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Symmetry breaking in the minimization of the second eigenvalue for composite membranes

Claudia Anedda and **Fabrizio Cuccu**
Mathematics and Computer Science Department,
University of Cagliari, Via Università 40, 09124 Cagliari, Italy
(canedda@unica.it; fcuccu@unica.it)

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Let $\Omega \subset \mathbb{R}^N$ be an open bounded connected set. We consider the eigenvalue problem $-\Delta u = \lambda \rho u$ in Ω with Dirichlet boundary condition, where ρ is an arbitrary function that assumes only two given values $0 < \alpha < \beta$ and is subject to the constraint $\int_\Omega \rho \, \mathrm{d}x = \alpha\gamma + \beta(|\Omega| - \gamma)$ for a fixed $0 < \gamma < |\Omega|$. Cox and McLaughlin studied the optimization of the map $\rho \mapsto \lambda_k(\rho)$, where λ_k is the kth eigenvalue. In this paper we focus our attention on the case when $N \geqslant 2$, $k = 2$ and Ω is a ball. We show that, under suitable conditions on α, β and γ, the minimizers do not inherit radial symmetry.

1. Introduction

We consider a vibrating membrane Ω with clamped boundary $\partial\Omega$ and density ρ. The displacement u satisfies the eigenvalue problem

$$\left.\begin{aligned} -\Delta u &= \lambda \rho u && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded connected set with C^2-boundary $\partial\Omega$, $\rho \in L^\infty(\Omega)$ and is positive, and $\lambda \in \mathbb{R}$; $u \in W_0^{1,2}(\Omega)$ is a weak solution of (1.1) if

$$\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x = \lambda \int_\Omega \rho u v \, \mathrm{d}x \quad \forall v \in W_0^{1,2}(\Omega).$$

Note that densities that differ from each other by a subset of null measure give the same eigenvalue problem. The constant λ associated with a non-trivial solution u is called an *eigenvalue*, and u is the corresponding *eigenfunction*. The set of all eigenfunctions of an eigenvalue λ is a linear space, called the *eigenspace* associated with λ; its dimension is the *multiplicity* of λ. If the eigenspace is one-dimensional, λ is said to be *simple*.

It is well known that the eigenvalues of (1.1), repeated according to their multiplicity, form a sequence

$$0 < \lambda_1(\rho) \leqslant \lambda_2(\rho) \leqslant \cdots,$$

and that $\lambda_1(\rho)$ is simple.

The physical meaning of $\lambda_k(\rho)$ is the kth natural frequency of vibration of the membrane.

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A detailed treatment of this classical argument can be found in [7] and [20].

We restrict our attention to membranes made with only two homogeneous materials of densities α and β, with $0 < \alpha < \beta$. Moreover, we require that the portion of the membrane with density α has a fixed Lebesgue measure $0 < \gamma < |\Omega|$, where $|\Omega|$ denotes the measure of Ω.

Mathematically, this means that we consider $\rho = \alpha\chi_E + \beta(1 - \chi_E)$, where χ_E is the characteristic function of a set $E \subset \Omega$ such that $|E| = \gamma$. Following Cox and McLaughlin [9, 10], we denote this class of densities by ad_γ.

An interesting problem is to understand how these two materials must be placed in order to minimize or maximize the kth eigenvalue. Again using the notation in [9, 10], $\check{\rho}_k$ ($\hat{\rho}_k$) stands for a minimizer (maximizer) of $\rho \mapsto \lambda_k(\rho)$.

The analogous problem in the case of a string (the one-dimensional case) was posed and fully solved in 1955 by Krein [18], who explicitly found the minimizers $\check{\rho}_k$ and the maximizers $\hat{\rho}_k$ for all eigenvalues. We remark that, for each k, these extremizers are unique and symmetric with respect to the midpoint of the string.

For $N \geqslant 2$, in 1977, Friedland (see [13, theorem 2]) proved the existence of $\check{\rho}_k$ for all k. In 1990, Cox and McLaughlin [9, 10] established the continuous dependence of eigenvalues with respect to ρ and some stability properties. Moreover, they gave a useful characterization of minimizers in terms of level sets of associated eigenfunctions, which is of fundamental importance in the present paper.

From results in [9, 10] (which are consistent with those of [13]) it is clear that, in order to minimize λ_k, we should place the more dense material where the membrane (or the string) displacement is greatest.

If Ω is a ball, the natural question about the radial symmetry of optimizers arises. In this case, the optimizers $\check{\rho}_1$ and $\hat{\rho}_1$ are unique and radially symmetric (see [11, 16]). In particular, [16] is a complete survey on the optimization of eigenvalues related to elliptic problems. Similar problems are also considered in [4–6, 8, 15].

In this paper we prove that, if $N \geqslant 2$, contrary to the one-dimensional case, for some values of α, β and γ every minimizer $\check{\rho}_2$ associated with the second eigenvalue is not radially symmetric (see theorem 3.5). This is suggested by the relation between the level sets of eigenfunctions and the density ρ, and by the fact that a second eigenfunction of the Dirichlet–Laplacian in a disk is not radially symmetric.

This article is organized as follows. In § 2 we give some preliminary tools that will be used in the following, and § 3 contains the main result.

2. Preliminaries

First, we list some further regularity properties of the eigenfunctions. If u is an eigenfunction of (1.1), then

$$u \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega) \cap C^1(\bar{\Omega})$$

(it is assumed that $\partial\Omega$ is C^2), and the equation in (1.1) holds almost everywhere (a.e.). Furthermore, $\lambda_1(\rho)$ is simple, every first eigenfunction u_1 is either positive or negative in Ω and every kth eigenfunction, with $k \geqslant 2$, changes sign in Ω (see [9, 10, 14]).

We denote by $\mathcal{S}_k(\rho)$ the eigenspace associated with $\lambda_k(\rho)$.

DEFINITION 2.1. Let $u \in \mathcal{S}_k(\rho)$, $k \geqslant 2$. The connected components of the open sets

$$\{x \in \Omega\colon u(x) > 0\} \quad \text{and} \quad \{x \in \Omega\colon u(x) < 0\}$$

are called the *nodal domains* of u.

We now recall some well-known variational characterizations of the eigenvalues.

Rayleigh's principle

Let $\lambda_1(\rho)$ and $\lambda_2(\rho)$ be the first two eigenvalues of (1.1). Then,

$$\lambda_1(\rho) = \min_{\substack{u \in W_0^{1,2}(\Omega) \\ u \neq 0}} \frac{\int_\Omega |\nabla u|^2 \,\mathrm{d}x}{\int_\Omega \rho u^2 \,\mathrm{d}x} \quad \text{and} \quad \lambda_2(\rho) = \min_{\substack{u \in W_0^{1,2}(\Omega) \\ \langle \rho u, u_1 \rangle = 0 \\ u \neq 0}} \frac{\int_\Omega |\nabla u|^2 \,\mathrm{d}x}{\int_\Omega \rho u^2 \,\mathrm{d}x},$$

where u_1 is an eigenfunction with respect to $\lambda_1(\rho)$ and $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $L^2(\Omega)$. The quantity $\int_\Omega |\nabla u|^2 \,\mathrm{d}x / \int_\Omega \rho u^2 \,\mathrm{d}x$ is called *Rayleigh's quotient*. Rayleigh's principle extends to each eigenvalue $\lambda_k(\rho)$ (see, for instance, [7, 9, 10]).

Auchmuty's principle

Let $\lambda_1(\rho)$ and $\lambda_2(\rho)$ be the first two eigenvalues of (1.1), let $\|\cdot\|_2$ be the usual norm in $L^2(\Omega)$ and let $\|u\|_\rho = \langle \rho u, u \rangle^{1/2}$. Then,

$$-\frac{1}{2\lambda_1(\rho)} = \inf_{u \in W_0^{1,2}(\Omega)} \mathcal{A}_1(\rho, u), \qquad \mathcal{A}_1(\rho, u) = \tfrac{1}{2}\|\nabla u\|_2^2 - \|u\|_\rho$$

and

$$-\frac{1}{2\lambda_2(\rho)} = \inf_{u \in W_0^{1,2}(\Omega)} \mathcal{A}_2(\rho, u), \qquad \mathcal{A}_2(\rho, u) = \tfrac{1}{2}\|\nabla u\|_2^2 - (\langle \rho u, u \rangle - \langle \rho u, u_1 \rangle^2)^{1/2}, \tag{2.1}$$

where the minimum is attained at a kth eigenfunction normalized according to

$$\|\nabla u_k\|_2^2 = \|u_k\|_\rho = \lambda_k^{-1}(\rho), \quad k = 1, 2, \tag{2.2}$$

and u_1 is a first eigenfunction such that $\|u_1\|_\rho = 1$. Auchmuty's principle also extends for all $\lambda_k(\rho)$ (see [9, 10]).

We denote by $\check{\rho}_k \in ad_\gamma = \{\alpha\chi_E + \beta(1 - \chi_E)\colon E \subset \Omega,\ |E| = \gamma\}$ a minimizer of the map $\rho \mapsto \lambda_k(\rho)$, that is,

$$\lambda_k(\check{\rho}_k) = \inf_{\rho \in ad_\gamma} \lambda_k(\rho). \tag{2.3}$$

From results in [13], such a minimizer exists. Actually, a stronger result holds; there exist minimizers in the wider class of densities

$$\left\{\rho \in L^\infty(\Omega)\colon \alpha \leqslant \rho(x) \leqslant \beta \text{ a.e. in } \Omega,\ \int_\Omega \rho \,\mathrm{d}x = \alpha\gamma + \beta(|\Omega| - \gamma)\right\},$$

and they all belong to ad_γ (see [16, theorem 9.2.3]). To emphasize the dependence of the minimum in (2.3) on the constants α, β and γ, we introduce the further notation

$$\check{\lambda}_k(\alpha, \beta, \gamma) = \lambda_k(\check{\rho}_k).$$

The following proposition of Cox and McLaughlin [9, 10, proposition 6.4] is crucial in proving symmetry breaking.

Proposition 2.2. *If $\partial\Omega$ is C^2 and there exists $u \in \mathcal{S}_k(\check{\rho}_k)$ with k nodal domains $\{\Omega_i\}_{i=1}^k$, then there exists a set $\{\check{l}_i\}_{i=1}^k$, $\check{l}_i \geqslant 0$, such that, for each $x \in \Omega_i$,*

$$\check{\rho}_k(x) = \begin{cases} \beta & \text{if } |u(x)| > \check{l}_i, \\ \alpha & \text{if } |u(x)| \leqslant \check{l}_i. \end{cases}$$

Remark 2.3. Consider $k = 2$. By Courant's nodal theorem the number of nodal domains of u_2 is less than or equal to 2. Therefore, since a second eigenfunction is sign changing in Ω, u_2 always has two nodal domains, and then proposition 2.2 applies to this case. By the unique continuation theorem (see [12, 17]) the set $\{x \in \Omega\colon u(x) = 0\}$ has null measure. Changing $\{x \in \Omega\colon \check{\rho}_2(x) = \beta\}$ for a set of null measure, we can assume that $\{x \in \Omega\colon \check{\rho}_2(x) = \beta\} \cap \{x \in \Omega\colon u(x) = 0\} = \emptyset$ and, by proposition 2.2, that $\{x \in \Omega\colon \check{\rho}_2(x) = \beta\}$ is open.

Proposition 2.2 states that, for each minimizing density $\check{\rho}_2$, the set with more dense material is a superlevel set of the modulus of a corresponding eigenfunction.

In order to prove our result we need to strengthen proposition 2.2 in the case $k = 2$.

Proposition 2.4. *If $\partial\Omega$ is C^2, $0 < \gamma < |\Omega|$, $u_2 \in \mathcal{S}_2(\check{\rho}_2)$ and Ω_1, Ω_2 are the nodal domains of u_2, then there exist positive numbers l_1, l_2 such that, for each $x \in \Omega_i$, $i = 1, 2$,*

$$\check{\rho}_2(x) = \begin{cases} \beta & \text{if } |u_2(x)| > l_i, \\ \alpha & \text{if } |u_2(x)| \leqslant l_i. \end{cases} \tag{2.4}$$

Proof. Note that, without loss of generality, we can assume that u_2 satisfies (2.2). By proposition 2.2 and the subsequent remark, there exist non-negative numbers l_1, l_2 that satisfy (2.4); we show that l_1 and l_2 are actually positive. Since $\gamma > 0$ and by remark 2.3, we can assume that $l_1 > 0$. By contradiction, suppose that $l_2 = 0$. Let $u_1 \in \mathcal{S}_1(\check{\rho}_2)$ be positive such that $\|u_1\|_{\check{\rho}_2} = 1$. Choose $n \in \mathbb{N}$ with $l_1 - 1/n > 0$; therefore, there exist two measurable sets

$$A_n \subseteq \left\{|u_2| < \frac{1}{n}\right\} \subset \Omega_2, \tag{2.5}$$

$$B_n \subseteq \left\{l_1 - \frac{1}{n} < |u_2| < l_1\right\} \subset \Omega_1 \setminus \{\check{\rho}_2 = \beta\}, \tag{2.6}$$

with $|A_n| = |B_n| > 0$. Note that $|A_n| \to 0$ as $n \to \infty$. We define

$$\rho_n(x) = \begin{cases} \check{\rho}_2(x), & x \notin A_n \cup B_n, \\ \alpha, & x \in A_n, \\ \beta, & x \in B_n, \end{cases}$$

and we observe that

$$\rho_n = \check{\rho}_2 + (\beta - \alpha)(\chi_{B_n} - \chi_{A_n}). \tag{2.7}$$

Moreover, $\rho_n \in ad_\gamma$ and $\rho_n \overset{*}{\rightharpoonup} \check{\rho}_2$ in $L^\infty(\Omega)$. Let $u_{1,n} \in \mathcal{S}_1(\rho_n)$ be positive such that $\|u_1\|_{\rho_n} = 1$; by [9, 10, proposition 4.3], $\lambda_1(\rho_n) \to \lambda_1(\check{\rho}_2)$ and $u_{1,n} \to u_1$ in $W_0^{1,2}(\Omega)$. Thus, it is enough to prove that, for n large,

$$\mathcal{A}_2(\check{\rho}_2, u_2) > \mathcal{A}_2(\rho_n, u_2) \tag{2.8}$$

holds, where $\mathcal{A}_2(\rho, u)$ is defined in (2.1). Indeed, if (2.8) is true, we have that

$$-\frac{1}{2\lambda_2(\check{\rho}_2)} = \mathcal{A}_2(\check{\rho}_2, u_2) > \mathcal{A}_2(\rho_n, u_2) \geqslant \inf_{u \in W_0^{1,2}(\Omega)} \mathcal{A}_2(\rho_n, u) = -\frac{1}{2\lambda_2(\rho_n)},$$

from which the contradiction (since $\lambda_2(\check{\rho}_2)$ is the minimum) that $\lambda_2(\check{\rho}_2) > \lambda_2(\rho_n)$ follows, which in turn proves the proposition.

We now prove (2.8). By (2.1), (2.8) is equivalent to

$$\int_\Omega \rho_n u_2^2 \,\mathrm{d}x - \left(\int_\Omega \rho_n u_2 u_{1,n} \,\mathrm{d}x\right)^2 > \int_\Omega \check{\rho}_2 u_2^2 \,\mathrm{d}x - \left(\int_\Omega \check{\rho}_2 u_2 u_1 \,\mathrm{d}x\right)^2.$$

Since u_1 and u_2 are orthogonal, $\int_\Omega \check{\rho}_2 u_2 u_1 \,\mathrm{d}x = 0$. Moreover, by using (2.7) and rearranging, we find that

$$(\beta - \alpha) \int_{B_n} u_2^2 \,\mathrm{d}x > (\beta - \alpha) \int_{A_n} u_2^2 \,\mathrm{d}x + \left(\int_\Omega \rho_n u_2 u_{1,n} \,\mathrm{d}x\right)^2.$$

Recalling that $|A_n| = |B_n| > 0$, the previous inequality is equivalent to

$$\frac{\beta - \alpha}{|B_n|} \int_{B_n} u_2^2 \,\mathrm{d}x > \frac{\beta - \alpha}{|A_n|} \int_{A_n} u_2^2 \,\mathrm{d}x + \frac{1}{|A_n|}\left(\int_\Omega \rho_n u_2 u_{1,n} \,\mathrm{d}x\right)^2. \tag{2.9}$$

If (2.9) is true for $n \to \infty$, then it can be proved for n large. We observe that, by (2.6),

$$\frac{1}{|B_n|} \int_{B_n} u_2^2 \,\mathrm{d}x \to l_1^2 \quad \text{as } n \to \infty$$

and, by (2.5),

$$\frac{1}{|A_n|} \int_{A_n} u_2^2 \,\mathrm{d}x \to 0 \quad \text{as } n \to \infty.$$

More work is needed to show that

$$\frac{1}{|A_n|}\left(\int_\Omega \rho_n u_2 u_{1,n} \,\mathrm{d}x\right)^2 \to 0 \quad \text{as } n \to \infty. \tag{2.10}$$

Writing the equation of (1.1) in weak form for u_2 and $u_{1,n}$ we find that

$$\int_\Omega \nabla u_2 \cdot \nabla u_{1,n} \,\mathrm{d}x = \lambda_2(\check{\rho}_2) \int_\Omega \check{\rho}_2 u_2 u_{1,n} \,\mathrm{d}x$$

and

$$\int_\Omega \nabla u_2 \cdot \nabla u_{1,n} \,\mathrm{d}x = \lambda_1(\rho_n) \int_\Omega \rho_n u_2 u_{1,n} \,\mathrm{d}x.$$

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Comparing these last two equations and using (2.7) we obtain that

$$(\lambda_2(\check{\rho}_2) - \lambda_1(\rho_n)) \int_\Omega \rho_n u_2 u_{1,n}\, \mathrm{d}x = \lambda_2(\check{\rho}_2)(\beta - \alpha) \int_\Omega (\chi_{B_n} - \chi_{A_n}) u_2 u_{1,n}\, \mathrm{d}x,$$

from which we have that

$$\int_\Omega \rho_n u_2 u_{1,n}\, \mathrm{d}x = \frac{\lambda_2(\check{\rho}_2)}{\lambda_2(\check{\rho}_2) - \lambda_1(\rho_n)}(\beta-\alpha)\biggl(\int_{B_n} u_2 u_{1,n}\, \mathrm{d}x - \int_{A_n} u_2 u_{1,n}\, \mathrm{d}x\biggr). \quad (2.11)$$

By the Sobolev imbedding theorem (see [3]) there exists $p > 2$ such that $W_0^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$. Therefore, $u_{1,n}, u_1 \in L^p(\Omega)$ and $u_{1,n} \to u_1$ in $L^p(\Omega)$. Let $1/p + 1/q = 1$; by the Hölder inequality we have that

$$\begin{aligned}\int_{A_n} u_2 u_{1,n}\, \mathrm{d}x &\leqslant \biggl(\int_{A_n} u_{1,n}^p\, \mathrm{d}x\biggr)^{1/p} \biggl(\int_{A_n} |u_2|^q\, \mathrm{d}x\biggr)^{1/q} \\ &\leqslant \|u_2\|_{L^\infty(\Omega)} \|u_{1,n}\|_{L^p(\Omega)} |A_n|^{1/q}; \end{aligned} \quad (2.12)$$

similarly,

$$\int_{B_n} u_2 u_{1,n}\, \mathrm{d}x \leqslant \|u_2\|_{L^\infty(\Omega)} \|u_{1,n}\|_{L^p(\Omega)} |A_n|^{1/q}. \quad (2.13)$$

From (2.11), (2.12) and (2.13) it follows that

$$\begin{aligned}&\frac{1}{|A_n|}\biggl(\int_\Omega \rho_n u_2 u_{1,n}\, \mathrm{d}x\biggr)^2 \\ &\quad \leqslant \biggl(\frac{2\lambda_2(\check{\rho}_2)}{\lambda_2(\check{\rho}_2) - \lambda_1(\rho_n)}\biggr)^2 (\beta - \alpha)^2 \|u_2\|_{L^\infty(\Omega)}^2 \|u_{1,n}\|_{L^p(\Omega)}^2 |A_n|^{(p-2)/p}.\end{aligned}$$

Since $\lambda_1(\rho_n) \to \lambda_1(\check{\rho}_2)$, $\|u_{1,n}\|_{L^p(\Omega)} \to \|u_1\|_{L^p(\Omega)}$ and $p > 2$, (2.10) follows. Then, as $n \to \infty$, (2.9) becomes $l_1^2 > 0$, i.e. it is true for $n \to \infty$. □

Remark 2.5. By the same argument used in the proof it can be shown that $l_1 = l_2$.

In the next section we use the following classical result (see, for example, [16]). Let $\lambda_1(\Omega)$ be the first eigenvalue of the Laplacian–Dirichlet in Ω.

Theorem 2.6 (Faber–Krahn theorem). *Let c be a positive number and let B be the ball of volume c. Then,*

$$\lambda_1(B) = \min\{\lambda_1(\Omega)\colon \Omega \subset \mathbb{R}^N \text{ open}, |\Omega| = c\}.$$

In the following we need some results about the singular Sturm–Liouville problem

$$\left.\begin{aligned} -u'' - \frac{N-1}{r}u' = \lambda\rho u \quad &\text{in } (0, R), \\ u(x) = O(1), \quad x \to 0, \quad &u(R) = 0, \end{aligned}\right\} \quad (2.14)$$

where $\rho = \rho(r)$ assumes only two positive values α and β, and $O(1)$ is a bounded quantity as x goes to 0.

Consider (2.14) in weak form. Let $W^{1,2}((0,R),\rho r^{N-1})$ be the weighted Sobolev space (with weight ρr^{N-1}) and let $H=\{u\in W^{1,2}((0,R),\rho r^{N-1})\colon u(R)=0\}$. We call $u\in H$ an eigenfunction, relative to λ, of (2.14) if

$$\int_0^R u'v'r^{N-1}\,\mathrm{d}r=\lambda\int_0^R uv\rho r^{N-1}\,\mathrm{d}r\quad\forall v\in H.$$

From general theory (see [3, 19–21, 23]) the existence of positive eigenvalues and their minimax characterization follows. The first eigenvalue is simple, and the corresponding eigenfunction does not change sign in $(0,R)$. Moreover, if E_2 denotes an arbitrary two-dimensional subspace of H, for the second eigenvalue we have

$$\eta_2(\rho)=\min_{E_2\subset H}\max_{\substack{u\in E_2\\ u\neq 0}}\frac{\int_0^R r^{N-1}(u')^2\,\mathrm{d}r}{\int_0^R \rho r^{N-1}u^2\,\mathrm{d}r}. \tag{2.15}$$

3. Symmetry breaking

Let $N\geqslant 2$. Let $\nu_1(\delta)$ and $\nu_2(\delta)$ be the first and the second eigenvalues, respectively, of the problem

$$\left.\begin{aligned}-\Delta u&=\lambda u\quad\text{in } B,\\ u&=0\quad\text{on }\partial B,\end{aligned}\right\} \tag{3.1}$$

where B is a ball of measure δ.

It is well known that $\nu_1(\delta)$ and $\nu_2(\delta)$ can be expressed in terms of δ and of zeros of Bessel functions of the first kind J_n (see [7, pp. 302–304] and [16, pp. 11]). Precisely, denoting by $k_{n,m}$ the mth zero of J_n, $n\in\mathbb{R}$ and $m=1,2,\dots$, we have that

$$\nu_1(\delta)=\left(\frac{\omega_N}{\delta}\right)^{2/N}k^2_{N/2-1,1}\quad\text{and}\quad\nu_2(\delta)=\left(\frac{\omega_N}{\delta}\right)^{2/N}k^2_{N/2,1},$$

where ω_N denotes the measure of the unit ball in $\mathbb{R}^N$. This result can be easily extended to the case

$$\left.\begin{aligned}-\Delta u&=\lambda\sigma u\quad\text{in } B,\\ u&=0\quad\text{on }\partial B,\end{aligned}\right\} \tag{3.2}$$

with $\sigma>0$. Note that $\lambda\mapsto\lambda/\sigma$ yields a bijection between the eigenvalues $\nu_k(\delta)$ of (3.1) and $\nu_k(\delta,\sigma)$ of (3.2), $k=1,2$. Consequently, we have that

$$\nu_1(\delta,\sigma)=\frac{1}{\sigma}\left(\frac{\omega_N}{\delta}\right)^{2/N}k^2_{N/2-1,1}\quad\text{and}\quad\nu_2(\delta,\sigma)=\frac{1}{\sigma}\left(\frac{\omega_N}{\delta}\right)^{2/N}k^2_{N/2,1}. \tag{3.3}$$

Recall that

$$ad_\gamma=\{\alpha\chi_E+\beta(1-\chi_E)\colon E\subset\Omega,\ |E|=\gamma\},$$

with $0<\alpha<\beta$ and $0<\gamma<|\Omega|$, and we consider densities $\rho\in ad_\gamma$.

From now on we assume that Ω is a ball. We prove that, for some values of α, β and γ, the eigenspace relative to $\check{\lambda}_2(\alpha,\beta,\gamma)$ is one dimensional.

We begin by proving the following.

LEMMA 3.1. *Let $\Omega = B(0,R) \subset \mathbb{R}^N$, $N \geqslant 2$, be the ball centred at the origin of radius R. Let $\nu_1(\delta,\sigma)$ and $\nu_2(\delta,\sigma)$ be as defined above. If α, β and γ are such that*

$$\frac{\beta}{\alpha} < \frac{\nu_1(|\Omega|-\gamma,1)}{\nu_2(|\Omega|,1)} = \left(\frac{|\Omega|}{|\Omega|-\gamma}\right)^{2/N} \frac{k^2_{N/2-1,1}}{k^2_{N/2,1}},$$

then $\beta\check{\lambda}_2(\alpha,\beta,\gamma)$ is less than the first eigenvalue of the problem

$$\left.\begin{aligned} -\Delta v &= \lambda v \quad \text{in } F,\\ v &= 0 \quad\ \ \text{on } \partial F, \end{aligned}\right\} \tag{3.4}$$

where F is an arbitrary domain with $|F| = |\Omega| - \gamma$.

Proof. Let μ_1 be the first eigenvalue of (3.4). Then, by the Faber–Krahn inequality, it follows that

$$\nu_1(|\Omega|-\gamma,1) \leqslant \mu_1.$$

On the other hand, from the assumption, we have that

$$\nu_1(|\Omega|-\gamma,1) > \frac{\beta}{\alpha}\nu_2(|\Omega|,1) = \beta\nu_2(|\Omega|,\alpha) = \beta\check{\lambda}_2(\alpha,\beta,|\Omega|) \geqslant \beta\check{\lambda}_2(\alpha,\beta,\gamma),$$

where in the last inequality we have used a monotonicity property of eigenvalues with respect to γ (see [9, 10, proposition 5.6]).

Then, $\mu_1 > \beta\check{\lambda}_2(\alpha,\beta,\gamma)$. The lemma follows. □

THEOREM 3.2. *Let $\Omega = B(0,R) \subset \mathbb{R}^N$, $N \geqslant 2$. Let $\check{\rho}_2$ be a minimizer in (2.3) with $k = 2$. If $0 < \gamma < |\Omega|$ and*

$$\frac{\beta}{\alpha} < \frac{\nu_1(|\Omega|-\gamma,1)}{\nu_2(|\Omega|,1)} = \left(\frac{|\Omega|}{|\Omega|-\gamma}\right)^{2/N} \frac{k^2_{N/2-1,1}}{k^2_{N/2,1}},$$

then $\lambda_2(\check{\rho}_2)$ is simple.

Proof. Let $u_2, \tilde{u}_2 \in \mathcal{S}_2(\check{\rho}_2)$. Let D be a connected component of the open set $\{x \in \Omega\colon \check{\rho}_2(x) = \beta\}$. Note that $|D| \leqslant |\Omega| - \gamma$. By proposition 2.4, there exists a nodal domain Ω_1 of u_2 containing D, and a constant $l > 0$ such that $|u_2(x)| > l$ in D and $|u_2(x)| = l$ in ∂D. Setting $t = l^{-1}\operatorname{sgn} u_2$, where $\operatorname{sgn} u_2$ is the sign of u_2 in Ω_1, we have $tu_2(x) = 1$ in ∂D. By the same argument we find a constant $\tilde{t}$ such that $\tilde{t}\tilde{u}_2(x) = 1$ in ∂D.

We observe that $v = tu_2 - \tilde{t}\tilde{u}_2$ is a solution of the eigenvalue problem

$$\left.\begin{aligned} -\Delta v &= \lambda v \quad \text{in } D,\\ v &= 0 \quad\ \ \text{on } \partial D, \end{aligned}\right\} \tag{3.5}$$

with $\lambda = \beta\check{\lambda}_2(\alpha,\beta,\gamma)$; we call η_1 the first eigenvalue of (3.5). Let F be a domain such that $F \supseteq D$ and $|F| = |\Omega| - \gamma$, and we call μ_1 the first eigenvalue of (3.4). If $u \in W_0^{1,2}(D)$, calling $\tilde{u}$ the extension of u in F obtained setting $u = 0$ in $F \setminus D$, we have that

$$\frac{\int_D |\nabla u|^2\,\mathrm{d}x}{\int_D u^2\,\mathrm{d}x} = \frac{\int_F |\nabla \tilde{u}|^2\,\mathrm{d}x}{\int_F \tilde{u}^2\,\mathrm{d}x} \geqslant \mu_1.$$

Taking the infimum in $W_0^{1,2}(D)$ we find that $\eta_1 \geqslant \mu_1$.

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Moreover, by lemma 3.1 it follows that $\mu_1 > \beta\check{\lambda}_2(\alpha,\beta,\gamma)$. Therefore, $\eta_1 > \beta\check{\lambda}_2(\alpha,\beta,\gamma)$, and then (3.5) with $\lambda = \beta\check{\lambda}_2(\alpha,\beta,\gamma)$ has only the trivial solution; thus, $tu_2 - \tilde{t}\tilde{u}_2 = 0$ in D. Now, by the unique continuation theorem (see [1,12,17]), it follows that $tu_2 - \tilde{t}\tilde{u}_2 = 0$ in Ω. This means that $\lambda_2(\check{\rho}_2)$ is simple. □

THEOREM 3.3. *Let $\Omega = B(0,R) \subset \mathbb{R}^N$, $N \geqslant 2$, and let $\rho \in ad_\gamma$ be radially symmetric. If*

$$\frac{\beta}{\alpha} < \frac{k^2_{N/2-1,2}}{k^2_{N/2,1}} \tag{3.6}$$

and $\lambda_2(\rho)$ is the second eigenvalue of (1.1), then $\lambda_2(\rho)$ has multiplicity greater than 1.

Proof. By contradiction, let $\lambda_2(\rho)$ be simple.

By [9, 10, lemma 3.2] we have that

$$\lambda_2(\rho) \leqslant \nu_2(|\Omega|,\alpha) = \frac{1}{\alpha}\left(\frac{\omega_N}{|\Omega|}\right)^{2/N} k^2_{N/2,1} = \frac{k^2_{N/2,1}}{\alpha R^2}, \tag{3.7}$$

where $\nu_2(|\Omega|,\alpha)$ denotes the second eigenvalue of the Dirichlet problem (3.2) with $B = \Omega$ and density $\sigma = \alpha$.

Now let $u_1 \in \mathcal{S}_1(\rho)$ and $u_2 \in \mathcal{S}_2(\rho)$ be, respectively, a first and a second eigenfunction of (1.1). Fix an oriented plane P passing through the origin. Let T_θ be the rotation of the angle θ, $0 \leqslant \theta < 2\pi$, that moves parallel to P. Note that $\rho \circ T_\theta = \rho$ for all θ. We define $u_{1,\theta} = u_1 \circ T_\theta$. By a change of variable we find that

$$\|\nabla u_{1,\theta}\|_2^2 = \|\nabla u_1\|_2^2 \quad \text{and} \quad \|u_{1,\theta}\|_\rho^2 = \|u_1\|_\rho^2.$$

The Rayleigh quotients of $u_{1,\theta}$ and u_1 are then the same, that is, $u_{1,\theta}$ is a first eigenfunction for each θ. By the simplicity of $\lambda_1(\rho)$, there exists a constant c_θ such that $u_{1,\theta} = c_\theta u_1$; therefore, $c_\theta u_1 = u_1 \circ T_\theta$. Fixing $x_0 \in \Omega$ with $u_1(x_0) \neq 0$, we obtain that $c_\theta = u_1 \circ T_\theta(x_0)/u_1(x_0)$ is a continuous function of θ. On the other hand, from $\|u_{1,\theta}\|_\rho^2 = \|u_1\|_\rho^2$, it follows that $c_\theta^2 = 1$. Note that $c_0 = 1$; then, $c_\theta = 1$ and $u_1 = u_1 \circ T_\theta$ for all θ. Since P is arbitrary, it follows that u_1 is radially symmetric.

Similarly, defining $u_{2,\theta} = u_2 \circ T_\theta$, observing that

$$\langle \rho u_{2,\theta}, u_1\rangle = \langle \rho u_2, u_1\rangle = 0$$

and using the simplicity of $\lambda_2(\rho)$, it can be shown that u_2 is also radially symmetric ($u_2 = u_2(r)$).

The pair $(\lambda_2(\rho), u_2(r))$ is then a solution of the problem

$$\left.\begin{aligned} -u'' - \frac{N-1}{r}u' = \lambda\rho u \quad \text{in } (0,R),\\ u(R) = u'(0) = 0. \end{aligned}\right\} \tag{3.8}$$

Note that, since u_2 is sign changing, $\lambda_2(\rho)$ cannot be the first eigenvalue of (3.8), i.e. $\lambda_2(\rho) \geqslant \eta_2(\rho)$, where $\eta_2(\rho)$ is the second eigenvalue of (3.8).

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We now compare the eigenvalues of (3.8) with those of the problem

$$\left.\begin{aligned} -u'' - \frac{N-1}{r}u' = \lambda\beta u \quad \text{in } (0,R), \\ u(R) = u'(0) = 0. \end{aligned}\right\} \tag{3.9}$$

We observe that this last equation can be transformed in a Bessel equation (see [2, p. 117]); it is not difficult to find that the second eigenvalue of (3.9) is equal to

$$\frac{k^2_{N/2-1,2}}{R^2\beta}.$$

By using the minimax characterization (2.15) we obtain that

$$\frac{k^2_{N/2-1,2}}{R^2\beta} \leqslant \eta_2(\rho) \leqslant \lambda_2(\rho).$$

Comparing this inequality with (3.7) we have that

$$\frac{\beta}{\alpha} \geqslant \frac{k^2_{N/2-1,2}}{k^2_{N/2,1}}.$$

This contradiction concludes the proof. □

REMARK 3.4. By the interlacing of zeros of Bessel functions (see [22, p. 479]), it follows that

$$\frac{k^2_{N/2-1,2}}{k^2_{N/2,1}} > 1,$$

and then condition (3.6) is not meaningless.

THEOREM 3.5. *Let* $\Omega = B(0,R) \subset \mathbb{R}^N$*,* $N \geqslant 2$*. If* $0 < \gamma < |\Omega|$*,*

$$\frac{\beta}{\alpha} < \min\left\{ \left(\frac{|\Omega|}{|\Omega| - \gamma}\right)^{2/N} \frac{k^2_{N/2-1,1}}{k^2_{N/2,1}}, \frac{k^2_{N/2-1,2}}{k^2_{N/2,1}} \right\}$$

and $\check{\rho}_2$ *is a minimizer of (3.1) with* $k = 2$*, then* $\check{\rho}_2$ *cannot be radially symmetric.*

Proof. Comparing theorem 3.2 and theorem 3.3, the assertion follows. □

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