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STATIONARY REFLECTION PRINCIPLES AND TWO CARDINAL TREE PROPERTIES

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Abstract We study the consequences of stationary and semi-stationary set reflection. We show that the semi-stationary reflection principle implies the Singular Cardinal Hypothesis, the failure of the weak square principle, etc. We also consider two cardinal tree properties introduced recently by Weiss, and prove that they follow from stationary and semi-stationary set reflection augmented with a weak form of Martin's Axiom. We also show that there are some differences between the two reflection principles, which suggests that stationary set reflection is analogous to supercompactness, whereas semi-stationary set reflection is analogous to strong compactness.

Keywords: stationary sets; reflection; tree property; large cardinals

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Introduction

Reflection principles are a way of transferring large cardinal properties to small cardinals. Over the years, a large number of such principles have been considered, and a rich theory has been developed with numerous applications not only to pure set theory but also to various other areas of mathematics. Some of the earliest and most important reflection principles concern reflection of various classes of stationary sets. In this paper, we will consider the stationary reflection principle SR, introduced by Foreman *et al.* [2], which asserts that, for every $\lambda \geqslant \omega_2$, the following statement $\mathrm{SR}(\lambda)$ holds.

If S is a stationary subset of $[\lambda]^\omega$, then there is $I \subseteq \lambda$ of cardinality ω_1 such that $\omega_1 \subseteq I$ and $S \cap [I]^\omega$ is stationary in $[I]^\omega$.

SR and its variations have been studied extensively by a number of authors, and it has been shown that it has important consequences in cardinal arithmetic, infinite

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combinatorics, topology, algebra, etc. One of the key observations of [2] is that SR implies the following principle (†).

Every poset preserving stationary subsets of ω_1 is semi-proper.

This allowed Foreman *et al.* [2] to show that, in the standard model for the Semi Proper Forcing Axiom (SPFA), a provably maximal forcing axiom, Martin's Maximum (MM), holds. Somewhat later, Shelah [8] showed that MM follows outright from SPFA. The principle (†) in itself has many important consequences; for instance, already in [2] it was shown that it implies that the nonstationary ideal NS_{ω_1} is precipitous, and that Strong Chang's Conjecture holds. It is therefore interesting in its own right. In [10, Chapter XIII, 1.7], Shelah showed that (†) is equivalent to a certain reflection principle. In order to explain this, we will introduce some notation.

For countable sets x and y, we say that y is an ω_1-*extension* of x if $x \subseteq y$ and $x \cap \omega_1 = y \cap \omega_1$. We will write $x \sqsubseteq y$ to say that y is an ω_1-extension of x. Given $S \subseteq [\lambda]^\omega$, for some $\lambda \geqslant \omega_1$, we will say that S is *full* if S is closed under ω_1-extensions. Shelah [10] showed that (†) is equivalent to the statement SSR, which says that, for every $\lambda \geqslant \omega_2$, the following statement SSR(λ) holds.

If S is a full stationary subset of $[\lambda]^\omega$, then there is $I \subseteq \lambda$ of cardinality ω_1 such that $\omega_1 \subseteq I$ and $S \cap [I]^\omega$ is stationary in $[I]^\omega$.

One may be tempted to conjecture that the assumption that S is full in the above statement is innocuous and that SSR is equivalent to SR. However, the first author [7] showed that this is not the case; indeed, SSR is strictly weaker than SR. One of the goals of the present paper is to show that SSR nevertheless has many of the consequences as SR; it implies the Singular Cardinal Hypothesis, the failure of a weak version of the square principle, etc.

Another topic of this paper has to do with two cardinal properties recently introduced and studied by Weiss [14]. We first recall the relevant definitions. Suppose that κ is a regular cardinal and that $\lambda \geqslant \kappa$. By $\mathrm{Fn}(\kappa, \lambda, 2)$ we denote the set of all partial functions of size $<\kappa$ from λ to $\{0, 1\}$. A (κ, λ)-*tree* is a family $\mathscr{F} \subseteq \mathrm{Fn}(\kappa, \lambda, 2)$ which is closed under restrictions and such that for every $u \in [\lambda]^{<\kappa}$ there is $f \in \mathscr{F}$ with $\mathrm{dom}(f) = u$. We denote by $\mathrm{lev}_u(\mathscr{F})$ the u-level of $\mathscr{F}$, i.e., the set $\{f \in \mathscr{F} : \mathrm{dom}(f) = u\}$. A (κ, λ)-tree is called *thin* if $\mathrm{lev}_u(\mathscr{F})$ is of size $< \kappa$, for all $u \in [\lambda]^{<\kappa}$. A *cofinal branch* through $\mathscr{F}$ is a function $b : \lambda \to \{0, 1\}$ such that $f \restriction u \in \mathscr{F}$, for every $u \in [\lambda]^{<\kappa}$. A *level sequence* of $\mathscr{F}$ is a sequence $\vec{f} = (f_u : u \in [\lambda]^{<\kappa})$ such that $f_u \in \mathrm{lev}_u(\mathscr{F})$ for all $u \in [\lambda]^{<\kappa}$. Given a (κ, λ)-tree and a level sequence $\vec{f}$ of $\mathscr{F}$, we will say that a branch b of $\mathscr{F}$ is *ineffable* for $\vec{f}$ if the set $\{u \in [\lambda]^{<\kappa} : b \restriction u = f_u\}$ is stationary in $[\lambda]^{<\kappa}$. Given a regular cardinal $\kappa \geqslant \omega_1$ and $\lambda \geqslant \kappa$, the two cardinal tree property $\mathrm{TP}(\kappa, \lambda)$ states that every thin (κ, λ)-tree has a cofinal branch. We say that κ has the strong tree property if $\mathrm{TP}(\kappa, \lambda)$ holds for every $\lambda \geqslant \kappa$. Given κ and λ as before, we let $\mathrm{ITP}(\kappa, \lambda)$ denote the statement that, for every thin (κ, λ)-tree and a level sequence $\vec{f}$ of $\mathscr{F}$, there is an ineffable branch for $\vec{f}$. We say that κ has the super tree property if $\mathrm{ITP}(\kappa, \lambda)$ holds for every $\lambda \geqslant \kappa$. Note that if κ is inaccessible then every (κ, λ)-tree is thin. With this in mind, we can now reinterpret

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classical results of Jech [3] and Magidor [6]. Namely, in our terminology, Jech [3] showed that an uncountable cardinal is strongly compact if and only if it is inaccessible and has the strong tree property. Similarly, Magidor [6] showed that an uncountable cardinal κ is supercompact if and only if it is inaccessible and has the super tree property. These results are analogous to the classical reformulation of weak compactness, which states that an uncountable cardinal κ is weakly compact if and only if it is inaccessible and the usual tree property holds for κ; see for instance [4]. Since all known proofs of the consistency of strong forcing axioms require supercompact cardinals, it was natural to expect that they would imply these two cardinal properties for $\kappa = \omega_2$. This was indeed confirmed by Weiss [14], who showed that the Proper Forcing Axiom (PFA) implies that ω_2 has the super tree property. Moreover, Viale and Weiss [15] showed that, if the universe V is obtained by forcing over some inner model M by a forcing notion which has the κ-chain condition and the κ-approximation property, then, if κ has the strong tree property in V, it also has the strong tree property in M. If, moreover, the forcing notion is proper, then the same holds for the super tree property. Since all known methods for producing a model of PFA start from an inaccessible cardinal κ in some universe M and produce a generic extension by a forcing notion which has the above property and in which κ becomes ω_2, it follows that they require at least a strongly compact cardinal. We will show that SR together with MA_{ω_1}(Cohen) implies the super tree property of ω_2 and that SSR together with MA_{ω_1}(Cohen) implies the strong tree property of ω_2. This suggests that SR + MA_{ω_1}(Cohen) should have the consistency strength of a supercompact cardinal whereas SSR + MA_{ω_1}(Cohen) should have the strength of a strongly compact cardinal. We also show that SSR + MA_{ω_1}(Cohen) does not imply the super tree property of ω_2.

This paper is organized as follows. In § 1, we present the notation and basic facts used in this paper. In § 2, we prove that SSR implies the failure of weak square principles. In § 3, we prove that SR + MA_{ω_1}(Cohen) and SSR + MA_{ω_1}(Cohen) imply the super and the strong tree properties, respectively. Finally, in § 4, we prove that SSR implies the SCH.

1. Preliminaries

In this section, we present the notation and basic facts used in this paper. For a set A of ordinals, let $\lim(A)$ be the set of all limit points in A. Moreover, let $\sup^+(A) = \sup\{\alpha + 1 : \alpha \in A\}$. We often use $\sup^+$ instead of sup, since it slightly simplifies our arguments. For an ordinal λ and a regular cardinal $\kappa < \lambda$, let $E^\lambda_\kappa = \{\alpha < \lambda : \mathrm{cof}(\alpha) = \kappa\}$.

Let A be a set and F be a function from $[A]^{<\omega}$ to A. We say that $x \subseteq A$ is closed under F if $F(a) \in x$, for all $a \in [x]^{<\omega}$. For each $x \subseteq A$, let $\mathrm{cl}_F(x)$ be the closure of x under F, i.e., the smallest subset y of A which contains x and is closed under F.

Let κ be a regular uncountable cardinal and A be a set including κ. Recall that a subset C of $[A]^{<\kappa}$ is said to be *club* if and only if it is $\subseteq$-cofinal in $[A]^{<\kappa}$, and closed under unions of $\subseteq$-increasing sequence of length $< \kappa$. $S \subseteq [A]^{<\kappa}$ is said to be *stationary* if it intersects all club subsets of $[A]^{<\kappa}$. We often use the well-known fact that S is stationary in $[A]^{<\kappa}$ if and only if for any function $F : [A]^{<\omega} \to A$ there exists a nonempty $x \in S$ which is closed under F and such that $x \cap \kappa \in \kappa$. We say that $X \subseteq [A]^{\kappa}$ is stationary (or club) if X is stationary (or club) in $[A]^{<\kappa^+}$.

For a set A and a limit ordinal η, we say that A is *internally approachable of length* η if there exists a $\subseteq$-increasing sequence $\langle x_\xi : \xi < \eta \rangle$ such that $\bigcup_{\xi<\eta} x_\xi = A$ and such that $\langle x_\xi : \xi < \zeta \rangle \in A$, for all $\zeta < \eta$.

Suppose that $\mathfrak{A} = \langle A, \trianglelefteq, \ldots \rangle$ is a structure in a countable first-order language and $\trianglelefteq$ is a well-ordering of A. Then $\mathfrak{A}$ has definable Skolem functions. For each $X \subseteq A$, let $\mathrm{Hull}^{\mathfrak{A}}(X)$ be the Skolem hull of X in $\mathfrak{A}$, i.e., $\mathrm{Hull}^{\mathfrak{A}}(X)$ is the smallest M elementary submodel of $\mathfrak{A}$ such that $X \subseteq M$. We say that a structure $\mathfrak{A} = \langle A, \ldots \rangle$ is an *expansion* of a structure $\mathfrak{A}' = \langle A, \ldots \rangle$ if $\mathfrak{A}$ is obtained by adding countably many constants, functions, and predicates to $\mathfrak{A}'$. We use the following fact due to Baumgartner [1].

Fact 1.1 (Baumgartner [1]). *Let θ be a regular uncountable cardinal and $\trianglelefteq$ a well-ordering of H_θ. Let $\mathfrak{A} = \langle H_\theta, \in, \trianglelefteq, \ldots \rangle$ be a structure in a countable language, and suppose that M is an elementary submodel of $\mathfrak{A}$ and λ is a regular uncountable cardinal with $\lambda \in M$. Let $\delta = \sup(M \cap \lambda)$. Then $\mathrm{Hull}^{\mathfrak{A}}(M \cup \delta) \cap \lambda = \delta$.*

Proof. It suffices to prove that $\mathrm{Hull}^{\mathfrak{A}}(M \cup \delta) \cap \lambda \subseteq \delta$. Take an arbitrary $\alpha \in \mathrm{Hull}^{\mathfrak{A}}(M \cup \delta) \cap \lambda$. Then there are a formula $\varphi(v_0, v_1, v_2)$, $b \in [\delta]^{<\omega}$, and $p \in M$ such that α is the unique element with $\mathfrak{A} \models \varphi[\alpha, b, p]$. Take $\gamma \in M \cap \lambda$ with $b \in [\gamma]^{<\omega}$, and for each $a \in [\gamma]^{<\omega}$ let $h(a)$ be the least $\xi < \lambda$ with $\mathfrak{A} \models \varphi[\xi, a, p]$ if such ξ exists. Then h is a partial function from $[\gamma]^{<\omega}$ to λ, and $h \in M$ by the definability of h and the elementarity of M. Then

$$\alpha = h(b) < \sup(\mathrm{ran}(h)) \in M \cap \lambda.$$

Hence $\alpha \in \delta$. □

Next, we give our notation and facts relevant to singular cardinal combinatorics. Recall that the SCH is the statement that $\lambda^{\mathrm{cof}(\lambda)} = \lambda^+$ for all singular cardinals λ with $2^{\mathrm{cof}(\lambda)} < \lambda$. We say that the SCH fails at a singular cardinal λ if $2^{\mathrm{cof}(\lambda)} < \lambda$, and $\lambda^{\mathrm{cof}(\lambda)} > \lambda^+$. We use the following well-known theorem.

Fact 1.2 (Silver [11]). *Suppose that λ is the least singular cardinal at which* SCH *fails. Then* $\mathrm{cof}(\lambda) = \omega$.

We also use Shelah's PCF theory. Since we will only be working with singular cardinals of cofinality ω, we make the relevant definitions only in this case. Let $\vec{\lambda} = \langle \lambda_n : n \in \omega \rangle$ be a strictly increasing sequence of regular cardinals, and let $\lambda = \sup_{n\in\omega} \lambda_n$. We let $\prod \vec{\lambda}$ denote $\prod_{n\in\omega} \lambda_n$. For a set x of ordinals with $|x| < \lambda_0$, let $\chi_x^{\vec{\lambda}} \in \prod \vec{\lambda}$ be the characteristic function of x, i.e., $\chi_x^{\vec{\lambda}}(n) = \sup^+(x \cap \lambda_n)$ for each $n \in \omega$. We will omit the superscript $\vec{\lambda}$ in $\chi_x^{\vec{\lambda}}$ if it is clear from the context.

For functions $f, g : \omega \to \mathrm{On}$, we use the following notation:

$$f < g \overset{\mathrm{def}}{\Leftrightarrow} \forall n\, f(n) < g(n)$$
$$f <^* g \overset{\mathrm{def}}{\Leftrightarrow} \exists m\, \forall n \geqslant m f(n) < g(n)$$
$$f =^* g \overset{\mathrm{def}}{\Leftrightarrow} \exists m\, \forall n \geqslant m f(n) = g(n).$$

Moreover, for $m < \omega$, we use the following:

$$f <_m g \overset{\text{def}}{\Leftrightarrow} \forall n \geqslant m f(n) < g(n)$$
$$f =_m g \overset{\text{def}}{\Leftrightarrow} \forall n \geqslant m f(n) = g(n).$$

$f \leqslant g, f \leqslant^* g$, and $f \leqslant_m g$ are defined in the same way as $f < g, f <^* g$, and $f <_m g$.

A $<^*$-increasing cofinal sequence in $\prod \vec{\lambda}$ of length λ^+ is called a *scale* on $\vec{\lambda}$. A scale $(f_\beta : \beta < \lambda^+)$ is called a *better scale* if for any $\alpha < \lambda^+$ of uncountable cofinality there exists a club $C \subseteq \alpha$ and $\sigma : C \to \omega$ such that, for any $\beta, \gamma \in C$ with $\beta < \gamma$, we have $f_\beta <_{\max\{\sigma(\beta),\sigma(\gamma)\}} f_\gamma$. We use the following fact.

Fact 1.3 (Shelah [9]). *Suppose that λ is a singular cardinal of cofinality ω such that $\mu^\omega < \lambda$ for all $\mu < \lambda$ and such that $\lambda^\omega > \lambda^+$. Then there exists a strictly increasing sequence of regular cardinals of length ω which converges to λ and on which a better scale exists.*

2. Failure of weak square

It is known, due to Veličković [13], that SR implies the failure of $\square(\lambda)$ for all regular $\lambda \geqslant \omega_2$. Recall that $\square(\lambda)$ says that there is a sequence $(C_\alpha : \alpha \in \lim(\lambda))$ such that:

(i) C_α is a club subset of α, for all α,
(ii) if $\alpha \in \lim(C_\beta)$, then $C_\alpha = C_\beta \cap \alpha$,
(iii) there are no club $C \subseteq \lambda$ with $C \cap \alpha = C_\alpha$ for all $\alpha \in \lim(C)$.

In this section, we prove that SSR also implies the negation of $\square(\lambda)$ for all regular $\lambda \geqslant \omega_2$.

Theorem 2.1. *Assume that λ is a regular cardinal $\geqslant \omega_2$ and that* SSR(λ) *holds. Then* $\square(\lambda)$ *fails.*

Our proof is based on that in [13]. To prove Theorem 2.1, we need several preliminaries.

First, we give a modification of SSR, which is also used in § 4. For countable sets x and y, we write $x \sqsubseteq^* y$ if

(i) $x \sqsubseteq y$,
(ii) $\sup^+(x) = \sup^+(y)$,
(iii) $\sup^+(x \cap \gamma) = \sup^+(y \cap \gamma)$ for all $\gamma \in E^\lambda_{\omega_1} \cap x$.

Given $X \subseteq [\lambda]^\omega$, for some $\lambda \geqslant \omega_1$, we say that X is *weakly full* if X is upward closed under $\sqsubseteq^*$.

Lemma 2.2. *Assume that $\lambda \geqslant \omega_2$ and that* SSR(λ) *holds. Then, for any weakly full stationary $X \subseteq [\lambda]^\omega$ there exists $I \in [\lambda]^{\omega_1}$ including ω_1 such that $X \cap [J]^\omega$ is stationary for all $J \subseteq \lambda$ such that $I \subseteq J$ and* $\sup^+(J) = \sup^+(I)$.

Proof. Let X be a weakly full stationary subset of $[\lambda]^\omega$. Take a sufficiently large regular cardinal θ and a well-ordering $\trianglelefteq$ of H_θ, and let $\mathfrak{A} = (H_\theta, \in, \trianglelefteq, \lambda)$. Let Y be the set of

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all $y \in X$ with $\mathrm{Hull}^{\mathfrak{A}}(y) \cap \lambda = y$, and let $\overline{Y}$ be the upward closure of Y under $\sqsubseteq$. By SSR(λ), there is $I' \in [\lambda]^{\omega_1}$ such that $\omega_1 \subseteq I'$ and $\overline{Y} \cap [I']^\omega$ is stationary. Let I be one of such I' with the least $\sup^+$. We show that I witnesses the lemma for X. To this end, we make a preliminary definition. Let

$$Z_0 = \{z \in [I]^\omega : \exists y \in Y, y \sqsubseteq z \wedge \sup^+(y) = \sup^+(z)\}.$$

Claim. *Z_0 is stationary in $[I]^\omega$.*

Proof. Assume not. Then

$$Z = \{z \in [I]^\omega : \exists y \in Y, y \sqsubseteq z \wedge \sup^+(y) < \sup^+(z)\}$$

is stationary in $[I]^\omega$. For each $z \in Z$, choose $y_z \in Y$ such that $y_z \sqsubseteq z$ and $\sup^+(y_z) < \sup^+(z)$, and let $\beta_z = \min(z \setminus \sup^+(y_z))$. Note that $\beta_z \geqslant \omega_1$. By Fodor's Lemma, we can find β such that $Z' = \{z \in Z : \beta_z = \beta\}$ is stationary in $[I]^\omega$. Let $I' = I \cap \beta$. Then $\{z \cap \beta : z \in Z'\}$ is stationary in $[I']^\omega$. Moreover, $z \cap \beta \in \overline{Y}$ for each $z \in Z'$, because $z \cap \beta \sqsupseteq y_z$. So $\overline{Y} \cap [I']^\omega$ is stationary. Note also that $\omega_1 \subseteq I'$ and $|I'| = \omega_1$. But $\sup^+(I') < \sup^+(I)$. This contradicts the choice of I. □

Now we prove that I witnesses the lemma for X. Take an arbitrary $J \subseteq \lambda$ with $I \subseteq J$ and $\sup^+(J) = \sup^+(I)$. Let

$$Z_1 = \{z \in [J]^\omega : z \cap I \in Z_0 \wedge \sup^+(z \cap I) = \sup^+(z) \wedge \mathrm{Hull}^{\mathfrak{A}}(z) \cap \omega_1 = z \cap \omega_1\}.$$

Then Z_1 is stationary in $[J]^\omega$, because Z_0 is stationary, $\sup^+(J) = \sup^+(I)$, and $\omega_1 \subseteq J$. We show that $Z_1 \subseteq X$. In order to see this, take an arbitrary $z \in Z_1$. We prove that $z \in X$. First, we can take $y \in Y$ with $y \sqsubseteq z$ and $\sup^+(y) = \sup^+(z)$. Recall that $Y \subseteq X$ and that X is closed under $\sqsubseteq^*$. So it suffices to prove that $y \sqsubseteq^* z$. For this, all we have to show is that $\sup^+(y \cap \gamma) \geqslant \sup^+(z \cap \gamma)$ for every $\gamma \in E^\lambda_{\omega_1} \cap y$. Suppose that $\gamma \in E^\lambda_{\omega_1} \cap y$. Let $M = \mathrm{Hull}^{\mathfrak{A}}(y)$ and $N = \mathrm{Hull}^{\mathfrak{A}}(z)$. Note that $M \cap \omega_1 = y \cap \omega_1 = z \cap \omega_1 = N \cap \omega_1$. Then, since $\gamma \in M \subseteq N$ and $\mathrm{cof}(\gamma) = \omega_1$, we have that $\sup^+(M \cap \gamma) = \sup^+(N \cap \gamma)$. Moreover, $\sup^+(z \cap \gamma) \leqslant \sup^+(N \cap \gamma)$, and $\sup^+(y \cap \gamma) = \sup^+(M \cap \gamma)$ by the definition of Y. Hence $\sup^+(y \cap \gamma) \geqslant \sup^+(z \cap \gamma)$. □

Next, we present a game which will be used to construct a weakly full stationary set. Let λ be a regular cardinal $\geqslant \omega_2$. For a function $F : [\lambda]^{<\omega} \to \lambda$, let $G_1(\lambda, F)$ be the following game of length ω:

I	α_0	γ_0	α_1	γ_1	$\cdots$	α_n	γ_n	$\cdots$
II	β_0		β_1		$\cdots$	β_n		$\cdots$

I and II in turn choose ordinals $< \lambda$. In the nth stage, first I chooses α_n, then II chooses β_n, and then I again chooses $\gamma_n > \alpha_n, \beta_n$ of cofinality ω_1. I wins if

$$\mathrm{cl}_F(\{\gamma_n : n \in \omega\}) \cap [\alpha_m, \gamma_m) = \emptyset$$

for every $m \in \omega$. Otherwise, II wins.

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Lemma 2.3. *Let λ be a regular cardinal $\geqslant \omega_2$ and F be a function from $[\lambda]^{<\omega}$ to λ. Then I has a winning strategy for the game $G_1(\lambda, F)$.*

Proof. Since $G_1(\lambda, F)$ is an open game for player I, by the Gale–Stewart Theorem, one of the players has a winning strategy. Assume towards contradiction that II has a winning strategy, say τ. We will find a play $(\alpha_n, \beta_n, \gamma_n : n \in \omega)$ in which II follows τ, but which is won by I.

Let θ be a sufficiently large regular cardinal. First, build an $\in$-chain $\{M_n : n < \omega\}$ of elementary submodels of H_θ containing F and τ as elements and such that $\gamma_n = M_n \cap \lambda$ is an ordinal $< \lambda$ of cofinality ω_1. Let $x = \mathrm{cl}_F(\{\gamma_n : n \in \omega\})$ and $\alpha_n = \sup(x \cap \gamma_n)$, for each n. Note that $\alpha_n < \gamma_n$, since x is countable and $\mathrm{cof}(\gamma_n) = \omega_1$. Finally, let $(\beta_n : n \in \omega)$ be a sequence of II's moves according to τ against $(\alpha_n, \gamma_n : n \in \omega)$. Note that $\beta_n < \gamma_n$, since $\alpha_0, \gamma_0, \ldots, \alpha_{n-1}, \gamma_{n-1}, \alpha_n \in M_n$ and M_n is an elementary submodel of H_θ containing τ. Now $(\alpha_n, \beta_n, \gamma_n : n \in \omega)$ is a legal play of $G_1(\lambda, F)$ in which II has followed τ. However, $x \cap [\alpha_n, \gamma_n) = \emptyset$, for each n, by the definition of the α_n. Therefore I wins this play, a contradiction. It follows that I has a winning strategy in $G_1(\lambda, F)$, as required. □

Now we prove Theorem 2.1.

Proof of Theorem 2.1. Assuming that $\square(\lambda)$ holds, we prove that $\mathrm{SSR}(\lambda)$ fails. Let $\vec{C} = (C_\alpha : \alpha \in \lim(\lambda))$ be a $\square(\lambda)$-sequence. Let X be the set of all $x \in [\lambda]^\omega$ which have limit order type and there is $\xi < \sup^+(x)$ such that

(1) $\sup(x \cap C_{\sup^+(x)}) \leqslant \xi$,

(2) $\mathrm{cof}(\min(x \setminus \beta)) = \omega_1$, for all $\beta \in C_{\sup^+(x)} \setminus \xi$.

Here, note that X is weakly full. So it suffices to prove the following claims.

Claim 1. *X is stationary in $[\lambda]^\omega$.*

Proof. Take an arbitrary function $F : [\lambda]^{<\omega} \to \lambda$. We find $x \in X$ closed under F. By Lemma 2.3, fix a winning strategy τ of I for $G_1(\lambda, F)$. Let C be the set of all limit ordinals $\beta < \lambda$ closed under τ and F. Note that C is club in λ.

Subclaim. *There is $\delta \in \lim(C) \cap E^\lambda_\omega$ such that $C \cap \delta \setminus C_\delta$ is unbounded in δ.*

Proof. Assume otherwise. Then, by the Pressing Down Lemma, we can find a stationary subset S of $\lim(C) \cap E^\lambda_\omega$ and $\xi < \lambda$ such that $C \cap (\xi, \delta) \subseteq C_\delta$, for all $\delta \in S$. Suppose that $\alpha, \beta \in \lim(C) \setminus (\xi + 1)$ and $\alpha < \beta$. Since S is unbounded in λ, there is $\delta \in S \setminus \beta$. Since $C \cap (\xi, \delta) \subseteq C_\delta$, it follows that both α and β are limit points of C_δ. By condition (ii) of the definition of $\square(\lambda)$, we have that $C_\alpha = C_\delta \cap \alpha$ and $C_\beta = C_\delta \cap \beta$. Therefore $C_\alpha = C_\beta \cap \alpha$. Let

$$D = \bigcup \{C_\alpha : \alpha \in C \setminus (\xi + 1)\}.$$

It follows that $D \cap \alpha = C_\alpha$, for all $\alpha \in \lim(D)$, which contradicts the fact that $\vec{C}$ is a $\square(\lambda)$-sequence. □

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Now, let us fix $\delta \in \lim(C) \cap E^\lambda_\omega$ such that $C \cap \delta \setminus C_\delta$ is unbounded in δ. Take a strictly increasing sequence $(\delta_n : n \in \omega)$ in $C \cap \delta \setminus C_\delta$ which is cofinal in δ. For each $n \in \omega$, we can take $\beta_n < \delta_n$ such that $[\beta_n, \delta_n) \cap C_\delta = \emptyset$, because δ_n is a limit ordinal which is not in C_δ. Then, let $(\alpha_n, \gamma_n : n \in \omega)$ be a sequence of I's moves according to τ against $(\beta_n : n \in \omega)$. Moreover, let $x = \mathrm{cl}_F(\{\gamma_n : n \in \omega\})$. It suffices to prove that $x \in X$.

To see this, first note that $\sup^+(x) = \delta$, because δ is closed under F. Next note that $\alpha_{n+1} < \delta_n$ for each $n \in \omega$, because δ_n is closed under τ. Moreover, $C_\delta \cap \delta_{n+1} \subseteq \beta_{n+1} \subseteq \gamma_{n+1}$ by the choice of β_{n+1}. Hence $C_\delta \cap [\delta_n, \delta_{n+1}) \subseteq [\alpha_{n+1}, \gamma_{n+1})$ for every $n \in \omega$. Note that $x \cap [\alpha_{n+1}, \gamma_{n+1}) = \emptyset$ for each $n \in \omega$, because I wins with the play $(\alpha_n, \beta_n, \gamma_n : n \in \omega)$. Thus $x \cap C_\delta \subseteq \delta_0$. Moreover, $\min(x \setminus \beta) = \gamma_{n+1}$ for all $\beta \in C_\delta \cap [\delta_n, \delta_{n+1})$, and $\mathrm{cof}(\gamma_{n+1}) = \omega_1$ by the rule of $G_1(\lambda, F)$. Therefore $\xi = \delta_0$ witnesses that $x \in X$. □

Claim 2. *The conclusion of Lemma 2.2 fails for X.*

Proof. It suffices to prove that $X \cap [\delta]^\omega$ is non-stationary for every ordinal $\delta \in \lambda \setminus \omega_1$. If δ is a successor ordinal, then $X \cap [\delta]^\omega$ is clearly non-stationary. Next, suppose that $\mathrm{cof}(\delta) = \omega$. Let Z_0 be the set of all $z \in [\delta]^\omega$ such that $\sup^+(z) = \delta$ and such that $z \cap C_\delta$ is unbounded in δ. Then Z_0 is club in $[\delta]^\omega$, and $X \cap Z_0 = \emptyset$. Thus $X \cap [\delta]^\omega$ is non-stationary.

Finally, suppose that $\mathrm{cof}(\delta) > \omega$. Let Z_1 be the set of all $z \in [\delta]^\omega$ such that $\sup^+(z) \in \lim(C_\delta)$ and such that $z \cap C_\delta$ is unbounded in $\sup^+(z)$. Then Z_1 is club in $[\delta]^\omega$. Here, note that $z \cap C_{\sup^+(z)}$ is unbounded in $\sup^+(z)$ for each $z \in Z_1$, because $C_\delta \cap \sup^+(z) = C_{\sup^+(z)}$. So $X \cap Z_1 = \emptyset$. □

This concludes the proof of Theorem 2.1. □

3. ITP and TP

In this section, we prove that SR + MA_{ω_1}(Cohen) implies the super tree property at ω_2 and that SSR + MA_{ω_1}(Cohen) implies the strong tree property at ω_2. Here, note that the tree property at ω_2 implies the failure of CH and that SR and SSR are consistent with CH. So SR or SSR alone does not imply the super or strong tree property at ω_2, respectively. We also prove that SSR + MA_{ω_1}(Cohen) does not imply the super tree property at ω_2.

Theorem 3.1. (a) *If* SR *and* MA_{ω_1}(Cohen) *hold, then ω_2 has the super tree property.*
(b) *If* SSR *and* MA_{ω_1}(Cohen) *hold, then ω_2 has the strong tree property.*

Before we prove Theorem 3.1, we introduce some notation. For an ordinal $\lambda \geqslant \omega_2$ and a set M, let

$$u^\lambda_M = \bigcup([\lambda]^{\omega_1} \cap M).$$

We will omit the superscript λ in u^λ_M if it is clear from the context. The following is a key lemma.

Lemma 3.2. *Assume* MA_{ω_1}(Cohen). *Let λ be an ordinal $\geqslant \omega_2$ and $\mathscr{F}$ be a thin (ω_2, λ)-tree. Let θ be a sufficiently large regular cardinal. Then there are stationarily many $M \in [H_\theta]^\omega$ such that, for all $f \in \mathrm{lev}_{u_M}(\mathscr{F})$, exactly one of the following holds.*

(1) *There exists* $b \in {}^{\lambda}2 \cap M$ *with* $b \restriction u_M = f$.
(2) *There exists* $u \in [\lambda]^{\omega_1} \cap M$ *with* $f \restriction u \notin M$.

To prove this lemma, we need two further lemmas. First, let θ be a regular cardinal $\geqslant \omega_2$. For a function $F : [H_\theta]^{<\omega} \to H_\theta$ and $\xi < \omega_1$, let $G_2(\theta, F, \xi)$ be the following game of length ω:

I	J_0	J_1	$\cdots$	J_n	$\cdots$
II	K_0	K_1	$\cdots$	K_n	$\cdots$

In the nth stage, first I chooses $J_n \in [H_\theta]^{\omega_1}$ and then II chooses $K_n \in [H_\theta]^{\omega_1}$ with $K_n \supseteq J_n$. II wins if and only if

$$\mathrm{cl}_F(\xi \cup \{K_n : n \in \omega\}) \cap \omega_1 = \xi.$$

Lemma 3.3. *Let* θ *be a regular cardinal* $\geqslant \omega_2$. *Then, for any function* $F : [H_\theta]^{<\omega} \to H_\theta$ *there exists* $\xi < \omega_1$ *such that* II *has a winning strategy for* $G_2(\theta, F, \xi)$.

Proof. Take an arbitrary function $F : [H_\theta]^{<\omega} \to H_\theta$. Towards contradiction, assume that II does not have a winning strategy for $G_2(\theta, F, \xi)$, for any $\xi < \omega_1$. Since each $G_2(\theta, F, \xi)$ is an open–closed game, I has a winning strategy, say τ_ξ, in $G_2(\theta, F, \xi)$, for each ξ. Let $\vec{\tau} = (\tau_\xi : \xi < \omega_1)$. Take a sufficiently large regular cardinal μ and a countable elementary submodel M of H_μ containing θ, F, and $\vec{\tau}$. Let $\zeta = M \cap \omega_1$. By induction on n, let

$$J_n = \tau_\zeta((K_m : m < n)),$$
$$K_n = \bigcup_{\xi < \omega_1} \tau_\xi((K_m : m < n)).$$

Then $(J_n, K_n : n \in \omega)$ is a legal play of $G_2(\theta, F, \zeta)$ in which I has moved according to τ_ζ. Here, note that $K_n \in M$, for each n, by the elementarity of M. Moreover, $\zeta = M \cap \omega_1$ and M is closed under F. Hence $\mathrm{cl}_F(\zeta \cup \{K_n : n \in \omega\}) \subseteq M$, and thus $\mathrm{cl}_F(\zeta \cup \{K_n : n \in \omega\}) \cap \omega_1 = \zeta$. Therefore II wins the play $(J_n, K_n : n \in \omega)$ in $G_2(\theta, F, \zeta)$, which contradicts the fact that τ_ζ is a winning strategy of I. □

The second one is a lemma on very thin (ω_2, λ)-trees.

Lemma 3.4. *Let* λ *be an ordinal* $\geqslant \omega_2$ *and* $\mathscr{F}$ *be an* (ω_2, λ)*-tree such that* $\mathrm{lev}_u(\mathscr{F})$ *is countable for every* $u \in [\lambda]^{\leqslant \omega_1}$. *Then there is a countable subset* $\mathscr{B}$ *of* ${}^{\lambda}2$ *and a club* C *in* $[\lambda]^{\leqslant \omega_1}$ *such that, for any* $u \in C$ *and* $f \in \mathrm{lev}_u(\mathscr{F})$, *there is a unique* $b \in \mathscr{B}$ *with* $b \restriction u = f$.

Proof. Let θ be a sufficiently large regular cardinal and W_0 be the set of all elementary submodels K of H_θ which have cardinality $\aleph_1$ and are internally approachable of length ω_1. Note that W_0 is stationary in $[H_\theta]^{\leqslant \omega_1}$. For each $K \in W_0$, we can find $x_K \in [K \cap \lambda]^{\leqslant \omega_1} \cap K$ such that $f_0 \restriction x_K \neq f_1 \restriction x_K$ for any distinct $f_0, f_1 \in \mathrm{lev}_{K \cap \lambda}(\mathscr{F})$. To see this, first note that, since $\mathrm{lev}_{K \cap \lambda}(\mathscr{F})$ is countable, there is a countable subset y_K of $K \cap \lambda$ such that $f_0 \restriction y_K \neq f_1 \restriction y_K$, for all distinct $f_0, f_1 \in \mathrm{lev}_{K \cap \lambda}(\mathscr{F})$. Then, since K is internally approachable of length ω_1, we can find $x_K \in K$ such that $y_K \subseteq x_K$. Now, by the Pressing Down Lemma, there is $x \in [\lambda]^{\leqslant \omega_1}$ such that $W_1 = \{K \in W_0 : x_K = x\}$ is

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stationary. Let $U = \{K \cap \lambda : K \in W_1\}$. Note that U is stationary in $[\lambda]^{\leqslant\omega_1}$. Let A be the set of all $h \in \mathrm{lev}_x(\mathscr{F})$ such that, for every $u \in U$, there is $f \in \mathrm{lev}_u(\mathscr{F})$ with $f \restriction x = h$. For each $h \in A$ and $u \in U$, let f^h_u be the unique element of $\mathrm{lev}_u(\mathscr{F})$ with $f^h_u \restriction x = h$. Here, note that, if $h \in A$, and $u, v \in U$, then $f^h_u \restriction (u \cap v) = f^h_v \restriction (u \cap v)$. This is because, if we take $w \in U$ with $u, v \subseteq w$, then $f^h_u = f^h_w \restriction u$, and $f^h_v = f^h_w \restriction v$. For each $h \in A$, let $b_h = \bigcup\{f^h_u : u \in U\} \in {}^\lambda 2$, and let $\mathscr{B} = \{b_h : h \in A\}$. Clearly, $\mathscr{B}$ is a countable subset of ${}^\lambda 2$. Note that $b_h \restriction u \in \mathrm{lev}_u(\mathscr{F})$ for all $u \in [\lambda]^{\leqslant\omega_1}$, since U is $\subseteq$-cofinal in $[\lambda]^{\leqslant\omega_1}$. Let D be the collection of all elementary submodels K of H_θ which have size $\aleph_1$ and contain all the relevant objects. Then D is a club in $[H_\theta]^{\omega_1}$ and $C = \{K \cap \lambda : K \in D\}$ is a club in $[\lambda]^{\omega_1}$. We claim that $\mathscr{B}$ and C are as desired. In order to see this, fix $K \in D$ and $f \in \mathrm{lev}_{K\cap\lambda}(\mathscr{F})$. Let $h = f \restriction x$. Then $h \in \mathrm{lev}_x(\mathscr{F}) \subseteq K$, and $f \restriction u$ witnesses that there is $g \in \mathrm{lev}_u(\mathscr{F})$ with $g \restriction x = h$, for every $u \in U \cap K$. So it follows from the elementarity of K that $h \in A$. Moreover, $f \restriction u = f^h_u$ for all $u \in U \cap K$, and $\bigcup(U \cap K) = K \cap \lambda$ by the elementarity of K. Therefore $f = \bigcup_{u \in U\cap K} f^h_u = b_h \restriction (K \cap \lambda)$. □

Proof of Lemma 3.2. Take an arbitrary function $F : [H_\theta]^{<\omega} \to H_\theta$. We find a countable elementary submodel M of $[H_\theta]$ closed under F such that, for any $f \in \mathrm{lev}_{u_M}(\mathscr{F})$, either (1) or (2) in Lemma 3.2 holds. Let $\trianglelefteq$ be a well-ordering of H_θ. By changing F if necessary, we may assume that, if a subset M of H_θ is closed under F, then M is an elementary submodel of $(H_\theta, \in, \trianglelefteq)$ and contains λ and $\mathscr{F}$. By Lemma 3.3, let $\xi < \omega_1$ be such that II has a winning strategy, say τ, for $G_2(\theta, F, \xi)$. Moreover, take a sufficiently large regular cardinal μ and a countable elementary submodel N of H_μ containing all the relevant objects. The desired M will be a subset of N and will be obtained by applying MA_{ω_1}(Cohen) to an appropriate poset.

Let $\mathscr{P}$ be the set of partial plays of the form $p = \langle J^p_0, K^p_0, \ldots, J^p_{n_p-1}, K^p_{n_p-1}\rangle$ in the game $G_2(\theta, F, \xi)$ in which II follows his/her winning strategy τ. We call the integer n_p the *length* of p. Moreover, let $u^p_i = K^p_i \cap \lambda$, for all $i < n_p$. We order $\mathscr{P}$ by reverse end extension. We will apply MA_{ω_1}(Cohen) to the poset $\mathscr{P}_N = \mathscr{P} \cap N$. Note that, since N is countable, so is $\mathscr{P}_N$.

Given $K \in [H_\theta]^{\omega_1} \cap N$, let

$$D_K = \{p \in \mathscr{P}_N : K \subseteq K^p_i, \text{ for some } i < n_p\}.$$

Then D_K is dense in $\mathscr{P}_N$, for all such K. Next, for $u \in [\lambda]^{\leqslant\omega_1}$, let $\langle f^u_\zeta : \zeta < \omega_1\rangle$ be the $\trianglelefteq$-least enumeration of $\mathrm{lev}_u(\mathscr{F})$, and let $A(u) = \{f^u_\zeta : \zeta < \xi\}$. Note that $\langle A(u) : u \in [\lambda]^{\leqslant\omega_1}\rangle \in N$. Then, for each $f \in \mathrm{lev}_{u_N}(\mathscr{F})$, let

$$E_f = \{p \in \mathscr{P}_N : f \restriction u^p_i \notin A(u^p_i), \text{ for some } i < n_p\}.$$

Claim. *Suppose that $f \in \mathrm{lev}_{u_N}(\mathscr{F})$ and that E_f is not dense in $\mathscr{P}_N$. Then there is $b \in {}^\lambda 2 \cap N$ such that $b \restriction u_N = f$.*

Proof. Let $f \in \mathrm{lev}_{u_N}(\mathscr{F})$ be such that E_f is not dense in $\mathscr{P}_N$. We find $b \in {}^\lambda 2 \cap N$ such that $b \restriction u_N = f$. Fix $p \in \mathscr{P}_N$ which has no extensions in E_f. Let

$$W = \{K \in [H_\theta]^{\omega_1} : p^\frown\langle J, K\rangle \in \mathscr{P}, \text{ for some } J\},$$

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and let $U = \{K \cap \lambda : K \in W\}$. Note that $W, U \in N$ and that W and U are $\subseteq$-cofinal in $[H_\theta]^{\omega_1}$ and $[\lambda]^{\omega_1}$, respectively. Note also that, by the choice of p, $f \restriction u \in A(u)$, for all $u \in U \cap N$. Let $\mathscr{G}$ be the set of all $g \in \mathscr{F}$ such that

(i) $g \restriction u \in A(u)$, for all $u \in U$ with $u \subseteq \mathrm{dom}(g)$,

(ii) for any $u \in U$ with $\mathrm{dom}(g) \subseteq u$, there exists $h \in A(u)$ with $g \subseteq h$.

Since all the parameters in the definition of $\mathscr{G}$ are in N and N is elementary in H_μ, it follows that $\mathscr{G} \in N$. Note also that $f \restriction v \in \mathscr{G}$ for all $v \in [\lambda]^{\leqslant\omega_1} \cap N$, by the fact that $f \restriction u \in A_u$ for all $u \in U \cap N$ and the elementarity of N. It follows that $\mathrm{lev}_u(\mathscr{G})$ is nonempty, for all $u \in [\lambda]^{\leqslant\omega_1}$, again by the elementarity of N. Clearly, $\mathscr{G}$ is closed under restrictions. So $\mathscr{G}$ is an (ω_2, λ)-tree. Moreover, all the levels of $\mathrm{lev}_u(\mathscr{G})$ are countable. Let $\mathscr{B} \subseteq^\lambda 2$ and $C \subseteq [\lambda]^{\leqslant\omega_1}$ be those obtained by applying Lemma 3.4 for $\mathscr{G}$. We may assume that $\mathscr{B}, C \in N$ by the elementarity of N. Take an $\subseteq$-increasing sequence $(u_n)_n$ of elements of $C \cap N$ such that $\bigcup\{u_n : n < \omega\} = u_N$. Moreover, for each n, let b_n be the unique element of $\mathscr{B}$ with $b_n \restriction u_n = f \restriction u_n$. Note that $b_m = b_n$ for each m, n by the uniqueness. Therefore $b_0 \restriction u_N = f$. Moreover, $b_0 \in N$, since $\mathscr{B} \in N$ and $\mathscr{B}$ is countable. Therefore b_0 is as desired. □

Now, let $\mathscr{D}$ be the set of the D_K, for $K \in [H_\theta]^{\omega_1} \cap N$, and $\mathscr{E}$ the set of the E_f, for $f \in \mathrm{lev}_{u_N}(\mathscr{F})$, such that there is no $b \in^\lambda 2 \cap N$ with $b \restriction u_N = f$. Then $\mathscr{D}$ and $\mathscr{E}$ are dense subsets of $\mathscr{P}_N$. Moreover, $\mathscr{D}$ is countable, and $\mathscr{E}$ has cardinality at most $\aleph_1$, since the cardinality of $\mathrm{lev}_{u_N}(\mathscr{F})$ is at most $\aleph_1$. By MA_{ω_1}(Cohen), we can find a filter G in $\mathscr{P}_N$ which meets all the sets of $\mathscr{D} \cup \mathscr{E}$. Let $r_G = \bigcup G$. Then r_G is an infinite run of the game $G_2(\theta, F, \xi)$ in which II follows τ and therefore wins. Let us say that $r_G = \langle J_0, K_0, J_1, K_1, \ldots\rangle$, and let $u_n = K_n \cap \lambda$, for all n. Let

$$M = \mathrm{cl}_F(\xi \cup \{K_n : n \in \omega\}).$$

We show that this M is as desired. Prior to this, note that M is an elementary submodel of $(H_\theta, \in, \trianglelefteq)$ and contains λ and $\mathscr{F}$. Moreover, $M \cap \omega_1 = \xi$, since II wins the play r_G. Moreover, $u_M = u_N$, since G meets all the dense sets in $\mathscr{D}$.

Now, $f \in \mathrm{lev}_{u_M}(\mathscr{F})$, and suppose first that (1) fails for f, i.e., there is no $b \in^\lambda 2 \cap M$ such that $b \restriction u_M = f$. Then $G \cap E_f \neq \emptyset$. Let n be such that $f \restriction u_n \notin A(u_n)$. Note that $A(u_n) = \mathrm{lev}_{u_n}(\mathscr{F}) \cap M$, since $u_n = K_n \cap \lambda \in M$ and $M \cap \omega_1 = \xi$. So $f \restriction u_n \notin M$. Thus u_n witnesses (2) for f. Next, suppose that there exists $b \in^\lambda 2 \cap N$ such that $b \restriction u_M = f$. Then we can find an integer n such that $b \in K_n$, by the $\mathscr{D}$-genericity of G. Since $K_n \in M \cap [H_\theta]^{\omega_1}$, we can find $u \in [\lambda]^{\leqslant\omega_1} \cap M$ such that $c \restriction u \neq d \restriction u$ for any distinct $c, d \in^\lambda 2 \cap K_n$. Then $c \mapsto c \restriction u$ is an injection from ${}^\lambda 2 \cap K_n$ to ${}^u 2$, and this injection belongs to M. Hence $c \in M$ if and only if $c \restriction u \in M$, for any $c \in^\lambda 2 \cap K_n$. Here, note that $b \notin M$ by our assumption that (1) fails for f. Therefore $f \restriction u = b \restriction u \notin M$. □

Using Lemma 3.2, we prove Theorem 3.1.

Proof of Theorem 3.1. (a) Assume SR and MA_{ω_1}(Cohen). Let λ be an ordinal $\geqslant \omega_2$, $\mathscr{F}$ be a thin (ω_2, λ)-tree, and $\vec{f} = (f_u : u \in [\lambda]^{\leqslant\omega_1})$ be a level sequence of $\mathscr{F}$. We will find an ineffable branch for $\vec{f}$. Let θ be a sufficiently large regular cardinal, let $\trianglelefteq$ be a

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well-ordering of H_θ, and let $\mathfrak{A} = (H_\theta, \in, \trianglelefteq)$. Moreover, let Z be the set of all countable elementary submodels M of $\mathfrak{A}$ containing λ and $\mathscr{F}$ and such that either (1) or (2) of Lemma 3.2 holds, for any $f \in \mathrm{lev}_{u_M}(\mathscr{F})$. Then Z is stationary in $[H_\theta]^\omega$, by Lemma 3.2. Let W be the set of all $K \in [H_\theta]^{\omega_1}$ such that $\omega_1 \subseteq K$ and $Z \cap [K]^\omega$ is stationary in $[K]^\omega$. By SR, it follows that W is stationary in $[H_\theta]^{\leqslant\omega_1}$.

Claim. *For any $K \in W$, there is $b_K \in {}^\lambda 2 \cap K$ such that $b_K \restriction (K \cap \lambda) = f_{K\cap\lambda}$.*

Proof. Fix $K \in W$, and let $f = f_{K\cap\lambda}$. Note that K is an elementary submodel of $\mathfrak{A}$, since $Z \cap [K]^\omega$ is stationary. Then $\mathrm{lev}_u(\mathscr{F})$ is a subset of K, for all $u \in [\lambda]^{\leqslant\omega_1} \cap K$, since $\mathrm{lev}_u(\mathscr{F})$ is an element of K of size $\aleph_1$ and $\omega_1 \subseteq K$. Since $Z \cap [K]^\omega$ is stationary in $[K]^\omega$, we can find $M \in Z \cap [K]^\omega$ such that, letting $N = \mathrm{Hull}^{\mathfrak{A}}(M \cup \{f\})$, we have that $N \cap K = M$. Note that $f \restriction u \in M$, for all $u \in [\lambda]^{\leqslant\omega_1} \cap M$, since $f \restriction u \in K \cap N$, for every such u. So (2) of Lemma 3.2 fails for $f \restriction u_M$ and M. Hence (1) holds for $f \restriction u_M$ and M; that is, there is $b \in {}^\lambda 2 \cap M$ with $b \restriction u_M = f \restriction u_M$. Note that $b \in K$. Hence it suffices to show that $b \restriction (K \cap \lambda) = f$. Note that both $b \restriction (K \cap \lambda)$ and f are functions on $K \cap \lambda$ which are in N. Moreover, $b \restriction (K \cap \lambda)$ and f coincide on $N \cap (K \cap \lambda)$, since $N \cap (K \cap \lambda) = M \cap \lambda$, and $b \restriction u_M = f \restriction u_M$. Hence $b \restriction (K \cap \lambda) = f$, by the elementarity of N. □

By the Pressing Down Lemma, we can find $b \in {}^\lambda 2$ such that $b_K = b$, for stationarily many $K \in W$. It follows that b is an ineffable branch for $\vec{f}$.

(b) Assume SSR and MA_{ω_1}(Cohen). Let λ be an ordinal $\geqslant \omega_2$ and $\mathscr{F}$ be a thin (ω_2, λ)-tree. We will find a cofinal branch for $\mathscr{F}$. Let θ, $\trianglelefteq$, $\mathfrak{A}$, and Z be as in the proof of (a). Moreover, let Z^* be the upward closure of Z under $\sqsubseteq$. By SSR, there is $K \in [H_\theta]^{\omega_1}$ such that $\omega_1 \subseteq K$ and $Z^* \cap [K]^\omega$ is stationary in $[K]^\omega$. Here, note that $Z^* \cap [K^*]^\omega$ is stationary, for any $K^* \supseteq K$. Hence, by replacing K with $\mathrm{Hull}^{\mathfrak{A}}(K)$ if necessary, we may assume that K is an elementary submodel of $\mathfrak{A}$ and contains λ and $\mathscr{F}$ as elements. Pick any $f \in \mathrm{lev}_{K\cap\lambda}(\mathscr{F})$. Then $f \restriction u \in K$, for all $u \in [\lambda]^{\leqslant\omega_1} \cap K$. So we can take $M^* \in Z^* \cap [K]^\omega$, which is an elementary submodel of $\mathfrak{A}$, contains λ and $\mathscr{F}$ as elements, and is such that $f \restriction u \in M^*$, for all $u \in [\lambda]^{\leqslant\omega_1} \cap M^*$. Let $M \in Z$ be such that $M \sqsubseteq M^*$. Here, note that $\mathrm{lev}_u(\mathscr{F}) \cap M = \mathrm{lev}_u(\mathscr{F}) \cap M^*$, for any $u \in [\lambda]^{\leqslant\omega_1} \cap M$, since both M and M^* are elementary submodels of $\mathfrak{A}$, $\mathrm{lev}_u(\mathscr{F})$ is of size $\aleph_1$, and $M \cap \omega_1 = M^* \cap \omega_1$. Hence $f \restriction u \in M$, for all $u \in [\lambda]^{\leqslant\omega_1} \cap M$, and so (2) of Lemma 3.2 fails for $f \restriction u_M$ and M. Thus there is $b \in {}^\lambda 2 \cap M$ with $b \restriction u_M = f \restriction u_M$. Then $b \restriction u \in \mathrm{lev}_u(\mathscr{F})$, for all $u \in [\lambda]^{\leqslant\omega_1} \cap M$. So $b \restriction u \in \mathrm{lev}_u(\mathscr{F})$, for all $u \in [\lambda]^{\leqslant\omega_1}$, by the elementarity of M. So, b is a cofinal branch of $\mathscr{F}$, as required. □

We now show that the conjuction of SSR and MA_{ω_1}(Cohen) is not sufficient to imply the super tree property for ω_2.

Theorem 3.5. *Assume that there exists a strongly compact cardinal. Then there exists a forcing extension in which* SSR *and* MA_{ω_1}(Cohen) *hold but ω_2 does not have the super tree property.*

Theorem 3.5 follows easily from the following facts.

Fact 3.6 (Magidor). *Assume that κ is a supercompact cardinal. Then there is a forcing extension in which κ is strongly compact but not supercompact.*

Fact 3.7 (Shelah [10, Chapter XIII, 1.6 and 1.10]). *Assume that κ is a strongly compact cardinal. Let $(\mathscr{P}_\alpha, \dot{\mathscr{Q}}_\beta : \alpha \leqslant \kappa, \beta < \kappa)$ be a revised countable support iteration of semi-proper posets of size $< \kappa$ such that $\kappa = \omega_2$ in $V^{\mathscr{P}_\kappa}$. Then* SSR *holds in $V^{\mathscr{P}_\kappa}$.*

Fact 3.8 (Viale–Weiss [15]). *Assume that κ is an inaccessible cardinal. Assume also that there exists a countable support iteration $(\mathscr{P}_\alpha, \dot{\mathscr{Q}}_\beta : \alpha \leqslant \kappa, \beta < \kappa)$ of proper posets of size $< \kappa$ such that κ has the super tree property in $V^{\mathscr{P}_\kappa}$. Then κ is supercompact in V.*

Proof of Theorem 3.5. Assume that κ is strongly compact in V. By Fact 3.6, we may assume that κ is not supercompact. Let $(\mathscr{P}_\alpha, \dot{\mathscr{Q}}_\beta : \alpha \leqslant \kappa, \beta < \kappa)$ be the countable support iteration of Cohen forcing. Here, recall that a revised countable support iteration coincides with a countable support iteration for proper posets. Note also that $\kappa = \omega_2$ in $V^{\mathbb{P}_\kappa}$. Hence SSR holds in $V^{\mathscr{P}_\kappa}$, by Fact 3.7. Moreover, MA_{ω_1}(Cohen) holds in $V^{\mathscr{P}_\kappa}$. By Fact 3.8, ω_2 does not have the super tree property in $V^{\mathscr{P}_\kappa}$. □

We end this section with some remarks. In Theorem 3.5, we have proved that $\mathrm{SSR} + \mathrm{MA}_{\omega_1}$(Cohen) does not imply the super tree property at ω_2. In fact, we can prove that it does not imply $\mathrm{ITP}(\omega_2, \omega_3)$. We outline the proof. For a regular uncountable cardinal κ, let

$$U_\kappa = \{u \in [\kappa^+]^{<\kappa} : u \cap \kappa \in \kappa \wedge \mathrm{otp}(u) = (u \cap \kappa)^+\}.$$

It is easy to see that, if κ is κ^+-supercompact, then U_κ is stationary in $[\kappa^+]^{<\kappa}$. On the other hand, Krueger [5] proved that this does not follow from the strong compactness of κ.

Fact 3.9 (Krueger [5]). *Assume that κ is a supercompact cardinal. Then there is a forcing extension in which κ is strongly compact, and U_κ is non-stationary.*

Moreover, by some extra work and using the ideas of [14, 15], we can prove the following.

Fact 3.10. *Assume that κ is an inaccessible cardinal. Assume also that there is a countable support iteration $(\mathscr{P}_\alpha, \dot{\mathscr{Q}}_\beta : \alpha \leqslant \kappa, \beta < \kappa)$ of proper posets of size $< \kappa$ such that $\mathrm{ITP}(\kappa, \kappa^+)$ holds in $V^{\mathscr{P}_\kappa}$. Then U_κ is stationary in $[\kappa^+]^{<\kappa}$ in V.*

Using Facts 3.9 and 3.10 instead of Facts 3.6 and 3.8, by the same argument as Theorem 3.5, we can prove that $\mathrm{SSR} + \mathrm{MA}_{\omega_1}$(Cohen) does not imply $\mathrm{ITP}(\omega_2, \omega_3)$.

On the other hand, we can also prove that $\mathrm{SSR} + \mathrm{MA}_{\omega_1}$(Cohen) implies $\mathrm{ITP}(\omega_2, \omega_2)$. Assume SSR, and suppose that $\mathscr{F}$ is a thin (ω_2, ω_2)-tree and that $\vec{f} = (f_u : u \in [\omega_2]^{\leqslant\omega_1})$ is a level sequence of $\mathscr{F}$. Let θ, $\trianglelefteq$, $\mathfrak{A}$, and Z be as in the proof of Theorem 3.1. Here, recall the fact, due to Foreman *et al.* [2], that SSR (equivalently (†)) implies Strong Chang's Conjecture. In fact it implies the following.

There is a club set of $M \in [H_\theta]^\omega$ such that $\mathrm{Hull}^{\mathfrak{A}}(M \cup \{\delta\}) \cap \delta = M \cap \omega_2$, for stationarily many $\delta \in \omega_2$.

Take such $M \in Z$, and let E be the set of all $\delta \in \omega_2$ with $\mathrm{Hull}^{\mathfrak{A}}(M \cup \{\delta\}) \cap \delta = M \cap \omega_2$. Then, by the same argument as in the proof of Theorem 3.1(a), for any $\delta \in E$ there is $b_\delta \in {}^{\omega_2}2 \cap M$ such that $b_\delta \restriction \delta = f_\delta$. Take $b \in {}^{\omega_2}2$ such that $\{\delta \in E : b_\delta = b\}$ is stationary. Then b is an ineffable branch for $\vec{f}$.

4. Singular Cardinal Hypothesis

In this section, we prove that SSR implies the SCH. In fact, we prove the following.

Theorem 4.1. *Assume that λ is a singular cardinal of cofinality ω and that* $\mathrm{SSR}(\lambda^+)$ *holds. Then, for any strictly increasing sequence $\vec{\lambda} = (\lambda_n : n < \omega)$ of regular cardinals converging to λ, there are no better scales on $\vec{\lambda}$.*

Assume Theorem 4.1 for a moment. We sketch how it is used to show that SSR implies the SCH. First, assuming SSR, we show by induction that $\kappa^\omega = \kappa$, for every regular $\kappa \geqslant \omega_2$. The base case of the induction follows from the fact, due to Foreman *et al.* [2], that SSR implies Strong Chang's Conjecture and the fact, due to Todorčević [12], that Strong Chang's Conjecture implies $2^\omega \leqslant \omega_2$. The only problem in the induction occurs if $\kappa = \lambda^+$, for some singular limit λ of cofinality ω. In this case, by the induction hypothesis we have that $\mu^\omega < \lambda$, for all $\mu < \lambda$. Then, by Theorem 4.1 and Fact 1.3, it follows that $\lambda^\omega = \lambda^+$. Therefore, $\kappa^\omega = \kappa$, for all regular $\kappa \geqslant \omega_2$. Now, combining this with Fact 1.2, we obtain the following.

Corollary 4.2. SSR *implies* SCH. □

To prove the theorem, we make some preliminary remarks. First, we present a game which is a variant of the game used in § 2.

Let $\vec{\lambda} = (\lambda_n : n \in \omega)$ be a strictly increasing sequence of regular cardinals $\geqslant \omega_2$, let $\lambda = \sup_{n\in\omega} \lambda_n$, and let $\vec{E} = (E_{n,i} : n \in \omega, i \in 2)$ be a sequence such that each $E_{n,i}$ is a stationary subset of $E^{\lambda_n}_{\omega_1}$. Moreover, let $\vec{b} = (b_\xi : \xi < \omega_1)$ be a sequence of functions from ω to 2. For a function $F : [\lambda^+]^{<\omega} \to \lambda^+$ and $\xi < \omega_1$, let $G_3(\vec{E}, \vec{b}, F, \xi)$ be the following game of length ω:

I	α_0	δ_0, ϵ_0	α_1	δ_1, ϵ_1	$\cdots$	α_n	δ_n, ϵ_n	$\cdots$
II	β_0, γ_0		β_1, γ_1		$\cdots$	β_n, γ_n		$\cdots$

In the nth stage, first I chooses $\alpha_n < \lambda_n$, then II chooses $\beta_n < \lambda_n$ and $\gamma_n < \lambda^+$. Then I again chooses $\delta_n > \alpha_n, \beta_n$ with $\delta_n \in E_{n,b_\xi(n)}$ and $\epsilon_n < \lambda^+$ with $\epsilon_n > \gamma_n$. I wins if, letting $x = \mathrm{cl}_F(\xi \cup \{\delta_n, \epsilon_n : n \in \omega\})$, we have

(i) $x \cap \omega_1 = \xi$,

(ii) $x \cap [\alpha_m, \delta_m) = \emptyset$, for every m.

Otherwise, II wins.

Lemma 4.3. *Let $\vec{\lambda}$, λ, $\vec{E}$, $\vec{b}$, and F be as above. Then there exists $\xi < \omega_1$ such that I has a winning strategy for $G_3(\vec{E}, \vec{b}, F, \xi)$.*

Proof. Towards a contradiction, assume that I does not have a winning strategy in $G_3(\vec{E}, \vec{b}, F, \xi)$, for every $\xi < \omega_1$. Since each $G_3(\vec{E}, \vec{b}, F, \xi)$ is an open–closed game, by the Gale–Stewart Theorem, II has a winning strategy, say τ_ξ, for all ξ. We will find $\zeta < \omega_1$ and a play $(\alpha_n, \beta_n, \gamma_n, \delta_n, \epsilon_n : n \in \omega)$ of $G_3(\vec{E}, \vec{b}, F, \zeta)$ in which II follows his/her strategy τ_ζ, yet I wins the game.

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Let $\vec{\tau} = (\tau_\xi : \xi < \omega_1)$. Take a sufficiently large regular cardinal θ. Then we can find a system $(K_{n,i} : n \in \omega, i \in 2)$ of elementary submodels of $(H_\theta, \in)$ containing $\vec{\lambda}$ and $\vec{\tau}$) such that $\delta_{n,i} = K_{n,i} \cap \lambda_n \in E_{n,i}$, for each n and i, and such that $K_{n,i} \in K_{n',i'}$ if $n < n'$ and $i, i' \in 2$. Let $\epsilon_{n,i} = \sup^+(K_{n,i} \cap \lambda^+)$, for each n and i. Then we can take $\zeta < \omega_1$ such that

$$\mathrm{cl}_F(\zeta \cup \{\delta_{n,i}, \epsilon_{n,i} : n \in \omega, i \in 2\}) \cap \omega_1 = \zeta.$$

For each n, let K_n, δ_n and ϵ_n be $K_{n,b_\zeta(n)}$, $\delta_{n,b_\zeta(n)}$ and $\epsilon_{n,b_\zeta(n)}$, respectively. Moreover, let

$$x = \mathrm{cl}_F(\zeta \cup \{\delta_n, \epsilon_n : n \in \omega\}),$$

and let $\alpha_n = \sup^+(x \cap \delta_n)$ for each n. Note that $\alpha_n < \delta_n$, since x is countable and $\mathrm{cof}(\delta_n) = \omega_1$. Finally, let $(\beta_n, \gamma_n : n \in \omega)$ be a sequence of II's moves according to τ_ζ against $(\alpha_n, \delta_n, \epsilon_n : n \in \omega)$. Note that $\zeta, \alpha_0, \delta_0, \epsilon_0, \ldots, \alpha_{n-1}, \delta_{n-1}, \epsilon_{n-1}, \alpha_n \in K_n$, and that K_n is an elementary submodel of H_θ, for all n. Hence $\beta_n \in K_n \cap \lambda_n = \delta_n$. Moreover, $\gamma_n \in K_n \cap \lambda^+$, and so $\gamma_n < \epsilon_n$. Thus $(\alpha_n, \beta_n, \gamma_n, \delta_n, \epsilon_n : n \in \omega)$ is a legal play of $G_3(\vec{E}, \vec{b}, F, \zeta)$ in which II moves according to his/her winning strategy τ_ζ. On the other hand, $x \cap [\alpha_m, \delta_m) = \emptyset$, for every m, by the choice of α_m and δ_m. Moreover, $x \cap \omega_1 = \zeta$ by the choice of ζ and x. Therefore I wins this play of the game. This is a contradiction. □

Next, we give a standard lemma on better scales.

Lemma 4.4. *Let $\vec{\lambda} = (\lambda_n : n < \omega)$ be a strictly increasing sequence of regular cardinals, and let $\lambda = \sup_{n\in\omega} \lambda_n$. Suppose that $\vec{f} = (f_\beta : \beta < \lambda^+)$ is a better scale on $\vec{\lambda}$. Then, for any regular $\theta > \lambda^+$, there are stationarily many $N \in [H_\theta]^\omega$ with $\chi_N \leqslant^* f_{\sup^+(N\cap\lambda^+)}$, where χ_N is the characteristic function of N (see § 1).*

Proof. Suppose that θ is a regular cardinal bigger than λ^+. It is sufficient to show that, for every expansion $\mathfrak{A}$ of $(H_\theta, \in)$, there is a countable elementary submodel N of $\mathfrak{A}$ such that $\chi_N \leqslant^* f_\rho$, where $\rho = \sup(N \cap \lambda^+)$. In order to find such an N, first take a continuous $\in$-chain $(N_\xi : \xi < \omega_1)$ of countable elementary submodels of $\mathfrak{A}$ containing all the relevant parameters. Let $\rho_\xi = \sup(N_\xi \cap \lambda^+)$. Then, since $\vec{f}$ is a better scale, we can find $m < \omega$ and a stationary $S \subseteq \omega_1$ such that $(f_{\rho_\xi} : \xi \in S)$ is $<_m$-increasing. Here, note that, if $\xi < \eta$, then $\chi_{N_\xi} <^* f_{\rho_\eta}$, since $N_\xi \in N_\eta$ and N_η is an elementary submodel of $\mathfrak{A}$. So, by shrinking S and increasing m if necessary, we may assume that $\chi_{N_\xi} <_m f_{\rho_\eta}$ for any $\xi, \eta \in S$ with $\xi < \eta$. Take $\eta \in \lim(S)$. Then $\chi_{N_\xi} <_m f_{\rho_\eta}$, for all $\xi \in S \cap \eta$. Moreover, $\chi_{N_\eta}(n) = \sup_{\xi\in S\cap\eta} \chi_{N_\xi}(n)$, for all n, since $N_\eta = \bigcup_{\xi\in S\cap\eta} N_\xi$. So $\chi_{N_\eta} \leqslant_m f_{\rho_\eta}$. Therefore $N = N_\eta$ is as desired. □

Now, we prove Theorem 4.1. In the proof, we will use Lemma 2.2 as well as Lemmas 4.3 and 4.4.

Proof of Theorem 4.1. Towards a contradiction, assume that $\vec{\lambda} = (\lambda_n : n < \omega)$ is a strictly increasing sequence of regular cardinals converging to λ, and that there is a better scale $\vec{f} = (f_\beta : \beta < \lambda^+)$ on $\vec{\lambda}$. We may also assume that $\lambda_0 \geqslant \omega_2$. Fix a sequence $\vec{E} = (E_{n,i} : n \in \omega, i \in 2)$ such that $E_{n,0}$ and $E_{n,1}$ are disjoint stationary subsets of $E^{\lambda_n}_{\omega_1}$, for all n, and fix a sequence $\vec{b} = (b_\xi : \xi < \omega_1)$ of functions from ω to 2 such that, if $\xi \neq \eta$, then $b_\xi \neq^* b_\eta$. Moreover, for each $x \subseteq \lambda^+$, let e_x be the function on ω defined by $e_x(n) = \min(x \setminus f_{\sup^+(x)}(n))$. (If $x \setminus f_{\sup^+(x)}(n) = \emptyset$, then let $e_x(n) = 0$.) Then let X be the

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set of all $x \in [\lambda^+]^\omega$ such that, letting $\xi = x \cap \omega_1 \in \omega_1$, we have

(i) $f_{\sup^+(x)} < \chi_x$,

(ii) $e_x(n) \in E_{n,b_\xi(n)}$, for all but finitely many n.

Note that X is weakly full. So, it suffices to prove the following two claims.

Claim 1. *X is stationary in $[\lambda^+]^\omega$.*

Proof. Take an arbitrary function $F : [\lambda^+]^{<\omega} \to \lambda^+$. We need to find $x \in X$ which is closed under F. By Lemma 4.3, fix $\xi < \omega_1$ such that there is a winning strategy τ of I in $G_3(\vec{E}, \vec{b}, F, \xi)$. Moreover, take a sufficiently large regular cardinal θ and a well-ordering $\trianglelefteq$ of H_θ, and let $\mathfrak{A} = (H_\theta, \in, \trianglelefteq, F, \tau)$. Then, by Lemma 4.4, we can find a countable elementary submodel N of $\mathfrak{A}$, containing F and τ, such that $\chi_N \leqslant^* f_\rho$, where $\rho = \sup(N \cap \lambda^+)$. Let $\beta_n = f_\rho(n)$, for each n, and take an increasing cofinal sequence $\langle \gamma_n : n \in \omega \rangle$ in $N \cap \lambda^+$. Then let $\langle \alpha_n, \delta_n, \epsilon_n : n \in \omega \rangle$ be a sequence of I's moves according to τ against $\langle \beta_n, \gamma_n : n \in \omega \rangle$, and let $x = \mathrm{cl}_F(\xi \cup \{\delta_n, \epsilon_n : n \in \omega\})$. It suffices to show that $x \in X$. In order to see this, first note that $x \cap \omega_1 = \xi$, since $\langle \alpha_n, \beta_n, \gamma_n, \delta_n, \epsilon_n : n \in \omega \rangle$ is a play of $G_3(\vec{E}, \vec{b}, F, \xi)$ in which I moves according to his/her winning strategy τ. Note also that $\sup(x) = \rho$. This is because $\sup(x) \geqslant \sup_{n\in\omega} \epsilon_n \geqslant \sup_{n\in\omega} \gamma_n = \rho$. On the other hand, note that $x \subseteq \mathrm{Hull}^{\mathfrak{A}}(\rho) \cap \lambda^+$. Indeed, $\beta_n, \gamma_n < \rho$, for all n, and $\tau, F \in \mathrm{Hull}^{\mathfrak{A}}(\rho)$. Moreover, $\mathrm{Hull}^{\mathfrak{A}}(\rho) \cap \lambda^+ = \rho$, by Fact 1.1. Therefore $\sup(x) \leqslant \rho$. It follows that $x \cap \omega_1 = \xi$. Also x satisfies (i), since $f_\rho(n) = \beta_n < \delta_n \in x \cap \lambda_n$, for every n. In order to check (ii), first note that $\alpha_n < \chi_N(n)$, for each n, since

$$\alpha_n \in \mathrm{Hull}^{\mathfrak{A}}(N \cup \chi_N(n)) \cap \lambda_n = \chi_N(n).$$

Here, the former $\in$-relation is because $\{\beta_m, \gamma_m : m < n\} \subseteq N \cup \chi_N(n)$, and the latter equality is by Fact 1.1. Then $\alpha_n < f_\rho(n)$, for all but finitely many n, since $\chi_N \leqslant^* f_\rho$. Note also that $\delta_n \in x$ and that $x \cap [\alpha_n, \delta_n) = \emptyset$, since I wins the play $\langle \alpha_n, \beta_n, \gamma_n, \delta_n, \epsilon_n : n \in \omega \rangle$ in $G_3(\vec{E}, \vec{b}, F, \xi)$. Hence $e_x(n) = \delta_n$, for all but finitely many n. Moreover, $\delta_n \in E_{n,b_\xi(n)}$ by the rules of $G_3(\vec{E}, \vec{b}, F, \xi)$. Thus x satisfies (ii). □

Claim 2. *The conclusion of Lemma 2.2 fails for X.*

Proof. Towards a contradiction, assume that the conclusion of Lemma 2.2 holds for X. Then we can find $u \in [\lambda]^{\omega_1}$ such that $\omega_1 \subseteq u$ and $X \cap [u]^\omega$ is stationary. Clearly, $\sup(u)$ is a limit ordinal. We consider two cases, according to whether the cofinality of $\sup(u)$ is ω or ω_1.

First, suppose that $\mathrm{cof}(\sup(u)) = \omega$. Then the set

$$Y = \{x \in [u]^\omega : \sup(x) = \sup(u) \text{ and } \mathrm{range}(e_u) \subseteq x\}$$

is club in $[u]^\omega$. Note that $e_x = e_u$ for all $x \in Y$. Take $x_0, x_1 \in X \cap Y$ with $x_0 \cap \omega_1 \neq x_1 \cap \omega_1$, and let $\xi_i = x_i \cap \omega_1$ for $i = 0, 1$. Then $e_{x_0} = e_u = e_{x_1}$, since $x_0, x_1 \in Y$, and so $b_{\xi_0} =^* b_{\xi_1}$, since $x_0, x_1 \in X$. This contradicts the choice of $\vec{b}$ and the fact that $\xi_0 \neq \xi_1$.

Next, suppose that $\mathrm{cof}(\sup(u)) = \omega_1$. Since $\vec{f}$ is a better scale, we can find a club C in $\sup(u)$ and $\sigma : C \to \omega$ such that $f_\beta <_{\max\{\sigma(\beta),\sigma(\gamma)\}} f_\gamma$ for any $\beta, \gamma \in C$ with $\beta < \gamma$. Let h

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and e be functions on ω defined by

$$h(n) = \sup\{f_\beta(n) : \beta \in C \wedge n \geqslant \sigma(\beta)\},$$
$$e(n) = \min(u \setminus h(n)).$$

Moreover, let Z be the set of all $x \in [u]^\omega$ such that

(iv) $\sup(x) \in C$;

(v) $x \cap h(n) \subseteq f_{\sup(x)}(n)$, for every $n \geqslant \sigma(\sup(x))$;

(vi) $\mathrm{range}(e) \subseteq x$.

Then it is easy to see that Z contains a club subset of $[u]^\omega$. Here, note that, if $x \in Z$, then $e_x(n) = e(n)$ for all $n \geqslant \sigma(\sup^+(x))$. Then we can get a contradiction by the same argument as in the case when $\mathrm{cof}(\sup(u)) = \omega$. □

This completes the proof of Theorem 4.1. □

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