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ON 2-HOLONOMY

HOSSEIN ABBASPOUR✉ **and FRIEDRICH WAGEMANN**

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Abstract

We construct a cycle in higher Hochschild homology associated to the two-dimensional torus which represents 2-holonomy of a nonabelian gerbe in the same way as the ordinary holonomy of a principal G-bundle gives rise to a cycle in ordinary Hochschild homology. This is done using the connection 1-form of Baez–Schreiber. A crucial ingredient in our work is the possibility to arrange that in the structure crossed module $\mu : \mathfrak{h} \to \mathfrak{g}$ of the principal 2-bundle, the Lie algebra $\mathfrak{h}$ is abelian, up to equivalence of crossed modules.

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1. Introduction

The notion of a principal 2-bundle grew out of the notion of a nonabelian gerbe, as defined by Giraud [16], Breen [6] and further studied by Breen and Messing [7] on the one hand and Laurent-Gengoux *et al.* [18] on the other hand.

Principal 2-bundles in the narrow sense have been studied in [4, 5, 14, 23, 30, 31, 36] and [8] (this list is by no means exhaustive). We will sketch the different approaches and explain our point of view, namely, we choose a framework at the intersection of gerbe theory and theory of principal 2-bundles. The structure 2-group of a principal 2-bundle is in our framework a strict 2-group (we refrain from considering more general structure groups like coherent 2-groups) and its Lie algebra a strict Lie 2-algebra, opening the way to using all information about strict Lie 2-algebras which we discuss in the first section.

The first (nongerbal) approach to principal 2-bundles is due to Bartels [5]. He defined 2-bundles by systematically categorifying spaces, groups and bundles. Bartels wrote down the necessary coherence relations for a locally trivial principal 2-bundle with structure group a coherent 2-group. This work has then been taken up by Baez and

Schreiber [4] in order to define connections for principal 2-bundles. In parallel work, Schreiber and Waldorf [30, 31] and Wockel [36] also took up Bartels' work in order to define holonomy (Schreiber–Waldorf) or to pass to gauge groups (Wockel). Baez and Schreiber described an approach using locally trivial 2-fibrations whose typical fiber is a strict 2-group.

Nonabelian gerbes and principal 2-bundles are two notions which are close, but have subtle differences. The cocycle data of the two notions has been compared in [4, Sections 2.1.4 and 2.2]. Baez and Schreiber showed that under certain conditions, the description in terms of local data of a principal 2-bundle with 2-connection is equivalent to the cocycle description of a (possibly twisted) nonabelian gerbe with *vanishing fake curvature*. This constraint is also shown to be sufficient for the existence of 2-holonomies, that is, the parallel transport over surfaces.

The approach of Schreiber and Waldorf [30, 31] is based on so-called transport functors. Schreiber and Waldorf pushed the equivalence between categories of principal G-bundles with connection over M and transport functors from the thin fundamental groupoid of M to the classifying stack of G to categorical dimension two. These transport functors can then be described in terms of differential forms, that is, for a trivial principal G-bundle, these transport functors correspond to $\Omega^1(M, \mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G. They showed similarly that 2-transport functors from the thin fundamental 2-groupoid correspond to pairs of differential forms $A \in \Omega^1(M, \mathfrak{g})$ and $B \in \Omega^2(M, \mathfrak{h})$ with vanishing fake curvature $F_A + \mu(B) = 0$, where $\mu : \mathfrak{h} \to \mathfrak{g}$ is the crossed module of Lie algebras corresponding to the strict Lie 2-group which comes into the problem. It is clear that this approach is based on the notion of holonomy.

Wockel [36] also took up Bartel's work. In order to make it more easily accessible, he formulated a principal 2-bundle over M in terms of spaces with a group action. A (semi-strict) principal 2-bundle over M is then a locally trivial $\mathcal{G}$-2-space. The 2-group $\mathcal{G}$ is strict and so is the action functor, but the local triviality requirement is not necessarily strict. Wockel showed that semi-strict principal 2-bundles over M are classified by nonabelian Čech cohomology.

The approach of Ginot and Stiénon [14] is based on looking at a principal G-bundle as a generalized morphism (in the sense of Hilsum and Skandalis) from M to G, both being considered as groupoids. In the same way they viewed principal 2-bundles as generalized morphisms from the manifold M (or in general some stack, represented by a Lie groupoid) to the 2-group $\mathcal{G}$, both being viewed as 2-groupoids. In this context, they exhibited a link to gerbes (in their incarnation as extensions of groupoids) and defined characteristic classes.

The particularity of Martins and Picken's approach [23] is that they considered special $\mathcal{G}$-2-bundles. For a strict 2-group $\mathcal{G}$ whose associated crossed module is $\mu : H \to G$, these bundles are obtained from a principal G-bundle P on M. The speciality requirement is that the principal $\mathcal{G}$-2-bundle is given by a nonabelian cocycle (g_{ij}, h_{ijk}) as below, but with $\mu(h_{ijk}) = 1$ in order to have a principal G-bundle P. Using the language which we will introduce below, Martins and Picken supposed that the band of the gerbe (which is in general a principal $G/\mu(H)$-bundle) lifts to a principal

G-bundle. Martins and Picken defined connections for these special $\mathcal{G}$-2-bundles and 2-holonomy 2-functors.

Chatterjee *et al.* [8] used in the first place a reference connection 1-form $\bar{A}$ in order to take for a fixed G-principal bundle $P \to M$ only $\bar{A}$-horizontal paths in the path space $\mathcal{P}_{\bar{A}}P$ they considered. Here $\mathcal{P}_{\bar{A}}P$ is a G-principal bundle over the usual path space $\mathcal{P}M$. Then, given a pair (A, B) as above, they constructed a connection 1-form $\omega_{(A,B)}$ on $\mathcal{P}_{\bar{A}}P$ using Chen integrals. Major issues are reparametrization invariance and the curvature. The authors switched to a categorical description motivated by their differential geometric study at the end of the article.

Let us summarize the different approaches in Table 1.

Let us also mention the more recent paper by Nikolaus and Waldorf [26], where the equivalences between some of the incarnations of nonabelian gerbes and principal 2-bundles are shown.

The goal of our article is to construct a cycle in higher Hochschild homology which represents 2-holonomy of a nonabelian gerbe as described above in the same way as the ordinary holonomy gives rise to a cycle in ordinary Hochschild homology; see [1]. This is done using the connection 1-form of Baez and Schreiber [4] which we construct here from the band of the nonabelian gerbe.

A crucial ingredient in our work is the possibility to arrange that in an arbitrary crossed module of Lie algebras $\mu : \mathfrak{h} \to \mathfrak{g}$, the Lie algebra $\mathfrak{h}$ is abelian, up to equivalence of crossed modules. This is shown in Section 1 (see [34]). The possibility to have $\mathfrak{h}$ abelian is used in order to obtain a *commutative* differential graded algebra

Table 1. Comparison of different approaches.

Author(s)	Concept
Bartels	Principal 2-bundles with coherent structure group
Baez–Schreiber	Global connection 1-form for principal 2-bundles
Schreiber–Waldorf	Holonomy in terms of transport functors
Wockel	Relation to nonabelian Čech cohomology
Ginot–Stiénon	2-bundles as Hilsum–Skandalis' generalized morphisms
Martins–Picken	Connections and holonomy for *special* principal 2-bundles
Chatterjee–Lahiri–Sengupta	Connections and holonomy using $\bar{A}$-horizontal paths for a reference 1-form $\bar{A}$

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$\Omega^* := \Omega^*(M, U\mathfrak{h})$ whose higher Hochschild homology $HH^{\mathbb{T}}_\bullet(\Omega^*, \Omega^*)$ associated to the two-dimensional torus $\mathbb{T}$ houses the holonomy cycle. We do not know of any definition of higher Hochschild homology for arbitrary differential graded algebras; therefore, we believe the reduction to abelian $\mathfrak{h}$ to be crucial when working with possibly nonabelian gerbes. Section 1 also provides a fundamental result on strict Lie 2-algebras directly inspired from [3], namely, we explicitly show that the two classifications of strict Lie 2-algebras in terms of skeletal models (of the associated semi-strict Lie 2-algebra), and in terms of the associated crossed modules, coincide.

Section 2 reports on crossed modules of Lie groups. These play a minor role in our study, because the main ingredient for the connection data is the infinitesimal crossed module, that is, the Lie algebra crossed module. Section 3 gives the definition of principal 2-bundles with which we work. It is taken from Wockel's article [36], together with restrictions from [4]. In Section 4, we discuss in general L_∞-valued differential forms on the manifold M, based on the article of Getzler [12]. We believe that this is the right generalization of the calculus of Lie algebra valued differential forms needed for ordinary principal G-bundles. We find a curious 3-form term (see Equation (5.2)) in the Maurer–Cartan equation for differential forms with values in a semi-strict Lie 2-algebra which also appears in [17]. In Section 5, we construct the connection 1-form $\mathcal{A}_0$ of Baez–Schreiber from the band of the nonabelian gerbe. It is not so clear in [4] on which differential geometric object the construction of $\mathcal{A}_0$ is carried out, and we believe that expressing it as the usual iterated integral construction on the band (which is an ordinary principal G-bundle) is of conceptual importance.

Section 6 is the heart of our article and explains the mechanism to transform the flat connection $\mathcal{A}_0$ into a Hochschild cycle for the differential graded algebra $CH_*(\Omega^*, \Omega^*)$. It lives therefore in the Hochschild homology of the algebra of Hochschild chains. Section 7 recalls from [15] that 'Hochschild of Hochschild'-homology is the higher Hochschild homology associated to the torus $\mathbb{T}^2$. We proceed with an explicit expression for (some terms arising in) the holonomy cycle in Section 8.

In this article too we try to give a more conceptual approach to the holonomy of gerbes rather than a computational one as is introduced in [11] and further developed in [33]. Another important difference is that our approach includes nonabelian gerbes as well.

The main theorem of the present article is the construction of the homology cycle representing the holonomy. It should be thought of as a 0-cochain on mapping surface (torus in this particular case) space.

THEOREM 1.1. *Consider a nonabelian principal* 2*-bundle with trivial band on a manifold M with a structure crossed module $\mu : \mathfrak{h} \to \mathfrak{g}$ such that the Lie algebra $\mathfrak{h}$ is abelian. Then the connection* 1*-form $\mathcal{A}_0$ of Baez–Schreiber gives rise to a cycle $P(\mathcal{A}_0)$ in the higher Hochschild homology $HH^{\mathbb{T}}_\bullet(\Omega^*, \Omega^*)$ which corresponds to the holonomy of the gerbe.*

As stated before, we do not consider the condition that $\mathfrak{h}$ is abelian as a restriction of generality, because, up to equivalence, it may be achieved for an arbitrary crossed module.

By construction, the cycle $P(\mathcal{A}_0)$ is not always trivial, that is, a boundary, because it represents the holonomy. Observe that for the crossed module $\mathrm{id} : \mathfrak{h} \to \mathfrak{h}$, we recover the result of [1]. The triviality condition on the band may be understood as expressing that the construction is local. The gluing of the locally defined connection 1-forms of Baez and Schreiber to a global connection 1-form (see [4]) should permit us to glue our Hochschild cycles.

Another subject of further research is to understand that the connection 1-form $\mathcal{A}_0$ does not only lead to a higher Hochschild cycle with respect to the two-dimensional torus, but actually to higher Hochschild cycles with respect to any compact topological surface. In fact, we believe that there is a way to recover $HH_\bullet^{\Sigma_g}$ for a connected compact surface Σ_g of genus g from $HH_\bullet^{\mathbb{T}}$.

2. Strict Lie 2-algebras and crossed modules

We gather in this section preliminaries on strict Lie 2-algebras and crossed modules, and their relation to semi-strict Lie 2-algebras. The main result is the possibility to replace a crossed module $\mu : \mathfrak{h} \to \mathfrak{g}$ by an equivalent one having abelian $\mathfrak{h}$. This will be important for defining holonomy as a cycle in higher Hochschild homology.

Lie 2-algebras have been the object of different studies; see [3] for semi-strict Lie 2-algebras or [28] for (general weak) Lie 2-algebras.

2.1. Strict 2-vector spaces. We fix a field $\mathbb{K}$ of characteristic 0; in geometrical situations, we will always take $\mathbb{K} = \mathbb{R}$. A 2-*vector space* V over $\mathbb{K}$ is simply a category object in `Vect`, the category of vector spaces (cf. [3, Definition 5]). This means that V consists of a vector space of *arrows* V_{-1}, a vector space of *objects* V_0, linear maps $V_{-1} \overset{s}{\underset{t}{\rightrightarrows}} V_0$, called *source* and *target*, a linear map $i : V_0 \to V_{-1}$, called *object inclusion*, and a linear map

$$m : V_{-1} \times_{V_0} V_{-1} \to V_{-1},$$

which is called the *categorical composition*. This data is supposed to satisfy the usual axioms of a category.

An equivalent point of view is to regard a 2-vector space as a two-term complex of vector spaces $d : C_{-1} \to C_0$. Pay attention to the change in degree with respect to Baez and Crans [3]. We use here a cohomological convention, instead of their homological convention, in order to have the right degrees for the differential forms with values in crossed modules later on.

The equivalence between 2-vector spaces and two-term complexes is spelt out in [3, Section 3]: one passes from a category object in `Vect` (given by $V_{-1} \overset{s}{\underset{t}{\rightrightarrows}} V_0$, $i : V_0 \to V_{-1}$ etc) to a two-term complex $d : C_{-1} \to C_0$ by taking $C_{-1} := \ker(s)$, $d := t|_{\ker(s)}$ and $C_0 = V_0$. In the reverse direction, to a given two-term complex $d : C_{-1} \to C_0$, one associates $V_{-1} = C_0 \oplus C_{-1}$, $V_0 = C_0$, $s(c_0, c_{-1}) = c_0$, $t(c_0, c_{-1}) = c_0 + d(c_{-1})$ and $i(c_0) = (c_0, 0)$. The only subtle point here is that the categorical composition m is

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already determined by $V_{-1} \overset{s}{\underset{t}{\rightrightarrows}} V_0$ and $i : V_0 \to V_{-1}$ (see [3, Lemma 6]). Namely, writing an arrow $c_{-1} =: f$ with $s(f) = x$, $t(f) = y$, that is, $f : x \mapsto y$, one denotes the *arrow part* of f by $\vec{f} := f - i(s(f))$ and, for two composable arrows $f, g \in V_{-1}$, the composition m is then defined by

$$f \circ g := m(f, g) := i(x) + \vec{f} + \vec{g}.$$

2.2. Strict Lie 2-algebras and crossed modules.

Definition 2.1. A strict Lie 2-algebra is a category object in the category `Lie` of Lie algebras over $\mathbb{K}$.

This means that it is the data of two Lie algebras, $\mathfrak{g}_0$, the *Lie algebra of objects*, and $\mathfrak{g}_{-1}$, the *Lie algebra of arrows*, together with morphisms of Lie algebras $s, t : \mathfrak{g}_{-1} \to \mathfrak{g}_0$, *source* and *target*, a morphism $i : \mathfrak{g}_0 \to \mathfrak{g}_{-1}$, the *object inclusion* and a morphism $m : \mathfrak{g}_{-1} \times_{\mathfrak{g}_0} \mathfrak{g}_{-1} \to \mathfrak{g}_{-1}$, the *composition of arrows*, such that the usual axioms of a category are satisfied.

Let us now come to crossed modules of Lie algebras. We refer to [34] for more details.

Definition 2.2. A crossed module of Lie algebras is a morphism of Lie algebras $\mu : \mathfrak{h} \to \mathfrak{g}$ together with an action of $\mathfrak{g}$ on $\mathfrak{h}$ by derivations such that for all $h, h' \in \mathfrak{h}$ and all $g \in \mathfrak{g}$:

(a) $\mu(g \cdot h) = [g, \mu(h)]$; and
(b) $\mu(h) \cdot h' = [h, h']$.

One may associate to a crossed module of Lie algebras a four-term exact sequence of Lie algebras

$$0 \to V \to \mathfrak{h} \overset{\mu}{\to} \mathfrak{g} \to \bar{\mathfrak{g}} \to 0,$$

where we used the notation $V := \ker(\mu)$ and $\bar{\mathfrak{g}} := \operatorname{coker}(\mu)$. It follows from the properties (a) and (b) of a crossed module that $\mu(\mathfrak{h})$ is an ideal, so $\bar{\mathfrak{g}}$ is a Lie algebra, and that V is a central ideal of $\mathfrak{h}$ and a $\bar{\mathfrak{g}}$-module (because the outer action, to be defined below, is a genuine action on the center of $\mathfrak{h}$).

Recall the definition of the *outer action* $s : \bar{\mathfrak{g}} \to \mathfrak{out}(\mathfrak{h})$ for a crossed module of Lie algebras $\mu : \mathfrak{h} \to \mathfrak{g}$. The Lie algebra

$$\mathfrak{out}(\mathfrak{h}) := \mathfrak{der}(\mathfrak{h})/\mathrm{ad}(\mathfrak{h})$$

is the Lie algebra of outer derivations of $\mathfrak{h}$, that is, the quotient of the Lie algebra of all derivations $\mathfrak{der}(\mathfrak{h})$ by the ideal $\mathrm{ad}(\mathfrak{h})$ of inner derivations, that is, those of the form $h' \mapsto [h, h']$ for some $h \in \mathfrak{h}$.

To define s, choose a linear section $\rho : \bar{\mathfrak{g}} \to \mathfrak{g}$ and compute its default to be a homomorphism of Lie algebras

$$\alpha(x, y) := [\rho(x), \rho(y)] - \rho([x, y])$$

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for $x, y \in \bar{\mathfrak{g}}$. As the projection onto $\bar{\mathfrak{g}}$ is a homomorphism of Lie algebras, $\alpha(x, y)$ is in its kernel and there exists therefore an element $\beta(x, y) \in \mathfrak{h}$ such that $\mu(\beta(x, y)) = \alpha(x, y)$.

We have for all $h \in \mathfrak{h}$,

$$(\rho(x) \circ \rho(y) - \rho(y) \circ \rho(x) - \rho([x, y])) \cdot h = \alpha(x, y) \cdot h = \mu(\beta(x, y)) \cdot h = [\beta(x, y), h]$$

and, in this sense, elements of $\bar{\mathfrak{g}}$ act on $\mathfrak{h}$ up to inner derivations. We obtain a well-defined homomorphism of Lie algebras

$$s : \bar{\mathfrak{g}} \to \mathfrak{out}(\mathfrak{h})$$

by $s(x)(h) = \rho(x) \cdot h$.

Strict Lie 2-algebras are in one-to-one correspondence with crossed modules of Lie algebras, like in the case of groups; cf. [19]. For the convenience of the reader, let us include this here.

THEOREM 2.3. *Strict Lie 2-algebras are in one-to-one correspondence with crossed modules of Lie algebras.*

PROOF. Given a Lie 2-algebra $\mathfrak{g}_{-1} \overset{s}{\underset{t}{\rightrightarrows}} \mathfrak{g}_0$, $i : \mathfrak{g}_0 \to \mathfrak{g}_{-1}$, the corresponding crossed module is defined by

$$\mu := t|_{\ker(s)} : \mathfrak{h} := \ker(s) \to \mathfrak{g} := \mathfrak{g}_0.$$

The action of $\mathfrak{g}$ on $\mathfrak{h}$ is given by

$$g \cdot h := [i(g), h]$$

for $g \in \mathfrak{g}$ and $h \in \mathfrak{h}$ (where the bracket is taken in $\mathfrak{g}_{-1}$). This is well defined and an action by derivations. Axiom (a) follows from

$$\mu(g \cdot h) = \mu([i(g), h]) = [\mu \circ i(g), \mu(h)] = [g, \mu(h)].$$

Axiom (b) follows from

$$\mu(h) \cdot h' = [i \circ \mu(h), h'] = [i \circ t(h), h']$$

by writing $i \circ t(h) = h + r$ for $r \in \ker(t)$ and by using that $\ker(t)$ and $\ker(s)$ in a Lie 2-algebra commute (shown in Lemma 2.4 after the proof).

On the other hand, given a crossed module of Lie algebras $\mu : \mathfrak{h} \to \mathfrak{g}$, associate to it

$$\mathfrak{h} \rtimes \mathfrak{g} \overset{s}{\underset{t}{\rightrightarrows}} \mathfrak{g}, \quad i : \mathfrak{g} \to \mathfrak{h} \rtimes \mathfrak{g}$$

by $s(h, g) = g$, $t(h, g) = \mu(h) + g$, $i(g) = (0, g)$, where the semi-direct product Lie algebra $\mathfrak{h} \rtimes \mathfrak{g}$ is built from the given action of $\mathfrak{g}$ on $\mathfrak{h}$. Let us emphasize that $\mathfrak{h} \rtimes \mathfrak{g}$ is built from the Lie algebra $\mathfrak{g}$ and the $\mathfrak{g}$-module $\mathfrak{h}$; the bracket of $\mathfrak{h}$ does not intervene here. The composition of arrows is already encoded in the underlying structure of 2-vector space, as remarked in the previous subsection. □

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Lemma 2.4. $[\ker(s), \ker(t)] = 0$ *in a Lie* 2*-algebra.*

Proof. The fact that the composition of arrows is a homomorphism of Lie algebras gives the following 'middle four exchange' (or functoriality) property

$$[g_1, g_2] \circ [f_1, f_2] = [g_1 \circ f_1, g_2 \circ f_2]$$

for composable arrows $f_1, f_2, g_1, g_2 \in \mathfrak{g}_1$. Now suppose that $g_1 \in \ker(s)$ and $f_2 \in \ker(t)$. Then denote by f_1 and by g_2 the identity (with respect to the composition) in $0 \in \mathfrak{g}_0$. As these are identities, we have $g_1 = g_1 \circ f_1$ and $f_2 = g_2 \circ f_2$. On the other hand, i is a morphism of Lie algebras and sends $0 \in \mathfrak{g}_0$ to $0 \in \mathfrak{g}_1$. Therefore, we may conclude that

$$[g_1, f_2] = [g_1 \circ f_1, g_2 \circ f_2] = [g_1, g_2] \circ [f_1, f_2] = 0. \qquad \square$$

Furthermore, it is well known (cf. [34]) that (equivalence classes of) crossed modules of Lie algebras with cokernel $\overline{\mathfrak{g}}$ and kernel V are classified by the third Chevalley–Eilenberg cohomology $H^3(\overline{\mathfrak{g}}, V)$.

Remark 2.5. It is implicit in the previous proof that starting from a crossed module $\mu : \mathfrak{h} \to \mathfrak{g}$, passing to the Lie 2-algebra $\mathfrak{g}_{-1} \overset{s}{\underset{t}{\rightrightarrows}} \mathfrak{g}_0$, $i : \mathfrak{g}_0 \to \mathfrak{g}_{-1}$ (and thus forgetting the bracket on $\mathfrak{h}$), one may finally reconstruct the bracket on $\mathfrak{h}$. This is due to the fact that it is encoded in the action and the morphism, using the property (b) of a crossed module.

2.3. Semi-strict Lie 2-algebras and two-term L_∞-algebras. An equivalent point of view is to regard a strict Lie 2-algebra as a Lie algebra object in the category Cat of (small) categories. From this second point of view, we have a functorial Lie bracket which is supposed to be antisymmetric and must fulfill the Jacobi identity. Weakening the antisymmetry axiom and the Jacobi identity up to coherent isomorphisms leads then to semi-strict Lie 2-algebras (here antisymmetry holds strictly, but Jacobi is weakened), hemi-strict Lie 2-algebras (here Jacobi holds strictly, but antisymmetry is weakened) or even to (general) Lie 2-algebras (both axioms are weakened). Let us record the definition of a semi-strict Lie 2-algebra (see [3, Definition 22]).

Definition 2.6. A semi-strict Lie 2-algebra consists a 2-vector space L together with a skew-symmetric, bilinear and functorial bracket $[,] : L \times L \to L$ and a completely antisymmetric trilinear natural isomorphism

$$J_{x,y,z} : [[x, y], z] \to [x, [y, z]] + [[x, z], y],$$

called the *Jacobiator*. The Jacobiator is required to satisfy the Jacobiator identity (see [3, Definition 22]).

Semi-strict Lie 2-algebras together with morphisms of semi-strict Lie 2-algebras (see [3, Definition 23]) form a strict 2-category (see [3, Proposition 25]). Strict Lie algebras form a full sub-2-category of this 2-category; see [3, Proposition 42]. In order to regard a strict Lie 2-algebra $\mathfrak{g}_{-1} \to \mathfrak{g}_0$ as a semi-strict Lie 2-algebra, the

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functorial bracket is constructed for $f : x \mapsto y$ and $g : a \mapsto b$, $f, g \in \mathfrak{g}_{-1}$ and $x, y, a, b \in \mathfrak{g}_0$ by defining its source $s([f, g])$ and its arrow part $[\vec{f, g}]$ to be $s([f, g]) := [x, a]$ and $[\vec{f, g}] := [x, \vec{g}] + [\vec{f}, b]$ (see [3, proof of Theorem 36]). By construction, it is compatible with the composition, that is, functorial.

Remark 2.7. One observes that the functorial bracket on a strict Lie 2-algebra $\mathfrak{g}_{-1} \to \mathfrak{g}_0$ is constructed from the bracket in $\mathfrak{g}_0$, and the bracket between $\mathfrak{g}_{-1}$ and $\mathfrak{g}_0$, but does not involve the bracket on $\mathfrak{g}_{-1}$ itself.

There is a 2-vector space underlying every semi-strict Lie 2-algebra; thus, one may ask which structure is inherited from a semi-strict Lie 2-algebra by the corresponding two-term complex of vector spaces. This leads us to two-term L_∞-algebras; see Baez and Crans [3, Theorem 36]. Our definition here differs from theirs as we stick to the cohomological setting and degree +1 differentials; see [12, Definition 4.1].

Definition 2.8. An L_∞-algebra is a graded vector space L together with a sequence $l_k(x_1, \ldots, x_k)$, $k > 0$, of graded antisymmetric operations of degree $2 - k$ such that the following identity is satisfied:

$$\sum_{k=1}^{n}(-1)^k \sum_{\substack{i_1<\cdots<i_k;\, j_1<\cdots<j_{n-k} \\ \{i_1,\ldots,i_k\}\cup\{j_1,\ldots,j_{n-k}\}=\{1,\ldots,n\}}} (-1)^{\epsilon} l_n(l_k(x_{i_1}, \ldots, x_{i_k}), x_{j_1}, \ldots, x_{j_{n-k}}) = 0.$$

Here the sign $(-1)^{\epsilon}$ equals the product of the sign of the shuffle permutation and the Koszul sign. We refer the reader to [32] for the definition of L_∞-morphism.

We will be mainly concerned with two-term L_∞-algebras. These are L_∞-algebras L such that the graded vector space L consists only of two components L_0 and L_{-1}. An L_∞-algebra $L = L_0 \oplus L_{-1}$ has at most l_1, l_2 and l_3 as its nontrivial 'brackets'. The bracket l_1 is a differential (that is, here just a linear map $L_0 \to L_{-1}$), l_2 is a bracket with components $[,] : L_0 \otimes L_0 \to L_0$ and $[,] : L_{-1} \otimes L_0 \to L_{-1}$, $[,] : L_0 \times L_{-1} \to L_{-1}$, and l_3 is some kind of 3-cocycle $l_3 : L_0 \otimes L_0 \otimes L_0 \to L_{-1}$. More precisely, in case $l_1 = 0$, L_0 is a Lie algebra, L_{-1} is an L_0-module and l_3 is then an actual 3-cocycle. This kind of two-term L_∞-algebra is called *skeletal*; see [3, Section 6] and the next subsection. The complete axioms satisfied by l_1, l_2 and l_3 in a two-term L_∞-algebra are listed in [3, Lemma 33].

As said before, the passage from a 2-vector space to its associated two-term complex induces a passage from semi-strict Lie 2-algebras to two-term L_∞-algebras, which turns out to be an equivalence of two categories (see [3, Theorem 36]).

Theorem 2.9. *The two categories of semi-strict Lie 2-algebras and of two-term L_∞-algebras are equivalent.*

Remark 2.10. In particular, restricting to the sub-2-category of strict Lie 2-algebras, there is an equivalence between crossed modules of Lie algebras and two-term L_∞-algebras with trivial l_3. In other words, there is an equivalence between crossed modules and differential graded Lie algebras $L_{-1} \oplus L_0$.

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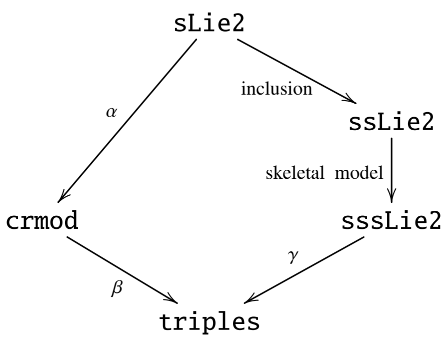
2.4. Classification of semi-strict Lie 2-algebras. Baez and Crans showed in [3] that every semi-strict Lie 2-algebra is equivalent to a skeletal Lie 2-algebra (that is, one where the differential d of the underlying complex of vector spaces is zero). Then they went on by showing that skeletal Lie 2-algebras are classified by triples consisting of an honest Lie algebra $\bar{\mathfrak{g}}$, a $\bar{\mathfrak{g}}$-module V and a class $[\gamma] \in H^3(\bar{\mathfrak{g}}, V)$. This is achieved using the homotopy equivalence of the underlying complex of vector spaces with its cohomology. In total, they got in this way a classification, up to equivalence, of semi-strict Lie 2-algebras in terms of triples $(\bar{\mathfrak{g}}, V, [\gamma])$.

On the other hand, strict Lie 2-algebras are in one-to-one correspondence with crossed modules of Lie algebras, as we have seen in a previous subsection. In conclusion, there are two ways to classify strict Lie 2-algebras: by the associated crossed module or, regarding them as special semi-strict Lie 2-algebras, by Baez–Crans classification. Let us show here that these two classifications are compatible, that is, that they lead to the same triple $(\bar{\mathfrak{g}}, V, [\gamma])$.

For this, let us denote by `sLie2` the class of strict Lie 2-algebras, by `ssLie2` the class of semi-strict Lie 2-algebras, by `sssLie2` the class of skeletal semi-strict Lie 2-algebras, by `triples` the class of triples of the above form $(\mathfrak{g}, V, [\gamma])$ and by `crmod` the class of crossed modules of Lie algebras.

Theorem 2.11. *The following diagram is commutative.*



The maps α and γ are bijections, while the map β induces a bijection when passing to equivalence classes.

Proof. Let us first describe the arrows. The arrow α : `sLie2` → `crmod` has been investigated in Theorem 2.3. The arrow β : `crmod` → `triplets` sends a crossed module $\mu : \mathfrak{h} \to \mathfrak{g}$ to the triple

$$(\mathrm{coker}(\mu) =: \bar{\mathfrak{g}}, \ker(\mu) =: V, [\gamma]),$$

where the cohomology class $[\gamma] \in H^3(\bar{\mathfrak{g}}, V)$ is defined choosing sections—the procedure is described in detail in [34]. The arrow `ssLie2` → `sssLie2` is the choice of a skeletal model for a given semi-strict Lie 2-algebra—it is given by the homotopy

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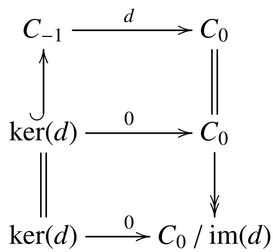
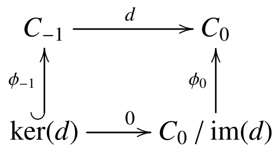
equivalence of the underlying two-term complex with its cohomology displayed in the extremal lines of the following diagram.



More precisely, choosing supplementary subspaces, one can define a map of complexes of vector spaces from $0 : \ker(d) \to C_0 \,/\, \mathrm{im}(d)$ to $d : C_{-1} \to C_0$ and give the structure of an L_∞-algebra to the latter in such a way that it becomes a morphism (with a new 3-cocycle l_3' that differs from the given 3-cocycle l_3 at most by a coboundary); see below.

The arrow $\gamma : \texttt{sssLie2} \to \texttt{triples}$ sends a skeletal 2-Lie algebra to the triple defined by the cohomology class of l_3 (cf. [3]).

Now let us show that the diagram commutes. For this, let $d : C_{-1} \to C_0$ with some bracket $[,]$ and $l_3 = 0$ be a two-term L_∞-algebra corresponding to seeing a strict Lie 2-algebra as a semi-strict Lie 2-algebra and build its skeletal model. The model comes together with a morphism of semi-strict Lie 2-algebras $(\phi_2, \phi_{-1}, \phi_0)$ given by



Here $\phi_0 =: \sigma$ is a linear section of the quotient map. The structure of a semi-strict Lie 2-algebra is transferred to the lower line in order to make $(\phi_2, \phi_{-1}, \phi_0)$ a morphism of semi-strict Lie 2-algebras. In order to compute now the l_3 term of the lower semi-strict Lie 2-algebra, denoted l_3', one first finds that (first equation in [3, Definition 34]) $\phi_2 : C_0 \,/\, \mathrm{im}(d) \times C_0 \,/\, \mathrm{im}(d) \to C_{-1}$ is such that

$$d\phi_2(x, y) = \sigma[x, y] - [\sigma(x), \sigma(y)],$$

the default of the section σ to be a homomorphism of Lie algebras. Then l_3' is related to ϕ_2 by the second formula in [3, Definition 34]. This gives

$$l_3'(x, y, z) = (d_{\mathrm{CE}}\phi_2)(x, y, z)$$

for $x, y, z \in C_0 \,/\, \mathrm{im}(d)$. Here d_{CE} is the formal Chevalley–Eilenberg differential of the cochain $\phi_2 : C_0 \,/\, \mathrm{im}(d) \times C_0 \,/\, \mathrm{im}(d) \to C_{-1}$ with values in C_{-1} as if C_{-1} was a $C_0 \,/\, \mathrm{im}(d)$-module (which is usually not the case). This is exactly the expression of the cocycle γ associated to the crossed module of Lie algebras $d : C_{-1} \to C_0$ obtained using the section σ; see [34]. □

Corollary 2.12. *Every semi-strict Lie 2-algebra is equivalent (as an object of the 2-category* `ssLie2`*) to a strict Lie 2-algebra.*

This corollary is already known because of abstract reasons. Here we have proved a result somewhat more refined: the procedure to strictify a semi-strict Lie 2-algebra is rather easy to perform. First one has to pass to cohomology by homotopy equivalence (that is, the arrows 'skeletal model' and γ in the diagram of Theorem 4) and then one has to construct the crossed module corresponding to a given cohomology class. This can be done in several ways, using free Lie algebras [22], using injective modules [34] (as described in the next subsection) etc and one may adapt the construction method to the problem at hand.

2.5. The construction of an abelian representative. We will show in this section that for a given class $[\gamma] \in H^3(\bar{\mathfrak{g}}, V)$, there exists a crossed module of Lie algebras $\mu : \mathfrak{h} \to \mathfrak{g}$ with class $[\gamma]$ (and $\ker(\mu) = V$ and $\mathrm{coker}(\mu) = \bar{\mathfrak{g}}$) such that $\mathfrak{h}$ is abelian. This will be important for the treatment in higher Hochschild homology of the holonomy of a gerbe.

Theorem 2.13. *For any $[\gamma] \in H^3(\bar{\mathfrak{g}}, V)$, there exists a crossed module of Lie algebras $\mu : \mathfrak{h} \to \mathfrak{g}$ with associated class $[\gamma]$ such that $\ker(\mu) = V$, $\mathrm{coker}(\mu) = \bar{\mathfrak{g}}$ and $\mathfrak{h}$ is abelian.*

Proof. This is [34, Theorem 3]. Let us sketch its proof here. The category of $\bar{\mathfrak{g}}$-modules has enough injectives; therefore, V may be embedded in an injective $\bar{\mathfrak{g}}$-module I. We obtain a short exact sequence of $\bar{\mathfrak{g}}$-modules

$$0 \to V \xrightarrow{i} I \xrightarrow{\pi} Q \to 0,$$

where $Q := I/V$ is the quotient. The module I being injective implies that $H^p(\bar{\mathfrak{g}}, I) = 0$ for all $p > 0$. Therefore, the short exact sequence of coefficients induces a connective homomorphism

$$\partial : H^2(\bar{\mathfrak{g}}, Q) \to H^3(\bar{\mathfrak{g}}, V),$$

which is an isomorphism. To $[\gamma]$ there thus corresponds a class $[\alpha] \in H^2(\bar{\mathfrak{g}}, Q)$ with $\partial[\alpha] = [\gamma]$. A representative $\alpha \in Z^2(\bar{\mathfrak{g}}, Q)$ gives rise to an abelian extension

$$0 \to Q \to Q \times_\alpha \bar{\mathfrak{g}} \to \bar{\mathfrak{g}} \to 0.$$

Now one easily verifies (see the proof of [34, Theorem 3]) that the splicing together of the short exact coefficient sequence and the abelian extension gives rise to a crossed module

$$0 \to V \to I \to Q \times_\alpha \bar{\mathfrak{g}} \to \bar{\mathfrak{g}} \to 0.$$

More precisely, the crossed module is $\mu : I \to Q \times_\alpha \bar{\mathfrak{g}}$ given by $\mu(x) = (\pi(x), 0)$, the action of $\mathfrak{g} := Q \times_\alpha \bar{\mathfrak{g}}$ on $\mathfrak{h} := I$ is induced by the action of $\bar{\mathfrak{g}}$ on I and the Lie bracket is trivial on I, that is, I is abelian.

One also easily verifies (see [34, proof of Theorem 3]) that the associated cohomology class for such a crossed module (which is the Yoneda product of a short

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exact coefficient sequence and an abelian extension) is $\partial[\alpha]$, the image under the connective homomorphism (induced by the short exact coefficient sequence) of the class defining the abelian extension. Therefore, the associated class is here $\partial[\alpha] = [\gamma]$, as required. □

We thus obtain the following refinement of Corollary 2.12.

COROLLARY 2.14. *Every semi-strict Lie 2-algebra is equivalent to a strict Lie 2-algebra corresponding to a crossed module $\mu : \mathfrak{h} \to \mathfrak{g}$ with abelian $\mathfrak{h}$ such that $\mathfrak{h}$ is a $\bar{\mathfrak{g}} := \mathfrak{g}/\mu(\mathfrak{h})$-module and such that the outer action is a genuine action.*

PROOF. This follows from Corollary 2.12 together with Theorem 2.13. The facts that $\mathfrak{h}$ is a $\bar{\mathfrak{g}} =: \mathfrak{g}/\mu(\mathfrak{h})$-module and that the outer action is a genuine action are equivalent. They are true either by inspection of the representative constructed in the proof of Theorem 2.13 or by the following argument.

The outer action s is an action only up to inner derivations. But these are trivial in case $\mathfrak{h}$ is abelian:

$$\mu(h) \cdot h' = [h, h'] = 0$$

for all $h, h' \in \mathfrak{h}$ by property (b) of a crossed module. □

REMARK 2.15. An analogous statement is true on the level of (abstract) groups and even topological groups [35]. Unfortunately, we ignore whether such a statement is true in the category of Lie groups, that is, given a locally smooth group 3-cocycle γ on $\bar{G}$ with values in a smooth $\bar{G}$-module V, is there a smooth (not necessarily split) crossed module of Lie groups $\mu : H \to G$ with H abelian and cohomology class $[\gamma]$? From the point of view of Lie algebras, there are two steps involved: having solved the problem on the level of Lie algebras (as above), one has to integrate the 2-cocycle α. This is well understood thanks to work of Neeb. The (possible) obstructions lie in $\pi_1(\bar{G})$ and $\pi_2(\bar{G})$, and thus vanish for simply connected, finite-dimensional Lie groups $\bar{G}$. The second step is to integrate the involved $\bar{\mathfrak{g}}$-module I to a $\bar{G}$-module. As I is necessarily infinite dimensional, this is the hard part of the problem.

Note however that for many interesting crossed modules it is not necessary to perform the construction using a genuine injective module I. The only thing we need about I is that $V \subset I$ and $H^3(\mathfrak{g}, I) = 0$, because then the connecting homomorphism is surjective. In concrete situations, there are often much easier modules which have this property. Note furthermore that for many interesting classes of crossed modules of Lie algebras $\mu : \mathfrak{m} \to \mathfrak{n}$, $\mathfrak{m}$ is already abelian. This is the case for $\mathrm{id} : \mathfrak{m} \to \mathfrak{m}$ with $\mathfrak{m}$ abelian or for $0 : V \to \mathfrak{g}$ where V is a $\mathfrak{g}$-module.

3. Crossed modules of Lie groups

In this section, we introduce the strict Lie 2-groups which will be the typical fiber of our principal 2-bundles. While the notion of a crossed module of groups is well understood and purely algebraic, the notion of a crossed module of Lie groups involves subtle smoothness requirements.

We will heavily draw on Neeb [25] and adopt Neeb's point of view, namely, we regard a crossed module of Lie groups as a central extension $\hat{N} \to N$ of a normal split Lie subgroup N in a Lie group G for which the conjugation action of G on N lifts to a smooth action on $\hat{N}$. This point of view is linked to the one regarding a crossed module as a homomorphism $\mu : H \to G$ by taking $H = \hat{N}$ and $\mathrm{im}(\mu) = N$.

Definition 3.1. A morphism of Lie groups $\mu : H \to G$, together with a homomorphism $\hat{S} : G \to \mathrm{Aut}(H)$ defining a smooth action $\hat{S} : G \times H \to H$, $(g, h) \mapsto g \cdot h = \hat{S}(g)(h)$ of G on H, is called a (split) crossed module of Lie groups if the following conditions are satisfied.

(1) $\mu \circ \hat{S}(g) = \mathrm{conj}_{\mu(g)} \circ \mu$ for all $g \in G$.
(2) $\hat{S} \circ \mu : H \to \mathrm{Aut}(H)$ is the conjugation action.
(3) $\ker(\mu)$ is a split Lie subgroup of H and $\mathrm{im}(\mu)$ is a split Lie subgroup of G for which μ induces an isomorphism $H/\ker(\mu) \to \mathrm{im}(\mu)$.

Recall that in a split crossed module of Lie groups $\mu : H \to G$, the quotient Lie group $\bar{G} := G/\mu(H)$ acts smoothly (up to inner automorphisms) on H. This outer action S of $\bar{G}$ on H is a homomorphism $S : \bar{G} \to \mathrm{Out}(H)$ which is constructed like in the case of Lie algebras. The smoothness of S follows directly from the splitting assumptions. Here $\mathrm{Out}(H)$ denotes the *group of outer automorphisms* of H, defined by

$$\mathrm{Out}(H) := \mathrm{Aut}(H)/\mathrm{Inn}(H),$$

where $\mathrm{Inn}(H) \subset \mathrm{Aut}(H)$ is the normal subgroup of automorphisms of the form $h' \mapsto hh'h^{-1}$ for some $h \in H$.

It is shown in [25] that one may associate to a (split) crossed module of Lie groups a locally smooth 3-cocycle γ (whose class is the obstruction against the realization of the outer action in terms of an extension).

It is clear that a (split) crossed module of Lie groups induces a crossed module of the corresponding Lie algebras.

Definition 3.2. Two crossed modules $\mu : M \to N$ (with action η) and $\mu' : M' \to N'$ (with action η') such that $\ker(\mu) = \ker(\mu') =: V$ and $\mathrm{coker}(\mu) = \mathrm{coker}(\mu') =: G$ are called *elementary equivalent* if there are group homomorphisms $\varphi : M \to M'$ and $\psi : N \to N'$ which are compatible with the actions, that is,

$$\varphi(\eta(n)(m)) = \eta'(\psi(n))(\varphi(m))$$

for all $n \in N$ and all $m \in M$, and such that the following diagram is commutative.

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & V & \xrightarrow{i} & M & \xrightarrow{\mu} & N & \xrightarrow{\pi} & G & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \mathrm{id}_V} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \mathrm{id}_G} & & \\ 0 & \longrightarrow & V & \xrightarrow{i'} & M' & \xrightarrow{\mu'} & N' & \xrightarrow{\pi'} & G & \longrightarrow & 1 \end{array}$$

We call *equivalence of crossed modules* the equivalence relation generated by elementary equivalence. One easily sees that two crossed modules are equivalent in case there exists a zigzag of elementary equivalences going from one to the other (where the arrows do not necessarily all go in the same direction).

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In the context of split crossed modules of Lie groups, all morphisms are supposed to be morphisms of Lie groups, that is, smooth, and to respect the sections.

Remark 3.3. It is elementary to show that a morphism of crossed modules of finite-dimensional Lie algebras $\mu : \mathfrak{m} \to \mathfrak{n}$ integrates to a morphism of crossed modules of Lie groups $M \to N$ for the 1-connected Lie groups M, N corresponding to $\mathfrak{m}$, $\mathfrak{n}$, respectively. This implies in particular that under this hypothesis equivalences of crossed modules of Lie algebras integrate to equivalences of crossed modules of Lie groups.

4. Principal 2-bundles and gerbes

In this section, we will start introducing the basic geometric objects of our study, namely principal 2-bundles and gerbes. We choose to work here with a strict Lie 2-group $\mathcal{G}$, that is, a split crossed module of Lie groups, and its associated crossed module of Lie algebras $\mu : \mathfrak{h} \to \mathfrak{g}$, and to consider principal 2-bundles and gerbes which are defined by nonabelian cocycles (or transition functions). The principal object which we will use later on is the band of a gerbe.

4.1. Definition. In order to keep notation and abstraction to a reasonable minimum, we will consider geometric objects like bundles, gerbes etc only over an honest (finite-dimensional) base manifold M, instead of considering a ringed topos or a stack.

Let $\mu : H \to G$ be a (split) crossed module of Lie groups. Let our base space M be an honest (ordinary) manifold and let $\mathcal{U} = \{U_i\}$ be a good open cover of M. The following definition dates back at least to [7]; in the present form, we took it from [4, page 29] (see also the corresponding presentation in [36]).

Definition 4.1. A nonabelian cocycle (g_{ij}, h_{ijk}) is the data of (smooth) transition functions

$$g_{ij} : U_i \cap U_j \to G$$

and

$$h_{ijk} : U_i \cap U_j \cap U_k \to H$$

which satisfy the nonabelian cocycle identities

$$\mu(h_{ijk}(x))g_{ij}(x)g_{jk}(x) = g_{ik}(x)$$

for all $x \in U_{ijk} := U_i \cap U_j \cap U_k$, and

$$h_{ikl}(x)h_{ijk}(x) = h_{ijl}(x)(g_{ij}(x) \cdot h_{jkl}(x))$$

for all $x \in U_{ijkl} := U_i \cap U_j \cap U_k \cap U_l$.

The Čech cochains g_{ij} and h_{ijk} are (by definition) ordered in the indices; one may then extend to antisymmetric indices. One may furthermore complete the set of indices to all pairs respectively triples by imposing the functions to be equal to 1_G respectively 1_H on repeated indices.

We go on by defining equivalence of nonabelian cocycles with values in the same crossed module of Lie groups $\mu : H \to G$.

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Definition 4.2. Two nonabelian cocycles (g_{ij}, h_{ijk}) and (g'_{ij}, h'_{ijk}) on the same cover are said to be *equivalent* if there exist (smooth) functions $\gamma_i : U_i \to G$ and $\eta_{ij} : U_{ij} \to H$ such that

$$\gamma_i(x)g'_{ij}(x) = \mu(\eta_{ij}(x))g_{ij}(x)\gamma_j(x)$$

for all $x \in U_{ij}$, and

$$\eta_{ik}(x)h_{ijk}(x) = (\gamma_i(x) \cdot h'_{ijk}(x))\eta_{ij}(x)(g_{ij}(x) \cdot \eta_{jk}(x))$$

for all $x \in U_{ijk}$.

In general, one should define equivalence for cocycles corresponding to different covers. Passing to a common refinement, one easily adapts the above definition to this framework (this is spelt out in [36]). Furthermore, one also defines equivalence for cocycles corresponding to different crossed modules; see, for example, [26].

Definition 4.3. A principal 2-bundle, also called a (nonabelian) gerbe and denoted $\mathcal{G}$, is the data of an equivalence class of nonabelian cocycles.

By abuse of language, we will also call a representative (g_{ij}, h_{ijk}) a principal 2-bundle or a (nonabelian) gerbe.

Lemma 4.4. *If the (split) crossed module of Lie groups $\mu : H \to G$ is replaced by an equivalent crossed module $\mu' : H' \to G'$, then the corresponding principal 2-bundles are equivalent.*

Proof. This follows from [26, Theorem 6.2.3]. □

This lemma is important for us, because in case we replace Lie group crossed modules by equivalent ones, we want to obtain an equivalent principal 2-bundle.

Recall that for a split crossed module of Lie groups $\mu : H \to G$, the image $\mu(H)$ is a normal Lie subgroup of G, and the quotient group $\bar{G} := G/\mu(H)$ is therefore a Lie group.

Lemma 4.5. *Let $\mathcal{G}$ be a gerbe defined by the cocycle (g_{ij}, h_{ijk}).*

Then one may associate to $\mathcal{G}$ an ordinary principal $\bar{G}$-bundle $\mathcal{B}$ on M which has as its transition functions the composition of the g_{ij} and the canonical projection $G \to G/\mu(H) = \bar{G}$.

Proof. This is clear. Indeed, passing to the quotient $G \to G/\mu(H)$, the identity

$$\mu(h_{ijk}(x))g_{ij}(x)g_{jk}(x) = g_{ik}(x)$$

becomes the cocycle identity

$$\bar{g}_{ij}(x)\bar{g}_{jk}(x) = \bar{g}_{ik}(x)$$

for a principal $G/\mu(H)$-bundle on M defined by the transition functions

$$\bar{g}_{ij} : U_{ij} \to G/\mu(H)$$

obtained from composing $g_{ij} : U_{ij} \to G$ with the projection $G \to G/\mu(H)$. □

Definition 4.6. The principal $\bar{G}$-bundle $\mathcal{B}$ on M associated to the gerbe $\mathcal{G}$ defined by the cocycle (g_{ij}, h_{ijk}) is called the band of $\mathcal{G}$.

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4.2. Connection data. Let, as before, M be a manifold and let $\mathcal{U} = \{U_i\}$ be a good open cover of M. Let $\mathcal{G}$ be a gerbe defined by the cocycle (g_{ij}, h_{ijk}). We associate to $\mathcal{G}$ now connection data like in [4, Section 2.1.4].

DEFINITION 4.7. *Connection data* for the nonabelian cocycle (g_{ij}, h_{ijk}) is the data of *connection* 1*-forms* $A_i \in \Omega^1(U_i, \mathfrak{g})$ and of *curving* 2*-forms* $B_i \in \Omega^2(U_i, \mathfrak{h})$, together with *connection transformation* 1*-forms* $a_{ij} \in \Omega(U_{ij}, \mathfrak{h})$ and *curving transformation* 2*-forms* $d_{ij} \in \Omega^2(U_{ij}, \mathfrak{h})$ such that the following laws hold:

(a) transition law for connection 1-forms on U_{ij}

$$A_i + \mu(a_{ij}) = g_{ij}A_j g_{ij}^{-1} + g_{ij}dg_{ij}^{-1};$$

(b) transition law for the curving 2-forms on U_{ij}

$$B_i = g_{ij} \cdot B_j + da_{ij};$$

(c) transition law for the curving transformation 2-forms on U_{ijk}

$$d_{ij} + g_{ij} \cdot d_{jk} = h_{ijk} d_{ik} h_{ijk}^{-1} + h_{ijk}(\mu(B_i) + F_{A_i})h_{ijk}^{-1};$$

(d) coherence law for the transformers of connection 1-forms on U_{ijk}

$$0 = a_{ij} + g_{ij} \cdot a_{jk} - h_{ijk} a_{ik} h_{ijk}^{-1} - h_{ijk} dh_{ijk}^{-1} - h_{ijk}(A_i \cdot h_{ijk}^{-1}).$$

In accordance with [4, Equation (2.73) on page 59], we will choose $d_{ij} = 0$ in the following. The transition law (c) for the curving transformation 2-forms reads then simply

$$0 = \mu(B_i) + F_{A_i},$$

which is the equation of *vanishing fake curvature*. In the following, we will always suppose that the fake curvature vanishes (cf. Section 5).

DEFINITION 4.8. Let $\mathcal{G}$ be a gerbe defined by the cocycle (g_{ij}, h_{ijk}) with connection data (A_i, B_i, a_{ij}). Then the *curvature* 3*-form* $H_i \in \Omega^3(U_i, \mathfrak{h})$ is defined by

$$H_i = d_{A_i} B_i,$$

that is, it is the covariant derivative of the curving 2-form $B_i \in \Omega^2(U_i, \mathfrak{h})$ with respect to the connection 1-form $A_i \in \Omega^1(U_i, \mathfrak{h})$.

Its transformation law on U_{ij} is

$$H_i = g_{ij} \cdot H_j$$

(because in our setting fake curvature and curving transformation 2-forms vanish).

Observe that only the crossed module of Lie algebras $\mu : \mathfrak{h} \to \mathfrak{g}$ plays a role as values of the differential forms A_i and B_i. According to Section 2, it constitutes no restriction of generality (up to equivalence) to consider $\mathfrak{h}$ abelian. If all components in the crossed module are finite dimensional, this equivalence induces even an equivalence between the corresponding crossed modules of 1-connected Lie groups. In our main application

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(construction of the holonomy higher Hochschild cycle), we will suppose $\mathfrak{h}$ to be abelian. Many steps on the way are true for arbitrary $\mathfrak{h}$. The property of being abelian simplifies the above coherence law (d) for the transformers of connection 1-forms on U_{ijk} for which we thus obtain in the abelian setting

$$0 = a_{ij} + g_{ij} \cdot a_{jk} - a_{ik} - h_{ijk} dh_{ijk}^{-1} - h_{ijk}(A_i \cdot h_{ijk}^{-1}).$$

We note in passing the following lemma.

Lemma 4.9. *The connection* 1*-forms induce an ordinary connection on the band* $\mathcal{B}$ *of the gerbe* $\mathcal{G}$.

Proof. This follows at once from equation (a) in the definition of connection data. □

On the other hand, we will always be in a local setting; therefore, in the following, we will drop the indices $i, j, k, \ldots$ which refer to the open set we are on.

5. L_∞-valued differential forms

In this section, we will associate to each principal 2-bundle an L_∞-algebra of L_∞-valued differential forms. This L_∞-algebra replaces the differential graded Lie algebra of Lie algebra valued forms which plays a role for ordinary principal G-bundles. Here the L_∞-algebra of values (of the differential forms) will be the two-term L_∞-algebra associated with the strict structure Lie 2-algebra of the principal 2-bundle. We follow closely [12, Section 4].

Given an L_∞-algebra $\mathfrak{g}_\infty$ and a manifold M, the tensor product $\Omega^*(M) \otimes \mathfrak{g}_\infty$ of $\mathfrak{g}_\infty$-valued smooth differential forms on M is an L_∞-algebra by prolonging the L_∞-operations of $\mathfrak{g}_\infty$ point by point to differential forms. The only point to notice is that the de Rham differential d_{deRham} gives a contribution to the first bracket $l_1 : \mathfrak{g}_\infty \to \mathfrak{g}_\infty$, which is also a differential of degree one.

We will apply this scheme to the two-term L_∞-algebras arising from a semi-strict Lie 2-algebra $\mathfrak{g}_\infty = (\mathfrak{g}_{-1}, \mathfrak{g}_0)$. The only (possibly) nonzero operations are the differential l_1, the bracket $[\,,\,] = l_2$ and the 3-cocycle l_3. Our choice of degrees is that an element $\alpha^k \in \Omega^*(M) \otimes \mathfrak{g}_\infty$ is of degree k in case $\alpha^k \in \bigoplus_{i \geq 0} \Omega^i(M) \otimes \mathfrak{g}_{k-i}$. An element of degree one is thus a sum $\alpha^1 = \alpha_1 + \alpha_2$ with $\alpha_1 \in \Omega^1(M) \otimes \mathfrak{g}_0$ and $\alpha_2 \in \Omega^2(M) \otimes \mathfrak{g}_{-1}$.

Recall the following definitions (cf. [12, Definition 4.2]).

Definition 5.1. The Maurer–Cartan set $\mathrm{MC}(\mathfrak{g}_\infty)$ of an L_∞-algebra $\mathfrak{g}_\infty$ is the set of $\alpha \in \mathfrak{g}^1$ satisfying the Maurer–Cartan equation $\mathcal{F}(\alpha) = 0$. More explicitly, this means that

$$\mathcal{F}(\alpha) := l_1 \alpha + \sum_{k=2}^{\infty} \frac{1}{k!} l_k(\alpha, \ldots, \alpha) = 0.$$

The Maurer–Cartan equations for the degree-one elements of the L_∞-algebra $\Omega^*(M) \otimes \mathfrak{g}_\infty$ (see [12, Definition 4.3]) with $\mathfrak{g}_\infty = (\mathfrak{g}_{-1}, \mathfrak{g}_0)$ read therefore

$$d_{\text{deRham}} \alpha_1 + \tfrac{1}{2}[\alpha_1, \alpha_1] + l_1 \alpha_2 = 0 \tag{5.1}$$

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and

$$d_{\text{deRham}}\alpha_2 + [\alpha_1, \alpha_2] + l_3(\alpha_1, \alpha_1, \alpha_1) = 0. \tag{5.2}$$

Equation (5.1) is an equation of 2-forms; in the gerbe literature it is known as the equation of the vanishing of the fake curvature. Equation (5.2) is an equation of 3-forms and seems to be (kind of) new in this context (it appears in [17, Section 8]). The special case $l_3 = 0$ corresponds to [29, Example 6.5.1.3], when one interprets $d_{\text{deRham}}\alpha_2 + [\alpha_1, \alpha_2]$ as the covariant derivative $d_{\alpha_1}\alpha_2$. When applied to connection data of a nonabelian gerbe (see Section 4.2), the vanishing of the covariant derivative means that the 3-curvature (cf. Definition 4.8) of the gerbe vanishes. This is sometimes expressed as being a *flat gerbe*.

Let us record the special case of a strict Lie 2-algebra $\mathfrak{g}_\infty$ given by a crossed module $\mu : \mathfrak{h} \to \mathfrak{g}$ for later use.

Lemma 5.2. *A degree-one element of the L_∞-algebra $\Omega^*(M) \otimes \mathfrak{g}_\infty$ is a pair (A, B) with $A \in \Omega^1(M) \otimes \mathfrak{g}$ and $B \in \Omega^2(M) \otimes \mathfrak{h}$.*

The element (A, B) satisfies the Maurer–Cartan equation if and only if

$$d_{\text{deRham}}A + \tfrac{1}{2}[A, A] + \mu B = 0 \quad \text{and} \quad d_{\text{deRham}}B + [A, B] = 0.$$

Elements of degree zero in $\Omega^*(M) \otimes \mathfrak{g}$ are sums $\alpha^0 = \beta_0 + \beta_1$ with $\beta_0 \in \Omega^0(M) \otimes \mathfrak{g}_0$ and $\beta_1 \in \Omega^1(M) \otimes \mathfrak{g}_{-1}$. These act by gauge transformations on elements of the Maurer–Cartan set. Namely, β_0 has to be exponentiated to an element $B_0 \in \Omega^0(M, G_0)$ (where G_0 is the connected, 1-connected Lie group corresponding to $\mathfrak{g}_0$) and leads then to gauge transformations of the first kind, in the sense of [4]. Elements β_1 lead directly to gauge transformations of the second kind, in the sense of [4]. The fact that they do not have to be exponentiated corresponds to the fact that there is no bracket on the $\mathfrak{g}_{-1}$-part of the L_∞-algebra. These gauge transformations will not play a role in the present paper, but will become a central subject when gluing the local expressions of the connection 1-form of Baez–Schreiber to a global connection.

Definition 5.3. Let $\mathfrak{g}$ be an L_∞-algebra. The Maurer–Cartan variety $\mathcal{MC}(\mathfrak{g})$ is the quotient of the Maurer–Cartan set $\mathrm{MC}(\mathfrak{g})$ by the exponentiated action of the infinitesimal automorphisms $\mathfrak{g}^0$ of $\mathrm{MC}(\mathfrak{g})$.

We do not assert that the quotient $\mathcal{MC}(\mathfrak{g})$ is indeed a variety. It is considered here as a set.

6. Path space and the connection 1-form associated to a principal 2-bundle

In this section we explain how the connective structure on a gerbe gives rise to a connection on path space.

6.1. Path space as a Fréchet manifold. We first recall some basic facts about path spaces which allow us to employ the basic notions of differential geometry, in particular differential forms and connections, to path spaces. For a manifold M, let $\mathcal{P}M := C^\infty([0,1], M)$ be the space of paths in M. Baez and Schreiber [4] fixed in their definition the starting point and the end point of the paths, that is, for two points s and t in M, $\mathcal{P}_s^t M$ denotes the space of paths from s ('source') to t ('target'). The space $\mathcal{P}M$ can made into a Fréchet manifold modeled on the Fréchet space $C^\infty([0,1], \mathbb{R}^n)$ in case n is the dimension of M. Similar constructions exhibit the loop space

$$LM := \{\gamma : [0,1] \overset{C^\infty}{\to} M \,|\, \gamma(0) = \gamma(1), \gamma^{(k)}(0) = \gamma^{(k)}(1) = 0 \,\forall k \geq 1\}$$

as a Fréchet manifold; see, for example, [24] for a detailed account of the Fréchet manifold structure on (this version of) LM in case M is a Lie group. The generalization to arbitrary M is quite standard. Let us emphasize that this version of LM comes to mind naturally when writing the circle S^1 as $[0,1]/\sim$ in $C^\infty(S^1, M)$. The fact that one demands $\gamma^{(k)}(0) = \gamma^{(k)}(1) = 0$ and not only $\gamma^{(k)}(0) = \gamma^{(k)}(1)$ for all $k \geq 1$ is sometimes expressed by saying that the loops have a 'sitting instant'.

For differentiable Fréchet manifolds, one can introduce differential forms, the de Rham differential and prove a de Rham theorem for smoothly paracompact Fréchet manifolds. The only thing beyond the necessary definitions that we need from Fréchet differential geometry is an expression of the de Rham differential on LM, an expression due to Chen [9] which will play its role in the proof of Proposition 7.2.

6.2. The connection 1-form of Baez–Schreiber. Let $\mu : H \to G$ be a split crossed module of Lie groups. Denote by S the outer action of $\bar{G}$ on H, that is, the homomorphism $S : \bar{G} \to \mathrm{Out}(H)$.

Composing the transition functions $\bar{g}_{ij} : U_{ij} \to \bar{G}$ with the homomorphism $S : \bar{G} \to \mathrm{Out}(H)$, we obtain the transition functions of an $\mathrm{Out}(H)$-principal bundle denoted $\mathcal{B}_S$. This is then an ordinary principal bundle and we may apply ordinary holonomy theory to the principal bundle $\mathcal{B}_S$.

Remark 6.1. Observe that a different choice of the outer action $S : \bar{G} \to \mathrm{Out}(H)$ results in an inner automorphism and thus in an isomorphic bundle $\mathcal{B}_S$. The same holds for a different choice of transition functions.

As we are only interested in these constructions and these constructions are purely on Lie algebra level, we will neglect now the crossed module of Lie groups and focus on the crossed module of Lie algebras. In doing so, we may assume (up to equivalence without loss of generality) that in the crossed module $\mu : \mathfrak{h} \to \mathfrak{g}$, $\mathfrak{h}$ is abelian and that the outer action $s : \bar{\mathfrak{g}} \to \mathfrak{out}(\mathfrak{h})$ (associated to $\mu : \mathfrak{h} \to \mathfrak{g}$ like in Section 1) is a genuine action (see Corollary 2.14). Note that we thus have $\mathfrak{out}(\mathfrak{h}) = \mathfrak{der}(\mathfrak{h}) = \mathrm{End}(\mathfrak{h})$, again because $\mathfrak{h}$ is abelian.

In the following, we will suppose that the principal bundle $\mathcal{B}_S$ is trivial (or, in other words, we will do a local construction). A connection 1-form on $\mathcal{B}_S$ is then simply a differential form $A_S \in \Omega^1(M, \mathrm{End}(\mathfrak{h}))$ and, given a 1-form $A \in \Omega^1(M, \bar{\mathfrak{g}})$, one obtains

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such a form A_S by $A_S := s \circ A$. We will suppose that $\mathcal{B}_S$ possesses a flat connection ∇, which will be our reference point in the affine space of connections.

Actually, in case the 1-form $A \in \Omega^1(M, \mathfrak{g})$ (and not in $\Omega^1(M, \bar{\mathfrak{g}})$!), there is no problem to define the action of A on B. We do not need $\mathfrak{h}$ to be abelian here (in case we do not want to use the band, for example).

Consider now the loop space LM of M. Let us proceed with the Wilson loop or iterated integral construction of [1, Section 6]. For every $n \geq 0$, consider the n-simplex

$$\triangle^n := \{(t_0, t_1, \ldots, t_n, t_{n+1}) \,|\, 0 = t_0 \leq t_1 \leq \cdots \leq t_n \leq t_{n+1} = 1\}.$$

Define the evaluation maps ev and $\mathrm{ev}_{n,i}$ as follows:

$$\begin{aligned}
&\mathrm{ev} : \triangle^n \times LM \to M,\\
&\mathrm{ev}(t_0, t_1, \ldots, t_n, t_{n+1}; \gamma) = \gamma(0) = \gamma(1),\\
&\mathrm{ev}_{n,i} : \triangle^n \times LM \to M,\\
&\mathrm{ev}_{n,i}(t_0, t_1, \ldots, t_n, t_{n+1}; \gamma) = \gamma(t_i).
\end{aligned}$$

Denote by $\mathrm{ad}\,\mathcal{B}_S$ the adjoint bundle associated to the principal bundle $\mathcal{B}_S$ using the adjoint action of $\mathrm{Out}(H)$ on $\mathfrak{out}(\mathfrak{h}) = \mathfrak{der}(\mathfrak{h}) = \mathrm{End}(\mathfrak{h})$. Let $T_i : \mathrm{ev}_{n,i}^*(\mathrm{ad}\,\mathcal{B}_S) \to \mathrm{ev}^*(\mathrm{ad}\,\mathcal{B}_S)$ denote the map, between pullbacks of adjoint bundles to $\triangle^n \times LM$, defined at a point $(0 = t_0, t_1, \ldots, t_n, t_{n+1} = 1; \gamma)$ by the parallel transport along and in the direction of γ from $\gamma(t_i)$ to $\gamma(t_{n+1}) = \gamma(1)$ in the bundle $\mathrm{ad}\,\mathcal{B}_S$ with respect to the flat connection ∇.

Denote by $\mathcal{B}_S^U$ the associated bundle to $\mathcal{B}_S$ with typical fiber the universal enveloping algebra $U\,\mathrm{End}(\mathfrak{h})$.

For $\alpha_i \in \Omega^*(M, \mathrm{ad}\,\mathcal{B}_S)$, $1 \leq i \leq n$, define $V^n_{\alpha_1,\ldots,\alpha_n} \in \Omega^*(LM, \mathrm{ev}^*\mathcal{B}_S^U)$ by

$$V^0_{\alpha_1,\ldots,\alpha_n} = 1,$$

$$V^n_{\alpha_1,\ldots,\alpha_n} = \int_{\triangle^n} T_1 \mathrm{ev}_{n,1}^*\alpha_1 \wedge \cdots \wedge T_n \mathrm{ev}_{n,n}^*\alpha_i \quad \text{for } n \geq 1$$

and set

$$V_\alpha = \sum_{n=0}^{\infty} V^n_{\alpha,\ldots,\alpha}.$$

It is noteworthy that this infinite sum is convergent. This is shown in [1, Appendix B]. Observe that for 1-forms $\alpha_1, \ldots, \alpha_n$, the loop space form $V^n_{\alpha_1,\ldots,\alpha_n}$ has degree zero for all n.

Furthermore, define for $B \in \Omega^2(M, \mathfrak{h})$ and $\sigma \in [0, 1]$ the 1-form $B^*(\sigma) \in \Omega^1(LM, \mathfrak{h})$ by

$$B^*(\sigma) = i_K \mathrm{EV}_\sigma^* B$$

for the evaluation map $\mathrm{EV}_\sigma : LM \to M$, $EV_\sigma(\gamma) := \gamma(\sigma)$ and the vector field K on LM, which is the infinitesimal generator of the S^1-action on LM by rigid rotations.

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Now fix an element (A, B) of the Maurer–Cartan set with respect to some Lie algebra crossed module $\mu : \mathfrak{h} \to \mathfrak{g}$. Evaluating elements of $U\,\mathrm{End}(\mathfrak{h})$ on $\mathfrak{h}$, we obtain a connection 1-form $\mathcal{A}_0$ on LM with values in $\mathfrak{h}$ given by

$$\mathcal{A}_0 = \int_0^1 V_A(B^*(\sigma))\,d\sigma.$$

(Indeed, as A is a 1-form, the loop space form $V_A \in \Omega^0(LM, \mathrm{ev}^*\mathcal{B}_S^U)$ is of degree zero, that is, a $\mathrm{ev}^*\mathcal{B}_S^U$-value function, and $V_A(B^*(\sigma))$ is of degree one and remains of degree one after integration with respect to σ.)

This gives the formula for the connection 1-form of Baez and Schreiber [4, page 43].

Proposition 6.2. *The constructed connection* 1*-form* $\mathcal{A}_0$ *on* LM *with values in* $\mathfrak{h}$ *coincides with the path space* 1*-form* $\mathcal{A}_{(A,B)} = \oint_A(B)$ *of* [4, Definition 2.23].

Proof. This follows from a step-by-step comparison. □

7. The holonomy cycle associated to a principal 2-bundle

A central construction of [2] associates to elements $\mathcal{A}$ in the Maurer–Cartan space a holonomy class $[P(\mathcal{A})]$ in $HH_*(\Omega^*, \Omega^*)$. This is done using the following proposition (cf. [1, Section 4]).

Proposition 7.1. *Suppose given a differential graded associative algebra* (Ω^*, d) *and an element* $\mathcal{A} \in \Omega^{\mathrm{odd}}$*, the following are equivalent:*

(a) $\mathcal{A}$ *is a Maurer–Cartan element, that is,* $d\mathcal{A} + \mathcal{A}\cdot\mathcal{A} = 0$*;*

(b) *the chain*

$$P(\mathcal{A}) := 1\otimes 1 + 1\otimes\mathcal{A} + 1\otimes\mathcal{A}\otimes\mathcal{A} + \cdots$$

in the Hochschild complex $CH_*(\Omega^*, \Omega^*)$ *is a cycle.*

Proof. We adopt the sign convention of Tsygan [10, page 78] for differential graded Hochschild homology. In this convention, the total differential d_{Hoch} is the sum of the internal differential

$$d(a_0\otimes\cdots\otimes a_p) = \sum_{i=0}^{p}(-1)^{1+\sum_{k<i}(|a_k|+1)}a_0\otimes\cdots\otimes da_i\otimes\cdots\otimes a_p$$

and of the (appropriately signed) Hochschild differential

$$b(a_0\otimes\cdots\otimes a_p) = \sum_{k=0}^{p-1}(-1)^{1+\sum_{i=0}^{k}(|a_i|+1)}a_0\otimes\cdots\otimes a_k a_{k+1}\otimes\cdots\otimes a_p \\ + (-1)^{|a_p|+(|a_p|+1)\sum_{i=0}^{p-1}(|a_i|+1)}a_p a_0\otimes\cdots\otimes a_{p-1}.$$

Note that there is no (additional relative) sign between d and b, that is, the total differential is $d_{\mathrm{Hoch}} = d + b$ (and satisfies $d_{\mathrm{Hoch}}^2 = 0$).

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We have to compute $d_{\mathrm{Hoch}}(P(\mathcal{A}))$. Let us only write down the terms contributing to one fixed tensor degree $p+1$. These are $d(1 \otimes \underbrace{\mathcal{A} \otimes \cdots \otimes \mathcal{A}}_{p \text{ times}})$ and $b(1 \otimes \underbrace{\mathcal{A} \otimes \cdots \otimes \mathcal{A}}_{p+1 \text{ times}})$. We obtain for the sum of these two terms,

$$\sum_{i=0}^{p} (-1)^{1+\sum_{k<i}(|\mathcal{A}|+1)} 1 \otimes \mathcal{A} \cdots \otimes \underbrace{d\mathcal{A}}_{i\text{th place}} \otimes \cdots \otimes \mathcal{A}$$

$$+ \sum_{i=0}^{p} (-1)^{1+\sum_{k=0}^{i}(|\mathcal{A}|+1)} 1 \otimes \mathcal{A} \cdots \otimes \underbrace{\mathcal{A} \cdot \mathcal{A}}_{i\text{th place}} \otimes \cdots \otimes \mathcal{A}$$

$$+ (-1)^{|\mathcal{A}|+(p+1)(|\mathcal{A}|+1)^2} \mathcal{A} \otimes \cdots \otimes \mathcal{A}.$$

Now in this sum there are exactly two terms of the form $\mathcal{A} \otimes \cdots \otimes \mathcal{A}$. In case the degree of $\mathcal{A}$, that is, $|\mathcal{A}|$, is odd, these two terms add up to zero. On the other hand, the other terms of the sum add to terms of the form

$$\sum \pm 1 \otimes \mathcal{A} \cdots \otimes (d\mathcal{A} + \mathcal{A} \cdot \mathcal{A}) \otimes \cdots \otimes \mathcal{A}.$$

Therefore, the Maurer–Cartan equation implies that $P(\mathcal{A})$ is a cycle.

In the reverse direction, if $P(\mathcal{A})$ is a cycle, then the terms of tensor degree two give exactly the Maurer–Cartan equation. Thus, the cycle property is equivalent to the Maurer–Cartan equation. □

The degrees are taken such that all terms in $P(\mathcal{A})$ are of degree zero in case $\mathcal{A}$ is of degree one, that is, the degrees of Ω^* are shifted by one. This is the correct degree when taking Hochschild homology as a model for loop space cohomology.

We will apply this proposition to the connection 1-form $\mathcal{A}_0$ on LM. The 1-form $\mathcal{A}_0$ is an element of $\Omega^1(LM, U\mathfrak{h})$. The condition that $\mathcal{A}_0$ is a Maurer–Cartan element is then that the curvature of $\mathcal{A}_0$ vanishes. This curvature has been computed in [4, page 43] to be given by the following formula (needless to say, no assumption is made on $\mathfrak{h}$ for this computation).

Proposition 7.2. *The curvature of the* 1*-form* $\mathcal{A}_0$ *is equal to*

$$\mathcal{F}_{\mathcal{A}_0} := -\oint_A (d_A B) - \oint_A (d\alpha(T_a)(B), (F_A + \mu(B))^a)$$

$$:= \int_0^1 V_A((d_A B)^*(\sigma))\, d\sigma - \int_0^1 V_A((F_A + \mu(B))^*(\sigma))\, d\sigma,$$

where $d_A B$ *is the covariant derivative of* B *with respect to* A *and* $F_A + \mu(B)$ *is the fake curvature of the couple* (A, B).

Proof. A detailed proof is given in [4, Proposition 2.7 and Corollary 2.2, pages 42–43]. Here we will only sketch the main steps of the proof.

First compute $d_{\text{deRham}}\mathcal{A}_0$ for the de Rham differential d_{deRham}. As explained in [4, Proposition 2.4, page 35], the action of the de Rham differential on a Chen form $\oint_A(\omega_1, \ldots, \omega_n)$ is given by two terms, namely,

$$\sum_k \pm \oint_A (\omega_1, \ldots, d_{\text{deRham}}\omega_k, \ldots \omega_n)$$

and

$$\sum_k \pm \oint_A (\omega_1, \ldots, \omega_{k-1} \wedge \omega_k, \ldots, \omega_n).$$

In our case, we thus get four terms, according to whether B is involved or not. The terms which do not involve B give a term involving the curvature $F_A = dA + A \cdot A$. The terms involving B give a term involving the covariant derivative $d_A B = d_{\text{deRham}} B + A \cdot B$ of B with respect to A.

Now the computation of the curvature of $\mathcal{A}_0$ adds to the de Rham derivative $d_{\text{deRham}}\mathcal{A}_0$ a term $\mathcal{A}_0 \cdot \mathcal{A}_0$. This term is easily seen to be the term involving $\mu(B)$. □

Remark 7.3. If $f(A, B)$ is a Maurer–Cartan element (in the sense of Section 5), then (by Lemma 5.2 and Proposition 7.2) $d_A B$ and the fake curvature vanish; therefore, $\mathcal{A}_0$ is a flat connection and $P(\mathcal{A}_0)$ is a Hochschild cycle (by Proposition 7.1).

Definition 7.4. The Hochschild cycle $P(\mathcal{A}_0)$ is the holonomy cycle associated to the given principal 2-bundle with connection.

Let us abbreviate $\Omega^*(M, U\mathfrak{h})$ to Ω^*. Our main point is now that the assumption that $\mathfrak{h}$ is abelian implies that Ω^* (and for the same reason also $\Omega^*(LM, U\mathfrak{h})$) is a *commutative* differential graded algebra; thus, the shuffle product endows the (ordinary) Hochschild complex $CH_*(\Omega^*, \Omega^*)$ with the structure of a differential graded commutative algebra (cf. [20, Corollary 4.2.7, page 125]). On the other hand, we have for a simply connected manifold M the following lemma.

Lemma 7.5. *There is a quasi-isomorphism of commutative differential graded algebras*

$$\Omega^*(LM, U\mathfrak{h}) \simeq CH_*(\Omega^*, \Omega^*).$$

Proof. Let us first observe that the loop space with sitting instant LM is homotopically equivalent to $C^\infty(S^1, M)$. A retraction is given by precomposing by a path in the circle with sitting instants and winding number one.

Therefore, our assertion is a version with coefficients in the graded associative algebra $U\mathfrak{h}$ of [21, Corollary 2.6, page 11], originally shown by Chen [9]. Observe that the coefficients do not contribute to the differentials. □

In conclusion, we obtain a homology class

$$[P(\mathcal{A}_0)] \in HH_*(CH_*(\Omega^*, \Omega^*), CH_*(\Omega^*, \Omega^*)).$$

In the next section, we will explain how to interpret

$$HH_*(CH_*(\Omega^*, \Omega^*), CH_*(\Omega^*, \Omega^*))$$

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in terms of higher Hochschild homology as $HH_*^{\mathbb{T}}(\Omega^*, \Omega^*)$, the *higher Hochschild homology* of the two-dimensional torus $\mathbb{T}$. We therefore obtain

$$[P(\mathcal{A}_0)] \in HH_*^{\mathbb{T}}(\Omega^*, \Omega^*).$$

8. Proof of the main theorem

In this section, we consider higher Hochschild homology. It has been introduced by Pirashvili in [27] and further developed by Ginot [13] and Ginot *et al.* in [15]. Here we follow closely [15].

In order to define higher Hochschild homology, it is essential to restrict to *commutative* differential graded associative algebras Ω^*. We will see below explicitly why this is the case.

Denote by $\triangle$ the (standard) category whose objects are the finite ordered sets $[k] = \{0, 1, \ldots, k\}$ and morphisms $f : [k] \to [l]$ are nondecreasing maps, that is, for $i > j$, one has $f(i) \geq f(j)$. Special nondecreasing maps are the injections $\delta_i : [k-1] \to [k]$ characterized by missing i (for $i = 0, \ldots, k$) and the surjections $\sigma_j : [k] \to [k-1]$ which send j and $j+1$ to j (equally for $j = 0, \ldots, k$).

Denote by $\mathtt{Sets_{fin}}$ the category of finite sets. A *finite simplicial set* $Y_\bullet$ is by definition a contravariant functor $Y_\bullet : \triangle^{\mathrm{op}} \to \mathtt{Sets_{fin}}$. The sets of k-simplices are denoted $Y_k := Y([k])$. The induced maps $d_i := Y_\bullet(\delta_i)$ and $s_j := Y_\bullet(\sigma_j)$ are called *faces* and *degeneracies*, respectively. Let $Y_\bullet$ be a pointed finite simplicial set. For $k \geq 0$, we put $y_k := |Y_k| - 1$, that is, one less than the cardinal of the finite set Y_k.

The *higher Hochschild chain complex* of Ω^* associated to the simplicial set $Y_\bullet$ (and with values in Ω^*) is defined by

$$CH_\bullet^{Y_\bullet}(\Omega^*, \Omega^*) := \bigoplus_{n \in \mathbb{Z}} CH_n^{Y_\bullet}(\Omega^*, \Omega^*),$$

where

$$CH_n^{Y_\bullet}(\Omega^*, \Omega^*) := \bigoplus_{k \geq 0} (\Omega^* \otimes (\Omega^*)^{\otimes y_k})_{n+k}.$$

In order to define the differential, define induced maps as follows. For any map $f : Y_k \to Y_l$ of pointed sets and any (homogeneous) element $m \otimes a_1 \otimes \cdots \otimes a_{y_k} \in \Omega^* \otimes (\Omega^*)^{\otimes y_k}$, we denote by $f_* : \Omega^* \otimes (\Omega^*)^{\otimes y_k} \to \Omega^* \otimes (\Omega^*)^{\otimes y_l}$ the map

$$f_*(m \otimes a_1 \otimes \cdots \otimes a_{y_k}) := (-1)^\epsilon n \otimes b_1 \otimes \cdots \otimes b_{y_l},$$

where $b_j = \Pi_{i \in f^{-1}(j)} a_i$ (or $b_j = 1$ in case $f^{-1}(j) = \emptyset$) for $j = 0, \ldots, y_l$, and $n = m \cdot \Pi_{i \in f^{-1}(\text{basepoint}), j \neq \text{basepoint}} a_i$. The sign ϵ is determined by the usual Koszul sign rule. The above face and degeneracy maps d_i and s_j induce thus boundary maps $(d_i)_* : CH_k^{Y_\bullet}(\Omega^*, \Omega^*) \to CH_{k-1} = k^{Y_\bullet}(\Omega^*, \Omega^*)$ and degeneracy maps $(s_j)_* : CH_{k-1}^{Y_\bullet}(\Omega^*, \Omega^*) \to CH_k^{Y_\bullet}(\Omega^*, \Omega^*)$, which are once again denoted d_i and s_j by abuse of notation. Using these, the differential $D : CH_\bullet^{Y_\bullet}(\Omega^*, \Omega^*) \to CH_\bullet^{Y_\bullet}(\Omega^*, \Omega^*)$ is defined by setting

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$D(a_0 \otimes a_1 \otimes \cdots \otimes a_{y_k})$ equal to

$$\sum_{i=0}^{y_k} (-1)^{k+\epsilon_i} a_0 \otimes \cdots \otimes d_i a_i \otimes \cdots \otimes a_{y_k} + \sum_{i=0}^{k} (-1)^i d_i (a_0 \otimes \cdots \otimes a_{y_k}),$$

where ϵ_i is again a Koszul sign (see the explicit formula in [15]). The simplicial relations imply that $D^2 = 0$ (this is the instance where one uses that Ω^* is graded *commutative*). These definitions extend by inductive limit to arbitrary (that is, not necessarily finite) simplicial sets.

The homology of $CH_\bullet^{Y_\bullet}(\Omega^*, \Omega^*)$ with respect to the differential D is by definition the *higher Hochschild homology* $HH_\bullet^{Y_\bullet}(\Omega^*, \Omega^*)$ of Ω^* associated to the simplicial set $Y_\bullet$. In fact, for two simplicial sets $Y_\bullet$ and $Y'_\bullet$ which have homeomorphic geometric realization, the complexes $(CH_\bullet^{Y_\bullet}(\Omega^*, \Omega^*), D)$ and $(CH_\bullet^{Y'_\bullet}(\Omega^*, \Omega^*), D)$ are quasi-isomorphic; thus, the higher Hochschild homology does only depend on the topological space which is the realization of $Y_\bullet$. Therefore, we will for example write $HH_\bullet^{\mathbb{T}}(\Omega^*, \Omega^*)$ for the higher Hochschild homology of Ω^* associated to the two-dimensional torus $\mathbb{T}$, inferring that it is computed with respect to some simplicial set having $\mathbb{T}$ as its geometric realization.

For the simplicial model of the circle S^1 given in [15, Example 2.3.1], one obtains the usual Hochschild homology. In this sense, $HH_\bullet^{Y_\bullet}$ generalizes ordinary Hochschild homology.

Example 2.4.5 in [15] gives the following result.

For the simplicial model of the 2-torus $\mathbb{T}$ given in Example 2.3.2 (of [15]), the algebra $CH_\bullet^{\mathbb{T}}(\Omega^*, \Omega^*)$ is quasi-isomorphic to

$$CH_\bullet(CH_*(\Omega^*, \Omega^*), CH_*(\Omega^*, \Omega^*)).$$

In this sense, the holonomy cycle $P(\mathcal{A}_0)$ (constructed in the previous section) may be regarded as living in the higher Hochschild complex $CH_\bullet^{\mathbb{T}}(\Omega^*, \Omega^*)$. This completes the proof of Theorem 1.1.

Remark 8.1. Observe that the element $P(\mathcal{A}_0)$ in $CH_\bullet^{\mathbb{T}}(\Omega^*, \Omega^*)$ is of total degree zero. Recall from [15, Corollary 2.4.7] the iterated integral map $It^{Y_\bullet}$ of [15] which provides a morphism of differential graded algebras

$$It^{Y_\bullet} : CH_\bullet^{Y_\bullet}(\Omega^*, \Omega^*) \to \Omega^\bullet(M^{\mathbb{T}}, U\mathfrak{h}).$$

Here $M^{\mathbb{T}} := C^\infty(\mathbb{T}, M)$. The image of $P(\mathcal{A}_0)$ in $\Omega^\bullet(M^{\mathbb{T}}, U\mathfrak{h})$ represents a degree-zero cohomology class which associates to each map $f : \mathbb{T} \to M$ an element of $U\mathfrak{h}$ which is interpreted as the gerbe holonomy taken over $f(\mathbb{T}) \subset M$. We believe that an explicit expression of this cohomology class (in the special case of an abelian gerbe where all forms are real-valued) is given exactly by Gawedzki–Reis' formula (2.14) [11]. The factors g_{ijk} do not appear in our formula, because we did not do the gluing yet and therefore everything is local.

Observe further that following the steps in [15, proof of Corollary 2.4.4], one may express $P(\mathcal{A}_0)$ in terms of matrices in A and B. This is what we do in the next section.

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9. Explicit expression for the holonomy cycle

In order to find an explicit expression for the holonomy cycle, we have to translate first the connection 1-form $\mathcal{A}_0 \in \Omega^*(LM, U\mathfrak{h})$ into an element of $CH_*(\Omega^*, \Omega^*)$ (with $\Omega^* = \Omega^*(M, U\mathfrak{h})$) using the quasi-isomorphism $\Omega^*(LM, U\mathfrak{h}) \simeq CH_*(\Omega^*, \Omega^*)$. The second step is then to translate the holonomy cycle $P(\mathcal{A}_0) \in CH_*(CH_*(\Omega^*, \Omega^*), CH_*(\Omega^*, \Omega^*))$ using this expression of $\mathcal{A}_0$.

The construction of $\mathcal{A}_0$ uses an iterated integral involving the 1-form $A \in \Omega^1(M, \mathfrak{g})$ and the 2-form $B \in \Omega^2(M, \mathfrak{h})$. Let us denote by $\widetilde{\mathcal{A}}_0$ the form $\widetilde{\mathcal{A}}_0 \in \Omega^1(LM, U\mathfrak{g} \otimes U\mathfrak{h})$ which arises before combining the $\mathfrak{g}$ coefficients of the A-components of the iterated integral with the $\mathfrak{h}$ coefficient of the B-component using the action of $\mathfrak{g}$ on $\mathfrak{h}$. This form is in fact in $\Omega^1(LM, U\mathfrak{g} \oplus \mathfrak{h})$, because the $\mathfrak{h}$ coefficients arise only at one place, namely in B. The transcription of $\widetilde{\mathcal{A}}_0$ into an element of $CH_*(\Omega^*(M, \mathfrak{g} \oplus \mathfrak{h}), \Omega^*(M, \mathfrak{g} \oplus \mathfrak{h}))$ is rather easy. The explicit expression is

$$\widetilde{\mathcal{A}}_0 = \sum_{n=0}^{\infty} 1 \otimes \underbrace{A \otimes \cdots \otimes A}_{n\,\text{times}} \otimes B.$$

The form $\mathcal{A}_0$ is derived from this using two morphisms. We use the fact that a Lie algebra homomorphism $\phi : \mathfrak{m} \to \mathfrak{n}$ induces a morphism of differential graded Lie algebras $\phi_* : \Omega^*(M, \mathfrak{m}) \to \Omega^*(M, \mathfrak{n})$. The first homomorphism of Lie algebras is the action $\alpha : U\mathfrak{g} \to \mathfrak{gl}(\mathfrak{h})$ sending $x_1 \cdot \ldots \cdot x_r$ to $\alpha(x_1, \alpha(x_2, \ldots, \alpha(x_r, -) \ldots))$. This is well defined, because α is zero on the ideal generated by $x \otimes y - y \otimes x - [x, y]$ in the tensor algebra $T\mathfrak{g}$ on $\mathfrak{g}$, and it is a Lie algebra homomorphism (using that $\mathfrak{g}$ acts on $\mathfrak{h}$ by derivations).

The second Lie algebra homomorphism is more involved. It is the evaluation morphism $\mathrm{ev} : \mathrm{End}(\mathfrak{h}) \oplus \mathfrak{h} \to \mathfrak{h}$. This is a Lie algebra homomorphism only if we consider the zero bracket on $\mathfrak{h}$. As we do have this restriction for other reasons at one place, we shall use it here also. This explains how to obtain $\mathcal{A}_0$ from $\widetilde{\mathcal{A}}_0$ by applying two Lie algebra homomorphisms on the coefficient side. This reasoning permits to split form part and coefficient part of $\mathcal{A}_0$.

From the above expression for $\widetilde{\mathcal{A}}_0$, we obtain easily an expression for $P(\widetilde{\mathcal{A}}_0)$:

$$P(\widetilde{\mathcal{A}}_0) = \sum_{l=0}^{\infty} 1 \boxtimes \underbrace{\left(\sum_{n=0}^{\infty} 1 \otimes \underbrace{A \otimes \cdots \otimes A}_{n\,\text{times}} \otimes B \right) \boxtimes \cdots \boxtimes \left(\sum_{n=0}^{\infty} 1 \otimes \underbrace{A \otimes \cdots \otimes A}_{n\,\text{times}} \otimes B \right)}_{l\,\text{times}}.$$

This is the element in $P(\widetilde{\mathcal{A}}_0) \in CH_*(CH_*(\Omega^*, \Omega^*), CH_*(\Omega^*, \Omega^*))$ where $\Omega^* := \Omega^*(M, \mathfrak{g} \oplus \mathfrak{h})$ and where $\boxtimes$ denotes the tensor product in the Hochschild complex, as opposed to the tensor product $\otimes$ which occurs in $\widetilde{\mathcal{A}}_0$ before applying the above homomorphisms to obtain $\mathcal{A}_0$. In order to obtain an explicit expression for $P(\widetilde{\mathcal{A}}_0)$ in the higher Hochschild complex with respect to the torus $\mathbb{T}^2$, we have to apply the quasi-isomorphisms of [15, Corollary 2.4.4]. The result will be displayed in terms of

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matrices, in accordance with the chain model for higher Hochschild homology of the torus $\mathbb{T}^2$ given in [15, Example 2.3.2].

As an example, let us treat one of the most simple terms, namely the form $1 \boxtimes (1 \otimes B) \boxtimes (1 \otimes B)$. The first step is to apply degeneracies in order to have the same degree in terms of ⊗'s and ⊠'s. This means we pass to

$$(1 \otimes 1 \otimes 1) \boxtimes (1 \otimes 1 \otimes B) \boxtimes (1 \otimes B \otimes 1).$$

We will write simply 1 for $1 \otimes 1 \otimes 1$ or $1 \otimes 1 \otimes 1 \otimes 1 \otimes 1$. The second step is to apply degeneracies according to shuffles. In our example, we have $p = 2$ (internal S^1) and $q = 2$ (external S^1), so we sum over all $(2, 2)$-shuffles. The 6 terms of the sum are thus (up to signs):

(1) $1 \boxtimes (1 \otimes 1 \otimes 1 \otimes 1 \otimes B) \boxtimes (1 \otimes 1 \otimes 1 \otimes B \otimes 1) \boxtimes 1 \boxtimes 1$;
(2) $1 \boxtimes (1 \otimes 1 \otimes 1 \otimes 1 \otimes B) \boxtimes 1 \boxtimes (1 \otimes 1 \otimes B \otimes 1 \otimes 1) \boxtimes 1$;
(3) $1 \boxtimes (1 \otimes 1 \otimes 1 \otimes B \otimes 1) \boxtimes 1 \boxtimes 1 \boxtimes (1 \otimes 1 \otimes B \otimes 1 \otimes 1)$;
(4) $1 \boxtimes 1 \boxtimes (1 \otimes 1 \otimes 1 \otimes 1 \otimes B) \boxtimes (1 \otimes B \otimes 1 \otimes 1 \otimes 1) \boxtimes 1$;
(5) $1 \boxtimes 1 \boxtimes (1 \otimes 1 \otimes 1 \otimes B \otimes 1) \boxtimes 1 \boxtimes (1 \otimes B \otimes 1 \otimes 1 \otimes 1)$;
(6) $1 \boxtimes 1 \boxtimes 1 \boxtimes (1 \otimes 1 \otimes B \otimes 1 \otimes 1) \boxtimes (1 \otimes B \otimes 1 \otimes 1 \otimes 1)$.

They correspond in this order to the shuffles $(1 < 2, 3 < 4)$, $(1 < 3, 2 < 4)$, $(1 < 4, 2 < 3)$, $(2 < 3, 1 < 4)$, $(2 < 4, 1 < 3)$ and $(3 < 4, 1 < 2)$. The first component is p and the second is q. The shuffles indicate where to place the 1's (internally and externally). These six terms may be translated into the following matrices:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & B \\ 1 & 1 & 1 & B & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & B \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & B & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & B & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & B & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & B \\ 1 & B & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & B & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & B & 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & B & 1 & 1 \\ 1 & B & 1 & 1 & 1 \end{pmatrix}.$$

Observe that in these matrices, only the first line and column are both intersecting the diagonal and also filled with 1's. If another pair line/column had this property, it would mean that the corresponding matrix is a degenerate Hochschild chain. A similar discussion holds for the general term in $P(\widetilde{\mathcal{A}}_0)$.

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Observe that matrices of the same type arose in the work of Tradler *et al.* [33, Remark 4.10], in the context of abelian gerbes.

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HOSSEIN ABBASPOUR, Université de Nantes,
2, rue de la Houssiniere, Nantes 44322, France
e-mail: hossein.abbaspour@univ-nantes.fr

FRIEDRICH WAGEMANN, Université de Nantes,
2, rue de la Houssiniere, Nantes 44322, France
e-mail: wagemann@math.univ-nantes.fr

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