) \tag{B.7}$$

so this term can be neglected. Then, returning to (B.5),

$$0 = (m-g)u' + \frac{\sigma^2 u''}{2} + O(h^3). \tag{B.8}$$

Again solving for g, we have

$$g = m + \frac{\sigma^2 u''}{2u'} = m - \sigma^2/\lambda + O(h^3). \tag{B.9}$$

Thus the sign of g is the sign of $\lambda m - \sigma^2$, which completes the proof.

Corollary 1. *The risk of a gain or loss of h with equal probability* ($\pm h$ *with probability* $\frac{1}{2}$) *is equivalent to the sure loss of* h^2/λ.

Proof:
Let

$$X_1 = \begin{cases} h & \text{with probability } 1/2 \\ -h & \text{with probability } 1/2. \end{cases}$$

Then

$$m = E(X_1) = 0$$

$$\sigma^2 = E(X_1^2) = h^2/2 + h^2/2 = h^2.$$

Then, using (B.9), $g = m - \sigma^2/\lambda = 0 - h^2/\lambda = -h^2/\lambda$.

Corollary 2. *A gain of* $\pm h$ *with respective probabilities* $(1 \pm d)/2$ *is indifferent if* $d = h/\lambda$.

Proof:
Let

$$X_2 = \begin{cases} h & \text{with probability } (1+d)/2 \\ -h & \text{with probability } (1-d)/2. \end{cases}$$

https://doi.org/10.1017/S0266267109990022 Published online by Cambridge University Press

Then

$$E(X_2) = h(1+d)/2 - h(1-d)/2 = hd/2 + hd/2 = hd$$
$$E(X_2^2) = h^2(1+d)/2 + h^2(1-d)/2 = h^2.$$

Since (B.9) implies that

$$\sigma^2 = E\left(X_2^2\right) + O(h^4), \text{ we have}$$
$$0 = g = m - \sigma^2/\lambda = hd - h^2/\lambda$$
$$\text{if } d = h/\lambda.$$

Corollary 3. *The price of a gain h (loss if $h < 0$) with probability p is* $ph(1-(1-p)h/\lambda)$.

Proof:
Let

$$X_3 = \begin{cases} h & \text{with probability } p \\ 0 & \text{with probability } 1-p, \end{cases}$$

Then X_3 is h times a Bernoulli (p) random variable, with mean $m = hp$ and variance $\sigma^2 = h^2 p(1-p)$.
Consequently

$$g = m - \sigma^2/\lambda = hp - h^2 p(1-p)/\lambda = hp(1-(1-p)h/\lambda).$$

De Finetti then shows how to derive $u(x)$ from $\lambda(x)$ as follows: since $\lambda = -2u'/u''$, we have $u''/u' = -2/\lambda$.
Then

$$\text{(B.10)} \qquad \frac{-2}{\lambda(x)} = \frac{u''}{u'} = \frac{d}{dx}\log u'.$$

Thence

$$\text{(B.11)} \qquad \log u' = \int \frac{-2}{\lambda(x)} dx + c_1,$$

so

$$\text{(B.12)} \qquad u' = e^{\int -2/\lambda(x)dx + c_1} = e^{c_1} e^{\int -2/\lambda(x)\,dx}$$

Hence

$$u = e^{c_1} \int e^{\int -2/\lambda(x)dx} + c_2$$
$$\text{(B.13)} \qquad = c_3 \int e^{\int -2/\lambda(x)dx} + c_2,$$

https://doi.org/10.1017/S0266267109990022 Published online by Cambridge University Press

where c_1 and c_2 are constants of integration and $c_3 = e^{c_1}$. He drops the constants c_2 and c_3 because utilities are defined only up to an affine transformation, which means that $u_1(x)$ and $u_2(x) = a u_1(x) + b$, where $a > 0$ order each pair of uncertain prospects identically.

The special cases mentioned by de Finetti are checked most easily by differentiation: If $u = K - e^{-2x/\lambda}$ then $u'(x) = 2/\lambda e^{-2x/\lambda}$ and $u''(x) = -4/\lambda^2 e^{-2x/\lambda}$, so $-2u'/u'' = -2(2/\lambda)e^{-2x/\lambda}/-4/\lambda^2 e^{-2x/\lambda} = \lambda$.

Similarly, if $u = \log x$, $u' = \frac{1}{x}$ and $u'' = -\frac{1}{x^2}$. Hence $\lambda(x) = -2u'/u'' = -2/x/-1/x^2 = 2x$. Also, if $u = x^{1-2c}$, $u' = (-2c)x^{-2c}$ and $u'' = 4c^2 x^{-1-2c}$, so

$$\lambda(x) = -2u'/u'' = \frac{-2(-2c)x^{-2c}}{4c^2 x^{-1-2c}} = x/c.$$

Thus, with minor amendments, de Finetti's statements are correct.

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