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Unique ergodicity of the automorphism group of the semigeneric directed graph

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Abstract. We prove that the automorphism group of the semigeneric directed graph (in the sense of Cherlin's classification) is uniquely ergodic.

Key words: unique ergodicity, ergodic decomposition, semigeneric directed graph.
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1. *Introduction*

One key notion in the study of dynamical properties of Polish groups is amenability. A topological group is amenable when every flow, that is, a continuous action on a compact space, admits a Borel probability measure that is invariant under the action of the group.

In recent years, the study of non-locally compact Polish groups has exhibited several refinements of this phenomenon. One of them is extreme amenability: a topological group is extremely amenable when every flow admits a fixed point (see [**KPT05**]). Another one is unique ergodicity: a topological group is uniquely ergodic if every minimal flow, that is, a flow where every orbit is dense, admits a unique Borel probability measure that is invariant under the action of the group. In this paper, all measures will be Borel probability measures.

Of course, extreme amenability implies unique ergodicity, but the converse is not true as, for instance, every compact group is uniquely ergodic. Beyond compactness, though, no example is known in the locally compact Polish case and Weiss proves in [**Wei12**] that there is no uniquely ergodic discrete group. In fact, it is suggested on page 5 in [**AKL12**] that, in the setting of locally compact groups, compactness is the only way to achieve unique ergodicity. However, some examples appear in the non-locally compact Polish case. The first of these examples was S_∞, the group of all permutations of $\mathbb{N}$ equipped with the pointwise convergence topology (this was done by Glasner and Weiss in [**GW02**]). Angel,

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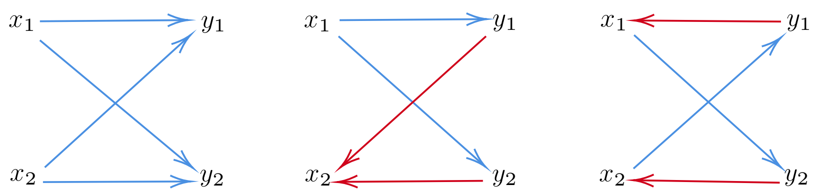

FIGURE 1. The three possible configurations (up to isomorphism) of two pairs of equivalent points respecting the parity condition.

Kechris and Lyons then showed, using probabilistic combinatorial methods, that several groups of the form Aut($\mathbb{F}$), where $\mathbb{F}$ is a particular kind of countable structure called a Fraïssé limit, are also uniquely ergodic (see [**AKL12**]).

A Fraïssé limit is a countable first-order homogeneous structure in the sense of model theory whose age, that is, the set of its finite substructures up to isomorphism, is a Fraïssé class. A class $\mathcal{F}$ of finite structures is a Fraïssé class if it contains structures of arbitrarily large (finite) cardinality and satisfies the following.

(HP) If $A \in \mathcal{F}$ and B is a substructure of A, then $B \in \mathcal{F}$.

(JEP) If $A, B \in \mathcal{F}$, then there exists $C \in \mathcal{F}$ such that A and B can be embedded in C.

(AP) If $A, B, C \in \mathcal{F}$ and $f : A \to B$, $g : A \to C$ are embeddings, then there exist $D \in \mathcal{F}$ and $h : B \to D$, $l : C \to D$ embeddings such that $h \circ f = l \circ g$.

Examples of Fraïssé classes include the class of finite graphs, the class of finite graphs omitting a fixed clique and the class of finite r-uniform hypergraphs for any $r \in \mathbb{N}$. The unique ergodicity of the automorphism groups of the limits of these classes was proved in [**AKL12**].

The Fraïssé limit of a Fraïssé class is unique up to isomorphism. By definition, Fraïssé limits are homogeneous, that is, any isomorphism between two finite parts of the structure can be extended in an automorphism of the structure. For more details on Fraïssé classes see [**Hod93**].

In [**PSar**], using methods from [**AKL12**], Pawliuk and Sokić extended the catalogue of uniquely ergodic automorphism groups with the automorphism groups of homogeneous directed graphs, which were all classified by Cherlin (see [**Che98**]), leaving as an open question only the case of the semigeneric directed graph.

This graph, which we denote $\mathbb{S}$, is the Fraïssé limit of the class $\mathcal{S}$ of simple, loopless, directed, finite graphs that verify the following conditions, where $\to$ denotes the directed edge.

(i) The relation $\perp$, defined by $x \perp y$ if and only if $\neg(x \to y \vee y \to x)$, is an equivalence relation.

(ii) For any $x_1 \neq x_2$, $y_1 \neq y_2$ such that $x_1 \perp x_2$ and $y_1 \perp y_2$, the number of (directed) edges from $\{x_1, x_2\}$ to $\{y_1, y_2\}$ is even (Fig. 1).

We will refer to $\perp$-equivalence classes as columns and to the second condition as the parity condition. The $\perp$-class of an element $a \in \mathbb{S}$ will be referred to as $a^\perp$.

More details on this structure will be given in the next section.

In this paper, we prove the following theorem.

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THEOREM 1.1. *The topological group* Aut($\mathbb{S}$) *is uniquely ergodic.*

The method we use is different from the one found in [**AKL12**, **PSar**] since we do not work with the so-called 'quantitative expansion property', but instead we show that an ergodic measure can only take certain values on a generating part of the Borel sets. It is also different from the approach in [**Tsa14**], Theorem 7.4 which only applies when the structure eliminates imaginaries. Our method relies on the idea that if there are equivalence classes in a structure and the universal minimal flow is essentially the convex orderings regarding the equivalence classes, then the ordering inside the equivalence classes and the ordering of the equivalence classes are independent, provided that the automorphism group behaves well enough.

2. *Preliminaries*

The starting point of our proof is common with that of [**AKL12**]: to prove that Aut($\mathbb{S}$) is uniquely ergodic, it suffices to show that one particular action is uniquely ergodic, namely, its universal minimal flow, Aut($\mathbb{S}$) $\curvearrowright$ M(Aut($\mathbb{S}$)). This is the unique minimal Aut($\mathbb{S}$)-flow that maps onto any minimal Aut($\mathbb{S}$)-flow (such a flow exists for any Hausdorff topological group by a classical result of Ellis; see [**Ell69**]); an explicit description was made by Jasiński, Laflamme, Nguyen Van Thé and Woodrow in [**JLNW14**]. It is the space of expansions of $\mathbb{S}$ whose age is a certain class $\mathcal{S}^*$.

Before describing this class, we give some more background on $\mathbb{S}$. Observe that the parity condition is equivalent to the fact that, for every $\mathbf{A} \in \mathcal{S}$ and two columns P, Q in $\mathbf{A}$, we have, for all $x, x' \in P$,

$$\text{(for all } y \in Q,\ ((x \to y) \Leftrightarrow (x' \to y))) \text{ or (for all } y \in Q,\ ((x \to y) \Leftrightarrow (y \to x'))).$$

This remark allows us to define the equivalence relation $\sim_Q$ on P as

$$x \sim_Q x' \Leftrightarrow \text{for all } y \in Q,\ (x \to y \Leftrightarrow x' \to y).$$

Note that, as a consequence of the parity condition, we get that, in $\mathbb{S}$,

$$\text{for all } y \in Q,\ (x \to y \Leftrightarrow x' \to y) \ \Leftrightarrow\ \text{there exists } y \in Q,\ (x \to y \text{ and } x' \to y).$$

We can now consider P^0 and P^1, the two $\sim_Q$ equivalence classes in P, and we have $P = P^0 \sqcup P^1$. Note that each of these class could be empty. Similarly, we have $Q = Q^0 \sqcup Q^1$, where Q^0 and Q^1 are $\sim_P$-equivalence classes. Note that, at this stage, the labelling of these classes is arbitrary, which is crucial to the construction and understanding of $\mathcal{S}^*$ below. Indeed, the language of $\mathcal{S}^*$ has a binary relation R whose interpretation is mainly to give a proper labelling of these equivalence classes.

This description has an interesting consequence when we recall that there must be an edge between any two points of P and Q. Denote $P^i \to Q^j$ to mean that, for all $x \in P^i$ and $y \in Q^j$, we have $x \to y$. Then $P^i \to Q^j$ implies that $Q^j \to P^{1-i}$, $P^{1-i} \to Q^{1-j}$ and $Q^{1-j} \to P^i$. In particular, this means that, for each $i \in \{0, 1\}$, there is a unique $j \in \{0, 1\}$ such that $P^i \to Q^j$ (Fig. 2).

The class $\mathcal{S}^*$ is the class of finite structures in the language $L = (\to, <, R)$, verifying the following.

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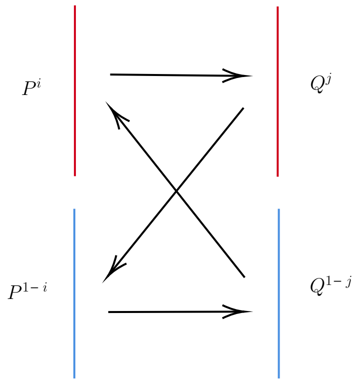

FIGURE 2. The different equivalent classes induced by two columns.

(A) $\mathcal{S}^*_{|\rightarrow} = \mathcal{S}$.

(B) $<$ is interpreted as a linear ordering convex with respect to the columns, that is, the columns are intervals for the ordering. For two columns P, Q, we will therefore write $P < Q$ to mean that, for all $x \in P$, $y \in Q$ we have $x < y$.

(C) For $\mathbf{A}^* \in \mathcal{S}^*$, the binary relation $R^{\mathbf{A}^*}$ verifies:

(a) for all $x, y \in \mathbf{A}^*$,

$$R^{\mathbf{A}^*}(x, y) \Rightarrow \neg x \perp y;$$

(b) for all $x, y, y' \in \mathbf{A}^*$,

$$(R^{\mathbf{A}^*}(x, y) \mathbin{\&} y \perp y') \Rightarrow R^{\mathbf{A}^*}(x, y');$$

(c) for all $x, x', y \in \mathbf{A}^*$,

$$(x \rightarrow y \mathbin{\&} y \rightarrow x' \mathbin{\&} x \perp x' \mathbin{\&} x <^{\mathbf{A}^*} y) \Rightarrow (R^{\mathbf{A}^*}(x, y) \Leftrightarrow \neg R^{\mathbf{A}^*}(x', y)).$$

Observe that in a structure $\mathbf{A}^* \in \mathcal{S}^*$, $R^{\mathbf{A}^*}$ gives us a proper labelling of the $\sim_Q$-equivalence classes in P when $P < Q$. In particular, we can render the arbitrary decomposition $P = P^0 \sqcup P^1$, $Q = Q^0 \sqcup Q^1$ canonical by setting

$$x \in P^1 \Leftrightarrow (\text{for all } y \in Q,\ R^{\mathbf{A}^*}(x, y))$$

and

$$y \in Q^1 \Leftrightarrow (\text{for all } x \in P,\ (y \rightarrow x \Leftrightarrow R^{\mathbf{A}^*}(x, y))).$$

A remarkable property of this decomposition is that the edge relation is actually entirely defined by it. Indeed, take two columns P, Q in $\mathbf{A}^*$ that we decompose as above in $P = P^0 \sqcup P^1$, $Q = Q^0 \sqcup Q^1$. We know, by construction of R on Q, that $Q^1 \rightarrow P^1$. As we observed before, this means that $P^1 \rightarrow Q^0$, $P^0 \rightarrow Q^1$ and $Q^0 \rightarrow P^0$.

Another point of view on this expansion is given in [JLNW14]. Take $\mathbf{A} \in \mathcal{S}$ with n columns $P_1, \ldots, P_n$ and an expansion $\mathbf{A}^* \in \mathcal{S}^*$. The expansion $\mathbf{A}^*$ is interdefinable with a structure $\mathbf{A}^{**}$ in the language $\{\rightarrow, <, L_{i,f}\}$, where $L_{i,f}$ is a unary predicate for all $i \in \{1, \ldots, n\} = [n]$ and $f \in 2^{[n]\setminus i}$. We have $\mathbf{A}^*_{|\rightarrow,<} = \mathbf{A}^{**}_{|\rightarrow,<}$. Assuming that

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$P_1 <^{\mathbf{A}^*} \cdots <^{\mathbf{A}^*} P_n$, we define

$$L^{\mathbf{A}^{**}}_{i,f} = \{x \in P_i : \text{for all } j \in [n]\backslash i,\ y \in P_j\ (f(j) = 1 \Leftrightarrow R^{\mathbf{A}^*}(x, y))\}.$$

Denote $\mathcal{M} \subset \{0, 1\}^{\mathbb{S}^2} \times \{0, 1\}^{\mathbb{S}^2}$, the space of expansions of $\mathbb{S}$ whose age is exactly $\mathcal{S}^*$. We will denote $E = (<^E, R^E)$, the elements of $\mathcal{M}$, by identification with the structure that can be inferred from the expansion. The following result is shown in [**JLNW14**].

THEOREM 2.1. *The universal minimal flow of* Aut($\mathbb{S}$) *is* Aut($\mathbb{S}$) $\curvearrowright \mathcal{M}$.

We are interested in showing that the Aut($\mathbb{S}$)-invariant measures on $\mathcal{M}$ are all equal. A useful tool of measure theory is the following lemma (see [**Gut05**, Theorem 3.5]).

LEMMA 2.2. *Let μ and ν be two probability measures defined on a σ-field $\mathcal{E}$. If there is a family $(A_n)_{n\in\mathbb{N}} \in \mathcal{E}^{\mathbb{N}}$, stable under intersection, that generates $\mathcal{E}$ and is such that, for all $n \in \mathbb{N}$, $\mu(A_n) = \nu(A_n)$, then $\mu = \nu$.*

The rest of this section is devoted to describing a family $\mathcal{P}$ of clopen sets that generate the Borel sets of $\mathcal{M}$. The sets of our family $\mathcal{P}$ are of the form

$$U_{(x_i)_{i=1}^n,(\varepsilon_i^j)_{1\le i<j\le n}} \cap V_{(a_1^1,\ldots,a_{i_1}^1),\ldots,(a_1^k,\ldots,a_{i_k}^k)} \subset \mathcal{M}.$$

They are defined as follows.

Let $(x_i)_{i=1}^n$ be in different columns. Let $(\varepsilon_i^j)_{i<j\le n} \in \{0, 1\}^{\binom{n}{2}}$. An element $E = (<^E, R^E) \in \mathcal{M}$ belongs to $U_{(x_i)_{i=1}^n,(\varepsilon_i^j)_{1\le i<j\le n}}$ if and only if the following conditions are satisfied.

(A) $(x_1^\perp <^E \cdots <^E x_n^\perp)$.

(B) For $k < l$,

$$R^E(x_k, x_l) \Leftrightarrow (x_k \to x_l)^{\varepsilon_k^l},$$

where, for all $x, y \in \mathbb{S}$ and $\varepsilon \in \{0, 1\}$, $(x \to y)^\varepsilon$ means that $(x \to y)$ if $\varepsilon = 1$ and $\neg(x \to y)$ otherwise.

The rest of R on those columns can be recovered from this by construction of $\mathcal{S}^*$. Indeed, observe that, for all $x \in x_k^\perp$, $y \in x_l^\perp$,

$$R^E(x, y) \Leftrightarrow ((x \sim^{\mathbb{S}}_{x_l^\perp} x_k \text{ and } R^E(x_k, x_l)) \text{ or } (x \nsim^{\mathbb{S}}_{x_l^\perp} x_k \text{ and } \neg R^E(x_k, x_l))).$$

An important remark is that if we have a different family $(x'_1, \ldots, x'_n)$ such that $x_i \perp x'_i$, then there is a family $(\alpha_i^j)_{1\le i<j\le n}$ such that

$$U_{(x_i)_{i=1}^n,(\varepsilon_i^j)_{1\le i<j\le n}} = U_{(x'_i)_{i=1}^n,(\alpha_i^j)_{1\le i<j\le n}}.$$

This can be achieved by taking $\alpha_i^j = \varepsilon_i^j$ if $x_i \sim_{x_j^\perp} x'_i$ and $\alpha_i^j = 1 - \varepsilon_i^j$ otherwise.

An additional remark that will be useful throughout the paper is that, for a given family $(x_1, \ldots, x_n)$ of elements taken in different columns,

$$\mathcal{M} = \bigsqcup_{\sigma\in S_n,(\varepsilon_i^j)_{1\le i<j\le n}} U_{(x_{\sigma(i)})_{i=1}^n,(\varepsilon_i^j)_{1\le i<j\le n}}.$$

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We also define

$$V_{(a_1^1,\ldots,a_{i_1}^1),\ldots,(a_1^k,\ldots,a_{i_k}^k)} = \{E \in \mathcal{M} : (a_1^1 <^E \cdots <^E a_{i_1}^1) \wedge \cdots \wedge (a_1^k <^E \cdots <^E a_{i_k}^k)\},$$

where $(a_i^j \perp a_{i'}^{j'})$ if and only if $j = j'$.

This collection of sets is a generating family for the open sets of our space, so it is also a generating family for the Borel sets.

To use Lemma 2.2, we would also need to know that this family is stable under intersection, but unfortunately this is not the case. However, the intersection of two sets in $\mathcal{P}$ is actually a disjoint union of sets in $\mathcal{P}$. Therefore, if we consider $\mathcal{P}'$ to be the collection of finite intersections of elements of $\mathcal{P}$, the evaluation of a measure on an element of $\mathcal{P}'$ is determined by the evaluation of the measure on $\mathcal{P}$. By Lemma 2.2, any measure is entirely characterized by its evaluation on elements of $\mathcal{P}'$, so it is characterized by its evaluation on elements of $\mathcal{P}$.

3. *Invariant measures*

From this point on, we denote $G = \mathrm{Aut}(\mathbb{S})$. Let us first define μ_0, which is a G-invariant probability measure on $\mathcal{M}$. We define μ_0 by

$$\mu_0\big(U_{(x_i)_{i=1}^n,(\varepsilon_i^j)_{1\le i<j\le n}} \cap V_{(a_1^1,\ldots,a_{i_1}^1),\ldots,(a_1^k,\ldots,a_{i_k}^k)}\big) = \frac{1}{n!\,2^{\binom{n}{2}}}\frac{1}{\prod_{j=1}^{k} i_j!}.$$

We call μ_0 the uniform measure. It is proved in [PSar] that this measure is well defined on all Borel sets and that it is G-invariant. We want to show that it is actually the only invariant measure. By Lemma 2.2, we only have to check that the invariant measures coincide on $U_{(x_i)_{i=1}^n,(\varepsilon_i^j)_{1\le i<j\le n}} \cap V_{(a_1^1,\ldots,a_{i_1}^1),\ldots,(a_1^k,\ldots,a_{i_k}^k)}$.

Before proving Theorem 1.1, we need to prove the following preliminary results.

PROPOSITION 3.1. *For all $(x_i)_{i=1}^n$ such that $\neg(x_i \perp x_j)$ for $i \neq j$ and $(\varepsilon_i^j)_{i<j\le n} \in 2^{\binom{n}{2}}$,*

$$\mu\big(U_{(x_i)_{i=1}^n,(\varepsilon_i^j)_{1\le i<j\le n}}\big) = \frac{1}{n!\,2^{\binom{n}{2}}}.$$

PROPOSITION 3.2. *For all $(a_1^1,\ldots,a_{i_1}^1,\ldots,a_1^k,\ldots,a_{i_k}^k)$ such that $a_i^j \perp a_{i'}^{j'}$ if and only if $j = j'$,*

$$\mu\big(V_{(a_1^1,\ldots,a_{i_1}^1),\ldots,(a_1^k,\ldots,a_{i_k}^k)}\big) = \frac{1}{\prod_{j=1}^{k} i_j!}.$$

Similar results were proved in [PSar]. We will prove those results using different methods. The proof of Proposition 3.2 is very similar to what we will do later on in order to conclude the paper and it contains the key argument.

For proofs of Proposition 3.2 and Theorem 1.1, we will need an ergodic decomposition theorem, and thus we need to define the notion of ergodicity.

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Definition 3.3. Let Γ be a Polish group acting continuously on a compact space X. A Γ-invariant measure ν is said to be Γ-*ergodic* if, for all A measurable such that

$$\text{for all } g \in \Gamma, \quad \nu(A \triangle g \cdot A) = 0,$$

we have $\nu(A) \in \{0, 1\}$.

We can now state the following theorem (see [**Phe01**, Proposition 12.4]).

THEOREM 3.4. *Let Γ be a Polish group acting continuously on a compact space X. Let $P(X)$ denote the space of probability measures on X and let $P_\Gamma(X) = \{\mu \in P(X) : \Gamma \cdot \mu = \mu\}$. Then, the extreme points of $P_\Gamma(X)$ are the Γ-ergodic invariant measures.*

We will also need to use Neumann's lemma (see [**Cam99**, Theorem 6.2]).

THEOREM 3.5. *Let H be a group acting on Ω with no finite orbit. Let Γ and Δ be finite subsets of Ω. Then there is $h \in H$ such that $h \cdot \Gamma \cap \Delta = \varnothing$.*

The remainder of the section will be divided in three subsections: one for the proof of Proposition 3.1, one for the proof of Proposition 3.2 and finally one for the proof of Theorem 1.1.

3.1. *Proof of Proposition 3.1.* For this proof, we will need the following technical lemma.

LEMMA 3.6. *Let $k < n$, let $P_1, \ldots, P_n$ be different columns in $\mathbb{S}$ and let $y_1 \in P_1, \ldots, y_k \in P_k$. Take a given family $\varepsilon_i^j \in \{0, 1\}$, where $1 \le i < j \le n$ and $k < j$. Then there exist $y_{k+1} \in P_{k+1}, \ldots, y_n \in P_n$ such that $(y_i \to y_j)^{\varepsilon_i^j}$ for all $i < j$ and $k < j$.*

Proof. Take $x_{k+1} \in P_{k+1}, \ldots, x_n \in P_n$. Consider the structure

$$\mathbf{A} = ((y_1^A, \ldots, y_n^A, x_{k+1}^A, \ldots, x_n^A), \to^{\mathbf{A}}),$$

where $(y_i^A \to^{\mathbf{A}} y_j^A) \Leftrightarrow (y_i \to y_j)$ if $i < j \le k$, $(y_i^A \to^{\mathbf{A}} y_j^A) \iff (\varepsilon_i^j = 1)$ if $1 \le i < j \le n$ and $k < j$. We also have $x_i^A \perp^{\mathbf{A}} y_i^A$ for $i > k$ and $(x_i^A \to^{\mathbf{A}} x_j^A \Leftrightarrow x_i \to x_j)$ for $k < i < j$.

We put edges between x_i^A and y_j^A in order for them to respect the parity condition. Notice that there is more than one way to do this: for instance, one can ask that when $k < i < j$, $(x_i^A \to^{\mathbf{A}} y_j^A) \Leftrightarrow (x_i^A \to^{\mathbf{A}} x_j^A)$ and $(x_j^A \to^{\mathbf{A}} y_i^A) \Leftrightarrow (y_j^A \to^{\mathbf{A}} y_i^A)$. The remaining edges can be added arbitrarily because they concern columns with only one vertex.

We make sure that $\mathbf{A} \in \mathcal{S}$. Indeed, note that, since there is one point in the first k columns and two in the remaining ones, it suffices to check the parity condition in the last $n - k$ columns. Take $k < j < i \le n$. We know that the edges between x_i^A and y_j^A and the edge between x_i^A and x_j^A go in the same direction. Similarly, the edge between x_j^A and y_i^A and the edge between y_j^A and y_i^A also go in the same direction. Therefore the parity condition must be respected.

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We remark that $((y_1^A, \ldots, y_k^A, x_{k+1}^A, \ldots, x_n^A), \to^{\mathbf{A}})$ and $((y_1, \ldots, y_k, x_{k+1}, \ldots, x_n), \to^{\mathbb{S}})$ are isomorphic, and hence $\mathbf{A}$ embeds in $\mathbb{S}$ in a way that extends this isomorphism. The image of $(y_{k+1}^A, \ldots, y_n^A)$ is as required. $\square$

The fundamental observation for the proof of Proposition 3.1 is that, if we take $x_1, \ldots, x_n \in \mathbb{S}$ all in different columns,

$$\overline{\mathrm{Aut}(\mathbb{S}) \cdot (<^*, R^*)} = \bigsqcup_{\sigma \in S_n,\ (\varepsilon_i^j)_{1 \leq i < j \leq n}} U_{(x_{\sigma(i)})_{i=1}^n, (\varepsilon_i^j)_{1 \leq i < j \leq n}}.$$

We will show that, for any two families $\varepsilon = (\varepsilon_i^j)_{i<j\leq n}$, $\alpha = (\alpha_i^j)_{i<j\leq n}$ and $\sigma \in S_n$, there is a $g \in G$ such that

$$U_{(x_i)_{i=1}^n, \varepsilon} = g \cdot U_{(x_{\sigma(i)})_{i=1}^n, \alpha}.$$

This means that all sets of this form have the same measure, and hence we will have the result because there are $n!\, 2^{\binom{n}{2}}$ such sets.

First, we construct $g' \in G$ such that

$$g' \cdot U_{(x_{\sigma(i)})_{i=1}^n, \alpha} = U_{x_1, \ldots, x_n, \beta}$$

for some $\beta = (\beta_i^j)_{1\leq i<j\leq n}$.

We want to prove that there is $g' \in G$ such that $g' \cdot x_i \in (x_{\sigma(i)})^{\perp}$. By Lemma 3.6, there exists $x_1', \ldots, x_n' \in \mathbb{S}$ such that $x_{\sigma(i)} \perp x_i'$ and $x_i \to x_j$ if and only if $x_i' \to x_j'$. Note that, by construction, there is a partial automorphism τ that sends $x_{\sigma(i)}$ to x_i'. By homogeneity, there is g', an automorphism of $\mathbb{S}$ that extends τ. We remark that

$$g' \cdot U_{(x_i)_{i=1}^n, \alpha} = U_{(x'_{\sigma(i)})_{i=1}^n, \alpha}$$

and, as we observed before, $U_{(x'_{\sigma(i)})_{i=1}^n, \alpha}$ does not depend on x_i', but on their columns. Thus, there exist a family $\beta = (\beta_i^j)_{1\leq i<j\leq n}$ such that

$$U_{(x'_{\sigma(i)})_{i=1}^n, \alpha} = U_{(x_i)_{i=1}^n, \beta}.$$

Next, we construct $h \in G$ such that

$$U_{(x_i)_{i=1}^n, \varepsilon} = h \cdot U_{(x_i)_{i=1}^n, \beta}.$$

Assume that there are $k < l$ such that $\beta_i^j = \varepsilon_i^j$ for all $(i, j) \neq (k, l)$ and $\beta_k^l \neq \varepsilon_k^l$. Note that taking care of this case will be enough to prove the result. If α and β disagree in more than one coordinate, iterating this process still allows us to modify coordinates one at a time.

Take $x_k' \perp x_k$ such that, for all $i \in [n]\backslash\{k, l\}$, $x_k' \to x_i$ if and only if $x_k \to x_i$ and $x_k' \to x_l$ if and only if $x_l \to x_k$. This is possible using Lemma 3.6, where $\{y_1, \ldots, y_{n-1}\} = \{x_1, \ldots, x_n\}\backslash\{x_k\}$ and $P_n = x_k^{\perp}$. We define $x_l' \perp x_l$ similarly.

We take $h \in G$ such that $h(x_i) = x_i$ for all $i \in [n]\backslash\{k, l\}$, $h(x_k') = x_k$ and $h(x_l') = x_l$. By homogeneity, such a h exists: indeed, by the parity condition, we have $(x_k \to x_l) \Leftrightarrow (x_k' \to x_l')$. Let us prove that h gives the result.

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Take $E \in U_{x_1,\ldots,x_n,\beta}$. We will prove that

$$h \cdot E \in U_{(x_i)_{i=1}^n,\varepsilon}.$$

For all $i < j$, we want to prove that

$$R^{h\cdot E}(x_i, x_j) \Leftrightarrow (x_i \to x_j)^{\varepsilon_i^j},$$

and since

$$R^{h\cdot E}(x_i, x_j) \Leftrightarrow R^E(h^{-1}(x_i), h^{-1}(x_j)),$$

we prove that

$$R^E(h^{-1}(x_i), h^{-1}(x_j)) \Leftrightarrow (x_i \to x_j)^{\varepsilon_i^j}.$$

If $\{i, j\} \cap \{k, l\} = \varnothing$, the result is obvious.

If $j = k$ and $i < k$,

$$\begin{aligned} R^{h\cdot E}(x_i, x_k) &\Leftrightarrow R^E(h^{-1}(x_i), h^{-1}(x_k)) \\ &\Leftrightarrow (x_i \to h^{-1}(x_k))^{\beta_i^k} \\ &\Leftrightarrow (x_i \to x_k')^{\beta_i^k} \\ &\Leftrightarrow (x_i \to x_k)^{\beta_i^k}, \end{aligned}$$

and since $\beta_i^k = \varepsilon_i^k$,

$$R^{h\cdot E}(x_i, x_k) \Leftrightarrow (x_i \to x_k)^{\varepsilon_i^k}.$$

The other cases where $|\{i, j\} \cap \{k, l\}| = 1$ are similar.

Finally, if $(i, j) = (k, l)$,

$$\begin{aligned} R^{h\cdot E}(x_k, x_l) &\Leftrightarrow R^E(h^{-1}(x_k), h^{-1}(x_l)) \\ &\Leftrightarrow (x_k \to h^{-1}(x_l))^{\beta_k^l} \\ &\Leftrightarrow (x_k \to x_l')^{\beta_k^l} \\ &\Leftrightarrow (x_k \to x_l)^{\varepsilon_k^l}. \end{aligned}$$

The last equivalence is a direct consequence of the definition of x_l' and the fact that $\beta_k^l = (1 - \varepsilon_k^l)$. □

3.2. *Proof of Proposition 3.2.* We prove the result by induction on the number k of columns.

By homogeneity, for any column $(a_1^j)^\perp$ and $\sigma \in S_{i_j}$, there exists $g \in G$ such that

$$g \cdot V_{(a_1^j,\ldots,a_{i_j}^j)} = V_{(a_{\sigma(1)}^j,\ldots,a_{\sigma(i_j)}^j)},$$

and thus

$$\mu\big(V_{(a_1^j,\ldots,a_{i_j}^j)}\big) = \frac{1}{i_j!}.$$

This proves the initial case.

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Let us now assume that, for all $(a_1^1, \ldots, a_{i_1}^1, \ldots, a_1^{k-1}, \ldots, a_{i_{k-1}}^{k-1})$ such that $a_i^j \perp a_{i'}^{j'}$ if and only if $j = j'$,

$$\mu\big(V_{(a_1^1,\ldots,a_{i_1}^1),\ldots,(a_1^{k-1},\ldots,a_{i_{k-1}}^{k-1})}\big) = \frac{1}{\prod_{j=1}^{k-1} i_j!}.$$

We consider $(a_1^k, \ldots, a_{i_k}^k)$ all in the same column and not in any $(a_1^i)^\perp$ for $i < k$. We remark that

$$V_{(a_1^1,\ldots,a_{i_1}^1),\ldots,(a_1^k,\ldots,a_{i_k}^k)} = V_{(a_1^1,\ldots,a_{i_1}^1),\ldots,(a_1^{k-1},\ldots,a_{i_{k-1}}^{k-1})} \cap V_{(a_1^k,\ldots,a_{i_k}^k)}.$$

We want to prove that the ordering of $(a_1^k)^\perp$ is independent of the ordering of the other columns.

Enumerate as $(V_1, \ldots, V_\tau)$ all the different sets of the form

$$V_{(a^1_{\sigma_1(1)},\ldots,a^1_{\sigma_1(i_1)}),\ldots,(a^{k-1}_{\sigma_{k-1}(1)},\ldots,a^{k-1}_{\sigma_{k-1}(i_{k-1})})},$$

where σ_j is a permutation of $\{1, \ldots, i_j\}$. Thus $\tau = \prod_{j=1}^{k-1} i_j!$.

For all $l \in \{1, \ldots, \tau\}$, we define

$$\mu_{V_l}(\cdot) = \frac{\mu(\cdot \cap V_l)}{\mu(V_l)}.$$

This is the conditional probability of μ given V_l. We remark that

$$\mu = \sum_{l=1}^{\tau} \mu(V_l)\mu_{V_l}.$$

Denote $\mathrm{LO}((a_1^k)^\perp)$, the space of linear orderings on $(a_1^k)^\perp$. There is a restriction map r from $\mathcal{M}$ to $\mathrm{LO}((a_1^k)^\perp)$. We denote by $V^r_{(a_1^k,\ldots,a_{i_k}^k)}$ the image of $V_{(a_1^k,\ldots,a_{i_k}^k)}$ by r. Let ν be the pushforward of μ on $\mathrm{LO}(a_1^{1\perp})$ by r, and let ν_{V_l} be the pushforward of μ_{V_l} by the same map. We have

$$\nu = \sum_{l=1}^{\tau} \mu(V_l)\nu_{V_l}.$$

Observe that the initial step of the induction implies that ν is the uniform measure on $\mathrm{LO}((a_1^k)^\perp)$.

We denote by $\mathrm{Stab}^{\mathrm{set}}_{(a_1^k)^\perp}$ the setwise stabilizer of $(a_1^k)^\perp$ and $\mathrm{Stab}^{\mathrm{pw}}_{(a_1^1,\ldots,a_{i_1}^1,\ldots,a_1^{k-1},\ldots,a_{i_{k-1}}^{k-1})}$ the pointwise stabilizer of $(a_1^1, \ldots, a_{i_1}^1, \ldots, a_1^{k-1}, \ldots, a_{i_{k-1}}^{k-1})$ and we set $H = \mathrm{Stab}^{\mathrm{set}}_{(a_1^k)^\perp} \cap \mathrm{Stab}^{\mathrm{pw}}_{(a_1^1,\ldots,a_{i_1}^1,\ldots,a_1^{k-1},\ldots,a_{i_{k-1}}^{k-1})}$. We remark that ν_{V_l} is H-invariant for all $l \in \{1, \ldots, \tau\}$.

Since $\mathrm{LO}(a_1^{1\perp})$ is compact, by Theorem 3.4, if we prove that ν is H-ergodic, then we have the result. Indeed, then ν is an extreme point of the H-invariant measures and all the

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ν_{V_l} are equal to ν. Thus for any l,

$$\begin{aligned}\mu\big(V_{(a_1^k,\ldots,a_{i_k}^k)} \cap V_l\big) &= \mu_{V_l}\big(V_{(a_1^k,\ldots,a_{i_k}^1)}\big)\mu(V_l)\\ &= \nu_{V_l}\big(V^r_{(a_1^k,\ldots,a_{i_k}^1)}\big)\mu(V_l)\\ &= \nu\big(V^r_{(a_k^1,\ldots,a_{i_k}^1)}\big)\mu(V_l)\\ &= \frac{1}{i_k!}\frac{1}{\prod_{j=1}^{k-1} i_j!}\end{aligned}$$

and this equality finishes the induction.

It only remains to prove the ergodicity of ν. The following lemma will allow us to conclude.

LEMMA 3.7. *Let K be a group acting on a set $\mathcal{N}$ with no finite orbits. Denote by* $\mathrm{LO}(\mathcal{N})$ *the space of linear orderings on $\mathcal{N}$. Then the uniform measure λ on* $\mathrm{LO}(\mathcal{N})$ *is K-ergodic.*

Proof. Suppose A is a Borel subset of $\mathrm{LO}(\mathcal{N})$ such that, for all $g \in K$, $\lambda(A \triangle g \cdot A) = 0$. We want to show that $\lambda(A) \in \{0, 1\}$. Let $\varepsilon > 0$. There is a cylinder, that is, a set depending only on a finite set of $\mathcal{N}$, $B = B(b_1, \ldots, b_k)$ such that $\mu(B \triangle A) \leq \varepsilon$. Using Neumann's lemma, we get that there exists $g \in K$ such that $\{b_1, \ldots, b_k\} \cap g \cdot \{b_1, \ldots, b_k\} = \varnothing$.

Moreover, since ν is uniform, the orderings of two disjoint sets of points are independent. Indeed, take $(a_1, \ldots, a_i)$ and $(c_1, \ldots, c_{i'})$ to be two disjoint families of points. Note that $\lambda(V_{(a_1,\ldots,a_i)} \cap V_{(c_1,\ldots,c_{i'})})$ is equal to the number of ways to insert $(c_1, \ldots, c_{i'})$ in $(a_1, \ldots, a_i)$ respecting both orderings times the weight of a given ordering of $(a_1, \ldots, a_i, c_1, \ldots, c_{i'})$. Therefore,

$$\begin{aligned}\lambda\big(V_{(a_1,\ldots,a_i)} \cap V_{(c_1,\ldots,c_{i'})}\big) &= \binom{i+i'}{i}\frac{1}{(i+i')!}\\ &= \frac{1}{i!}\frac{1}{i'!}.\end{aligned}$$

This means that B and $g \cdot B$ are independent. We can now write

$$\begin{aligned}|\lambda(A) - \lambda(A)^2| &= |\lambda(A \cap g \cdot A) - \lambda(A)^2|\\ &\leq |\lambda(A \cap g \cdot A) - \lambda(B \cap g \cdot A)| + |\lambda(B \cap g \cdot A) - \lambda(B \cap g \cdot B)|\\ &\quad + |\lambda(B \cap g \cdot B) - \lambda(B)^2| + |\lambda(B)^2 - \nu(A)^2|\\ &\leq 4\varepsilon.\end{aligned}$$

The last inequality comes from the inequalities

$$\begin{aligned}|\lambda(A \cap g \cdot A) - \lambda(B \cap g \cdot A)| &\leq \lambda((A \triangle B) \cap g \cdot A) \leq \varepsilon,\\ |\lambda(B \cap g \cdot A) - \lambda(B \cap g \cdot B)| &\leq \lambda(g \cdot (A \triangle B) \cap B) \leq \varepsilon,\end{aligned}$$

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$$\lambda(B \cap g \cdot B) = \lambda(B)^2$$

and

$$|\lambda(B)^2 - \lambda(A)^2| = (\lambda(A) + \lambda(B))|\lambda(A) - \lambda(B)| \leq 2\varepsilon.$$

This proves that λ is K-ergodic. □

We only have to prove that H has no finite orbits on $(a_1^1)^\perp$. It suffices to remark that, for all $a \in \mathbb{S}$, $(u_1, \ldots, u_i) \in \mathbb{S}$, there are infinitely many $b \in a^\perp$ such that $a \to u_j$ if and only if $b \to u_j$ for all $1 \leq j \leq i$.

Indeed, take $k \in \mathbb{N}$. Consider the structure $((a_1, \ldots, a_k, v_1, \ldots, v_i), \to)$, where $a_l \perp a_j$, $a_l \to v_k$ if and only if $a \to u_k$ and $v_m \to v_{m'}$ if and only if $u_m \to u_{m'}$ for all $l, j \leq k$ and $m, m' \leq i$. It is obvious that this structure verifies the parity condition. Therefore, in $\mathbb{S}$, we can find k copies of a in its column for any $k > 0$.

This is enough to conclude that ν is indeed H-ergodic. □

3.3. *Proof of Theorem 1.1.* In what follows, we will show that

$$\mu(U \cap V) = \mu(U)\mu(V)$$

for all $U = U_{(x_i)_{i=1}^n, (\varepsilon_i^j)_{1 \leq i < j \leq n}}$ and $V = V_{(a_1^1, \ldots, a_{i_1}^1), \ldots, (a_1^k, \ldots, a_{i_k}^k)}$. It will follow that $\mu = \mu_0$.

Let us take a certain set $\{x_1, \ldots, x_n\}$, where none of the x_i are in the same column. We denote by m the number of sets U, as above, associated to this family. We consider $(U_i)_{i=1}^m$ the disjoint sets of $\mathcal{M}$ corresponding to the ways of defining a relation R and an order on the columns $x_1^\perp, \ldots, x_n^\perp$, that is, $U_i = U_{(x_{\sigma(i)})_{i=1}^n, \varepsilon}$ for some $\sigma \in S_n$ and $\varepsilon \in 2^{\binom{n}{2}}$. Proposition 3.1 tells us that

$$\text{for all } i, j \in \{1, \ldots, m\}, \quad \mu(U_i) = \mu(U_j).$$

We remark that this quantity is $(1/m)$. We now define, for all $i \in \{1, \ldots, m\}$,

$$\mu_{U_i}(\cdot) = \frac{\mu(\cdot \cap U_i)}{\mu(U_i)}.$$

This is the conditional probability of μ given U_i. Denote by H the subgroup of G that stabilizes $x_i^\perp$ for all $1 \leq i \leq n$ and each $\sim_{x_j^\perp}$-equivalence class in $x_i^\perp$ for $i \neq j$. We remark that H stabilizes U_i, by construction, and hence μ_{U_I} is H-invariant.

A simple but fundamental remark is that since $\bigsqcup_{i=1}^m U_i = \mathcal{M}$ and all the U_i have the same measure under μ,

$$\mu = \frac{1}{m} \sum_{i=1}^m \mu_{U_i}.$$

Let $\mathrm{LO}_p(\mathbb{S})$ denote the space of partial orders that are total on each column and do not compare elements of different columns. There is a restriction map from $\mathcal{M}$ to $\mathrm{LO}_p(\mathbb{S})$. We consider λ, the pushfoward of μ on $\mathrm{LO}_p(\mathbb{S})$ by this map. Similarly, we consider λ_{U_i}, the

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pushfoward of μ_{U_i} on $\mathrm{LO}_p(\mathbb{S})$. We have

$$\lambda = \frac{1}{m}\sum_{i=1}^{m} \lambda_{U_i}.$$

The rest of the proof is similar to the proof of Proposition 3.2: we prove that λ is H-ergodic. Take A to be a Borel subset of $\mathrm{LO}_p(\mathbb{S})$ such that, for all $h \in H$, $\lambda(A \triangle h \cdot A) = 0$. For any $\varepsilon > 0$, there is a cylinder B that depends only on finitely many points $(b_1, \ldots, b_k)$ such that $\lambda(A \triangle B) \leq \varepsilon$. We now want to find an element $g \in H$ such that B and $g \cdot B$ are λ-independent.

Take $\{b_1, \ldots, b_k\} \subset \mathbb{S}$. We remark that there is $\{b'_1, \ldots, b'_k\} \subset \mathbb{S}$ disjoint from $\{b_1, \ldots, b_k\}$ such that $b_l \perp b'_l$ and $b_l \sim_{x_j^{\perp}} b'_l$ for all $1 \leq l \leq k$ and $1 \leq j \leq n$. Therefore there is an element of H that sends $\{b_1, \ldots, b_k\}$ to $\{b_1, \ldots, b_k\}$, as required.

As in the proof of Proposition 3.2,

$$\begin{aligned}
|\lambda(A) - \lambda(A)^2| &= |\lambda(A \cap g \cdot A) - \lambda(A)^2| \\
&\leq |\lambda(A \cap g \cdot A) - \lambda(B \cap g \cdot A)| + |\lambda(B \cap g \cdot A) - \lambda(B \cap g \cdot B)| \\
&\quad + |\lambda(B \cap g \cdot B) - \lambda(B)^2| + |\lambda(B)^2 - \lambda(A)^2| \\
&\leq 4\varepsilon.
\end{aligned}$$

Thus $\lambda(A) \in \{0, 1\}$.

Since $\mathrm{LO}_p(\mathbb{S})$ is compact, we have the result: λ is an extreme point of the H-invariant measures and all the λ_{U_i} are equal. Therefore,

$$\begin{aligned}
\mu(V \cap U_i) &= \mu_{U_i}(V)\mu(U_i) \\
&= \lambda_{U_i}(V)\mu(U_i) \\
&= \lambda(V)\mu(U_i) \\
&= \mu(V)\mu(U_i)
\end{aligned}$$

for all $i \in \{1, \ldots, m\}$, and $V = V_{(a^1_1,\ldots,a^1_{i_1}),\ldots,(a^k_1,\ldots,a^k_{i_k})}$. This finishes the proof of Theorem 1.1.

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