

Q-FANO THREEFOLDS AND LAURENT INVERSION

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Abstract. We construct families of non-toric \mathbb{Q} -factorial terminal Fano (\mathbb{Q} -Fano) threefolds of codimension ≥ 20 corresponding to 54 mutation classes of rigid maximally mutable Laurent polynomials. From the point of view of mirror symmetry, they are the highest codimension (non-toric) \mathbb{Q} -Fano varieties for which we can currently establish the Fano/Landau–Ginzburg correspondence. We construct 46 additional \mathbb{Q} -Fano threefolds with codimensions of new examples ranging between 19 and 5. Some of these varieties are presented as toric complete intersections, and others as Pfaffian varieties.

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§1. Introduction

In this paper, we use mirror symmetry methods to construct a number of new deformation families of three-dimensional Fano varieties with terminal orbifold singularities. Our main technique, Laurent inversion [15], is an inverse of the celebrated Givental/Hori-Vafa construction that associates a Laurent polynomial mirror to a Fano toric complete intersection. We construct our examples by applying Laurent inversion to specific class of Laurent polynomials. The resulting deformation families of \mathbb{Q} -Fano threefolds are complete intersections or Pfaffians, of low codimension, in singular toric varieties. Classically, Fano varieties conjecturally embed inside (weighted) projective spaces via their anticanonical

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ring, yet such considerations rarely lead to effective constructions of the varieties themselves. The crux of this issue is that when *codimensions* are higher than 3, which is most often the case for the anticanonical embedding, we lack structure theorems that describe the embedded object. This paper constructs new embeddings which significantly lower this number and thus go well beyond the reach of classical constructions. Since the new ambient spaces are highly singular toric varieties, these examples also fall outside the range where Laurent inversion was previously known to work. Thus this paper has several implications:

- it systematically explores part of the landscape of possible \mathbb{Q} -Fano threefolds that so far has been out of reach;
- it is one of the first occasions where mirror symmetry has been used to construct previously unknown algebraic varieties;
- it gives further evidence for a surprising phenomenon: that terminal singularities, which arise so naturally in birational geometry, are also singled out by mirror symmetry.

Outline

\mathbb{Q} -factorial terminal threefolds appear naturally in the minimal model program. Among these, we can hope to classify Fano varieties, as they come in a finite number of families. For simplicity, we only consider orbifold terminal quotient singularities, that is, cyclic quotient singularities of the form $\frac{1}{r}(1, a, -a)$ with $(a, r) = 1$. Possible Hilbert series of these objects have been inventoried in the Graded Ring Database (GRDB for short), yielding 54,610 cases in the “Fano 3-folds” list [6] (see [5], [3], [35], [34], [33] for the theorems on which it is based and [8] for the data repository). Any polarized variety can be embedded inside a weighted projective space, and the GRDB also predicts the number of generators of such an embedding. The *Fano index* of a variety X is the largest positive index $r > 0$ such that $-K_X = rA$ in the class group $\text{Cl } X$. Throughout this paper, we refer to the codimension of X with respect to the predicted A -embedding as its *GRDB-codimension*. We do not know how many entries in the “Fano 3-folds” list correspond to existing varieties, especially since constructions using graded ring methods have had limited success in high codimension (i.e., ≥ 5). This paper shows that a previously inaccessible part of the database is in fact constructible in a practical way.

The examples are situated within the more general framework of providing evidence for the Fano/Landau–Ginzburg correspondence as formulated in [12]. This can be represented as a conjectural bijective correspondence between the two sets:

$$\left\{ \begin{array}{l} \text{qG-deformation equivalence classes of} \\ \mathbb{Q}\text{-Fano threefolds of class TG} \end{array} \right\} \xleftrightarrow{\text{F/LG}} \left\{ \begin{array}{l} \text{mutation classes of} \\ \text{rigid MMLPs} \end{array} \right\},$$

where MMLP stands for *maximally mutable Laurent polynomial* (see Section 2.1). A qG or \mathbb{Q} -Gorestein deformation of a \mathbb{Q} -Gorenstein variety X is a flat family $\pi: \mathcal{X} \rightarrow B$ over a scheme B such that there exist $r \in \mathbb{N}$ and a Cartier divisor L on \mathcal{X} with $L|_F = -r \cdot K_F$ for any fiber F of π and $b \in B$ such that $X \simeq \pi^{-1}(b)$. If X is a Fano variety, the Hilbert series is locally constant on the fibers of such deformations (see [25] for a more detailed discussion).

DEFINITION 1.1. Two \mathbb{Q} -Fano varieties X and X' are *qG-deformation equivalent* if there exists a qG deformation $\pi: \mathcal{X} \rightarrow B$ with terminal fibers such that B is connected and there exist $b, b' \in B$ with $X \simeq \pi^{-1}(b)$ and $X' \simeq \pi^{-1}(b')$.

The F/LG correspondence can be understood more geometrically in terms of *toric degenerations*: the Newton polytope P of a rigid MMLP f spans a fan from which we build a toric Fano variety X_P . This is in general far from being terminal or \mathbb{Q} -factorial, yet we expect X_P to qG-deform to a \mathbb{Q} -Fano variety X . Working with toric Fano varieties is a crucial advantage, as they can be described via lattice polytopes, of which 674,688 appear in a list of toric *canonical* varieties in the GRDB [7], [24], [26]. We assume that our varieties admit a toric degeneration, and call this property TG (from *toric generization*). We further assume that such a degeneration contains a canonical toric variety in its mutation class.

We say a \mathbb{Q} -Fano X and a Laurent polynomial f are mirror to each other (via the F/LG correspondence) if the regularized quantum period of X and the classical period of f coincide (see [10] for the general picture). We refer to the coefficients of the classical period when expanded as a series as the *period sequence* of the Laurent polynomial.

In upcoming work, Coates, Kasprzyk, and Pitton classify period sequences of rigid MMLPs supported on canonical polytopes. For our first result, we extract from the GRDB the Hilbert series of the candidate \mathbb{Q} -Fano varieties of codimension ≥ 20 . We find all the canonical polytopes for each Hilbert series and look for rigid MMLPs supported on them. At the time of writing, there are 54 unique period sequences that can be obtained in this way [14].

THEOREM 1.2. *We construct all 54 \mathbb{Q} -Fano families with GRDB-codimension at least 20 that correspond to a period sequence as described above. They are listed in Table 2 together with their embedding data inside a toric variety. We verify that the examples constructed as complete intersections correspond under mirror symmetry to the Laurent polynomials in the “Rigid MMLP” column.*

In Theorem 1.2, we construct a family for every period sequence arising from a set of rigid MMLPs, thus providing strong evidence for the Fano/LG correspondence. However, we are not defining a set-theoretic function, in the sense that in principle there could exist more than one such family. We have no knowledge of this phenomenon actually occurring, neither in our work or in the literature.

Another consideration was the *Fano index* of the examples that we construct (see Section 2.6 for a detailed discussion). Once we have an embedding inside an ambient space, the adjunction theorem (see Sections 2.2.1 and 2.3.1 for the complete intersection and the Pfaffian cases, respectively) yields a weight vector corresponding to the anticanonical divisor. One can then directly determine the Fano index by checking whether this vector is divisible in the class group. Higher Fano index varieties occupy a distinct place in this paper because some of their GRDB matching (between [6], [7]) is inaccurate—in other words: one cannot determine exclusively from numerical considerations whether $-K_X$ will be prime. This aspect is only clarified *a posteriori* when we have constructed the varieties, and an indication that this is to be expected is found in the period sequence of the corresponding Laurent polynomial.

There are many benefits to working in dimension three, from visualizing polytopes to the specific form of singularities, which make it similar to the surface case. Nevertheless, there is an increased difficulty from the point of view of deformation theory (see [27] for a survey on the topic). In particular, we are missing two key ingredients: a concrete result in the style of [2, Proposition 2.7], which is a recipe for determining which toric surface singularities are smoothable, and [25, Lemma 6], a useful local-to-global glueing result

that is central when obtaining classifications. There have been attempts at settling these issues (e.g., [17, Conjectures A&B] provide a conjectural answer to the first question in the Gorenstein case), yet it is apparent that they are relevant even in mildly singular cases (see [28, Theorem 1.2] for an example where expected local-to-global glueing fails). Fortunately, the strategy we employ in this paper bypasses most of these subtle discussions by constructing the deformations explicitly.

We are able to obtain our constructions precisely because we shift perspective from the high codimension embedding inside a projective space to a codimension ≤ 2 complete intersection or a codimension 3 Pfaffian inside a higher-rank toric variety, as we explain below.

The method we use to put this in practice is *Laurent inversion*: in [15], Coates, Kasprzyk, and Prince reverse-engineer of the so-called “Przjalkowski trick” [32]. Their algorithm is recalled in Section 2.9: starting from a Laurent polynomial, it produces an embedded Fano variety. In [20], Doran and Harder obtain similar constructions in the complete intersection case, however, the present article uses the language and notation in [15]—the correspondence between the two is discussed in [15, Section 12].

Instead of considering its entire deformation family, the authors of [15] construct a toric embedding of a Fano toric variety X_P into another toric variety Y , either as a complete intersection or as a Pfaffian variety. Using the Cox coordinates on Y , one deforms the equations that give X_P to obtain a less singular variety X . In general, Y is not an orbifold (neither is X_P), and moreover its non-orbifold locus need not be isolated. However, if chosen conveniently, Y is large enough so that X completely avoids this locus and only has \mathbb{Q} -factorial terminal singularities. Very explicit such computations are found in Sections 3.1 and 4.2. Crucially, the codimension of X inside Y in the examples of this paper ranges from 1 to 3, dramatically decreasing the GRDB-codimension and thus allowing for more control on our variety.

One of our aims is to see how far we are able to go with these constructions, and to determine this we dive deeper into the GRDB. The key object in the Laurent inversion algorithm is the *shape*, a smooth toric variety which encodes a decomposition of a given Laurent polynomial into summands. Its geometry rules many of the properties of Y and the embedding of X_P inside it. The dimension of the shape can in principle range from 1 to $\dim X_P$. For all but two varieties in Table 2, this shape is two-dimensional. We further explored the database in order to determine if the predominance of 2D shapes is characteristic to varieties of high codimension.

THEOREM 1.3. *In Table 3, we construct 46 \mathbb{Q} -Fano threefolds with GRDB-codimensions of new examples ranging from 19 to 5, together with their singularity baskets and embedding data. We verify that the examples presented as complete intersections correspond under mirror symmetry to the Laurent polynomials in the “Rigid MMLP” column.*

REMARK 1.4. Eleven examples in Table 3 use 3D shapes in their construction, and we discuss the most complicated case in Section 4.2.

Theorem 1.3 extracts more information on the structure of the deformation spaces involved than Theorem 1.2, in the following sense.

Mutating a polynomial implies mutating its Newton polytope, a combinatorial operation described in [1, Section 3]. Geometrically, this operation produces a pencil deforming a toric Fano variety into another. This pencil was introduced in [23], was formulated in the setting

of toric pairs in [29, Theorem 1.3], and the deformation was shown to be \mathbb{Q} -Gorenstein in the surface case in [25, Lemma 7].

The notion of mutation for polynomials is finer than its combinatorial counterpart, which can be interpreted in the following geometric way from the point of view of mirror symmetry. Mutation-equivalent *polytopes* determine toric varieties with the same Hilbert series. If both toric varieties deform to \mathbb{Q} -Fano varieties, their Hilbert series will also coincide. Note that the converse is not true: two polytopes with the same Hilbert series are not necessarily mutation-equivalent. An easy example are the polytopes of \mathbb{F}_1 and $\mathbb{P}^1 \times \mathbb{P}^1$, which are the two smooth toric del Pezzo surfaces of degree 8.

A refinement of the statement above is that mutation-equivalent *polynomials* correspond to \mathbb{Q} -Fano varieties with the same Gromov–Witten invariants. Since the latter are deformation-invariant, this is a rephrasing of the F/LG correspondence as stated above: when we fix the mutation class of a polynomial, we are singling out a component of a deformation space.

The Laurent inversion process starts from a Laurent polynomial f and creates an embedding of a toric Fano variety X_f . In the case of toric complete intersections, we are able to check that the partial smoothing of X_f to a \mathbb{Q} -Fano inside this embedding is mirror to f : it has the same regularised quantum period as the classical period of f .

Furthermore, if two different rigid MMLPs f and g with different period sequences are supported on the same polytope and they deform to \mathbb{Q} -Fano threefolds X_f and X_g , then these two varieties are not deformation-equivalent. In other words, from the point of view of mirror symmetry we have found different Fano varieties. Some such examples appear in Table 3, and crucially do not occur in the surface case (see [25, Lemma 6]) nor in the high codimension examples in Table 2. Combinatorial mutations alone cannot detect this phenomenon.

Input of the construction.

1. Parse the list of canonical polytopes in GRDB which have the same Hilbert series as a (conjectural) \mathbb{Q} -Fano threefold.
2. Find rigid MMLPs on the polytopes. In particular, the highest codimension in which these exist is 23.
3. Some of these polynomials have the same period sequence, conjecturally they deform in the same family of \mathbb{Q} -Fano threefolds. We treat these redundancies as follows:
 - (a) Given a fixed period sequence, if there is a terminal, \mathbb{Q} -factorial toric variety supporting one of its polynomial representatives, we skip this case as this variety exists in the toric canonical 3-fold database. See Section 2.5 for a precise description of what is discarded in the systematic codimension ≥ 20 case.
 - (b) Otherwise, we choose a single polynomial representative of this class.

Sketch of the construction.

4. For such a Laurent polynomial, use Laurent inversion to construct the \mathbb{Q} -Fano threefold. This is either a complete intersection in a toric variety, or a codimension 3 Pfaffian variety.
5. Verify that the object we found has precisely the singularities that GRDB predicts for the \mathbb{Q} -Fano threefold. If it does not, mutate the polynomial (see Section 2.1 for the precise definition) and retry step 4 with its mutations.

REMARK 1.5. • Steps 1–3 are the starting data of this paper, and are recorded in the database [14]. The difficulty of the rest of the procedure is found in step 5, as we often construct varieties that are not \mathbb{Q} -Fano (i.e., they have worse singularities). There are results in the literature [30], [31] which guarantee that the setup will produce an orbifold ambient space, however they are not general enough to produce the examples of this paper. While these restrictions make sense when aiming for smooth varieties, they are no longer needed here and allow for a broader search.

- Another significant advantage is implicit within step 5: the access to the mutation graph of a given polynomial is entirely automatised [13] in the computational algebra system (or CAS) Magma [4], and each corresponding polytope can be visualized. We record in Tables 2 and 3 the iteration in the mutation graph (in the column MG#), denoting the amount of times we have mutated before being successful with our construction. It ranges between 1 and 31.

There is of course no guarantee that a given \mathbb{Q} -Fano threefold can be realized as a toric complete intersection or as a toric Pfaffian variety, yet it is very encouraging that many of them are, and is certainly a good indication of which entries in GRDB correspond to actual geometric objects. It is likely that some varieties in Table 3 are birationally linked to one another via projections/unprojections (e.g., Q33018 and Q33019), as suggested in the GRDB. We chose not to explore this route, as our construction embeds them in ambient spaces which seem unrelated. Nonetheless, it is entirely conceivable to correlate blow-ups/contractions on shapes with projections/unprojections of X inside Y .

Our constructions have been especially easy in the case of varieties whose baskets contain only $\frac{1}{2}(1, 1, 1)$ singularities, which could be explored in the future as a systematic classification. It is unclear how much such an endeavour would overlap with Takagi's lists in [36], because computing the Picard rank of a complete intersection inside a non-orbifold toric variety can be subtle.

A natural direction to pursue is the automation of this process: indeed, many of the steps involved rely heavily on a CAS. We are however far from a systematic implementation: choosing a shape which yields a good construction of the ambient space is still very much a matter of experience as opposed to brute computational force. Our attempts at writing a program that generates all possible shapes (and then scaffoldings) on various iterated mutations of a given rigid MMLP currently yield overwhelmingly many pathological cases. Though ultimately this is our desired direction, we cannot yet propose an algorithm that is applicable at the large scale needed to construct all varieties in the GRDB.

§2. Preliminaries

2.1 Laurent polynomials and mutations

Our starting point is a class of Laurent polynomials intrinsically linked with the concept of mutation. For the purpose of self-containment, we reproduce the terminology below, most of which we extract from [16], a paper entirely dedicated to these objects.

Let P be a full-dimensional polytope in a lattice N . It is a *Fano polytope* if $0 \in P \setminus \partial P$ and its vertices are primitive elements in N . This implies that X_P , the toric variety whose fan is the *spanning fan* of P , is a (usually very singular) Fano variety. We consider Laurent polynomials $f = \sum_{v \in P \cap N} c_v x^v$ with coefficients $c_v \in \mathbb{N}$ such that $\text{Newt}(f) = P$, and we say they are *supported on P* . We always set $c_0 = 0$, as two polynomials which differ by a constant

have the same mirror (see [15, Remark 2.9]). We remember this convention when using Laurent inversion, where we allow the origin to be part of any strut.

Lastly, we assume, as in [16, Convention 2.3], that all Laurent polynomials (and mutation factors) have non-negative integer coefficients and that if $f \in \mathbb{C}[N]$ then the exponents of monomials in f generate N .

DEFINITION 2.1 [16, Definition 1.6.]. Let N be a lattice and let $w \in M$ be a primitive vector in the dual lattice. Then w induces a grading on $\mathbb{C}[N]$. Let $a \in \mathbb{C}[w^\perp \cap N]$ be a Laurent polynomial in the zeroth piece of $\mathbb{C}[N]$, where $w^\perp \cap N = \{v \in N | w(v) = 0\}$. The pair (w, a) defines an automorphism of $\mathbb{C}(N)$ given by

$$\mu_{w,a}: \mathbb{C}(N) \rightarrow \mathbb{C}(N), x^v \mapsto x^v a^{w(v)}.$$

Let $f \in \mathbb{C}[N]$. We say that f is *mutable with respect to* (w, a) if

$$g := \mu_{w,a}(f) \in \mathbb{C}[N],$$

in which case we call g a mutation of f and a a factor.

DEFINITION 2.2 [16, Definition 2.4.]. Given a Laurent polynomial f , consider the graph G with vertex labels that are Laurent polynomials and edge labels that are pairs $L(w, a)$, defined as follows. Write $l(v)$ for the label of a vertex $v \in V(G)$, and $l(e)$ for the label of an edge $e \in E(G)$.

- (i) Begin with a vertex labelled by the Laurent polynomial f .
- (ii) Given a vertex v , set $g := l(v)$. For each (w, a) , $\deg a > 0$, such that g is mutable with respect to (w, a) and either:
 - (a) there does not exist an edge with endpoint v and label $L(w, a)$; or
 - (b) for every edge $e = \overline{vv'}$ with $l(e) = L(w, a)$ we have that

$$l(v') \notin \mu_{a, x^{w^\perp \cap N}}(g);$$

pick a representative $g' \in \mu_{a, x^{w^\perp \cap N}}(g)$ and add a new vertex v' and edge $\overline{vv'}$ labelled by g' and $L(w, a)$, respectively.

The mutation graph \mathcal{G}_f of f is defined by removing the labels from the edges of G and changing the labels of the vertices from g to the $\text{GL}(N)$ -equivalence class of $\text{Newt}(g)$.

DEFINITION 2.3 [16, Definition 2.5.]. We partially order the mutation graphs of Laurent polynomials by saying that $\mathcal{G}_f \prec \mathcal{G}_g$ whenever there is a label-preserving injection $\mathcal{G}_f \hookrightarrow \mathcal{G}_g$. A Laurent polynomial f is *maximally mutable* (or for short, f is an MMLP) if $\text{Newt}(f)$ is a Fano polytope, the constant term of f is zero, and \mathcal{G}_f is maximal with respect to \prec .

DEFINITION 2.4 [16, Definition 2.6.]. An MMLP f is *rigid* if the following holds: for all g such that the constant term of g is zero and $\text{Newt}(f) = \text{Newt}(g)$, if $\mathcal{G}_f = \mathcal{G}_g$ then $f = g$.

2.2 Toric varieties, weight matrices and complete intersections

The \mathbb{Q} -Fanos that we are interested in live inside toric ambient spaces, either as complete intersections or as Pfaffian varieties. As in the original Laurent inversion paper [15, Section 2], the most useful description of these ambient spaces is via their GIT data: having global coordinates and equations then enables us to compute singularities. This characterisation is equivalent to the description via fans and cones and it is the main output of our paper.

The Laurent inversion construction typically involves three incarnations of toric varieties: the starting object X_P is given by a fan, the shape Z appears (almost exclusively) via the polytopes of its nef divisors, and the ambient space Y appears as a GIT quotient, all of which are necessary when discussing examples. We present a minimal set of preliminaries, adapted from [18, Section 3.1] to include the non-orbifold case.

Let $\mathbb{T} \simeq \mathbb{C}^{\times n}$ be an n -dimensional torus, $M = \text{Hom}(\mathbb{T}, \mathbb{C}^{\times})$ its character lattice and $N = \text{Hom}(M, \mathbb{Z})$ the lattice dual to M .

Let P be a Fano polytope with vertices in N , and Σ its spanning fan. We construct the projective toric variety $F := X_{\Sigma}$ by glueing together the affine open subsets $X_{\sigma} = \text{Spec} \mathbb{C}[M \cap \sigma^{\vee}]$, where $\sigma \in \Sigma$. Denote by $\rho_1, \dots, \rho_m \in N$ the primitive generators of the rays of Σ , which we assume generate N as a group. Note that F is not necessarily an orbifold, as the fan need not be simplicial, and could have at worst klt singularities.

We have the following two dual exact sequences:

$$\begin{aligned} 0 \longrightarrow \mathbb{L} \longrightarrow \mathbb{Z}^m \xrightarrow{\rho} N \longrightarrow 0 \\ 0 \longleftarrow \mathbb{L}^* \xleftarrow{D} (\mathbb{Z}^m)^* \xleftarrow{\rho^T} M \longleftarrow 0. \end{aligned} \tag{2.1}$$

The lattice \mathbb{L}^* is in fact the \mathbb{T} -class group of F , we call ρ the *ray map* (which in this paper will always be surjective) and D the *divisor homomorphism* of F . If \mathbb{G} is the torus with character group \mathbb{L}^* , then D is dual to a morphism $\mathbb{G} \rightarrow \mathbb{C}^{\times m}$ and it is via this morphism that $F = \mathbb{C}^m //_{\omega} \mathbb{G}$. In other words, a matrix of the morphism D is a *weight matrix* of the \mathbb{G} -action on \mathbb{C}^m .

More precisely, let x_i be the standard basis of $(\mathbb{Z}^m)^*$. Note that since F is projective, the cone $\mathcal{C} \subset \mathbb{L}_{\mathbb{R}}^*$ spanned by $D_i = D(x_i)$ is strictly convex. There is a wall-and-chamber decomposition of \mathcal{C} called the *secondary fan*, and we choose ω inside a (not necessarily full-dimensional) cone of this fan. Now set $V_{\omega} = V(\text{Irr})$, where

$$\text{Irr} = (x_{i_1} \cdots x_{i_k} \mid \omega \in \text{RelInt}\langle D_{i_1}, \dots, D_{i_k} \rangle)$$

is the *irrelevant ideal*. We then write F as a quotient

$$F = (\mathbb{C}^m \setminus V_{\omega}) / \mathbb{G},$$

by the \mathbb{G} -action given by D .

REMARK 2.5 (Birational point of view).

1. The choice of ω coincides with choosing the ample cone inside the secondary fan—every divisor in the same cone-interior as ω is ample. When working with toric Fano varieties, we usually bypass this discussion by imposing that $\omega = \sum_i D(x_i) = -K_F$, and the stability cone is the smallest-dimensional cone that contains it. When working with Fano complete intersections or codimension 3 Pfaffian varieties, we impose similar conditions by using adjunction-type formulas (see Sections 2.2.1 and 2.3.1, respectively).
2. The rays of Σ are determined by the weight matrix and vice-versa, modulo appropriate basis changes. The quotient $\mathbb{C}^m //_{\omega} \mathbb{G}$ also depends on the chamber in which ω lies, and two stability conditions that belong to the same chamber induce isomorphic varieties.

Varying ω between two adjacent chambers inside the secondary fan produces varieties that are related either by small or divisorial contractions. In terms of the fan, choosing ω in a maximal-dimensional chamber implies all cones in Σ are simplicial, that is, F is an orbifold. Choosing ω on a wall may produce a non- \mathbb{Q} -Gorenstein variety.

The choice of a stability condition determines the \mathbb{G} -invariant open sets $V_{i_1 \dots i_k} = \{x_{i_1} \neq 0, \dots, x_{i_k} \neq 0\} \subset \mathbb{C}^m$ such that

$$F = \bigcup_{\{(i_1, \dots, i_k) | \omega \in \text{RelInt}(\langle D_{i_1}, \dots, D_{i_k} \rangle)\}} [V_{i_1, \dots, i_k} / \mathbb{G}].$$

Since we do not restrict to the orbifold case, we do not impose that $k = r$, therefore, the toric charts $V_{i_1 \dots i_k}$ are not necessarily quotients by finite groups. Indeed, when working inside these charts we write $[V_{i_1 \dots i_k} / \mathbb{G}] = [U_{i_1 \dots i_k} / \mu]$, where $U_{i_1 \dots i_k} = \{x_{i_1} = \dots = x_{i_k} = 1\}$ and μ is the subgroup of \mathbb{G} which fixes $U_{i_1 \dots i_k}$. More precisely, μ is the group whose character group is the cokernel of the homomorphism

$$D_{i_1 \dots i_k} : (\mathbb{Z}^k)^* \rightarrow \mathbb{L}^*.$$

2.2.1. Complete intersections

We consider complete intersections $X \subset F$, that is, general elements of linear systems $|L_1|, \dots, |L_c|$ on F , where $L_i \in \mathbb{L}^*$ are \mathbb{G} -linearized line bundles on F . The space of sections $H^0(F, L_i)$ is a vector subspace of $\mathbb{C}[x_1, \dots, x_m]$ with basis consisting of monomials of homogeneity type L_i . Let $f_i \in H^0(F, L_i)$, then $V(f_1, \dots, f_c)$ is stable under the action of \mathbb{G} and we consider the subvariety $X = (V(f_1, \dots, f_c) \setminus V_\omega) / \mathbb{G} \subset F$.

DEFINITION 2.6.

1. Assume $X \subset F$ avoids the non- \mathbb{Q} -Gorenstein locus of F . Then X is *quasi-smooth* if

$$V(f_i, \dots, f_c) \setminus V_\omega \subset \mathbb{C}^m \setminus V_\omega.$$

2. Suppose $X \subset F$ is quasi-smooth. We say that X is *well-formed* if for every toric stratum $S \subset F$ with nontrivial stabiliser, $S \subset X$ implies $\text{codim}_X D \geq 2$.

If $X \subset F$ is a complete intersection as above, its canonical class is

$$-K_X = -K_F - \sum L_i.$$

There is a similar formula for Pfaffian varieties, which we work out in detail in Section 2.3.1.

2.3 Laurent inversion

Finally, we recall the algorithm of Laurent inversion in [15], whose purpose is to provide an embedding $X_P \hookrightarrow Y$, where X_P and Y are toric varieties, constructed using a decomposition into summands of a rigid MMLP supported on P . We then use the coordinates on this new ambient space to smooth X_P into a \mathbb{Q} -Fano variety X .

DEFINITION 2.7 [15, Definition 3.1.]. Fix the following data:

- (i) a lattice N together with a splitting $N = \overline{N} \oplus N_U$;
- (ii) the dual lattice $M := \text{Hom}(N, \mathbb{Z})$, with the dual splitting $M = \overline{M} \oplus M_U$;
- (iii) a Fano polytope $P \subset N_{\mathbb{Q}}$;

- (iv) a projective toric variety Z given by a fan in \overline{M} whose rays span the lattice \overline{M} . Given such data, a *scaffolding* S of P is a set of pairs (D, χ) where D is a nef divisor on Z and χ is an element of N_U , such that

$$P = \text{conv}(P_D + \chi \mid (D, \chi) \in S).$$

We refer to Z as the *shape* of the scaffolding, and the elements $(D, \chi) \in S$ as *struts*.

The algorithm in [15] directly produces the weight matrix of Y and assumes the following.

REMARK 2.8 [15, Remark 3.3]. Given the $N = \overline{N} \oplus N_U$ splitting in Definition 2.7, there are $u = |U|$ struts of type $(0, e_i)$ where $\{e_i\}_{i=1}^u$ is a basis of N_U .

What happens when this assumption is not fulfilled appears in [30] and a detailed example can be found in Section 3.1.2 of this paper. Many entries in the systematic Table 2 use this modification, and it is likely that an automatization of Laurent inversion should include it.

We now state the algorithm, which appears in [15] with a sign typo at step (ii) (as confirmed by [15], which are correct constructions). To follow along its steps in a simple example, go to Section 3.1.1.

ALGORITHM 2.9 [15, Algorithm 5.1]. Let S be a scaffolding of a Fano polytope P with shape Z . Let $u = \dim N_U$, $r = |S| - u$ and assume Remark 2.8. Let R be the sum of $|S|$ and the number z of rays of Z . We determine an $r \times R$ matrix \mathbb{M} , which is the weight matrix of the toric variety Y , as follows. Let $m_{i,j}$ denote the (i, j) entry of \mathbb{M} . Fix an identification of the rows of \mathbb{M} with the r elements (D_i, χ_i) of S which do not correspond to the basis of N_U , and an ordering $\Delta_1, \dots, \Delta_z$ of the toric divisors in Z .

- (i) For $1 \leq j \leq r$ and any i , let $m_{i,j} = \delta_{i,j}$.
- (i) For $1 \leq j \leq u$ and any i , let $m_{i,r+j}$ be determined by the expansion

$$\chi_i = - \sum_{j=1}^u m_{i,r+j} e_j.$$

- (iii) For $1 \leq j \leq z$, let $m_{i,|S|+j}$ be determined by the expansion

$$D_i = \sum_{j=1}^z m_{i,|S|+j} \Delta_j.$$

The weight matrix \mathbb{M} alone does not determine a unique toric variety—we also need to choose a stability condition ω . Let Y denote the toric variety determined by this choice. Unless otherwise stated, we take ω to be the sum of the first $|S|$ columns in \mathbb{M} . This translates to the adjunction-type formulas we mentioned earlier, to ensure that the complete intersection inside Y is a Fano variety.

Warning: This last remark does *not* mean all varieties in Tables 2 and 3 have stability conditions given by the sum of the first $|S|$ columns of the ambient weight matrix, as we may have solved for one of those columns to eliminate an equation. When reading data from the tables, it is safer to apply the adjunction formulas in Sections 2.2.1 or 2.3.1 to determine ω .

The construction of the line bundles L_i is described explicitly in [15, Proposition 12.2], which connects the algorithm to the method in [20] in the case of complete intersections. Each “level” of the tower determines a corresponding line bundle: note that by (iii) in 2.9 the last z columns of \mathbb{M} correspond to rays of Z . Since the rays in the fan of Z are partitioned into the c factors of the tower, so are then the Cox coordinates in this block. The \mathbb{G} -weight of a line bundle will then be the same as that of $\text{div}(\prod x_j)$, where x_j are Cox coordinates on Y corresponding to a factor of the partition. In [20], the embedding construction of X_P inside Y depends on the order of the L_i , and implicitly on the order of the factors. As we are looking to construct the general section of $\oplus_{i=1}^c L_i$ (and are not particularly concerned with a specific toric section), it is then enough to add all monomials in the same $(\mathbb{C}^\times)^r$ -eigenspace as L_i for each $i = 1 \dots c$. In other words, as soon as we choose Z and S , we directly determine the equations of the smoothest possible X given by this scaffolding.

2.3.1. Pfaffians

By the Buchsbaum-Eisenbud structure theorem [9, Theorem 2.1], a codimension three ideal inside a Gorenstein ring is given by the $2n \times 2n$ diagonal Pfaffians of a skew $(2n + 1) \times (2n + 1)$ matrix. In all such examples in this paper, $n = 2$.

Some entries in Tables 2 and 3 are in this format. Additionally, the codimension 4 object in Section 4.2 is a complete intersection between a Pfaffian variety and a hyperplane, which we eventually reduce to the Pfaffian case. We now recall the information the format encodes in the case of weight matrices obtained by Laurent inversion.

To understand the singularities and degree of such a variety, we determine the eigentype of the anticanonical divisor of a variety embedded as a Pfaffian inside a toric variety; this is the equivalent of an adjunction theorem in the case of complete intersections. We then impose that the stability condition of the ambient space be in the same cone as the anticanonical divisor, ensuring that the resulting variety is Fano.

We follow the reasoning in [18, Section 3.3], which treats a surface example. In our context, X is typically embedded inside an ambient space Y with weight matrix:

$$\frac{U \quad t_1 \quad t_2 \quad t_3 \quad t_4 \quad t_5}{I_r \quad C_U \quad C_1 \quad C_2 \quad C_3 \quad C_4 \quad C_5}$$

where $C_U, C_i \in \mathbb{C}^r$ are vectors representing the weights of the \mathbb{G} -action on the column corresponding to N_U (there will be only one, for dimension reasons) and the Cox coordinates

t_i , respectively. Then $-K_Y = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} + C_U + \sum C_i$. The antisymmetric matrix we consider is

of the form

$$A = \begin{pmatrix} 0 & * & t_1 + * & t_2 + * & * \\ & 0 & * & t_3 + * & t_4 + * \\ & -sym & 0 & * & t_5 + * \\ & & & 0 & * \\ & & & & 0 \end{pmatrix}$$

and its 4×4 Pfaffians, that is, the equations of X , are

$$\begin{cases} t_3t_5 + t_4 * + \dots = 0 \\ t_2t_5 + t_1 * + \dots = 0 \\ t_2t_4 + t_3 * + \dots = 0 \\ t_1t_4 + t_5 * + \dots = 0 \\ t_1t_3 + t_2 * + \dots = 0 \end{cases} \tag{2.2}$$

where the forms “ $*$ ” are such that the entries of A are of fixed weights and the Pfaffian equations are homogeneous. We denote the line bundles whose sections are given by these equations by L_i , $i = 1 \dots 5$, respectively. Note that this determines the weights of the \mathbb{G} -action on the line bundles (e.g. $L_1 \sim C_3 + C_5$).

Let $E = \bigoplus_{i=1}^5 L_i$. Then there exists a linearised line bundle L on Y such that $X \subset Y$ is the degeneracy locus of a general antisymmetric homomorphism $s : E \otimes L \rightarrow E^\vee$ defined by the matrix A .¹ The specific format of A implies that $L = -\frac{\sum_{i=1}^5 L_i}{2} = -\sum_{i=1}^5 C_i$.

CLAIM (The adjunction formula for Pfaffians). The weights of the \mathbb{G} -action on $-K_X$ are given by

$$-K_X = (-K_Y + L)|_X = (-K_Y - \frac{\sum_{i=1}^5 L_i}{2})|_X.$$

This computation requires that $-K_Y$ be locally free. We denote by ω_Y its restriction to the \mathbb{Q} -factorial locus of Y . The locus which we avoid is at most of dimension one in all examples in this paper. A posteriori, X avoids it completely (if we follow through with the computation and X intersects it, then it is not \mathbb{Q} -factorial and we discard the example).

We use the homomorphism A to construct a resolution \mathcal{O}

$$0 \longrightarrow L \xrightarrow{\text{Pf}^\vee} E \otimes L \xrightarrow{A} E^\vee \xrightarrow{\text{Pf}} \mathcal{O}_Y \longrightarrow \mathcal{O}_X \longrightarrow 0. \tag{2.3}$$

We want to compute $\omega_X = \underline{\text{Ext}}_{\mathcal{O}_Y}^3(\mathcal{O}_X, \omega_Y)$, so we use this resolution to induce a complex:

$$0 \rightarrow \underline{\text{Hom}}_{\mathcal{O}_Y}(\mathcal{O}_Y, \omega_Y) \rightarrow \underline{\text{Hom}}_{\mathcal{O}_Y}(E^\vee, \omega_Y) \rightarrow \underline{\text{Hom}}_{\mathcal{O}_Y}(E \otimes L, \omega_Y) \rightarrow \underline{\text{Hom}}_{\mathcal{O}_Y}(L, \omega_Y) \rightarrow 0$$

whose third cohomology is ω_X . This is the same as the self-dual complex

$$0 \longrightarrow L \xrightarrow{\text{Pf}^\vee} E \otimes L \xrightarrow{A} E^\vee \xrightarrow{\text{Pf}} \mathcal{O}_Y \longrightarrow 0 \tag{2.4}$$

tensored by $L^\vee \otimes \omega_Y$. The complex (2.4) is everywhere exact except at \mathcal{O}_Y , where the cohomology is $\mathcal{O}_Y/\text{Pf}(E^\vee) = \mathcal{O}_X$ (since (2.3) is a resolution), thus we conclude that $\omega_X = L^\vee \otimes \omega_Y \otimes \mathcal{O}_X$. This proves the claim.

2.4 The tables

We give a few remarks on how to read the tables appearing in this paper.

Table 2 is ordered by codimension and \mathbb{Q} -Fano ID in the Graded Ring Database, that is, the IDs which appear in the “Fano 3-folds or big table” list. Table 3 is ordered only by the

¹ the order of the L_i factors in E is important: it must be compatible with the columns of A .

latter. We also include the polytope ID (from the “Toric canonical Fano 3-folds” list) to which we initially associate a rigid MMLP, and the expected singularities of each \mathbb{Q} -Fano.

NOTATION 2.10. Since we regularly refer to objects in two different databases, to avoid confusion we refer to GRDB IDs in the \mathbb{Q} -Fano threefolds database using the prefix Q (e.g., Q38989) and the GRDB IDs in the toric canonical Fano threefolds using the prefix P (e.g., P519468). We sometimes directly refer to such a variety by its polytope, which uniquely determines it.

For simplicity, we do not record the original rigid MMLP (i.e. that is supported on a *canonical* toric variety), as this may not give the successful partial smoothing. Instead, what we record is its mutation for which Laurent Inversion yields a \mathbb{Q} -Fano threefold, in the “Rigid MMLP” column. Its corresponding toric Fano might therefore not be canonical, but klt.

NOTATION 2.11. When discussing Laurent inversion, we always denote the toric variety which we embed as X_P , the ambient space which we embed it in by Y , the shape of the embedding by Z and the (partial) \mathbb{Q} -Fano smoothing by X . If X_P is embedded as a toric complete intersection given by line bundles L_1, \dots, L_c , then X is a general section of these line bundles. We use the coefficient convention for complete intersections in toric varieties: all coefficients of monomials in any equation equal 1.

REMARK 2.12. In the “Weight matrix and bundles” column we record the GIT weights of the $(\mathbb{C}^*)^k$ -action on the ambient space $Y = \mathbb{C}^n // (\mathbb{C}^*)^k$ in which X is embedded, followed by a vertical line and the weights of the $(\mathbb{C}^*)^k$ -linearised line bundles L_1, \dots, L_c of which X is a general section (all of this is determined by the Laurent inversion construction). We insist that such an X is a Fano variety and therefore impose that $-K_X$, which is computed by adjunction from $-K_Y$, is ample. This automatically chooses the ample cone of Y , which becomes *a posteriori* well-defined as a GIT quotient.

Indeed, if a table entry is a complete intersection given by $(\mathbb{C}^*)^k$ -linearized line bundles L_1, \dots, L_c , the stability condition is necessarily $\omega = -K_Y - \sum L_i = \sum \text{div}(x_j) - \sum L_i$, where $\{x_j\}$ are the Cox coordinates on Y . We discuss how to obtain the weight matrices in this case, as well as how to analyse the singularities of a general section of $\oplus L_i$ in 3.1. If a table entry is a codimension 3 Pfaffian variety given by the line bundles L_1, \dots, L_5 , then cf. Section 2.3.1, the stability condition is $\omega = -K_Y - \frac{1}{2} \sum_{i=1}^5 L_i = \sum \text{div}(x_j) - \frac{1}{2} \sum_{i=1}^5 L_i$, with the same notation for the x_j . A Pfaffian example is discussed in Section 4.2.

The shape Z we use to produce these weight matrices using Laurent inversion is also included in Tables 2 and 3. We mention if we use point-struts as a basis of N_U , and the difference of how we obtain the ambient space in these cases is discussed in Examples 3.1.1 and 3.1.2.

REMARK 2.13. As a sanity check after producing a weight matrix, we verify that the period sequence of the general element in the complete intersection (i.e., that obtained from its regularized quantum period) is the same as the period sequence of the Laurent polynomial (obtained by computing its classical period). The theory for computing the former exists only in the context of orbifold ambient spaces, however since we ultimately check that $X \subset Y$ completely avoids the non-orbifold locus, we can assume Y is an orbifold by taking a partial crepant resolution if necessary. For the theory, we require an equivalent

of [11, Corollary D.5] in the orbifold case. This result is a combination of Givental’s mirror theorem [11, Theorem C.1] and the Quantum Lefschetz Theorem [11, Theorem D.1], the appropriate generalisations of which can be found in [12, Theorem 31] and [37, Theorem 1.1] respectively. We verify that the first 10 terms of both period sequences coincide using Magma, and we include the results in Tables 2 and 3.

REMARK 2.14. A formula for computing the regularised quantum period of a codimension 3 Pfaffian variety is not yet known. We however compute the anticanonical degree and singularities in each case (an example of this is discussed in detail in Section 4.2.3), which, along with [15, Theorem 5.5], we consider as sufficient evidence of having constructed the correct objects.

2.5 How Table 2 is constructed

We now sketch how the number 54 is obtained in Theorem 1.2. Assume we have followed Steps 1–3 in the “Input of the construction” in the introduction, but only starting from polytopes P such that there exists a candidate \mathbb{Q} -Fano threefold of GRDB— codimension ≥ 20 with the same Hilbert series as X_P . In the “Fano 3-folds” list, no Fano index > 1 varieties have GRDB-codimension ≥ 20 , and we look to translate this into a condition on the polynomials we obtain after Steps 1–3.

In order to extract the correct period sequences we use the criterion below. This is not proved in general, but we can verify it by hand in our situation - there is a handful of cases to examine, and they are either trivial² or included in Table 1.

Table 1. High Fano index varieties.

Polytope ID	Weight matrix & bundles	Fano index	\mathbb{Q} -Fano ID	
1	P543951	$\begin{array}{cccccc c} 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 3 & 1 & 5 & 2 & 3 & 6 \end{array}$	2	Q40971 codim 9
2	P534760	$\begin{array}{cccccc c} 1 & -3 & -5 & -1 & -2 & 2 & -6 \\ 0 & 3 & 5 & 1 & 2 & -1 & 6 \end{array}$	2	Q40948 codim 9
3	P473887	$\begin{array}{cccccc c} 1 & 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 3 & 2 & 1 \end{array}$	2	Q40993 codim 6
4	P413267	$\begin{array}{cccccc c} 1 & 0 & 1 & -1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 2 & -1 & 1 & 2 \end{array}$	2	Q40988 codim 6
5	P402202	$\begin{array}{cccccc c} 1 & 0 & 2 & -1 & -1 & 4 & 2 \\ 0 & 1 & -1 & 1 & 1 & -2 & 0 \end{array}$	3	Q41251 codim 5
6	P547328	$\begin{array}{cccccc c} 1 & 0 & 2 & 2 & 1 & 3 & 1 \\ 0 & 1 & -1 & 1 & -1 & -1 & 1 \end{array}$	3	Q41200* codim 4
7	P544064	$\begin{array}{cccccc c} 1 & 0 & 3 & 0 & 1 & -1 & 2 \\ 0 & 1 & -1 & 1 & 1 & 1 & -1 \end{array}$	3	Q41218* codim 4
8	P543852	$\begin{array}{cccccc c} 1 & 0 & -1 & 1 & 1 & 1 & -1 \\ 0 & 1 & 4 & 0 & 1 & -1 & 3 \end{array}$	4	Q41334* codim 4

² High Fano index varieties tend to have low GRDB-codimension. For these examples, the embedding into weighted projective space is enough to establish the F/LG correspondence.

CRITERION 2.15. Assume a rigid MMLP f is mirror to a \mathbb{Q} -Fano threefold X and let $(p_n)_{n \geq 0}$ be their common period sequence. Then X is of Fano index r iff $p_k = 0$ for all $k \not\equiv 0 \pmod{r}$.

In other words, periodic zeroes in the period sequence of such an f suggests that a corresponding \mathbb{Q} -Fano built via Laurent inverting f does not belong to Table 2. After discarding high Fano index cases (which we construct and find their GRDB-codimension is strictly below 20; we nevertheless store these examples in Table 1), we finally obtain 54 period sequences which can be found in Table 2. Note that some of these appear to have a periodicity of zeroes, however the periodic zeroes stop if we compute more than 10 terms of its period sequence.

2.6 High Fano index entries in Table 3

We discuss one of the non-trivial examples illustrating Criterion 2.15. The other cases are similar and can be found in Table 1.

The polynomial

$$f = \frac{x^6}{y^4 z^5} + \frac{2x^3}{y^2 z^2} + \frac{2x^2}{y^3 z^2} + x + y + z + \frac{1}{y} + \frac{2z}{xy} + \frac{z}{x^2 y^2}$$

deceptively appears as an output of Steps 1-3 after the restriction to codimension ≥ 20 . It is the unique rigid MMLP supported on a mutation of the canonical polytope P543951. Its period sequence suggests a higher Fano index:

$$[1, 0, 2, 0, 6, 0, 20, 0, 3430, 0, 75852, \dots].$$

When Laurent inverting f we obtain a hypersurface X in the toric variety with weight matrix

$$\begin{matrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 3 & 1 & 5 & 2 & 3 \end{matrix},$$

where X is given by a general section of degree $(0, 6)$ and as $-K_X = (2, 4)$, indeed the variety is of Fano index 2.

We now need to find the correct GRDB ID of the \mathbb{Q} -Fano threefold we have built, which we deduce will appear polarized by $A = -K_X/2$. Keeping in mind that $-K_X = -K_{X_f} = -K_{X_{P543951}} = 168/5$, we search by the Fano index and the degree of the polarization $A^2 = 1/4(-K_X)^2 = 21/5$ and find Q40971 as the only candidate.³ For this reason, the object we build from P543951 appears as variety 39 in Table 3 and does not belong in Table 2 of high codimension objects.

There are a total of five new Laurent inversion constructions in this situation among the ones in our classification, and three varieties (marked with an asterisk) that had previously appeared in [19], albeit in a different format (in fact, Brown and Suzuki already suggest Q41200 can be constructed in [34]). All of these entries are now part of Table 3.

³ If more precision is necessary, we may also search by a subsequence – in this case, every second term – of the Hilbert series associated to A : $[1, *, 19, *, 87, *, 239, *, 509, \dots]$ which we deduce from the Hilbert series of X_f .

Table 2. Systematic constructions of high codimension examples.

\mathbb{Q} -Fano ID	Polytope ID	Singularities	MG #	Rigid MMLP	Weight Matrix and Bundles	Period Sequence	Shape	
Codim 23								
1	Q38950	P519483	$\frac{1}{4}(1,1,3),$ $\frac{1}{5}(1,2,3)$	2	$xy^8z^5 + 2xy^6z^4 + xy^4z^3 +$ $x + 3y^4z^3 + 4y^2z^2 + y + z +$ $3z/x + 2/(xy^2) + 1/(x^2y^4z)$	$\begin{array}{ccc ccc} 1 & 0 & 2 & -2 & 1 & 1 & 2 \\ 0 & 1 & -1 & 3 & 1 & -1 & 0 \end{array}$	[1,0, 0, 0, 24, 0, 0, 0, 2520, 7560, 0]	$\mathbb{F}_1 + 0\text{-st.}$
2	Q38985	P519424	$\frac{1}{2}(1,1,1),$ $\frac{1}{3}(1,1,2),$ $\frac{1}{5}(1,2,3)$	7	$x + y + z + 1/y + z/(xy) +$ $1/(xy^5z^2) + 3/(x^2y^5z) +$ $3/(x^3y^5) + z/(x^4y^5)$	$\begin{array}{ccc ccc} 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 2 & 2 & 1 & -3 & 3 & 3 \end{array}$	[1, 0, 2, 0, 6, 0, 20, 0, 70, 7560, 252]	$\mathbb{P}^1 \times \mathbb{P}^1 +$ 0-st.
3	Q38989	P519468	$2 \times \frac{1}{2}(1,1,1),$ $\frac{1}{3}(1,1,2),$ $\frac{1}{4}(1,1,3)$	2	$x + y + z + y/(x^4z^3) +$ $2/(x^4z^2) + 1/(x^4yz) +$ $y/(x^5z^4) + 3/(x^5z^3) +$ $3/(x^5yz^2) + 1/(x^5y^2z)$	$\begin{array}{ccc ccc} 1 & 1 & 1 & -1 & 1 & 1 & 2 \\ 0 & 2 & -2 & 3 & 1 & -1 & 0 \end{array}$	[1, 0, 0, 0, 0, 0, 0, 420, 0, 7560, 0]	$\mathbb{F}_1 + 0\text{-st.}$
4	Q39260	P516023	$\frac{1}{5}(1,2,3)$	9	$x + y + z + y^2/(x^2z^3) +$ $2y^2/(x^2z^4) + y^2/(x^2z^5) +$ $y/(x^2z^2) + 4y/(x^2z^3) +$ $3y/(x^2z^4) + 2/(x^2z^2) +$ $3/(x^2z^3) + 1/(x^2yz^2)$	$\begin{array}{ccc ccc} 1 & 0 & 1 & -1 & 1 & 1 & 2 \\ 0 & 1 & -1 & 3 & 2 & -2 & 0 \end{array}$	[1, 0, 0, 0, 0, 60, 360, 0, 0, 0, 18900]	$\mathbb{F}_1 + 0\text{-st.}$
5	Q39266	P541268	$2 \times \frac{1}{3}(1,1,2)$	2	$xyz^2 + x + y^2z + y + 2/(xz) +$ $1/(x^2y^2z^3)$	$\begin{array}{ccc ccc} 0 & 1 & 0 & 1 & 1 & -1 & 0 \\ 1 & -4 & 2 & -2 & -1 & 5 & 4 \end{array}$	[1, 0, 0, 0, 0, 120, 0, 420, 0, 0, 88200]	$\mathbb{F}_2 + 0\text{-st.}$
6		P541702	$2 \times \frac{1}{3}(1,1,2)$	2	$xyz^3 + x + 2yz + y + 1/y +$ $y/(xz)$	$\begin{array}{ccc ccc} 1 & 0 & 2 & -2 & 3 & 1 & 4 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{array}$	[1, 0, 2, 0, 6, 120, 20, 1680, 70, 15120, 88452]	$\mathbb{F}_2 + 0\text{-st.}$
7	Q39329	P429996	$\frac{1}{2}(1,1,1),$ $\frac{1}{3}(1,1,2)$	3	$xy^3z^2 + x + 2yz + y + z +$ $1/(xy) + 2/(xy^2) +$ $1/(x^2y^4z)$	$\begin{array}{ccc ccc} 1 & 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 2 & -1 & 3 & 2 \end{array}$	[1, 0, 0, 6, 24, 0, 90, 1680, 2520, 1680, 75600]	$\mathbb{P}^1 \times \mathbb{P}^1 +$ 0-st.
8		P516841		1	$x + y + z + 1/(y^2z^3) +$ $1/(xyz^3) + 1/(xy^2z^2) +$ $1/(xy^2z^3)$	$\begin{array}{ccc ccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 & 3 & -5 & 1 \end{array}$	[1, 0, 0, 0, 0, 0, 360, 420, 0, 0, 0]	\mathbb{P}^3
Codim 22								
9	Q38746	P517792	$\frac{1}{4}(1,1,3),$ $\frac{1}{5}(1,1,4)$	11	$xy^4z^5 + xy^3z^4 + x + 3y^3z^3 +$ $2y^2z^2 + y + 1/y + 3y^2z/x +$ $y/x + y/(x^2z)$	$\begin{array}{ccc ccc} 1 & 0 & -1 & -2 & 2 & -1 & -1 \\ 0 & 0 & 0 & -2 & 3 & -1 & 1 \\ 0 & 1 & 1 & 2 & -2 & 1 & 1 \end{array} \left \begin{array}{c} -2 \\ 0 \\ 2 \end{array} \right.$	[1, 0, 2, 6, 6, 60, 110, 420, 5110, 4200, 94752]	$\mathbb{F}_1 + 0\text{-st.}$

(Continued)

Table 2. Continued

Q-Fano ID	Polytope ID	Singularities	MG #	Rigid MMLP	Weight Matrix and Bundles	Period Sequence	Shape	
10	P518340		1	$xy^4z^5 + xy^3z^4 + x + 2yz^2 + y + z + 1/y + 1/(xy^2z)$	$\begin{array}{ccccccc c} 1 & 0 & 0 & 3 & -3 & 1 & 2 & 3 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 2 & 1 & -1 & 0 \end{array}$	[1, 0, 2, 0, 6, 60, 20, 840, 3430, 7560, 94752]	$\mathbb{F}_1 + 0\text{-st.}$	
Q38775	P519409	$\frac{1}{2}(1,1,1),$ $\frac{1}{3}(1,1,2),$ $\frac{1}{5}(1,1,4)$	3	$xyz^2 + x + 2x/(yz) + x/(y^2z^3) + x/(y^3z^4) + y + 2z/y + 1/(y^2z) + 2/(y^3z^2) + 1/(xy^3)$	$\begin{array}{ccccccc c} 1 & 3 & -3 & 0 & 1 & 2 & 3 & 3 \\ 0 & -4 & 5 & 1 & -1 & -3 & -3 & -3 \end{array}$	[1, 0, 0, 0, 0, 60, 0, 0, 3360, 0, 18900]	Z_1	
12	Q38791	P519951	$2 \times \frac{1}{3}(1,1,2),$ $\frac{1}{4}(1,1,3)$	1	$x + y + z + 1/(xz) + 1/(x^2y^2z^3) + 1/(x^3y^2z^2)$	$\begin{array}{ccccccc c} 1 & 0 & 0 & 2 & 1 & 1 & 2 & 2 \\ 0 & 1 & 2 & 5 & 3 & 3 & 6 & 6 \end{array}$	[1, 0, 0, 6, 0, 0, 90, 0, 3360, 1680, 0]	\mathbb{F}_1 (no 0-st.)
13	Q38946	P427943	$\frac{1}{3}(1,1,2),$ $\frac{1}{5}(1,1,4)$	1	$x + y^3z + y + z + y^3z^2/x + 1/x + 1/(xyz)$	$\begin{array}{ccccccc c} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & -3 & -1 & 1 & 1 & -2 & -1 \\ 0 & 1 & -8 & -3 & 3 & 2 & -4 & -2 \end{array}$	[1, 0, 2, 0, 30, 0, 380, 0, 5950, 7560, 101052]	$\mathbb{F}_1 + 0\text{-st}$
14	Q38948	P428047	$\frac{1}{2}(1,1,1),$ $\frac{1}{3}(1,1,2),$ $\frac{1}{4}(1,1,3)$	1	$x + xz^3/y^2 + y^2/z + y + y/(xz) + y/(xz^2) + y^3/(x^2z^3)$	$\begin{array}{ccccccc c} 1 & 0 & 1 & 1 & -1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 & 1 & -1 & 0 & -1 \\ 0 & 0 & -2 & -1 & 2 & -3 & 2 & -1 \end{array}$	[1, 0, 0, 0, 24, 0, 0, 420, 2520, 7560, 0]	$\mathbb{F}_1 + 0\text{-st}$
15	Q39259	P505841	$\frac{1}{2}(1,1,1),$ $\frac{1}{3}(1,1,2)$	2	$x^3y/z^2 + 2xy/z + x + y + z + 1/y + y/x$	$\begin{array}{ccccccc c} 1 & 0 & 0 & 2 & -2 & 1 & 3 & 4 \\ 0 & 1 & 0 & 1 & -1 & 0 & 2 & 2 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{array}$	[1, 0, 2, 6, 6, 180, 110, 2100, 8470, 19320, 258552]	$\mathbb{F}_2 + 0\text{-st.}$
16		P507415		2	$x^2yz^3 + xyz^2 + x + y + 2z + 1/x + 1/(x^2yz)$	$\begin{array}{ccccccc c} 1 & 0 & 2 & -2 & 1 & 1 & 2 & 2 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \end{array}$	[1, 0, 2, 0, 6, 120, 20, 2100, 70, 22680, 88452]	$\mathbb{F}_1 + 0\text{-st.}$
17	Q39277	P520149	$3 \times \frac{1}{2}(1,1,1)$	1	$xyz^2 + x + yz + y + z/x + 1/(xyz)$	$\begin{array}{ccccccc c} 0 & -2 & 0 & 1 & 1 & -1 & 0 & 0 \\ 1 & 3 & 1 & -1 & -1 & 3 & 2 & 2 \end{array}$	[1, 0, 0, 6, 0, 120, 90, 0, 6720, 1680, 88200]	$\mathbb{F}_1 + 0\text{-st.}$
18	Q39321	P255545	$\frac{1}{3}(1,1,2)$	2	$xy^3/z + xy^2/z + x + y + y/z + z + 1/(xy)$	$\begin{array}{ccccccc c} 1 & 0 & 0 & 0 & -2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 & 1 & 1 \end{array}$	[1, 0, 0, 6, 24, 60, 90, 1680, 5880, 16800, 94500]	$\mathbb{P}^2 + 0\text{-st.}$

(Continued)

Table 2. Continued

Q-Fano ID	Polytope ID	Singularities	MG #	Rigid MMLP	Weight Matrix and Bundles	Period Sequence	Shape	
19	Q39328	P254809	$2 \times \frac{1}{2}(1,1,1)$	1	$x^2z^3/y^2 + x^2z^4/y^3 + xz + x + xz^2/y + xz^3/y^2 + y + y/z + y/(xz^2)$	$\begin{array}{ccc ccc} 1 & 0 & 0 & -1 & 1 & 2 & -1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & -1 & 0 & 1 & 1 \end{array}$	[1, 0, 2, 6, 30, 60, 470, 2100, 7630, 42000, 195552]	$\mathbb{P}^1 \times \mathbb{P}^1 + 0\text{-st.}$
Codim 21								
20	Q38745	P515798	$2 \times \frac{1}{4}(1,1,3)$	2	$x + y + z + 1/y + 1/(xy) + 1/(xy^2z) + 2/(x^2y^3z^2) + 1/(x^3y^5z^4)$	$\begin{array}{ccc ccc} 1 & 0 & 0 & 2 & -2 & -1 & 3 & 2 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 2 & 1 & -1 & 0 \end{array}$	[1, 0, 2, 6, 6, 120, 110, 1260, 8470, 11760, 189252]	$\mathbb{F}_1 + 0\text{-st.}$
21	Q38764	P429611	$2 \times \frac{1}{2}(1,1,1),$ $\frac{1}{5}(1,1,4)$	2	$xy^4z^5 + 2xy^3z^4 + xy^2z^3 + x + 2yz^2 + y + 2z + 1/(xy^2z)$	$\begin{array}{ccc ccc} 1 & 0 & 0 & 1 & -3 & 2 & 0 & 0 \\ 0 & 1 & 1 & 1 & 2 & -1 & 2 & 2 \end{array}$	[1, 0, 0, 0, 0, 120, 0, 0, 3360, 0, 88200]	$\mathbb{P}^2 + 0\text{-st.}$
22	Q38772	P428621	$\frac{1}{2}(1,1,1),$ $\frac{1}{3}(1,1,2),$ $\frac{1}{4}(1,1,3)$	2	$xy^3z^4 + x + 2yz^2 + y + z + 1/(xy) + 1/(xy^2z)$	$\begin{array}{ccc ccc} 1 & 0 & 2 & -2 & -1 & 3 & 2 & 0 \\ 0 & 1 & -1 & 2 & 1 & -1 & 0 & 0 \end{array}$	[1, 0, 0, 6, 0, 60, 90, 0, 6720, 1680, 18900]	$\mathbb{F}_1 + 0\text{-st.}$
23	Q38917	P512391	$\frac{1}{3}(1,1,2),$ $\frac{1}{4}(1,1,3)$	2	$x^3y^4z + x^2y^4z + 2xy^2z + x + y + z + 2y/x + 2/(x^2y) + 1/(x^4y^2z)$	$\begin{array}{ccc ccc} 1 & -3 & -1 & 3 & 2 & -2 & 2 & 0 \\ 0 & 2 & 1 & -1 & -1 & 2 & 0 & 0 \end{array}$	[1, 0, 0, 0, 24, 120, 0, 0, 2520, 37800, 88200]	\mathbb{F}_1 (no 0-st.)
24	Q38945	P399587	$\frac{1}{2}(1,1,1),$ $2 \times \frac{1}{3}(1,1,2)$	1	$x^3z^4/y^3 + x^2z + x + xz^3/y^2 + xz^2/y^2 + y + y/z + y/(xz^2)$	$\begin{array}{ccc ccc} 1 & 0 & 1 & -3 & -1 & -2 & -2 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \end{array}$	[1, 0, 0, 12, 0, 0, 540, 420, 0, 33600, 75600]	\mathbb{F}_1 (no 0-st.)
25	Q39250	P428844	$\frac{1}{3}(1,1,2)$	1	$x^2z^2/y + 2xz + x + xz^3/y^2 + y + y/z + 1/z + y/(xz^2)$	$\begin{array}{ccc ccc} 1 & 0 & 1 & -1 & 1 & 1 & 2 & 0 \\ 0 & 1 & -1 & 2 & 1 & -1 & 0 & 0 \end{array}$	[1, 0, 0, 6, 48, 0, 90, 2520, 11760, 1680, 100800]	$\mathbb{F}_1 + 0\text{-st.}$
26		P430410		1	$xyz^2 + xyz + x + y + z/x + 1/x + 1/(xyz)$	$\begin{array}{ccc ccc} 1 & 2 & 1 & 1 & 1 & -1 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 2 & 0 \end{array}$	[1, 0, 4, 0, 36, 120, 400, 4200, 4900, 105840, 151704]	$\mathbb{P}^1 \times \mathbb{P}^1$ (no 0-st.)
27	Q39258	P401207	$2 \times \frac{1}{2}(1,1,1)$	3	$x + y + z + 1/y + 1/(xy) + 2/(xy^2z) + 1/(xy^3z^2)$	$\begin{array}{ccc ccc} 1 & 0 & 1 & -1 & -1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \end{array}$	[1, 0, 2, 6, 6, 180, 110, 2520, 8470, 26880, 283752]	$\mathbb{F}_1 + 0\text{-st.}$

(Continued)

Table 2. Continued

\mathbb{Q} -Fano ID	Polytope ID	Singularities	MG #	Rigid MMLP	Weight Matrix and Bundles	Period Sequence	Shape	
28	Q39320	P430098	$\frac{1}{2}(1, 1, 1)$	1	$x^3yz + x^2yz + x + y + z + 1/(x^2z) + 1/(x^2y)$	$\begin{array}{ccc ccc} 1 & 0 & 1 & 2 & 3 & 1 & 4 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \end{array}$	[1, 0, 0, 0, 48, 60, 0, 0, 11760, 30240, 18900]	$\mathbb{P}^1 \times \mathbb{P}^1$ (no 0-st.)
Codim 20								
29	Q37923	P544136	$\frac{1}{2}(1, 1, 1),$ $\frac{1}{4}(1, 1, 3),$ $\frac{1}{5}(1, 1, 4)$	5	$x + y + z + 1/(yz^2) + 1/(xz^2) + 1/(xy^2z^3) + 3/(x^2yz^3) + 3/(x^3z^3) + y/(x^4z^3)$	$\begin{array}{ccc ccc} 1 & 2 & 1 & -1 & 3 & 1 & 4 \\ 0 & -3 & -2 & 3 & -5 & -1 & -6 \end{array}$	[1, 0, 0, 0, 24, 0, 0, 2100, 2520, 0, 0]	\mathbb{F}_2 (no 0-st)
30	Q37992	P519452	$\frac{1}{2}(1, 1, 1),$ $\frac{1}{4}(1, 1, 3),$ $\frac{1}{5}(1, 1, 4)$	2	$x + x/(yz) + y + z + z/(xy^2) + 2/(xy^3) + 1/(xy^4z) + z^3/(x^2y) + 4z^2/(x^2y^2) + 3z/(x^2y^3) + 2z^4/(x^3y) + 3z^3/(x^3y^2) + z^5/(x^4y)$	$\begin{array}{ccc ccc} 1 & 0 & 2 & -1 & 1 & 1 & 2 \\ 0 & 1 & -1 & 3 & 1 & -1 & 0 \end{array}$	[1, 0, 0, 0, 0, 60, 0, 1680, 0, 7560, 18900]	$\mathbb{F}_1 + 0$ -st.
31		P544275		4	$x^3yz^5 + 2xyz^2 + x + y + 2z^2 + y/(xz) + 2/(x^2z) + 1/(x^3yz)$	$\begin{array}{ccc ccc} 0 & 3 & 4 & 1 & 3 & 2 & 6 \\ 1 & 0 & 1 & 0 & 1 & -1 & 0 \end{array}$	[1, 0, 0, 0, 0, 0, 0, 2100, 0, 15120, 0]	\mathbb{P}^2 (no 0-st.)
32	Q38005	P544272	$2 \times \frac{1}{2}(1, 1, 1),$ $2 \times \frac{1}{4}(1, 1, 3)$	2	$xz^2 + 2xz + x + y + z^2/y + 1/x + z^2/(x^3y^3) + 2/(x^4y^2) + 1/(x^5yz^2)$	$\begin{array}{ccc ccc} 0 & 3 & 4 & 2 & 3 & 1 & 6 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}$	[1, 0, 2, 0, 6, 0, 20, 2100, 70, 37800, 252]	\mathbb{F}_1 (no 0-st.)
33	Q38329	P480334	$\frac{1}{2}(1, 1, 1),$ $\frac{1}{3}(1, 1, 2),$ $\frac{1}{5}(1, 1, 4)$	4	$xyz^2 + x + x/(yz^3) + y + 2/(yz^2) + 2/(y^3z^5) + 1/(xyz) + 1/(xyz^2) + 2/(xy^3z^4) + 1/(xy^5z^7)$	$\begin{array}{ccc ccc} 0 & 0 & 3 & 5 & 1 & -1 & 2 & 6 \\ 1 & 0 & 2 & 4 & 0 & -1 & 1 & 4 \\ 0 & 1 & -3 & -5 & -1 & 2 & -2 & -6 \end{array}$	[1, 0, 2, 0, 6, 60, 380, 1260, 10150, 15120, 170352]	\mathbb{F}_1 (no 0-st.)
34		P480338		5	$x^4y^7z^2 + 2x^3y^4z + 2x^2y^4z + x^2y^3z + x^2y + 2xy + x + y + z + 1/z + 1/(xy^3z^2)$	$\begin{array}{ccc ccc} 0 & 0 & 3 & 1 & 5 & -1 & 2 & 6 \\ 1 & 0 & -3 & -1 & -5 & 2 & -2 & -6 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \end{array}$	[1, 0, 2, 0, 6, 0, 380, 420, 10150, 15120, 151452]	\mathbb{F}_1 (no 0-st.)
35	Q38517	P518018	$2 \times \frac{1}{4}(1, 1, 3)$	3	$x + y + y/z + z + 1/z + 2/z^2 + 1/(yz^3) + 2y/x^2 + 2/(x^2z) + yz/x^4$	$\begin{array}{ccc ccc} 0 & 0 & -2 & 1 & -1 & 0 & 1 & 0 \\ 1 & 0 & 2 & -1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 4 & 0 & 2 & 1 & -1 & 2 \end{array}$	[1, 0, 2, 6, 30, 60, 470, 2940, 7630, 64680, 271152]	$\mathbb{F}_1 + 0$ -st.

(Continued)

Table 2. Continued

Q-Fano ID	Polytope ID	Singularities	MG #	Rigid MMLP	Weight Matrix and Bundles	Period Sequence	Shape	
36	Q38737	P511389	$\frac{1}{3}(1,1,2),$ $\frac{1}{4}(1,1,3)$	31	$x^3yz^4 + 3x^3z^3 + 3x^3z^2/y +$ $x^3z/y^2 + x^2yz^3 + 4x^2z^2 +$ $5x^2z/y + 2x^2/y^2 + xz + x +$ $2x/y + x/(y^2z) + y + 1/x$	$\begin{matrix} 1 & 0 & 0 & -1 & 1 & 1 & -1 & -2 & -1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 2 & 3 & 1 & -1 & -1 & 3 & 2 \\ 0 & 0 & 1 & 1 & -1 & -1 & 1 & 2 & 1 & 0 & 1 \end{matrix}$	[1, 0, 2, 12, 6, 180, 560, 1680, 20230, 46200, 359352]	dP ₇ + 0-st
37		P429716		3	$xy^3z^4 + xy^2z^3 + x + 2yz^2 +$ $y + 2z + 1/y + 1/(xy) +$ $1/(xy^2z)$	$\begin{matrix} 1 & 0 & -1 & 1 & -1 & 0 & 2 & 2 \\ 0 & 1 & 0 & -1 & 1 & -2 & 1 & -1 \\ 0 & 0 & 1 & 0 & 1 & 2 & -2 & 0 \end{matrix}$	[1, 0, 2, 6, 6, 180, 110, 2100, 11830, 19320, 334152]	F ₁ + 0-st.
38	Q38763	P254789	$2 \times$ $\frac{1}{2}(1,1,1),$ $\frac{1}{4}(1,1,3)$	1	$x + y + z + 1/(xz) +$ $1/(xyz^2) + 1/(x^2yz) +$ $1/(x^2y^2z^3) + 1/(x^3y^2z^2)$	$\begin{matrix} 1 & 0 & -1 & 1 & -2 & 2 & 0 \\ 0 & 1 & 1 & 1 & 2 & -1 & 2 \end{matrix}$	[1, 0, 0, 6, 0, 120, 90, 0, 10080, 1680, 88200]	F ₁ (no 0-st.)
39		P519989		3	$x + xz^2/y + y + z + 2/y^2 +$ $1/(xyz) + 1/(xy^3z^2)$	$\begin{matrix} 1 & -3 & 2 & -2 & -1 & -1 & -2 \\ 0 & 2 & 0 & 1 & 1 & 1 & 2 \end{matrix}$	[1, 0, 0, 6, 24, 0, 90, 2520, 2520, 1680, 151200]	F ₁ + 0-st.
40	Q38769	P429717	$\frac{1}{2}(1,1,1),$ $2 \times$ $\frac{1}{3}(1,1,2)$	24	$x + y + z + y^4/(xz^3) +$ $3y^3/(xz^2) + 3y^2/(xz) +$ $y/x + 2y^2/(x^2z^2) +$ $5y/(x^2z) + 4/x^2 +$ $z/(x^2y) + 1/(x^3z) +$ $2/(x^3y) + z/(x^3y^2)$	$\begin{matrix} 1 & 0 & 1 & 1 & -1 & -1 & 1 & 2 & 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 & 2 & 1 & 1 & 1 & 2 & 3 & 2 & 2 & 3 \end{matrix}$	[1, 0, 0, 12, 0, 60, 540, 0, 10080, 33600, 18900]	dP ₇ + 0-st.

(Continued)

Table 2. Continued

	\mathbb{Q} -Fano ID	Polytope ID	Singula- rities	MG #	Rigid MMLP	Weight Matrix and Bundles	Period Sequence	Shape
41	Q38916	P246021	$2 \times \frac{1}{3}(1, 1, 2)$	2	$xy^3z^2 + x + y^3z + yz + y + z + y/x + 1/x + 1/(xyz)$	$\begin{array}{ccccccc c} 1 & 0 & 0 & -1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & -1 & -1 & 3 & 2 \end{array}$	[1, 0, 2, 6, 30, 120, 470, 2940, 10990, 72240, 290052]	$\mathbb{F}_1 + 0\text{-st.}$
42		P387779		2	$x + x/(y^2z) + yz^2 + y + z + y^2z^2/x + 2yz/x + y^2z/x^2$	$\begin{array}{ccccccc c} 1 & 0 & 2 & -1 & 3 & 1 & & 4 \\ 0 & 1 & -1 & 1 & -2 & 1 & & -1 \end{array}$	[1, 0, 0, 18, 0, 0, 1350, 420, 0, 141120, 100800]	\mathbb{F}_1 (no 0-st.)
43		P411025		2	$x^2y^3z + xy^3z + xy^2z + x + 2yz + y + z + 2y/x + 1/x + z/(xy) + 2/(x^2y) + 1/(x^3yz)$	$\begin{array}{ccccccc c} 1 & 0 & 0 & 2 & -1 & 2 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 & 2 & 2 \end{array}$	[1, 0, 2, 0, 30, 120, 380, 2100, 5950, 60480, 189252]	\mathbb{F}_1 (no 0-st.)
44	Q38909	P412951	$\frac{1}{2}(1, 1, 1),$ $\frac{1}{4}(1, 1, 3)$	3	$x^4y^3z^4 + 2x^3y^2z^3 + 2x^2y^2z^2 + x^2yz^2 + x^2yz + 2xyz + x + x/z + y + z + 1/(xy)$	$\begin{array}{ccccccc c} 1 & 0 & 0 & 2 & -1 & 2 & -1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 2 & 2 \end{array}$	[1, 0, 0, 6, 24, 120, 90, 1680, 9240, 39480, 163800]	\mathbb{F}_1 (no 0-st.)
45	Q38936	P220424	$2 \times \frac{1}{2}(1, 1, 1),$ $\frac{1}{3}(1, 1, 2)$	2	$xy^3z^2 + x + y^2z + yz + y + z + 1/x + 1/(xz) + 1/(xy)$	$\begin{array}{ccccccc c} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 2 & -1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & -1 & 1 \end{array}$	[1, 0, 2, 12, 6, 180, 560, 2100, 16870, 53760, 359352]	$\mathbb{P}^1 \times \mathbb{P}^1 + 0\text{-st.}$
46		P231730		2	$x^2yz + x^2z + x + x/y + y + z + z/y + y/(xz)$	$\begin{array}{ccccccc c} 1 & 0 & 4 & -3 & 3 & -2 & -1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & -2 & 1 & 1 & 0 \end{array}$	[1, 0, 0, 12, 24, 0, 540, 2940, 2520, 33600, 327600]	$\mathbb{P}^2 \times \mathbb{P}^1$
47	Q39052	P336592	$\frac{1}{2}(1, 1, 1),$ $\frac{1}{3}(1, 1, 2)$	6	$x^3z/y^2 + x + 2x/y + y + z + 1/z + 1/(xz)$	$\begin{array}{ccccccc c} 1 & 0 & -1 & 2 & -2 & 1 & 3 & 4 \\ 0 & 1 & 0 & 1 & -1 & 0 & 2 & 2 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{array}$	[1, 0, 2, 6, 54, 60, 830, 2940, 20230, 64680, 523152]	$\mathbb{F}_2 + 0\text{-st.}$
48		P336593		2	$x + y + z + 2y^2z^2/x + y^2z/x + 2yz/x + y/x + 1/x + 1/(xyz) + y^4z^3/x^2 + 2y^3z^2/x^2 + y^2z/x^2$	$\begin{array}{ccccccc c} 0 & 0 & 0 & -1 & 1 & -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & -1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 2 & -1 & 1 & 2 \end{array}$	[1, 0, 2, 0, 54, 60, 740, 1260, 18550, 45360, 422352]	$\mathbb{F}_1 + 0\text{-st.}$

(Continued)

Table 2. Continued

	\mathbb{Q} -Fano ID	Polytope ID	Singularities	MG #	Rigid MMLP	Weight Matrix and Bundles	Period Sequence	Shape
49	Q39159	P255193	$\frac{1}{3}(1,1,2)$	2	$x+y+z+y/(xz)+2y/(xz^2)+y/(xz^3)+2/(xz)+2/(xz^2)+1/(xyz)$	$\begin{array}{ccc cc} 1 & 0 & 0 & -1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 0 & 1 & -1 \end{array} \left \begin{array}{c} 0 & 2 \\ 2 & 0 \end{array} \right.$	[1, 0, 0, 12, 48, 0, 540, 5040, 11760, 33600, 504000]	$\mathbb{F}_1 + 0\text{-st.}$
50	Q39168	P515512	$2 \times \frac{1}{2}(1,1,1)$	2	$x+x/(yz)+x/(y^2z)+y+z+2/(yz)+1/(xz)$	$\begin{array}{ccc cc} 1 & 0 & 1 & -1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 & -1 \end{array} \left \begin{array}{c} 3 \\ 0 \end{array} \right.$	[1, 0, 0, 18, 0, 60, 1350, 0, 10080, 141120, 18900]	$\mathbb{F}_2 + 0\text{-st.}$
51		P516400		2	$x+y+z+1/z+1/(yz)+2/(xz)+y/(x^2z)$	$\begin{array}{ccc ccc} 1 & 0 & 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 & -1 & -2 \end{array} \left \begin{array}{c} 3 \\ -3 \end{array} \right.$	[1, 0, 2, 18, 6, 180, 1370, 1260, 25270, 148680, 283752]	\mathbb{F}_1 (no 0-st.)
52	Q39249	P254856	$\frac{1}{2}(1,1,1)$	5	$x+y+z+1/(xy)+1/(xyz)+z/(x^2y)+2/(x^2y)+1/(x^2yz)$	$\begin{array}{ccc ccc} 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & -1 & 2 & 1 & -1 \end{array} \left \begin{array}{c} 1 \\ 1 \end{array} \right.$	[1, 0, 0, 6, 48, 60, 90, 2520, 15120, 31920, 119700]	$\mathbb{F}_1 + 0\text{-st.}$
53		P255732		2	$x^2y/z+x+x/z+y+z+1/y+1/x$	$\begin{array}{ccc ccc} 1 & 0 & -1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{array} \left \begin{array}{c} 0 \\ 2 \end{array} \right.$	[1, 0, 4, 6, 36, 240, 490, 6300, 14980, 142800, 592704]	$\mathbb{P}^1 \times \mathbb{P}^1$ (no 0-st.)
54		P255860		1	$xy/z+x+x/z+x/y+yz+y+z+1/x$	$\begin{array}{ccc ccc} 1 & 0 & 0 & 1 & -1 & -1 & 2 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{array} \left \begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right.$	[1, 0, 2, 12, 6, 240, 560, 2940, 23590, 61320, 554652]	$\mathbb{P}^1 \times \mathbb{P}^1 + 0\text{-st.}$

Table 3. Exploring the GRDB.

\mathbb{Q} -Fano ID	Polytope ID	Codimand Singularities	MG #	Rigid MMLP	Weight Matrix and Bundles	Period Sequence	Shape
1 Q29792	P542987	codim. 13; $\frac{1}{2}(1, 1, 1)$, $2 \times$ $\frac{1}{3}(1, 1, 2)$, $\frac{1}{5}(1, 1, 4)$	28	$x^2y^3z^4 + xy^3z^4 +$ $2xy^2z^2 + x +$ $y^2z^3 + 2y^2z^2 + y +$ $z + 2yz/x + y/x +$ $z/x + 3/(xyz) +$ $1/(x^2z) +$ $1/(x^2y) +$ $2/(x^2yz) +$ $1/(x^2y^2z^3) +$ $2/(x^3y^2z^2) +$ $1/(x^3y^2z^3) +$ $1/(x^4y^3z^4)$	$\begin{array}{ccccccc cc} 1 & 0 & 0 & 1 & 1 & -3 & 2 & 2 & 2 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & -1 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}$	[1, 0, 0, 0, 576, 2820, 4320, 2100, 2540160, 26490240, 110508300]	$\mathbb{P}^2 \times \mathbb{P}^1$
2 Q29911	P542990	codim. 13; $3 \times$ $\frac{1}{3}(1, 1, 2)$, $\frac{1}{4}(1, 1, 3)$	5	$xy/z^2 + xy/z^3 + xz +$ $x/z^2 + x/z^3 +$ $xz/y + yz + y/z^2 +$ $y/z^3 + 3z + z/y +$ $yz/x + z/x$	$\begin{array}{ccccccc c} 0 & 0 & 0 & 3 & -2 & 1 & -1 & 1 & 1 \\ 1 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 & 1 & 1 \end{array}$	[1, 0, 0, 54, 576, 0, 14850, 408660, 2540160, 5821200, 268077600]	$\mathbb{P}^2 \times \mathbb{P}^1$
3 Q31617	P543084	codim. 11; $2 \times$ $\frac{1}{2}(1, 1, 1)$, $\frac{1}{5}(1, 2, 3)$	19	$x^2z + x^2z/y + xyz +$ $2xz + xz/y + yz +$ $y/z + y/z^2 + z +$ $1/z + 2/z^2 +$ $1/(yz^2) + y/(xz) +$ $y/(xz^2) + 1/(xz) +$ $2/(xz^2) + 1/(xyz^2)$	$\begin{array}{ccccccc cc} 0 & 0 & 2 & -1 & 1 & 0 & -1 & 1 & 1 & 0 \\ 1 & 0 & -1 & 1 & 0 & 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & 1 & -1 & 2 & 2 & 0 & 1 & 2 \end{array}$	[1, 0, 8, 72, 264, 6900, 44360, 552720, 7081480, 63161280, 887727708]	$\mathbb{F}_1 \times \mathbb{P}^1$
4 Q33018	P31470	codim. 10; $2 \times$ $\frac{1}{2}(1, 1, 1)$, $\frac{1}{3}(1, 1, 2)$	4	$xy^2 + xy^2/z + 2xy +$ $4xy/z + xy/z^2 +$ $x + 4x/z + 2x/z^2 +$ $x/(yz) + x/(yz^2) +$ $y^2/z + 2y/z + z +$ $1/z + 2/y +$ $1/(yz) + 1/(y^2z) +$ $1/x + 1/(xy)$	$\begin{array}{ccccccc cccc} 1 & 0 & 0 & 1 & 0 & -1 & 3 & 2 & -1 & 2 & 3 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 & 1 & 2 & -1 & -1 & 1 & 1 & -1 & 0 & 1 & 1 \end{array}$	[1, 0, 8, 108, 720, 8760, 111500, 1218000, 15156400, 193520880, 2416466808]	\mathbb{Z}_2

(Continued)

Table 3. Continued

Q-Fano ID	Polytope ID	Codimand Singularities	MG # Rigid MMLP	Weight Matrix and Bundles	Period Sequence	Shape	
5 Q33019	P542859	codim. 11; $\frac{1}{2}(1, 1, 1)$, $\frac{1}{5}(1, 2, 3)$	6	$x^3y/z^3 + x^2y^2/z^3 + x^2y/z^2 + x + 2x/z + y + 2y/z + z + y^2/(xz) + y/x + z^2/(xy) + z/(xy) + z/x^2$	$\begin{array}{cccccc cc} 1 & 0 & 0 & 1 & 1 & -2 & 2 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 & 1 & 1 & 1 & 1 \end{array}$	[1, 0, 0, 72, 432, 1020, 33840, 406140, 2222640, 28113120, 412416900]	Z_3
6 Q33189	P543701	codim. 11; $3 \times$ $\frac{1}{2}(1, 1, 1)$, $\frac{1}{3}(1, 1, 2)$	6	$xy/z^2 + xz + x + x/z^2 + xz/y + x/y + y/z^2 + 2z + 2z/y + z/y^2 + z/x + z/(xy)$	$\begin{array}{cccccc cc} 1 & 0 & 2 & -2 & 1 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 & 0 & -1 & 2 & 1 & 1 & 1 \end{array}$	[1, 0, 0, 72, 192, 120, 32760, 176400, 342720, 21354480, 171397800]	Z_3
7 Q33470	P518676	codim. 13; $3 \times$ $\frac{1}{2}(1, 1, 1)$, $\frac{1}{4}(1, 1, 3)$	3	$xyz^2 + x/z + yz^2 + y/z + z + 3/z + 1/(yz) + y/(xz) + 2/(xz) + 1/(xyz)$	$\begin{array}{cccccc cc} 1 & 0 & 0 & -2 & 1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 2 \end{array}$	[1, 0, 6, 54, 90, 2880, 16710, 119700, 1679370, 10402560, 115479756]	$\mathbb{P}^1 \times \mathbb{P}^1 + 0\text{-st.}$
8 Q34466	P515086	codim. 12; $3 \times$ $\frac{1}{2}(1, 1, 1)$, $\frac{1}{3}(1, 1, 2)$	5	$xyz^3 + xyz^2 + xz^3 + xz + x + xz/y + yz^3 + yz + y + 2z + 1/z + 1/(yz) + yz/x + 1/(xz)$	$\begin{array}{cccccc cccc} 1 & 0 & 0 & 0 & -2 & 1 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 1 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 2 & 1 & 1 & 1 \end{array}$	[1, 0, 8, 42, 264, 3360, 21590, 245280, 2202760, 20993280, 218056608]	dP ₇ + 0-st
9		codim. 12; $3 \times$ $\frac{1}{2}(1, 1, 1)$, $\frac{1}{3}(1, 1, 2)$	2	$x + x/z + x/(yz) + y + y/z + y/z^3 + z + 3/z + z/y + y^2/(xz^3) + 2y/(xz) + z/x$	$\begin{array}{cccccc cccc} 1 & 0 & 0 & 0 & -2 & 1 & -1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 1 & -1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 2 & 1 & 1 & 1 & 1 \end{array}$	[1, 0, 10, 54, 342, 4680, 31150, 371700, 3524710, 34957440, 383835060]	$\mathbb{P}^1 \times \mathbb{P}^1 + 0\text{-st}$

(Continued)

Table 3. Continued

Q-Fano ID	Polytope ID	Codimand Singularities	MG #	Rigid MMLP	Weight Matrix and Bundles	Period Sequence	Shape	
10	Q35361	P516572	codim. 12; $2 \times \frac{1}{2}(1, 1, 1), \frac{1}{3}(1, 1, 2)$	9	$x + x/(y^2z) + y + z + 1/y + 3/(yz) + 1/(y^2z^2) + yz/x + y/x + 1/x + 2/(xz) + 1/(xz^2)$	$\begin{array}{cccccccc cc} 1 & 0 & -1 & 1 & -1 & 2 & 1 & -2 & 1 & -1 \\ 0 & 1 & -1 & 1 & 0 & 1 & 2 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \end{array}$	[1, 0, 4, 60, 204, 2040, 24340, 160440, 1547980, 16366560, 139003704]	$\mathbb{F}_1 \times \mathbb{P}^1$
11	Q35364	P516506	codim. 13; $2 \times \frac{1}{2}(1, 1, 1), \frac{1}{4}(1, 1, 3)$	7	$xy/z^2 + xy/z^3 + xz + x + x/z^2 + xz/y + yz + y + y/z^2 + 3z + z/y + yz/x + z/x$	$\begin{array}{cccccccc cc} 1 & 0 & 0 & -2 & 1 & 1 & 2 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 & -1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}$	[1, 0, 0, 54, 168, 120, 14850, 100380, 239400, 6063120, 61021800]	Z_4
12	Q35470	P519948	codim. 19; $2 \times \frac{1}{4}(1, 1, 3), \frac{1}{6}(1, 1, 5)$	10	$x + y^3z^4 + 2y^2z^2 + y + y^2z^3/x^3 + 3yz^2/x^3 + 2yz/x^3 + 3z/x^3 + 6/x^3 + 1/(x^3z) + 1/(x^3y) + 6/(x^3yz) + 3/(x^3y^2) + 2/(x^3y^2z^2) + 3/(x^3y^2z^3) + 1/(x^3y^3z^4)$	$\begin{array}{cccccc cc} 1 & 0 & 2 & -2 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 & 1 & 1 & 2 \end{array}$	[1, 0, 0, 0, 24, 120, 0, 0, 2520, 75600, 88200]	$\mathbb{P}^1 \times \mathbb{P}^1 + 0\text{-st}$
13	Q35496	P518648	codim. 13; $3 \times \frac{1}{2}(1, 1, 1), \frac{1}{3}(1, 1, 2)$	6	$xyz^3 + xyz^2 + xz^3 + xz + x + xz/y + yz^3 + yz + y + 3z + 1/(yz) + yz/x + 1/(xz)$	$\begin{array}{cccccc cc} 0 & 0 & 2 & -1 & 2 & -1 & 1 & 1 & 1 & 2 \\ 1 & 0 & -1 & 1 & 1 & 0 & 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}$	[1, 0, 4, 54, 36, 1800, 15250, 44100, 992740, 6773760, 40547304]	$\mathbb{F}_1 \times \mathbb{P}^1$
14	Q36639	P254485	codim. 15; $6 \times \frac{1}{2}(1, 1, 1)$	1	$xy^2z^3 + xy^2z^2 + xyz^2 + 2xyz + xy + x + x/z + y + 1/y + 1/(xyz)$	$\begin{array}{cccccc cc} 1 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 & -1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{array}$	[1, 0, 6, 0, 114, 120, 3300, 7560, 114450, 393120, 4461156]	$\mathbb{P}^1 \times \mathbb{P}^1 + 0\text{-st}$

(Continued)

Table 3. Continued

\mathbb{Q} -Fano ID	Polytope ID	Codimand Singularities	MG #	Rigid MMLP	Weight Matrix and Bundles	Period Sequence	Shape	
15		codim. 15; $6 \times \frac{1}{2}(1, 1, 1)$	1	$xy^2z^3 + xy^2z^2 +$ $xyz^2 + 3xyz + xy +$ $x + x/z + y + 1/y +$ $1/(xyz)$	$\begin{array}{cccc ccc} 1 & 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{array} \mid \begin{array}{c} -2 \\ 2 \\ 1 \\ 0 \end{array}$	[1, 0, 8, 0, 168, 120, 5120, 9240, 190120, 559440, 7976808]	$\mathbb{F}_1 + 0\text{-st.}$	
16	Q37383	P516493	codim. 13; $\frac{1}{3}(1, 1, 2)$	4	$xy/z + xz + yz + y +$ $y/z + 2z + 1/z +$ $z/y + yz/x + y/x +$ $2z/x + 1/x +$ $z/(xy)$	$\begin{array}{cccc ccc} 1 & 0 & 1 & -1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 & 0 & 1 & 0 & 0 \end{array} \mid \begin{array}{c} 1 & 2 \\ 1 & 0 \\ 1 & 0 \end{array}$	[1, 0, 8, 18, 264, 1200, 12950, 82320, 778120, 5856480, 52669008]	Z_3
17	37832	P220586	codim. 13; $\frac{1}{2}(1, 1, 1)$	2	$x + x/(y^2z) + y + z +$ $1/y + 1/(yz) +$ $2/(y^2z) + y/x +$ $2/x + 1/(xy) +$ $1/(xyz) + 1/(xy^2z)$	$\begin{array}{cccc ccc} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 & 2 & -1 \end{array} \mid \begin{array}{c} 2 \\ 0 \\ 0 \end{array}$	[1, 0, 6, 18, 186, 1200, 9330, 69720, 553770, 4450320, 36502956]	$\mathbb{F}_1 + 0\text{-st}$
18	Q37847	P515113	codim. 15; $\frac{1}{2}(1, 1, 1),$ $\frac{1}{3}(1, 1, 2)$	3	$x + x/(yz) + y + z +$ $z/y + 2yz/x +$ $2z/x + 2y^2z/x^2 +$ $yz/x^2 + y^2z^2/x^3 +$ y^3z^2/x^4	$\begin{array}{cccc ccc} 1 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & -1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \end{array} \mid \begin{array}{c} 2 & 2 & 1 & 1 & 0 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{array}$	[1, 0, 6, 18, 90, 960, 3210, 39900, 197610, 1668240, 12109356]	$dP_7 + 0\text{-st}$
19			codim. 15; $\frac{1}{2}(1, 1, 1),$ $\frac{1}{3}(1, 1, 2)$	3	$x + x/(yz) + y + z +$ $z/y + 3yz/x +$ $2z/x + 2y^2z/x^2 +$ $yz/x^2 + y^2z^2/x^3 +$ y^3z^2/x^4	$\begin{array}{cccc ccc} 1 & 0 & 0 & 0 & 2 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \end{array} \mid \begin{array}{c} 2 & 0 \\ 1 & 0 \\ 1 & 2 \end{array}$	[1, 0, 4, 18, 60, 780, 2470, 27720, 151900, 1063440, 8348004]	$\mathbb{P}^1 \times \mathbb{P}^1 + 0\text{-st}$
20		P516442	codim. 15 $\frac{1}{2}(1, 1, 1),$ $\frac{1}{3}(1, 1, 2)$	6	$x^2z/y + x^2z/y^2 +$ $xy + x + 3xz/y +$ $xz/y^2 + y + z +$ $2z/y + y/(xz) + z/x$	$\begin{array}{cccc ccc} 1 & 0 & 0 & 0 & 2 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \end{array} \mid \begin{array}{c} 1 & 2 \\ 1 & 1 \\ 1 & 1 \end{array}$	[1, 0, 4, 18, 60, 720, 2470, 25200, 141820, 972720, 7447104]	$\mathbb{F}_1 + 0\text{-st.}$

(Continued)

Table 3. Continued

Q-Fano ID	Polytope ID	Codimand Singularities	MG #	Rigid MMLP	Weight Matrix and Bundles	Period Sequence	Shape	
21	P516475	codim. 15; $\frac{1}{2}(1, 1, 1)$, $\frac{1}{3}(1, 1, 2)$	11	$xy^3z^2 + 2xy^2z^2 +$ $xyz^2 + x + 2y^2z +$ $4yz + y + 3z +$ $z/y + y/x + 2/x +$ $1/(xz) + 1/(xy)$	$\begin{array}{cccccccc cccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 2 & 1 & 1 & 2 & 2 & 2 \\ 0 & 0 & 1 & 1 & 2 & 1 & 0 & 0 & 1 & 2 & 1 & 1 & 2 & 2 \end{array}$	[1, 0, 4, 24, 60, 780, 4000, 24360, 209020, 1228920, 9356004]	dP ₇ +0-st.	
22	Q37859	P424999	codim. 16; $2 \times \frac{1}{3}(1, 1, 2)$	15	$x + x/(yz) + y + z +$ $2/z + 1/(yz) +$ $y^2/(xz) +$ $3y/(xz) + 1/(xz) +$ $2y^3/(x^2z) +$ $2y^2/(x^2z) +$ $y^4/(x^3z)$	$\begin{array}{cccccccc cccc} 1 & 0 & 0 & 0 & -1 & 1 & 2 & 1 & -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 1 & 0 & 0 & 1 & 2 & 1 & 1 & 2 & 2 \end{array}$	[1, 0, 4, 12, 60, 480, 1660, 17220, 74620, 610680, 3641904]	dP ₇ + 0-st.
23	Q37894	P425006	codim. 16; $2 \times$ $\frac{1}{2}(1, 1, 1)$, $\frac{1}{3}(1, 1, 2)$	14	$x + y + z + y/(xz) +$ $2/x + z/(xy) +$ $1/(xy) + z/(xy^2) +$ $1/(x^2z) +$ $1/(x^2y) +$ $1/(x^2yz) +$ $2/(x^2y^2) +$ $z/(x^2y^3)$	$\begin{array}{cccccccc cccc} 1 & 0 & 0 & 0 & 2 & 1 & -1 & -1 & 1 & 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 2 & 1 & 1 & 1 \end{array}$	[1, 0, 4, 6, 60, 360, 1210, 14280, 41020, 505680, 1966104]	dP ₇ + 0-st.
24		codim. 16; $2 \times$ $\frac{1}{2}(1, 1, 1)$, $\frac{1}{3}(1, 1, 2)$	1	$x^3/(yz^2) + x^2/(yz) +$ $x + 2x/(yz) + y +$ $z + 3/y + z/(xy) +$ $1/(xy) + z/(x^2y)$	$\begin{array}{cccc ccc} 1 & 0 & 0 & 0 & 2 & -1 & -1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \end{array}$	[1, 0, 6, 6, 90, 420, 1950, 19740, 63210, 810600, 2968056]	$\mathbb{P}^1 \times \mathbb{P}^1$ + 0-st.	
25	Q37905	P515878	codim. 17; $\frac{1}{2}(1, 1, 1), 2 \times$ $\frac{1}{3}(1, 1, 1)$	2	$x + xz^2/y + y + z +$ $z/y + 2/x +$ $2/(x^2z) +$ $y/(x^3z^2) +$ $y/(x^4z^3)$	$\begin{array}{cccc ccc} 1 & 0 & 0 & 2 & -2 & -1 & 3 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{array}$	[1, 0, 4, 0, 60, 120, 1120, 6300, 24220, 249480, 655704]	\mathbb{F}_1 + 0-st

(Continued)

Table 3. Continued

Q-Fano ID	Polytope ID	Codimand Singularities	MG #	Rigid MMLP	Weight Matrix and Bundles	Period Sequence	Shape	
26		codim. 17; $\frac{1}{2}(1,1,1), 2 \times$ $\frac{1}{3}(1,1,1)$	2	$x+xz^2/y+y+z+$ $z/y+3/x+$ $2/(x^2z)+$ $y/(x^3z^2)+$ $y/(x^4z^3)$	$\begin{array}{cccc ccc} 1 & 0 & 0 & 2 & -2 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 2 \end{array}$	[1, 0, 6, 0, 90, 120, 1860, 7980, 44730, 378000, 1260756]	$\mathbb{P}^1 \times \mathbb{P}^1 + 0\text{-st.}$	
27	Q38250	P254482	1	$x^2z^3/y^2+x^2z^4/y^3+$ $xz+x+2xz^2/y+$ xz^3/y^2+y+z+ $y/(xz)+y/(xz^2)$	$\begin{array}{cccc ccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}$	[1, 0, 4, 6, 84, 300, 2290, 11760, 80500, 483000, 3182004]	$\mathbb{P}^1 \times \mathbb{P}^1 + 0\text{-st.}$	
28		codim. 16; $3 \times \frac{1}{2}(1,1,1)$	1	$x^2z^3/y^2+x^2z^4/y^3+$ $xz+x+3xz^2/y+$ xz^3/y^2+y+z+ $y/(xz)+y/(xz^2)$	$\begin{array}{cccc ccc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & -1 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{array}$	[1, 0, 6, 6, 114, 360, 3390, 16800, 126210, 765240, 5368356]	$\mathbb{F}_1 + 0\text{-st}$	
29	Q38484	P515100	14	$x+y+z+z/y+y/x+$ $y/(xz)+2/x+$ $1/(xz)+z/(xy)+$ $2/(xy)+z/(xy^2)$	$\begin{array}{cccc cccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \end{array}$	$\begin{array}{cccc} 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{array}$	[1, 0, 4, 24, 84, 780, 4360, 29400, 214900, 1433040, 10540404]	dP ₇ + 0-st.
30		codim. 15; $\frac{1}{2}(1,1,1)$	4	$x+y+z+z/y+y/x+$ $y/(xz)+3/x+$ $1/(xz)+z/(xy)+$ $2/(xy)+z/(xy^2)$	$\begin{array}{cccc ccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \end{array}$	$\begin{array}{cc} 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{array}$	[1, 0, 6, 24, 114, 1020, 5460, 42000, 299250, 2100840, 16160256]	$\mathbb{P}^1 \times \mathbb{P}^1 + 0\text{-st}$

(Continued)

Table 3. Continued

	\mathbb{Q} -Fano ID	Polytope ID	Codimand Singularities	MG #	Rigid MMLP	Weight Matrix and Bundles	Period Sequence	Shape
31	Q38496	P427129	codim. 16; $2 \times \frac{1}{2}(1, 1, 1)$	1	$x + y + z + y/x +$ $y/(xz) + 2/x +$ $1/(xz) + z/(xy) +$ $2/(xy) + z/(xy^2)$	$\begin{array}{cccc ccc} 1 & 0 & 0 & 1 & -1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{array}$	[1, 0, 4, 18, 60, 600, 2470, 18900, 118300, 723240, 5242104]	$\mathbb{F}_1 + 0\text{-st}$
32			codim. 16; $2 \times \frac{1}{2}(1, 1, 1)$	1	$x + y + z + y/x +$ $y/(xz) + 3/x +$ $1/(xz) + z/(xy) +$ $2/(xy) + z/(xy^2)$	$\begin{array}{cccc ccc} 1 & 0 & 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 2 \end{array}$	[1, 0, 6, 18, 90, 780, 3210, 28560, 164010, 1146600, 8247456]	$\mathbb{P}^1 \times \mathbb{P}^1 + 0\text{-st}$
33	Q38895	P425118	codim. 17; $\frac{1}{2}(1, 1, 1)$	1	$x^2z^3/y^2 + xz + x +$ $3xz^2/y + xz^3/y^2 +$ $y + y/z + 2z +$ $z^2/y + y/(xz^2)$	$\begin{array}{cccc ccc} 1 & 0 & 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \end{array} \left \begin{array}{c} 1 & 0 \\ 1 & 0 \\ 1 & 2 \end{array} \right.$	[1, 0, 6, 12, 90, 540, 2400, 20160, 95130, 751800, 4291056]	$\mathbb{P}^1 \times \mathbb{P}^1 + 0\text{-st}$
34			codim. 17; $\frac{1}{2}(1, 1, 1)$	4	$x + yz + y + z + y/x +$ $y/(xz) + z/x +$ $2/x + 1/(xz) +$ $z/(xy) + 1/(xy)$	$\begin{array}{cccc ccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{array} \left \begin{array}{c} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{array} \right.$	[1, 0, 4, 12, 60, 420, 1660, 13440, 64540, 451920, 2665404]	$\mathbb{P}^1 \times \mathbb{P}^1 + 0\text{-st}$
35	Q38906	P516432	codim. 18; $2 \times \frac{1}{2}(1, 1, 1)$	2	$x + xz^2/y + y + z +$ $2z/y + 2/x +$ $1/(xy) + y/(x^2z) +$ $1/(x^2z)$	$\begin{array}{cccc ccc} 1 & 0 & 0 & 1 & -1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \end{array} \left \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right.$	[1, 0, 4, 6, 60, 240, 1210, 7980, 34300, 256200, 1172304]	$\mathbb{F}_1 + 0\text{-st}$
36			codim. 18; $2 \times \frac{1}{2}(1, 1, 1)$	2	$x + xz^2/y + y + z +$ $2z/y + 3/x +$ $1/(xy) + y/(x^2z) +$ $1/(x^2z)$	$\begin{array}{cccc ccc} 1 & 0 & 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 2 \end{array} \left \begin{array}{c} 0 \\ 0 \\ 2 \end{array} \right.$	[1, 0, 6, 6, 90, 300, 1950, 11760, 56490, 432600, 2023056]	$\mathbb{P}^1 \times \mathbb{P}^1 + 0\text{-st}$

(Continued)

Table 3. Continued

	Q-Fano ID	Polytope ID	Codim and Singularities	MG #	Rigid MMLP	Weight Matrix and Bundles	Period Sequence	Shape
37	Q38935	P424987	codim. 19; $3 \times \frac{1}{2}(1, 1, 1)$	1	$x^2 z^3 / y^2 +$ $2x^2 z^4 / y^3 +$ $x^2 z^5 / y^4 + xz + x +$ $2xz^2 / y + xz / y +$ $xz^3 / y^2 + y +$ $y / (xz^2)$	$\begin{array}{ccccccc c} 1 & 0 & 0 & -1 & 1 & 2 & -1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 2 \end{array}$	[1, 0, 4, 0, 60, 60, 1120, 2520, 24220, 90720, 586404]	$\mathbb{P}^1 \times \mathbb{P}^1 + 0\text{-st.}$
38			codim. 19; $3 \times \frac{1}{2}(1, 1, 1)$	1	$x^2 z^3 / y^2 +$ $2x^2 z^4 / y^3 +$ $x^2 z^5 / y^4 + xz + x +$ $3xz^2 / y + xz / y +$ $xz^3 / y^2 + y +$ $y / (xz^2)$	$\begin{array}{ccccccc c} 1 & 0 & 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \end{array}$	[1, 0, 6, 0, 90, 60, 1860, 3360, 44730, 143640, 1191456]	$\mathbb{F}_1 + 0\text{-st.}$
39	Q40948	P534760	codim 9; $2 \times \frac{1}{5}(1, 2, 4)$	5	$xy^3 z^5 + x + 2y^2 z^5 +$ $2y^2 z^2 + y + yz^5 / x +$ $2z^2 / x + 1 / x +$ $1 / (xyz)$	$\begin{array}{cccccc c} 1 & -3 & -5 & -1 & -2 & 2 & -6 \\ 0 & 3 & 5 & 1 & 2 & -1 & 6 \end{array}$	[1, 0, 2, 0, 6, 0, 380, 0, 10150, 0, 151452]	\mathbb{F}_1 (no 0-st.)
40	Q40971	P543951	codim 9; $2 \times \frac{1}{5}(2, 2, 3)$	2	$x^6 / (y^4 z^5) +$ $2x^3 / (y^2 z^2) +$ $2x^2 / (y^3 z^2) + x +$ $y + z + 1 / y +$ $2z / (xy) + z / (x^2 y^2)$	$\begin{array}{cccccc c} 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 3 & 1 & 5 & 2 & 3 & 6 \end{array}$	[1, 0, 2, 0, 6, 0, 20, 0, 3430, 0, 75852]	\mathbb{F}_1 (no 0-st.)
41	Q40988	P413267	codim 6; $\frac{1}{5}(1, 2, 4)$	1	$x + y + z + 1 / (yz^2) +$ $1 / (y^2 z^3) +$ $1 / (xz^2) +$ $1 / (xyz) +$ $1 / (xyz^3) +$ $1 / (xy^2 z^2)$	$\begin{array}{cccc c} 1 & 0 & 1 & -1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 2 & -1 & 1 & 2 \end{array}$	[1, 0, 0, 0, 48, 0, 360, 0, 11760, 0, 226800]	\mathbb{P}^2 (no 0-st)
42	Q40993	P473887	codim 6; $3 \times \frac{1}{3}(1, 2, 2)$	2	$x + xz / y^3 + y + z +$ $2z / y^2 + 1 / x +$ $z / (xy) + 2 / (xy^2) +$ $2 / (x^2 y) + 1 / (x^3 yz)$	$\begin{array}{cccc c} 1 & 0 & 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 3 & 2 & 1 & 2 \end{array}$	[1, 0, 2, 0, 54, 0, 740, 0, 18550, 0, 403452]	$\mathbb{F}_1 + 0\text{-st.}$

(Continued)

Table 3. Continued

Q-Fano ID	Polytope ID	Codim and Singularities	MG #	Rigid MMLP	Weight Matrix and Bundles	Period Sequence	Shape
43 Q41200	P547328	codim. 4; $2 \times \frac{1}{2}(1, 1, 1), \frac{1}{5}(1, 3, 4)$	6	$xyz^2 + x + xz/y + x/(y^2z) + y + z + 2/(yz) + 2/(xz) + 1/(xyz^3) + 1/(x^2z^3)$	$\begin{array}{ccc ccc} 2 & 1 & 3 & 1 & 2 & 1 & 0 \\ 1 & -1 & -1 & 1 & -1 & 0 & 1 \end{array} \mid \begin{array}{cc} 3 & 4 \\ 0 & 0 \end{array}$	[1, 0, 0, 72, 0, 0, 32760, 0, 0, 21067200, 0]	\mathbb{F}_1 (no 0-st.)
44 Q41218	P544064	codim. 4; $3 \times \frac{1}{2}(1, 1, 1), \frac{1}{4}(1, 1, 3)$	3	$x + x/(y^2z) + y + z + 1/(yz) + 3yz/x + 2y^2z/x^2 + y^4z^3/x^3 + y^5z^3/x^4$	$\begin{array}{ccc ccc} 1 & 0 & 3 & 0 & 1 & -1 & 2 \\ 0 & 1 & -1 & 1 & 1 & 1 & -1 \end{array} \mid \begin{array}{cc} 3 & 0 \\ 0 & 2 \end{array}$	[1, 0, 0, 54, 0, 0, 14850, 0, 0, 5821200, 0]	$\mathbb{P}^1 \times \mathbb{P}^1$ (no 0-st.)
45 Q41251	P402202	codim 5; $2 \times \frac{1}{2}(1, 1, 1), \frac{1}{4}(1, 3, 3)$	5	$xyz^2 + x + y + z + z^2/y + 1/(xz) + 1/(xy)$	$\begin{array}{ccc ccc} 1 & 0 & 2 & -1 & -1 & 4 \\ 0 & 1 & -1 & 1 & 1 & -2 \end{array} \mid \begin{array}{c} 2 \\ 0 \end{array}$	[1, 0, 0, 12, 0, 0, 900, 0, 0, 86520, 0]	\mathbb{P}^2 (no 0-st.)
46 Q41334	P543852	codim. 4; $3 \times \frac{1}{3}(1, 1, 2), \frac{1}{5}(1, 4, 4)$	3	$x^2/(y^3z^2) + x + x/(y^2z^2) + y + z + 3yz/x + 2y^2z/x^2 + y^5z^4/x^4 + y^6z^4/x^5$	$\begin{array}{ccc ccc} 1 & 0 & -1 & 1 & 1 & 1 & -1 \\ 0 & 1 & 4 & 0 & 1 & -1 & 3 \end{array} \mid \begin{array}{cc} 0 & 2 \\ 4 & 0 \end{array}$	[1, 0, 0, 0, 576, 0, 0, 0, 2540160, 0, 0]	$\mathbb{P}^1 \times \mathbb{P}^1$ (no 0-st.)

§3. A systematic approach in high codimension

In Table 2, we provide a complete list of non-toric Fano varieties of codimensions between 20 and 23, which are mirror to rigid maximally-mutable Laurent polynomials, modulo the considerations in Section 2.5.

3.1 Typical examples

Recall Algorithm 2.9 directly produces the weight matrix of the ambient space Y , provided that one assumes Remark 2.8. Throughout [15], the point-struts considered in Remark 2.8 are referred to as “uneliminated variables” In [30, Section 3] Prince proves the embedding theorem without requiring the condition in Remark 2.8. We discuss how to adjust the construction, as well as a basic incarnation of Algorithm 2.9, each on an example from Table 2.

3.1.1. Laurent inversion with point-struts at the basis of N_U .

As a warm-up, we demonstrate the construction in detail for threefold #4 in Table 2. In other words, we start with a polynomial supported on the toric canonical Fano polytope P519468 and construct the \mathbb{Q} -Fano threefold Q38989. The target variety has four isolated singular points: $2 \times \frac{1}{2}(1, 1, 1)$, $\frac{1}{3}(1, 1, 2)$ and $\frac{1}{4}(1, 1, 3)$.

The only rigid MMLP supported on P519468 is:

$$f = x + \frac{xz^3}{y^2} + \frac{y^2}{z} + y + \frac{y}{xz^2} + \frac{y^3}{x^2z^3}.$$

We use Magma to show that f is mutation-equivalent (via 9 mutation steps) to:

$$g = \frac{xy}{z} + \frac{x}{y^2z^2} + \frac{x}{y^3z^2} + \frac{y}{z} + z + \frac{2}{y^2z^2} + \frac{3}{y^3z^2} + \frac{1}{xy^2z^2} + \frac{3}{xy^3z^2} + \frac{1}{x^2y^3z^2},$$

which is supported on the polytope in Figure 1.

Despite the existence of various lattice points inside this polytope, the only non-vanishing coefficients of g are along its edges, suggesting the use of two-dimensional struts. Moreover,

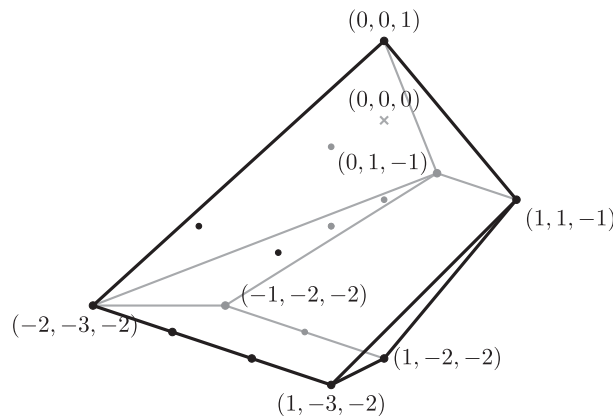


Figure 1.
The Newton Polytope of g .

The situation is quite tame, as Y is an orbifold—this is the case for many of the ambient spaces in Table 2, yet for very few in Table 3. The main issue is that C_1 or C_2 could be in the base locus of L : once we establish that a general section of L has isolated singularities, they typically coincide with the singularities predicted by the GRDB [5].

A general section $s \in H^0(Y, L)$ is

$$x_1x_3x_4x_6 + x_3^2x_4x_5 + x_3^4x_4^3x_6 + x_4x_6^3 + x_2x_3 + x_1^2 + x_5x_6 + x_1x_3^3x_4^2 + x_3^6x_4^4 + x_3^2x_4^2x_6^2.$$

This is a partial smoothing obtained by adding monomials in $\mathcal{O}_Y(2, 0)$ to the toric equation $x_1^2 + x_5x_6 = 0$. Thus $X = V(s)$ is quasi-smooth: it is enough to check this using the Jacobian criterion, making sure its potential singularities all lie in the unstable locus.

The curve C_1 belongs to the chart U_{46} , which is of type $\frac{1}{2}(1, 1, 1, 0)_{1235}$. We see that the $\frac{1}{2}(1, 1, 1)$ -action propagates along the x_5 -axis, that is, C_1 in this chart (so that C_1 is a curve of $\frac{1}{2}(1, 1, 1)$ singularities). If X intersects it transversely, it acquires that precise type of singularity.

When restricting s to C_1 we obtain $x_4x_6^3 + x_5x_6 = 0$, thus X contains the origin of U_{45} with multiplicity one (this is a $\frac{1}{4}(1, 1, 3)$ point), and any other singularity belongs to U_{46} , the only other chart containing C_1 . On U_{46} , the equation becomes $1 + x_5 = 0$, whose solution is an isolated point, and (when comparing Jacobian matrices) it is clear that X and C_1 intersect transversely here.

Similarly, the curve C_2 belongs to the charts U_{12} and U_{23} . When restricting s to C_2 we are left with $x_2x_3 + x_1^2 = 0$, that is, a point solving $x_3 + 1 = 0$ in the chart U_{12} . This chart is $U_{12} = \frac{1}{2}(0, 1, 1, 1)_{x_3x_4x_5x_6}$ so we find another singularity of type $\frac{1}{2}(1, 1, 1)$.

According to the GRDB prediction, we should obtain another $\frac{1}{3}(1, 1, 2)$ point, which we find while examining the singular 0-strata of Y : there are only two such points disjoint from $C_1 \cup C_2$: $P_1 = \{x_1 = x_3 = x_4 = x_5 = 0\}$ and $P_2 = \{x_2 = x_3 = x_5 = x_6 = 0\}$. The equation of s vanishes at P_1 , which is the origin of the chart U_{26} (a point of type $\frac{1}{3}(1, 1, 2)$), and does not vanish at P_2 .

REMARK 3.1. When analyzing the singularity on C_1 , we make a point of examining it in the chart U_{46} , where we clearly see that the $\frac{1}{2}(1, 1, 1)$ singularity propagates. The origin of U_{45} is a more singular point with a μ_4 -action, and in this chart the matter of determining which singularity C_1 acquires is more subtle. In general it may occur that both charts containing a curve such as C_1 have such points at the origin, though this is never true for the examples in this paper.

3.1.2. Laurent inversion without point-struts at the basis of N_U

We explain how to build the weight matrix of Y without the assumption in Remark 2.8. The key fact is that the lattice of its fan is precisely $\text{Div}_{T_{\overline{M}}}(Z) \oplus N_U$ (see discussion following [15, Remark 5.4]). In order to determine the weight matrix we need to specify the rays of this fan and write the sequences (2.1) for Y .

Suppose $S = \{(D_i, \chi_i)\}_{i=1}^s$ is a scaffolding of P with shape Z . Recall from Definition 2.7 that this fixes a splitting of the ambient lattice of P as $N = \overline{N} \oplus N_U$, where \overline{N} is the character lattice of Z . Denote by $u = \dim(N_U)$, $\overline{M} = \text{Hom}(\overline{N}, \mathbb{Z})$ and recall that the toric divisors $\Delta_1, \dots, \Delta_z$ on Z are a \mathbb{Z} -basis of its divisor group $\text{Div}_{T_{\overline{M}}}(Z)$.

THEOREM 3.2 [15]. *The fan of Y has ambient lattice $\text{Div}_{T_{\overline{M}}}(Z) \oplus N_U \simeq \bigoplus_{i=1}^z \mathbb{Z}\Delta_i \oplus \mathbb{Z}^u$. It has the following $r + s$ rays:*

- $(\Delta_i, 0)$, with $i = 1, \dots, z$ and
- $(-D_i, \chi_i)$, since $D_i \in \bigoplus_{i=1}^z \mathbb{Z}\Delta_i$ by construction.

It is inconvenient to think of high-dimensional toric varieties in terms of fans and cones as opposed to GIT quotients, which is why we usually avoid this description. The rows of the weight matrix output of Algorithm 2.9 span the kernel of the ray matrix ρ appearing in the dual exact sequences (2.1) written for Y . This kernel can be computed directly because the ray matrix contains an $r \times r$ identity block if we have u 0-struts at the basis of N_U . This is ultimately the same as imposing that the weight matrix begin with the identity block in the algorithm.

We illustrate this by computing the weight matrix in the example in Section 3.1.1 using its ray map. Recall $z = 4$, $u = 1$, $r = s - u = 2$, and $N_U = \langle (0, 0, 1) \rangle$. There is one 0-strut at the origin of N_U , i.e. $(D_0, \chi_0) = (0, (0, 0, 1))$, and two other struts corresponding to $(D_1, \chi_1) = (\Delta_1 - \Delta_2 + \Delta_3 + \Delta_4, -(0, 0, 1))$ and $(D_2, \chi_2) = (-2\Delta_1 + 3\Delta_2 + \Delta_3 - \Delta_4, -2(0, 0, 1))$, therefore in the basis $\{(\Delta_1, 0), \dots, (\Delta_z, 0), (0, (0, 0, 1))\}$ the rays are:

Δ_1	Δ_2	Δ_3	Δ_4	$-D_0$	$-D_1$	$-D_2$
1	0	0	0	0	-1	2
0	1	0	0	0	1	-3
0	0	1	0	0	-1	-1
0	0	0	1	0	-1	1
0	0	0	0	1	-1	-2

Modulo a reordering of the rays, we obtain that its kernel is spanned precisely by the weight matrix in Example 3.1.1. It is easy to see that the 0-strut condition is not necessary in order to have an identity block. We illustrate this in the example below.

Consider variety #21 in Table 2, which is a codimension 21 \mathbb{Q} -Fano threefold with two singular points, of type $\frac{1}{3}(1, 1, 2)$ and $\frac{1}{4}(1, 1, 3)$, respectively. We construct this as a deformation of the toric canonical P512391 to the \mathbb{Q} -Fano Q38917. The polytope of P512391 supports a single rigid MMLP:

$$f = x + y + z + \frac{1}{xyz} + \frac{2}{xyz^2} + \frac{1}{xy^3z^4} + \frac{1}{x^2y^3z^3} + \frac{1}{x^2y^3z^4}$$

whose 1-step mutation (modulo a change of basis)

$$h = \frac{xz^2}{y} + \frac{x}{z} + \frac{1}{z} + \frac{y}{z^3} + \frac{2y}{xz^3} + \frac{y}{x^2z^3} + \frac{2y^2}{xz^3} + \frac{2y^2}{x^2z^3} + \frac{y^3}{x^2z^3}$$

is supported on the Newton polytope P in Figure 3.

Notice the bottom triangle suggests the shape Z is a variety admitting a contraction to \mathbb{P}^2 (the polytope of one of its nef divisors is this triangle, and Z is a tower of projective bundles). As in the previous case, the coefficients of h suggest a two-dimensional shape as

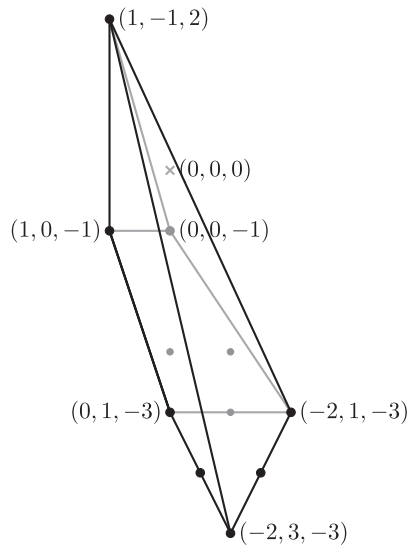


Figure 3.
Newton polytope of h .

h is entirely supported on the edges of P . We aim for a complete intersection, therefore the two options are $Z = \mathbb{P}^2$ and $Z = \mathbb{F}_1$.

REMARK 3.3. The key issue here is that no single point proves to be a good candidate as the basis of N_U : the best option is $(1, -1, 2)$, however, N does not decompose as $\overline{N} \oplus N_U$, where \overline{N} is the two-dimensional ambient lattice of the bottom triangle, and N_U the lattice spanned by $(1, -1, 2)$ (since $\overline{N} \oplus N_U$ is of index two inside N). The cases of $N_U = \langle (0, 0, -1) \rangle$ and $N_U = \langle (1, 0, -1) \rangle$ are unsuccessful — here the lattice decomposes as needed but when analyzing the weight matrix obtained as in Section 3.1.1 we obtain that X_P does not deform to an orbifold. It is in general desirable to use a minimal amount of struts, and this is exactly why the example below works.

Consider the scaffolding in Figure 4, with shape \mathbb{F}_1 (so $z = 4$ just as before) and three struts S_i . Decompose $N = \overline{N} \oplus N_U$ with $N_U = \langle (0, 0, 1) \rangle$ and follow the above method for producing the ray map of Y : the ambient space is of dimension $z + u = 5$ and its fan is given by $z + s = 4 + 3 = 7$ rays in the lattice $\bigoplus_i \mathbb{Z}\Delta_i \oplus N_U$. Proceeding exactly as in the previous case, we obtain:

Δ_1	Δ_2	Δ_3	Δ_4	$-D_1$	$-D_2$	$-D_3$
1	0	0	0	1	0	-3
0	1	0	0	-1	0	1
0	0	1	0	1	0	-2
0	0	0	1	0	-1	-1
0	0	0	0	-2	1	3

and the weight matrix is simply the two-dimensional kernel of the ray map.

The homogeneity type of the two line bundles which give X_P are again given by the relations in the fan of Z , written in terms of the Cox coordinates. One obtains the following, modulo a reordering of the rays:

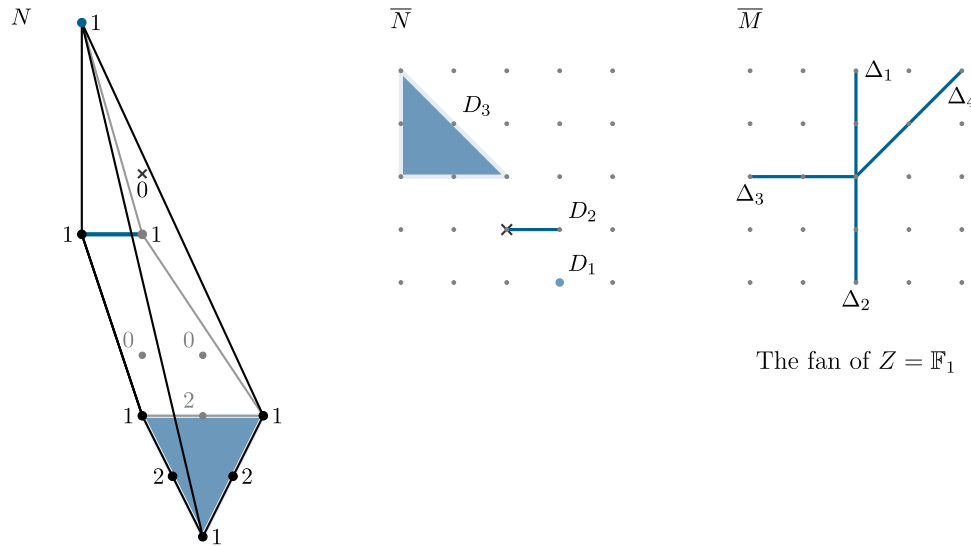


Figure 4.
A scaffolding of f_2 .

x_1	x_2	x_3	x_4	x_5	x_6	x_7	L_1	L_2
1	0	-3	3	-1	2	-2	2	0
0	1	2	-1	1	-1	2	0	1

with the stability condition $\omega = (-2, 3)$. We can eliminate L_2 by solving for x_2 and the rest of the verification is almost identical to the case in Section 3.1.1, down to the fact that the ambient space is an orbifold which is singular along two curves (in this case, $C_1 = \{x_4 = x_5 = x_6 = 0\}$ and $C_2 = \{x_1 = x_3 = x_6 = 0\}$). We leave the details of this calculation to the reader.

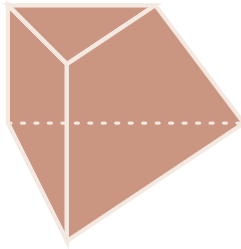
§4. Exploring the database to diversify examples

Constructing varieties using Laurent inversion appears to work best for varieties in higher GRDB-codimension, at least given the set of shapes we have considered so far. In this section, we explore examples in GRDB-codimensions ranging between 19 and 5, as well as three examples in codimension 4 which have been constructed previously in [19], albeit in a different format.

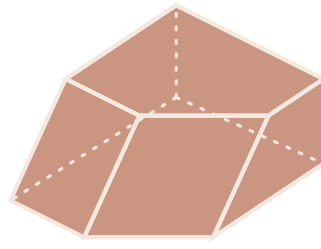
In some cases, several three-dimensional shapes appear, with arguably the most difficult example we obtain being discussed at length in Section 4.2, a \mathbb{P}^1 -bundle over a Pfaffian. The remaining cases are all towers of projective bundles.

4.1 Interesting 3D shapes

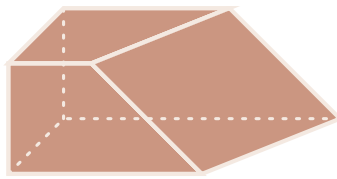
The shape Z can be any smooth toric variety, and so far surfaces have been sufficient in the construction of all but two of the examples of Table 2: variety 8 was scaffolded with shape \mathbb{P}^3 and variety 11 was scaffolded with shape Z_1 , which we introduce below. We also list the 3D polarising polytopes of the toric varieties Z_i which occur as shapes for the constructions in Table 3.



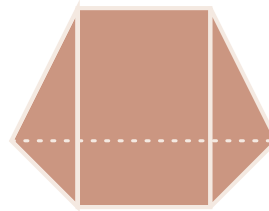
Z_1 is a \mathbb{P}^1 -bundle over \mathbb{P}^2 obtained as a toric blow up of a point on \mathbb{P}^3 . Shape for 11 in Table 2



Z_2 is a \mathbb{P}^1 -bundle over dP_7 . Can be obtained by blowing up a curve on Z_3 . Shape for 4 in Table 3, discussed in Section 4.2



Z_3 is a \mathbb{P}^1 -bundle over \mathbb{F}_1 and a $\mathbb{P}^1 \times \mathbb{P}^1$ -bundle on \mathbb{P}^1 . Shape for 5, 6 and 16 in Table 3



Z_4 is a \mathbb{P}^2 -bundle over \mathbb{P}^1 obtained by blowing up a toric curve on \mathbb{P}^3 . Shape for 11 in Table 3.

Other 3D shapes that occur come from products: $\mathbb{P}^2 \times \mathbb{P}^1$ and $\mathbb{F}_1 \times \mathbb{P}^1$. We also encounter many of the 2D shapes in Table 2.

4.2 The 3D Pfaffian: Q33018

Consider the rigid MMLP

$$f = xy^2z^3 + xy^2z^2 + xyz^2 + 2xyz + x + 3yz^2 + 2yz + y + 4z + \frac{1}{y} + 3\frac{z}{x} + \frac{3}{x} + \frac{1}{xz} + \frac{4}{xy} + \frac{1}{xy^2z} + \frac{1}{x^2y} + \frac{3}{x^2yz} + \frac{1}{x^2y^2z} + \frac{1}{x^3y^2z^2}$$

supported on the polytope P31470. Our attempts of applying Laurent inversion to f (and P31470) itself did not yield sufficiently non-singular Fano threefolds, therefore, in order to construct its mirror we consider its mutations. We find that f is mutation-equivalent (via 4 one-step mutations) to the Laurent polynomial:

$$g = \frac{x}{yz} + \frac{x}{yz^2} + yz + 2z + \frac{z}{y} + \frac{1}{y} + \frac{1}{yz} + \frac{2}{yz^2} + \frac{1}{yz^3} + \frac{yz^2}{x} + \frac{2yz}{x} + \frac{y}{x} + \frac{z^2}{x} + \frac{4z}{x} + \frac{4}{x} + \frac{1}{xz} + \frac{z}{xy} + \frac{2}{xy} + \frac{1}{xyz}$$

with Newton polytope:

4.2.1. The weight matrix and the Pfaffian format

Figure 6 contains a scaffolding compatible with the polynomial g : two 2-dimensional quadrilateral struts and one full-dimensional strut. Note that we cannot choose the 3-dimensional strut itself to be the polarizing polytope of the shape Z , as it has a non- \mathbb{Q} -factorial singularity (coming from the 4-valent vertex of the strut). To obtain the

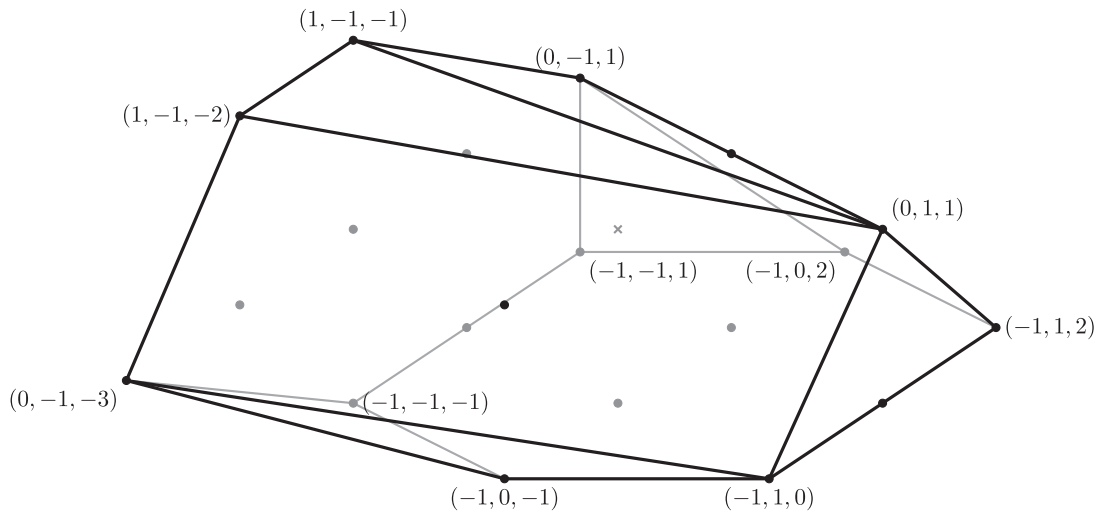
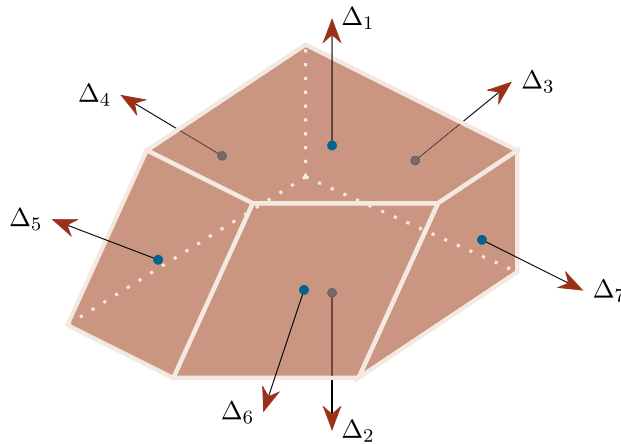


Figure 5.
Newton polytope of g .

correct object we make a small resolution and we recover Z_2 below, described by its ample polytope. Indeed, this shape is a \mathbb{P}^1 -bundle over dP_7 , whose torus-invariant divisors are labeled as in the picture

The variety we obtain is of codimension 4, albeit in a special configuration because of the \mathbb{P}^1 -bundle structure, which corresponds to the vertical directions Δ_1 and Δ_2 on the shape.



The weight matrix and line bundles are:

	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7	H	L_1	L_2	L_3	L_4	L_5		
1	0	0	1	0	-1	1	3	2	-1	1	2	0	3	1	2
0	1	0	0	1	1	1	0	0	1	1	1	2	1	1	1
0	0	1	-1	1	2	0	-1	-1	1	0	0	1	-1	1	1
x_1	x_2	x_3	x_4	x_5	x_6	y	x_7	x_8	x_9						

where H corresponds to a hypersurface on Y which intersects a Pfaffian variety given by the L_i . The chosen ample class is $\omega = (1, 1, 1) = -K_X = -K_Y + L$, as discussed in Section 2.3.1

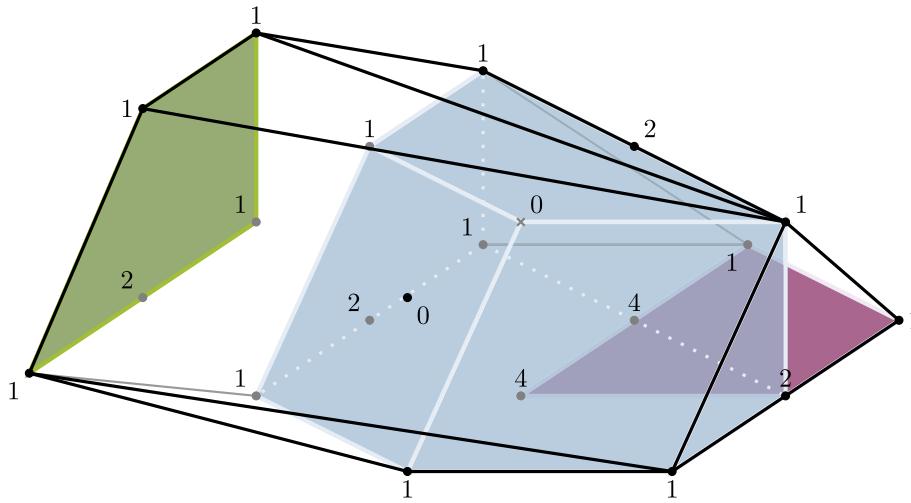


Figure 6.
a scaffolding of f_4 .

(here $U = \emptyset$). Despite the fact that the chosen Z_2 is not a tower of projective bundles, we obtain X_P as a toric Pfaffian complete intersection (TPCI). More precisely, in these coordinates X_P is given by the following six equations:

$$x_4x_5 = x_1x_2 \text{ and} \tag{4.1}$$

$$x_7x_9 = x_2x_3x_8, \quad yx_9 = x_2^2x_3, \quad yx_8 = x_2x_7, \quad x_6x_8 = x_1x_3x_4x_9, \text{ and } x_6x_7 = x_1x_2x_3^2x_4. \tag{4.2}$$

We briefly mention that we have obtained (4.1) and (4.2) from the 6 primitive relations (in the sense of Batyrev) between the Δ_i in the fan of Z , which we rewrote in terms of x_4, \dots, x_9 and homogenized using x_1, x_2 and x_3 . The equation (4.2) corresponds to sections of L_1, L_2, L_3, L_4 , and L_5 , respectively, and fit into the format:

$$\begin{pmatrix} 0 & x_1x_3x_4 & x_6 & y & x_2 \\ & 0 & 0 & x_7 & x_8 \\ & -\text{sym} & 0 & x_2x_3 & x_9 \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix},$$

that is, they are the 4×4 Pfaffians of this 5×5 antisymmetric matrix. The format is a tool for deforming this toric variety, however, before we proceed we simplify the situation: since the first deformed equation is linear in y , we can solve for this coordinate, eliminate equation (4.1) and obtain the weight matrix and line bundles appearing in Table 2:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	L_1	L_2	L_3	L_4	L_5
1	0	0	1	0	-1	3	2	-1	2	0	3	1	2
0	1	0	0	1	1	0	0	1	1	2	1	1	1
0	0	1	-1	1	2	-1	-1	1	0	1	-1	1	1

where this is now a bona fide toric Pfaffian variety. In what follows, we slightly abuse notation by still referring to the ambient space given by the new matrix as Y and to the toric variety inside it as X_P .

X_P is given by 5 binomial equations, obtained by substituting y in (4.2). To aid in determining the deformation, we record the weights above each monomial and obtain the following for X_P :

$$A_P = \begin{pmatrix} \boxed{200} & \boxed{-112} & \boxed{110} & \boxed{010} \\ 0 & x_1x_3x_4 & x_6 & x_4x_5 + \dots & x_2 \\ & & \boxed{101} & \boxed{30-1} & \boxed{20-1} \\ & 0 & 0 & x_7 & x_8 \\ & \text{-sym} & 0 & \boxed{011} & \boxed{-111} \\ & & & x_2x_3 & x_9 \\ & & & & \boxed{11-1} \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix}$$

The weights of the matrix entries remain fixed when moving from X_P to X . We deform these equations by adding all monomials of each homogeneity type in the splitting (maintaining the coefficient convention throughout). In other words, deforming a Pfaffian variety is done not by adding all monomials in $H^0(Y, L_i)$ until we obtain general sections, as is the case for complete intersections, but by only including monomials which respect the splitting as below:

$$A = \begin{pmatrix} \boxed{200} & \boxed{-112} & \boxed{110} & \boxed{010} \\ 0 & x_1^2 + x_3x_8 + x_1x_3x_4 + x_3^2x_4^2 & x_6 + x_3x_9 & x_1x_2 + x_4x_5 + x_8x_9 + x_4^2x_6x_1x_4x_9 + x_2x_3x_4 + x_3x_4^2x_9 & x_2 + x_4x_9 \\ & & \boxed{101} & \boxed{30-1} & \boxed{20-1} \\ & 0 & x_1x_3 + x_3^2x_4 & x_7 + x_1x_8 + x_1^2x_4 + x_3^2x_4^3 + x_3x_4x_8 + x_1x_3x_4^2 & x_8 + x_1x_4 + x_3x_4^2 \\ & \text{-sym} & 0 & \boxed{011} & \boxed{-111} \\ & & & x_2x_3 + x_5 + x_4x_6 + x_1x_9 + x_3x_4x_9 & x_9 \\ & & & & \boxed{11-1} \\ & & & 0 & x_2x_4 + x_4^2x_9 \\ & & & & 0 \end{pmatrix}$$

The five equations of X are the Pfaffians of this matrix, which we omit writing down as all the required information can be determined directly from the more compact form of A .

4.2.2. Singularities

GRDB suggests that we should find three singular points: $2 \times \frac{1}{2}(1, 1, 1)$ and $1 \times \frac{1}{3}(1, 1, 2)$.

The ambient space Y is not an orbifold, its non- \mathbb{Q} -factorial locus consisting of both curves and points. Given the stability condition is $\omega = (1, 1, 1)$ following the discussion in Section 2.3.1, Y is covered by three non-orbifold charts: U_{15}, U_{19}, U_{68} and 17 orbifold charts:

$U_{123}, U_{126}, U_{146}, U_{234}, U_{237}, U_{238}, U_{267}, U_{345}, U_{346}, U_{349}, U_{357}, U_{358}, U_{379}, U_{389}, U_{467}, U_{567}$, and U_{679} . We recall the notation $U_{i_1 \dots i_k} := \{x_{i_1} = \dots = x_{i_k} = 1\}$.

We first discuss the six singular curves:

- $C_1 = \{x_2 = x_3 = x_4 = x_6 = x_7 = x_8 = 0\}$. Note that this is a non- \mathbb{Q} -factorial curve, belonging to the charts U_{15} and U_{19} (notice in particular that x_1 never vanishes here). We restrict the equations of X to C_1 , which become the Pfaffians of

$$A|_{C_1} = \begin{pmatrix} 0 & \boxed{200} & \boxed{-112} & \boxed{110} & \boxed{010} \\ & x_1^2 & 0 & 0 & 0 \\ & 0 & \boxed{101} & \boxed{30-1} & \boxed{20-1} \\ & -\text{sym} & 0 & \boxed{011} & \boxed{-111} \\ & & & x_5 + x_1 x_9 & x_9 \\ & & & 0 & \boxed{11-1} \\ & & & & 0 \end{pmatrix},$$

that is, all but two equations disappear and we obtain the system

$$\begin{cases} x_9 x_1^2 = 0 \\ x_1^2 (x_5 + x_1 x_9) = 0 \end{cases}$$

This system has no solutions, as $x_5 = x_9 = 0$ does not belong to either chart containing the curve C_1 .

- $C_2 = \{x_1 = x_2 = x_4 = x_5 = x_7 = 0\}$. This belongs to U_{68} and U_{389} , therefore $x_8 \neq 0$. The matrix A becomes:

$$A|_{C_2} = \begin{pmatrix} 0 & \boxed{200} & \boxed{-112} & \boxed{110} & \boxed{010} \\ & x_3 x_8 & x_6 + x_3 x_9 & x_8 x_9 & 0 \\ & 0 & \boxed{101} & \boxed{30-1} & \boxed{20-1} \\ & -\text{sym} & 0 & \boxed{011} & \boxed{-111} \\ & & & 0 & x_8 \\ & & & & x_9 \\ & & & & \boxed{11-1} \\ & & & & 0 \end{pmatrix}$$

and we have the three surviving equations

$$\begin{cases} x_8(x_6 + x_3 x_9) - x_3 x_8 x_9 = 0 \\ x_8^2 x_9 = 0 \\ x_8 x_9^2 = 0 \end{cases}$$

This time the contradiction is $x_6 = x_9 = 0$.

- $C_3 = \{x_1 = x_2 = x_4 = x_5 = x_8 = 0\}$ belongs to U_{379} and U_{679} , therefore both x_7 and x_9 do not vanish here. The matrix is

$$A|_{C_3} = \begin{pmatrix} 0 & \boxed{200} & \boxed{-112} & \boxed{110} & \boxed{010} \\ & 0 & x_6 + x_3x_9 & 0 & 0 \\ & & \boxed{101} & \boxed{30-1} & \boxed{20-1} \\ & 0 & 0 & x_7 & 0 \\ -\text{sym} & & 0 & \boxed{011} & \boxed{-111} \\ & & & 0 & x_9 \\ & & & & \boxed{11-1} \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix}$$

and among the equations notice $x_7x_9 = 0$, which is impossible in the above charts.

- $C_4 = \{x_1 = x_4 = x_6 = x_7 = x_9 = 0\}$ belongs to U_{238} and U_{358} and

$$A|_{C_4} = \begin{pmatrix} 0 & \boxed{200} & \boxed{-112} & \boxed{110} & \boxed{010} \\ x_3x_8 & 0 & 0 & 0 & x_2 \\ & \boxed{101} & \boxed{30-1} & \boxed{20-1} & \\ 0 & 0 & 0 & 0 & x_8 \\ -\text{sym} & 0 & x_5 + x_2x_3 & \boxed{011} & \boxed{-111} \\ & & & 0 & 0 \\ & & & & \boxed{11-1} \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix}$$

There are three remaining equations:

$$\begin{cases} x_8(x_5 + x_2x_3) = 0 \\ x_2(x_5 + x_2x_3) = 0 \\ x_3x_8(x_5 + x_2x_3) = 0 \end{cases}$$

whose only solution is $x_5 + x_3x_2 = 0^4$. This is one point belonging to both charts but which is not the origin of either. Since $U_{238} = \frac{1}{2}(1, 1, 0, 1, 1, 1)_{145679}$, and C_4 is given by the x_5 -axis which X intersects transversally, the point is of type $\frac{1}{2}(1, 1, 1)$.

- $C_5 = \{x_1 = x_4 = x_6 = x_8 = x_9 = 0\}$ is almost identical to C_4 , except that we find a single $\frac{1}{3}(1, 1, 2)$ point.
- $C_6 = \{x_1 = x_2 = x_3 = x_8 = x_9 = 0\}$ is similar to C_3 , and X does not intersect it.

The remaining singular points of Y which are not contained in C_1, \dots, C_6 are $P_1 = \{x_1 = x_3 = x_4 = x_5 = x_8 = x_9 = 0\}$, $P_2 = \{x_1 = x_2 = x_3 = x_4 = x_5 = x_7 = x_9 = 0\}$,

⁴ This binomial appears in all three equations, with arbitrary but fixed coefficients — sometimes expressions look the same because of the coefficient convention, but may lead to a contradiction. That is not the case in this example.

and $P_3 = \{x_3 = x_4 = x_5 = x_7 = x_8 = x_9 = 0\}$, that is, the origins of U_{267} , U_{68} , and U_{126} , respectively. The latter is the only point belonging to X , as all 4×4 Pfaffians of the matrix

$$A|_{P_2} = \begin{pmatrix} 0 & \boxed{200} & \boxed{-112} & \boxed{110} & \boxed{010} \\ & x_1^2 & x_6 & x_1x_2 & x_2 \\ & & \boxed{101} & \boxed{30-1} & \boxed{20-1} \\ & 0 & 0 & 0 & 0 \\ -\text{sym} & & & \boxed{011} & \boxed{-111} \\ & 0 & 0 & 0 & 0 \\ & & & & \boxed{11-1} \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix}$$

vanish. It is the second expected point of type $\frac{1}{2}(1, 1, 1)$.

4.2.3. Anticanonical degree

Since the shape Z is neither a tower of projective bundles [20, Theorem 2.21] [15, Section 8] nor a product of projective spaces [30, Proposition 5.1], existing theoretical results⁵ do not determine the equations of X_P in Y . As a cross-check that we have constructed the correct variety X (i.e., a deformation of X_P), we therefore compute the anticanonical degree of X and verify that it coincides with the prediction from GRDB, that is, $47/3$.

We compute this degree inside an orbifold \tilde{Y} quasi-isomorphic to Y , which is an obtained by slightly changing the stability condition from $\omega = (1, 1, 1)$ to (for example) $\tilde{\omega} = (1 + \varepsilon, 1 + \frac{\varepsilon}{2}, 1 + \frac{\varepsilon}{3})$, where ε is small, which is now contained in a maximal-dimensional chamber of the secondary fan. Since X avoids the non-orbifold locus of Y , it is isomorphic to $\tilde{X} \subset \tilde{Y}$ given by the same equations and in particular it has the same degree.

Write the Chow ring of \tilde{Y} in terms of its generators $M = (1, 0, 0)$, $N = (0, 1, 0)$, and $P = (0, 0, 1)$. The relations are given by writing the components of the irrelevant ideal in terms of M, N and P . In our case

$$I_{\tilde{\omega}} = (x_1, x_3, x_6) \cdot (x_2, x_4, x_5, x_8, x_9) \cdot (x_2, x_5, x_6, x_9) \cdot (x_3, x_5, x_6, x_9) \cdot (x_1, x_3, x_7) \cdot (x_1, x_4, x_7, x_8) \cdot (x_2, x_4, x_7, x_8),$$

resulting in the relations:

$$\begin{aligned} MP(-M + N + 2P) &= 0 \\ N(M - P)(N + P)(2M - P)(-M + N + P) &= 0 \\ N(N + P)(-M + N + 2P)(-M + N + P) &= 0 \\ P(N + P)(-M + N + 2P)(-M + N + P) &= 0 \\ MP(3M - P) &= 0 \\ M(M - P)(3M - P)(2M - P) &= 0 \\ N(M - P)(3M - P)(2M - P) &= 0. \end{aligned}$$

⁵ Note that there are already some examples of successful Pfaffian and TCPI constructions. However, the shapes involved are simpler: either 2D [15] [18] or a product [31].

A key ingredient in finding $(-K_{\tilde{X}})^3$ is computing the fundamental class of the determinantal variety \tilde{X} by using the squaring principle [22, Proposition 9] and the corresponding Porteous formula [22, Theorem 10], for which the set-up is as follows:

THEOREM 4.1 (Porteous formula, [22]). *Let E be a rank 5 homogeneous vector bundle and L a line bundle on the simplicial toric variety \tilde{Y} . If $s : E \otimes L \rightarrow E^\vee$ is a general skew-symmetric map, and \tilde{X} the degeneracy locus inside \tilde{Y} where s drops rank by 2, then the cohomology class of \tilde{X} is:*

$$\mathbb{1}_{\tilde{X}} = \begin{vmatrix} c_2 & c_3 \\ 1 & c_1 \end{vmatrix}$$

where $c_i = c_i(E^\vee \otimes \sqrt{L}^\vee)$.

The squaring principle guarantees that we are allowed to perform computations involving \sqrt{L} , as there exists a variety $\sigma : \hat{Y} \rightarrow \tilde{Y}$ on which \sqrt{L} is a line bundle and the induced map $\sigma^* : H^*(\tilde{Y}, \mathbb{Z}) \rightarrow H^*(\hat{Y}, \mathbb{Z})$ is injective.

Recall that the matrix A represents a map

$$s : E \otimes L = \bigoplus_{i=1}^5 L_i \otimes L \rightarrow E^\vee = \bigoplus_{i=1}^5 L_i^\vee$$

and the variety \tilde{X} is the degeneracy locus of the antisymmetric homomorphism of vector bundles s on \tilde{Y} . As all vector bundles involved are split, the Chern classes c_i are easy to determine in terms of M, N , and P .

From Section 2.3.1, we have $L = -\frac{\sum_{i=1}^5 L_i}{2} = \mathcal{O}(-4, -3, -1)$, so that $E \otimes L = \mathcal{O}(-2, -2, -1) \oplus \mathcal{O}(-4, -1, 0) \oplus \mathcal{O}(-1, -2, -2) \oplus \mathcal{O}(-3, -2, 0) \oplus \mathcal{O}(-2, -2, 0)$. Using the adjunction formula $-K_{\tilde{X}} = -K_{\tilde{Y}} + L = \mathcal{O}_{\tilde{Y}}(1, 1, 1) = M + N + P$, we compute the degree as follows (using Macaulay2 [21] for the very last equality):

$$\begin{aligned} (-K_{\tilde{X}})^3 &= (M + P + N)^3 \cdot \mathbb{1}_{\tilde{X}} = (M + P + N)^3 \cdot (c_1 c_2 - c_3) = \\ &= (M + P + N)^3 \left(\sum_i \hat{L}_i \cdot \sum_{i < j} \hat{L}_i \hat{L}_j - \sum_{i < j < k} \hat{L}_i \hat{L}_j \hat{L}_k \right) = \frac{564}{2213} N^6, \end{aligned}$$

where

$$\begin{aligned} \hat{L}_1 &= L_1^\vee \otimes \sqrt{L}^\vee = (-2M - N) + \frac{1}{2}(4M + 3N + P) = \frac{1}{2}(N + P), \\ \hat{L}_2 &= L_2^\vee \otimes \sqrt{L}^\vee = \frac{1}{2}(4M - N - P) \\ \hat{L}_3 &= L_3^\vee \otimes \sqrt{L}^\vee = \frac{1}{2}(-2M + N + 3P) \\ \hat{L}_4 &= L_4^\vee \otimes \sqrt{L}^\vee = \frac{1}{2}(2M + N - P) \\ \hat{L}_5 &= L_5^\vee \otimes \sqrt{L}^\vee = \frac{1}{2}(N - P). \end{aligned}$$

Finally, we look at any chart, for example U_{678} , and determine that $\prod_{i \notin \{6,7,8\}} D_i = \frac{36}{2213} N^6$. Since $\prod_{i \notin \{6,7,8\}} D_i = 1$ (the chart is smooth), we have that $(-K_X)^3 = \frac{564}{36} = \frac{47}{3}$.

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Data availability statement. All cited Databases are available freely either as data repositories on Zednodo or at GRDB.co.uk, to which more precise references can be found in the bibliography. Commands used to manipulate and visualise the polytopes in Magma are available at [13].

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