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Ordinary differential equations and systems with time-dependent discontinuity sets

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In this paper we prove new existence results for non-autonomous systems of first order ordinary differential equations under weak conditions on the nonlinear part. Discontinuities with respect to the unknown are allowed to occur over general classes of time-dependent sets which are assumed to satisfy a kind of inverse viability condition.

1. Introduction and preliminaries

We are concerned with the existence of Carathéodory solutions for

$$x'(t) = f(t, x(t)) \quad \text{for almost all (a.a.) } t \in I := [t_0, t_0 + L], \quad x(t_0) = x_0, \tag{1.1}$$

where $L > 0$, $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^m$ and $f : I \times \mathbb{R}^m \to \mathbb{R}^m$ may be discontinuous. We recall that Carathéodory solutions are absolutely continuous functions on I that satisfy (1.1). We shall denote by $\mathcal{C}$ the set of all Carathéodory solutions of (1.1).

This paper's point of view somewhat recaptures the spirit of [18]: we pass from problem (1.1) to a solvable differential inclusion, and then we look for solutions of (1.1) among those of the inclusion. This process of 'passing from the equation to the inclusion and back again' has a twofold interest: first, it leads to new existence results for (1.1); and, second, it provides us with a bridge between two different approaches to discontinuous differential equations.

To start introducing some necessary preliminaries, let us say that the main idea consists of replacing f by a suitable multi-valued mapping $F : I \times \mathbb{R}^m \to \mathcal{P}(\mathbb{R}^m)$ and then searching for solutions of the initial-value problem

$$x'(t) \in F(t, x(t)) \quad \text{for a.a. } t \in I, \quad x(t_0) = x_0. \tag{1.2}$$

One can find in the literature different Fs, which lead to different notions of a solution (see [1, 9, 10, 12, 18, 21] and the references therein). We shall consider Krasovskij solutions, which are absolutely continuous functions that satisfy (1.2) with

$$F(t, x) := \bigcap_{\varepsilon > 0} \overline{\text{co}}\, f(t, x + \varepsilon B), \quad (t, x) \in I \times \mathbb{R}^m. \tag{1.3}$$

Here, $\overline{\text{co}}$ means closed convex hull, $B = \{y \in \mathbb{R}^m : \|y\| \leqslant 1\}$ is the unit closed ball centred at the origin and $x + \varepsilon B$ is the closed ball of radius $\varepsilon > 0$ and centre $x \in \mathbb{R}^m$. Unless stated otherwise, we shall use the maximum norm

$$\|x\| = \max\{|x_i| : 1 \leqslant i \leqslant m\} \quad \text{for each } x = (x_1, x_2, \dots, x_m) \in \mathbb{R}^m.$$

We shall denote by $\mathcal{K}$ the set of all Krasovskij solutions of (1.1).

Plainly, the definition of F guarantees that $f(t,x) \in F(t,x)$ for all (t,x), and therefore $\mathcal{C} \subset \mathcal{K}$. Now we reduce our problem to obtain conditions on f which imply that $\mathcal{K}$ is non-empty and, on the other hand, that $\mathcal{K} \subset \mathcal{C}$. It is well known that continuity with respect to x is enough, but we are precisely interested in discontinuous differential equations and thus we are forced to improve that.

In order to achieve our goal, we shall introduce conditions on the sets where f is discontinuous, so that Krasovskij solutions either become Carathéodory solutions whenever their graphs lie on those sets, or they are simply pushed away from them. There exist previous mathematical formulations of this idea, as the reader can see in [18]. Here we use an 'inverse-viability' approach. The high development reached by viability theory makes it easy to find in the literature very general conditions that imply that the graphs of all solutions of a given differential inclusion are forced to lie on a certain set. We are interested in the opposite kind of result, but the necessary (and sharp!) theoretical background already exists.

The main elements in viability theory are contingent cones and derivatives: for a given set $A \subset \mathbb{R}^m$, the Bouligand's contingent cone at $x \in A$ is defined as

$$T_A(x) := \bigcap_{\varepsilon>0} \bigcap_{\alpha>0} \bigcup_{0<h<\alpha} \left(\frac{1}{h}(A - x) + \varepsilon B\right).$$

An analytical description of Bouligand's contingent cone is established in the following proposition.

Proposition 1.1 (cf. proposition 2, p. 177, of [1]). *$v \in T_A(x)$ if and only if there exists sequences of strictly positive numbers h_n and of elements $u_n \in \mathbb{R}^m$ satisfying the following.*

(i) $\lim_{n\to\infty} u_n = v$.

(ii) $\lim_{n\to\infty} h_n = 0$.

(iii) $x + h_n u_n \in A \ \forall n \geqslant 0$.

For an interval $I \subset \mathbb{R}$ and a set-valued map $K : I \to \mathcal{P}(\mathbb{R}^m)$, we recall the notion of graph of K, which is the set $\text{graph}(K) := \{(t,x) \in I \times \mathbb{R}^m : x \in K(t)\}$. For the case when K is strict, i.e. $K(t) \neq \emptyset$ for each $t \in I$, the contingent derivative of K at a point $(t,x) \in \text{graph}(K)$ is defined as the mapping $DK(t,x) : \mathbb{R} \to \mathcal{P}(\mathbb{R}^m)$, whose graph is the contingent cone $T_{\text{graph}(K)}(t,x)$, i.e.

$$v_0 \in DK(t,x)(t_0) \quad \Leftrightarrow \quad (t_0, v_0) \in T_{\text{graph}(K)}(t,x).$$

Just for notational purposes, if $K(t) = \emptyset$, then we shall write $DK(t,x)(t_0) = \emptyset$ for all $t_0 \in \mathbb{R}$.

For the case when K is single- and scalar-valued, we have the following results.

LEMMA 1.2. *Let $J \subset \mathbb{R}$ be an interval and let $\gamma : J \to \mathbb{R}$. Then the mapping $K(t) := \{\gamma(t)\}$, $t \in J$, satisfies the following conditions.*

(a) *$DK(t,\gamma(t))(1)$ lies between $D_+\gamma(t)$ and $D^+\gamma(t)$ for all $t \in J$, where $D_+\gamma$ and $D^+\gamma$ denote the lower-right and the upper-right Dini derivatives, respectively.*

In particular, if γ is right differentiable at some $t \in J$, then we have that

$$DK(t,\gamma(t))(1) = \{\gamma'_+(t)\}.$$

(b) *$-DK(t,\gamma(t))(-1)$ lies between $D_-\gamma(t)$ and $D^-\gamma(t)$ for all $t \in J$, where $D_-\gamma$ and $D^-\gamma$ denote the lower-left and the upper-left Dini derivatives, respectively.*

In particular, if γ is left differentiable at some $t \in J$, then we have that

$$-DK(t,\gamma(t))(-1) = \{\gamma'_-(t)\}.$$

Proof. By definition, $\xi \in DK(t,\gamma(t))(1)$ if and only if $(1,\xi) \in T_{\mathrm{graph}(K)}(t,\gamma(t))$. Then, by proposition 1.1, we have that $\xi \in DK(t,\gamma(t))(1)$ if and only if there exist a sequence of strictly positive numbers $\{h_n\}_n$ and another sequence $\{u_n\}_n = \{(t_n,w_n)\}_n \subset \mathbb{R}^2$ such that $\{h_n\}_n \to 0$, $\{u_n\}_n \to (1,\xi)$ and $(t,\gamma(t)) + h_n u_n \in \mathrm{graph}(K)$ for all $n \in \mathbb{N}$. Therefore, for each n, we have

$$(t,\gamma(t)) + h_n u_n = (t + h_n t_n, \gamma(t + h_n t_n)),$$

and then (a) follows from the expression

$$\xi = \lim_{n\to\infty} w_n = \lim_{n\to\infty} w_n t_n^{-1} = \lim_{n\to\infty} \frac{\gamma(t+h_n t_n) - \gamma(t)}{h_n t_n}.$$

The proof of (b) is similar. □

Notice that γ need not be continuous in lemma 1.2.

This paper is organized as follows. In § 2 we study non-autonomous equations and systems. In § 3 we prove an alternative result concerning the scalar case. Examples and comparison with the literature are provided throughout the paper.

2. Existence results for systems

Let us consider problem (1.1) and assume that, for $f : I \times \mathbb{R}^m \to \mathbb{R}^m$, there exists a null-measure set $N \subset I$ such that the following conditions hold.

(i) There exists $\psi \in L^1(I)$ such that, for all $t \in I \setminus N$ and all $x \in \mathbb{R}^m$, we have $\|f(t,x)\| \leqslant \psi(t)(1 + \|x\|)$.

(ii) For all $x \in \mathbb{R}^m$, $f(\cdot,x)$ is measurable.

We say that a (Carathéodory or Krasovskij) solution x^* of (1.1) is the maximal solution if $x^*(t) \geqslant x(t)$ for all $t \in I$ and for any other solution x (here, '$\geqslant$' must be understood component wise). The minimal solution is defined analogously: when both the minimal and the maximal solutions exist, we call them the extremal solutions.

We have the following result about Krasovskij solutions. By $AC(I)$ we denote the set of all real-valued functions that are absolutely continuous on I.

Proposition 2.1. *If f satisfies (i) and (ii), then $\mathcal{K}$ is a non-empty, compact and connected subset of $\mathcal{C}(I,\mathbb{R}^m)$.*

Moreover, in the scalar case ($m=1$), we have the following.

(1) *$\mathcal{K}$ has pointwise maximum x^* and minimum x_*, which are the extremal solutions of (1.2). Moreover, for each $t \in I$, we have*

$$x^*(t) = \max\{v(t) : v \in AC(I),\ v'(s) \in F(s,v(s)) - \mathbb{R}_+ \ a.e.,\ v(t_0) \leqslant x_0\}, \tag{2.1}$$

$$x_*(t) = \min\{v(t) : v \in AC(I),\ v'(s) \in F(s,v(s)) + \mathbb{R}_+ \ a.e.,\ v(t_0) \geqslant x_0\}. \tag{2.2}$$

(2) *$\mathcal{K}$ is a funnel, i.e. for all $\bar{t} \in I$ and $c \in [x_*(\bar{t}), x^*(\bar{t})]$, there exists $x \in \mathcal{K}$ such that $x(\bar{t}) = c$.*

Proof. It is clear that $F(t,x)$, defined in (1.3), is closed, convex and non-empty for all $(t,x) \in I \times \mathbb{R}^m$. Moreover, for each $t \in I$, $F(t,\cdot)$ is upper semicontinuous and, by (i), we have, for all $t \in I \setminus N$, that

$$\sup\{\|y\| : y \in F(t,x)\} \leqslant \psi(t)(1 + \|x\|) \quad \text{for all } x \in \mathbb{R}^m.$$

Finally, condition (ii) implies that $f(\cdot,x)$ is a measurable selection of $F(\cdot,x)$ for each $x \in \mathbb{R}^m$, and then it follows from [9, corollary 5.1 and theorem 7.2] that $\mathcal{K}$ is a non-empty compact and connected subset of $\mathcal{C}(I,\mathbb{R}^m)$.

In the scalar case ($m=1$), the existence of extremal solutions follows from a similar argument to that in the proof of [8, theorem 3].

We are going to prove (2.2) using a slight modification of that of [8, theorem 4] (such a modification is necessary because, in our case, $F(\cdot,x)$ needs not be measurable, as we shall show in § 3.1). Let $v \in AC(I)$ be such that

$$v'(t) \in F(t,v(t)) + \mathbb{R}_+ \quad \text{for a.a. } t \in I, \quad v(t_0) \geqslant x_0.$$

On the exceptional null set, we (re)define $v'(t)$ as any element of $F(t,v(t))$. Since $F(t,\cdot)$ is upper semicontinuous (USC) and $F(\cdot,x)$ has a measurable selection, it follows from [9, proposition 3.5] that there exists a measurable selection $w : I \to \mathbb{R}$ of $F(\cdot,v(\cdot))$. Then we have that

$$v'(t) \in F(t,v(t)) + y(t) \quad \text{for all } t \in I,$$

where $y(t) := \max\{0, v'(t) - w(t)\}$, $t \in I$ (note that y is measurable).

For each $n \geqslant 1$, let $\lambda_n : I \times \mathbb{R} \to [0,1]$ be continuous and such that $\lambda_n(t,x) = 1$ for $x \leqslant v(t)$ and $\lambda_n(t,x) = 0$ for $x \geqslant v(t) + 1/n$. Consider, for all $(t,x) \in I \times \mathbb{R}$,

$$F_n(t,x) = \lambda_n(t,x)F(t,\min\{x,v(t)\}) + (1 - \lambda_n(t,x))(v'(t) - y(t)).$$

For each $x \in \mathbb{R}$, the mapping

$$\lambda_n(\cdot,x)(f(\cdot,x)\chi_A(\cdot) + w(\cdot)\chi_B(\cdot)) + (1 - \lambda_n(\cdot,x))(v'(\cdot) - y(\cdot))$$

is a measurable selection of $F_n(\cdot,x)$, where

$$A = \{t \in I : x \leqslant v(t)\} \quad \text{and} \quad B = \{t \in I : x > v(t)\}.$$

Whence, since $F_n(t,\cdot)$ is USC and satisfies

$$\sup\{|z| : z \in F_n(t,x)\} \leqslant \psi(t)(1+|v(t)|)(1+|x|) \quad \text{a.e.},$$

the problem

$$z_n'(t) \in F_n(t, z_n(t)) \quad \text{for a.a. } t \in I, \quad z_n(t_0) = x_0,$$

has a solution z_n and we have that $z_n \leqslant v + 1/n$ on I. By a standard argument, we deduce that a subsequence of $\{z_n\}_n$ converges uniformly to a solution of (1.2), $z \leqslant v$. Then, since $x_* \leqslant z \leqslant v$, we obtain (2.2). The proof of (2.1) is similar.

Finally, let $\bar{t} \in I$ be fixed. Since $\mathcal{K}$ is connected and the function $\pi_{\bar{t}} : \mathcal{K} \to \mathbb{R}$ defined as $\pi_{\bar{t}}(x) = x(\bar{t})$ is continuous, we have that $\pi_{\bar{t}}(\mathcal{K})$ is also connected. Then, for all $c \in [x_*(\bar{t}), x^*(\bar{t})] = [\pi_{\bar{t}}(x_*), \pi_{\bar{t}}(x^*)]$, there exists $x \in \mathcal{K}$ such that $\pi_{\bar{t}}(x) = x(\bar{t}) = c$. □

Following the sketch that we outlined in § 1, we now have to re-enforce the assumptions required in proposition 2.1 in order to also obtain that $\mathcal{K} \subset \mathcal{C}$. A first result in this direction is the following theorem.

THEOREM 2.2. *Assume that, for a null-measure set $N \subset I$, the mapping $f : I \times \mathbb{R}^m \to \mathbb{R}^m$ satisfies conditions (i), (ii) and, for each $t \in I \setminus N$, $f(t,\cdot)$ is continuous on $\mathbb{R}^m \setminus N_1 \times \cdots \times N_m$, where $N_i \subset \mathbb{R}$ is a null-measure set for $i = 1, \dots, m$.*

Moreover, if, for each $t \in I \setminus N$ and each $x \in N_1 \times \cdots \times N_m$, we have that

$$\bigcap_{\varepsilon>0} \overline{\text{co}}\, f(t, x + \varepsilon B) \cap \{0\} \subset \{f(t,x)\}, \tag{2.3}$$

then $\mathcal{C} = \mathcal{K}$ for each $x_0 \in \mathbb{R}^m$ (and thus $\mathcal{C}$ enjoys all the properties established for $\mathcal{K}$ in proposition 2.1).

Proof. For $x \in \mathcal{K}$, we define $A := \{t \in I : x(t) \in N_1 \times \cdots \times N_m\}$. If we put $x(t) = (x_1(t), x_2(t), \dots, x_m(t))$, then $A = \bigcap_{i=1}^m A_i$, where $A_i := \{t \in I : x_i(t) \in N_i\}$. By [19, theorem 38.2], we have that $x_i'(t) = 0$ for a.a. $t \in A_i$ and thus $x'(t) = 0$ for a.a. $t \in A$. Hence $0 \in F(t, x(t))$ for a.a. $t \in A$. Our hypothesis implies then that $f(t, x(t)) = 0$ for a.a. $t \in A$ and, consequently, $x'(t) = f(t, x(t))$ for a.a. $t \in A$. Since $F(t, x(t)) = \{f(t, x(t))\}$ for all $t \in I \setminus (A \cup N)$, we conclude that $x'(t) = f(t, x(t))$ for a.a. $t \in I$ and therefore $x \in \mathcal{C}$. □

REMARKS TO THEOREM 2.2.

1. When specialized to the autonomous case, it can be proven exactly as in [21, theorem 1] that the condition '$\mathcal{K} \subset \mathcal{C}$ for all $x_0 \in \mathbb{R}^m$' implies (2.3). In doing so, we would have a generalization of [18, theorems 2.2 and 3.11]. Remember, however, that in the scalar autonomous case, necessary and sufficient conditions for the existence of Carathéodory solutions are known (see [5]).

2. Theorem 2.2 also improves the results in [18] for non-autonomous problems.

3. We emphasize that the assumptions do not imply that the set of discontinuity points of $f(t,\cdot)$ is equal to $N_1 \times \cdots \times N_m$, but it only needs to be contained in $N_1 \times \cdots \times N_m$. Therefore, the set of discontinuity points of $f(t,\cdot)$ is not explicitly

prescribed, and thus such sets need not be the same for all values of t. However, such a simple case as that of a nonlinear f which is discontinuous with respect to x exactly at the points of the line $x_1 = \cdots = x_m = t$ falls outside the scope of theorem 2.2. This is a severe limitation that we avoid in our next result (which the reader should compare with example 4.1 in [18], which shows that existence may fail if discontinuities depend on t).

To deal with more complicated types of time-dependent discontinuity sets, we shall impose conditions (i), (ii) and the following.

(iii) For all $t \in I \setminus N$, $f(t, \cdot)$ is continuous in $\mathbb{R}^m \setminus K(t)$, where $K(t) = \bigcup_{n=1}^{\infty} K_n(t)$, and, for each $n \in \mathbb{N}$ and $x \in K_n(t)$, we have

$$\bigcap_{\varepsilon>0} \overline{\mathrm{co}}\, f(t, x + \varepsilon B) \cap DK_n(t, x)(1) \subset \{f(t, x)\}. \tag{2.4}$$

Next we show how condition (iii) implies that $\mathcal{K} \subset \mathcal{C}$.

Lemma 2.3. *Let $f : I \times \mathbb{R}^m \to \mathbb{R}^m$ satisfy (i) and (ii) for some null-measure set $N \subset I$. The following results hold.*

(a) *If there exist multi-valued mappings $K_n : I \to \mathcal{P}(\mathbb{R}^m)$, $n \in \mathbb{N}$, such that, for all $t \in I \setminus N$, all $n \in \mathbb{N}$ and all $x \in K_n(t)$, we have*

$$\bigcap_{\varepsilon>0} \overline{\mathrm{co}}\, f(t, x + \varepsilon B) \cap DK_n(t, x)(1) \subset \{f(t, x)\}, \tag{2.5}$$

then every $x \in \mathcal{K}$ satisfies

$$x'(t) = f(t, x(t)) \quad \text{a.e. in } \left\{ t \in I : x(t) \in \bigcup_{n \in \mathbb{N}} K_n(t) \right\}.$$

(b) *If condition (iii) is satisfied, then $\mathcal{K} \subset \mathcal{C}$.*

Proof. Let $x \in \mathcal{K}$ and put

$$\begin{aligned} I_x &= \{t \in [t_0, t_0 + L) \setminus N : x'(t) \text{ exists and } x'(t) \in F(t, x(t))\}, \\ A &= \left\{ t \in I_x : x(t) \in \bigcup_{n \in \mathbb{N}} K_n(t) \right\}, \\ A_n &= \{t \in I_x : x(t) \in K_n(t)\}, \\ B_n &= \{t \in A_n : (t, t + \varepsilon_t) \subset I \text{ and } (t, t + \varepsilon_t) \cap A_n = \emptyset \text{ for some } \varepsilon_t > 0\}. \end{aligned}$$

To establish part (a), we have to show that $x'(t) = f(t, x(t))$ for a.a. $t \in A$. Since $A = \bigcup_{n \in \mathbb{N}} A_n$, it suffices to prove that $x'(t) = f(t, x(t))$ for a.a. $t \in A_n$ and all $n \in \mathbb{N}$. This will be proven in the next two steps.

Step 1 (for each $t \in A_n \setminus B_n$, we have that $x'(t) = f(t, x(t))$). For $t_1 \in A_n \setminus B_n$, there exists a sequence of strictly positive numbers $\{h_i\}_i$ that converges to 0 and is such that $t_1 < t_1 + h_i < t_1 + L$ and $(t_1 + h_i, x(t_1 + h_i)) \in \operatorname{graph}(K_n)$. Now we define $u_i = (1, h_i^{-1}(x(t_1 + h_i) - x(t_1))) \in \mathbb{R}^{m+1}$ for $i \in \mathbb{N}$, and we have

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(1) $\lim_{i\to\infty} u_i = (1, x'(t_1))$;

(2) $\lim_{i\to\infty} h_i = 0$;

(3) $(t_1, x(t_1)) + h_i u_i \in \operatorname{graph}(K_n)\ \forall i \in \mathbb{N}$,

which, by proposition 1.1, implies that $(1, x'(t_1)) \in T_{\operatorname{graph}(K_n)}(t_1, x(t_1))$, or, equivalently, that $x'(t_1) \in DK_n(t_1, x(t_1))(1)$. Moreover, $x'(t_1) \in F(t_1, x(t_1))$, and then equation (2.5) implies that $x'(t_1) = f(t_1, x(t_1))$.

Step 2 (B_n is denumerable for each $n \in \mathbb{N}$). For each $t \in B_n$, take the number $\varepsilon_t > 0$ associated to it by the definition of B_n. Since the intervals $(t, t+\varepsilon_t)$, $t \in A_n$, do not overlap, the sum of each denumerable subfamily of $\{\varepsilon_t : t \in B_n\}$ is finite and bounded above by $L > 0$. Hence the sum $\sum_{t\in B_n} \varepsilon_t$ is finite and therefore B_n can be, at most, denumerable.

To prove (b), we have to show that for a.a. $t \in I_x$ we have $x'(t) = f(t, x(t))$. This follows directly from part (a) and the fact that $F(t, x(t)) = \{f(t, x(t))\}$ whenever $t \in I_x \setminus A$, as $f(t, \cdot)$ is continuous at $x(t)$ for $t \in I_x \setminus A$. □

Now we establish this section's main result, which follows immediately from lemma 2.3 and proposition 2.1.

Theorem 2.4. *If f satisfies (i), (ii) and (iii), then $\mathcal{C}$ is a non-empty, compact and connected subset of $\mathcal{C}(I, \mathbb{R}^m)$.*

Moreover, in the scalar case ($m = 1$), we have the following.

(1) *$\mathcal{C}$ has pointwise maximum x^* and minimum x_*, which are the extremal solutions of (1.1). Furthermore, for each $t \in I$, we have*

$$x^*(t) = \max\{v(t) : v \in AC(I),\ v'(s) \leqslant f(s, v(s))\ \text{a.e.},\ v(t_0) \leqslant x_0\}, \quad (2.6)$$

$$x_*(t) = \min\{v(t) : v \in AC(I),\ v'(s) \geqslant f(s, v(s))\ \text{a.e.},\ v(t_0) \geqslant x_0\}. \quad (2.7)$$

(2) *$\mathcal{C}$ is a funnel, i.e. for all $\bar{t} \in I$ and $c \in [x_*(\bar{t}), x^*(\bar{t})]$, there exists $x \in \mathcal{C}$ such that $x(\bar{t}) = c$.*

Example 2.5. Consider the problem $x'(t) = f(t, x(t))$ a.e. in $[0, 1]$, $x(0) = 0$, where $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a changing-sign nonlinearity given by

$$f(t, x) = \begin{cases} \frac{1}{2} & \text{if } x \leqslant -t, \\ \dfrac{\arctan(n-3)}{\pi} & \text{if } -t + \dfrac{1}{n+1} < x \leqslant -t + \dfrac{1}{n},\ n \in \mathbb{N}, \\ -\frac{1}{2} & \text{if } -t + 1 < x. \end{cases}$$

It is obvious that f satisfies conditions (i) and (ii). Moreover, we have that $f(t, \cdot)$ is continuous in $\mathbb{R} \setminus K(t)$, where $K(t) = \bigcup K_n(t)$ and $K_n(t) = \{-t + 1/n\}$ for all $t \in [0, 1]$. Then $DK_n(t, x)(1) = -1$ for all $(t, x) \in \operatorname{graph}(K_n)$ and all $n \in \mathbb{N}$. On the other hand, $f(t, x) \geqslant -\frac{1}{2}$ for all $(t, x) \in [0, 1] \times \mathbb{R}$, and therefore

$$\bigcap_{\varepsilon>0} \overline{\text{co}}\, f(t, x + \varepsilon B) \cap DK_n(t, x)(1) = \emptyset,$$

which implies that f also satisfies (iii). Thus theorem 2.4 ensures the existence of the extremal solutions for this problem. Furthermore, a standard uniqueness result (see [13]) implies that there exists a unique solution because $f(t,\cdot)$ is non-increasing.

We remark that the results established in [6,14,20] do not apply in this example.

Remarks to theorem 2.4.

1. We cannot expect to have extremal solutions in the conditions of theorem 2.4 when $m \geqslant 2$. In fact, the continuous system

$$x_1' = t^3 - x_2, \quad t \in [0,1], \quad x_1(0) = 0,$$
$$x_2' = 3x_2^{2/3}, \quad t \in [0,1], \quad x_2(0) = 0,$$

has neither a maximal solution nor a minimal one in the sense defined at the beginning of this section. Adding a standard quasi-monotonicity assumption over f and re-enforcing the measurability conditions as in [14, theorem 5.1] is probably the first step towards an extremality result, which we hope to consider elsewhere.

2. We can improve theorem 2.4, weakening hypothesis (iii) until we have the following.

($\tilde{\text{iii}}$) For all $t \in I \setminus N$, we have that $f(t,\cdot)$ is continuous in $\mathbb{R}^m \setminus K(t)$, where $K(t) = \bigcup_{n=1}^{\infty} K_n(t)$, and for each $n \in \mathbb{N}$ and $x \in K_n(t)$, we have

$$\bigcap_{\varepsilon>0} \overline{\text{co}}\, f(t, x+\varepsilon B) \cap DK_n(t,x)(1) \cap -DK_n(t,x)(-1) \subset \{f(t,x)\},$$

but we have preferred to use (iii) for simplicity.

Using the standard change of variables $y(t) = x(2t_0 - t)$, it is easy to check that (i), (ii) and ($\tilde{\text{iii}}$), with the obvious modifications, guarantee an analogous to theorem 2.4 for solutions defined on $[t_0 - L, t_0]$. Since (iii) implies ($\tilde{\text{iii}}$), theorem 2.4 holds valid for the interval $[t_0 - L, t_0 + L]$.

We also note that when K_n is single and scalar valued, then ($\tilde{\text{iii}}$) is trivially fulfilled at those points t where the left and right derivatives exist and they are different (see lemma 1.2).

3. Carathéodory's existence result is covered by theorem 2.4 with $K(t) = \emptyset$ for all $t \in I$. Even Goodman's characterization of the maximal and minimal solution [11] as the greatest subfunction and the least superfunction is also included in theorem 2.4.

4. The existence result is not guaranteed, in general, in case the condition '$F(t,x) \cap DK_n(t,x)(1) \subset \{f(t,x)\}$' fails just for a single x and all t in a subinterval of I. The following standard example shows this.

Example 2.6. The problem $x'(t) = f(t, x(t))$, $x(0) = 0$, for

$$f(t,x) = \begin{cases} 1 & \text{if } x < 0, \\ -1 & \text{if } x \geqslant 0, \end{cases}$$

has no solution defined on, say, $I = [0,1]$. Here, $K(t) = \{0\}$ for all $t \in I$, and thus $DK(t,0)(1) = \{0\}$ for all t.

On the other hand, $F(t,0) := \bigcap_{\varepsilon>0} \overline{\text{co}}\, f(t, 0 + \varepsilon B) = [-1, 1]$, and then

$$F(t,0) \cap DK(t,0)(1) = \{0\} \not\subset \{f(t,0)\}.$$

5. The condition '$F(t,x) \cap DK_n(t,x)(1) = \emptyset$ for all $n \in \mathbb{N}$', which implies (2.4), is a type of transversality (or in-viability) condition, and it prevents the solutions from touching 'tangentially' the discontinuity set graph(K_n). The geometrical idea behind this condition is not new at all, and can be traced back to Filippov's discontinuity surfaces described in [10]. Similar conditions for scalar problems were introduced in [20].

6. Most existence results for inclusions of the type of (1.2) require the multi-valued mapping $F(\cdot, x)$ be measurable for each x, i.e. that $\{t \in I : F(t,x) \cap A \neq \emptyset\}$ be Lebesgue measurable for each open $A \subset \mathbb{R}^m$ (see [16] or [9, definition 3.1]). It seems that Davy in [7] was the first author who realized that, in many situations, the existence of a measurable selection of $F(\cdot, x)$ is enough. This is exploited in, for instance, the proofs of corollary 5.1 and theorem 7.2 in [9], which play a central role in the proof of our proposition 2.1. Davy's observation appears to be crucial in this paper, as the multi-valued mapping $F(t,x) := \bigcap_{\varepsilon>0} \overline{\text{co}}\, f(t, x + \varepsilon B)$ may fail to be measurable in t, even though f satisfies (i), (ii) and $f(t, \cdot)$ is continuous everywhere except, at most, on a countable and nowhere dense subset. This is the case in the following example.

Example 2.7. Let $\mathcal{S} \subset (0, 1]$ be a non-measurable set and define the function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ as

$$f(t,x) = \begin{cases} 1 & \text{if } t = s \text{ and } x = s/n \text{ for some } s \in \mathcal{S} \text{ and some } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that, for each $x \in \mathbb{R}$, there is, at most, a finite number of points $s \in \mathcal{S}$ and a finite number of positive integers n such that $x = s/n$. Therefore, the function $t \mapsto f(t,x)$ is continuous everywhere except, at most, on a finite set of ts. Hence $f(\cdot, x)$ is measurable for each $x \in \mathbb{R}$.

On the other hand, for each $t \in [0,1]$, the function $f(t, \cdot)$ is continuous everywhere except, at most, on the points of the set $K(t) = \{t/n : n \in \mathbb{N}\}$.

It is easy to see that $F(t,0) = \{0\}$ if $t \notin \mathcal{S}$ and $F(t,0) = [0,1]$ if $t \in \mathcal{S}$. Hence $F(\cdot, 0)$ is not measurable, since, for instance, $\{t : F(t,0) \cap (\frac{1}{2}, 2) \neq \emptyset\} = \mathcal{S}$.

3. Another existence result for the scalar case

It is proven in [14] that problem (1.1) with $m = 1$ has extremal solutions provided that $f : I \times \mathbb{R} \to \mathbb{R}$ satisfies (ii) and

(iii)* for all $t \in I \setminus N$ and all $x \in \mathbb{R}$, we have

$$\limsup_{y \to x^-} f(t,y) \leqslant f(t,x) \leqslant \liminf_{y \to x^+} f(t,y),$$

together with a boundedness condition similar to (i).

In this part, we shall focus on the right-hand sides f that satisfy (iii)* outside a certain set of the type of graph(K) in condition (iii), but first we shall prove some

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technical results on superpositional measurability that will be needed to establish our existence results.

3.1. Conditions for superpositional measurability

It is not clear whether the technique employed in [14] may be adapted to this new setting, and there is a main difficulty that we have to overcome in a different way: compositions $f(\cdot, x(\cdot))$ may be non-measurable, even for $x \in \mathcal{C}(I)$ (see [14]). We shall use an obvious way to wipe this problem out, which consists in explicitly requiring something like

(ii)* $f(\cdot, x(\cdot))$ is measurable for each $x \in \mathcal{C}(I)$.

Although (ii)* is commonplace in the current literature of discontinuous differential equations (see [2–4]), it is not a completely satisfactory assumption. First, despite the fact that everyone agrees that measurability is quite a weak condition, it is easy to find elementary examples of solvable Cauchy problems satisfying (ii) and (iii)*, but not (ii)* (see [14]). On the other hand, condition (ii)* is stronger, and hence harder to check, than the classical (ii). Thus we consider that it is interesting to investigate which types of f that satisfy (ii) and (iii)* also satisfy (ii)*.

We shall also show that, loosely speaking, the gap between those f fulfilling (ii) and (iii)* and the ones satisfying (ii)* and (iii)* is occupied by functions that are discontinuous with respect to x on curves of the (t, x)-plane such that the restriction of f to those curves is not a measurable function.

First, we need the following lemma, which is a slight extension of lemma 2.1 in [14] for real-valued f.

Lemma 3.1. *Let $N \subset I$ be a null-measure set and let $f : I \times \mathbb{R} \to \mathbb{R}$ be such that $f(\cdot, q)$ is measurable for each $q \in \mathbb{Q}$. Then we have the following.*

(a) *If, for all $t \in I \setminus N$ and all $x \in \mathbb{R}$, we have*

$$\min\Big\{\limsup_{y\to x^-} f(t,y), \limsup_{y\to x^+} f(t,y)\Big\} \leqslant f(t,x),$$

then the mapping $t \in I \mapsto \inf\{f(t,y) : x_1(t) < y < x_2(t)\}$ is measurable for each pair $x_1, x_2 \in \mathcal{C}(I)$ such that $x_1(t) < x_2(t)$ for all $t \in I$.

(b) *If, for all $t \in I \setminus N$ and all $x \in \mathbb{R}$, we have*

$$\max\Big\{\liminf_{y\to x^-} f(t,y), \liminf_{y\to x^+} f(t,y)\Big\} \geqslant f(t,x),$$

then the mapping $t \in I \mapsto \sup\{f(t,y) : x_1(t) < y < x_2(t)\}$ is measurable for each pair $x_1, x_2 \in \mathcal{C}(I)$ such that $x_1(t) < x_2(t)$ for all $t \in I$.

Proof. We shall only prove part (a), since (b) is similar.

We denote by $\mathcal{S}$ the following set of step functions: $v : [t_0, t_0 + L) \to \mathbb{R}$ belongs to $\mathcal{S}$ if v assumes only rational values, $x_1(t) < v(t) < x_2(t)$ on $[t_0, t_0 + L)$ and there exists $j \in \mathbb{N}$ such that v is constant on every interval

$$\Big[t_0, t_0 + \frac{L}{j}\Big), \Big[t_0 + \frac{L}{j}, t_0 + \frac{2L}{j}\Big), \dots, \Big[t_0 + \frac{(j-1)L}{j}, t_0 + L\Big).$$

As x_1, x_2 are continuous on $[t_0, t_0+L]$, then $\mathcal{S}$ is not empty. Note, moreover, that for each $q \in (x_1(t), x_2(t)) \cap \mathbb{Q}$, there exists $v \in \mathcal{S}$ such that $v(t) = q$.

Since $\mathcal{S}$ is a countable family and any composition $f(\cdot, v(\cdot))$ with $v \in \mathcal{S}$ is measurable on $[t_0, t_0+L)$, it suffices to prove that

$$\iota(t) := \inf_{y \in (x_1(t), x_2(t))} f(t, y) = \inf_{v \in \mathcal{S}} f(t, v(t)) =: \iota_0(t)$$

a.e. on $[t_0, t_0+L)$ to deduce that ι is measurable.

Clearly, $\iota(t) \leqslant \iota_0(t)$ on $[t_0, t_0+L)$. To prove that $\iota(t) \geqslant \iota_0(t)$ on $[t_0, t_0+L) \setminus N$, we fix an arbitrary $t \in [t_0, t_0+L) \setminus N$ and we take a sequence $\{y_n\}_n \subset (x_1(t), x_2(t))$ such that

$$\lim_{n \to \infty} f(t, y_n) = \iota(t). \tag{3.1}$$

Our assumptions guarantee that, for each n, we have

$$\limsup_{y \to y_n^-} f(t, y) \leqslant f(t, y_n) \quad \text{or} \quad \limsup_{y \to y_n^+} f(t, y) \leqslant f(t, y_n),$$

and therefore there exists $q_n \in (x_1(t), y_n) \cap \mathbb{Q}$ (or $q_n \in (y_n, x_2(t)) \cap \mathbb{Q}$) such that $f(t, q_n) \leqslant f(t, y_n) + 1/n$. Since there exists $v_n \in \mathcal{S}$ such that $v_n(t) = q_n$, we have, for all n, that

$$\iota_0(t) = \inf_{v \in \mathcal{S}} f(t, v(t)) \leqslant f(t, v_n(t)) \leqslant f(t, y_n) + \frac{1}{n}$$

and, using (3.1), we conclude that

$$\iota_0(t) \leqslant \lim_{n \to \infty} \left[f(t, y_n) + \frac{1}{n} \right] = \iota(t).$$

□

It is known that a function $g : \mathbb{R} \to \mathbb{R}$ such that

$$\limsup_{y \to x^-} g(y) \leqslant g(x) \leqslant \liminf_{y \to x^+} g(y) \quad \text{for all } x \in \mathbb{R},$$

can have at most a countable set of discontinuity points (as a consequence of Young's theorem [17, p. 287]). Therefore, for each mapping $f : I \times \mathbb{R} \to \mathbb{R}$ for which there exists a null-measure set $N \subset I$ such that for all $t \in I \setminus N$ we have

$$\limsup_{y \to x^-} f(t, y) \leqslant f(t, x) \leqslant \liminf_{y \to x^+} f(t, y) \quad \text{for all } x \in \mathbb{R},$$

there must exist a countable set of mappings $j_n : I_n \subset I \to \mathbb{R}$, $n \in \mathbb{N}$, such that the set of discontinuity points of $f(t, \cdot)$ is exactly $\bigcup_{n/t \in I_n} \{j_n(t)\}$ for each $t \in I \setminus N$.

Bearing these considerations in mind, the assumptions required in the following proposition are natural.

Proposition 3.2. *Let $N \subset I$ be a null-measure set and let $f : I \times \mathbb{R} \to \mathbb{R}$ be such that the following hold.*

(1) *$f(\cdot, q)$ is measurable for each $q \in \mathbb{Q}$.*

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(2) *Either for all* $t \in I \setminus N$ *and all* $x \in \mathbb{R}$ *we have*

$$\min\left\{\limsup_{y\to x^-} f(t,y), \limsup_{y\to x^+} f(t,y)\right\} \leqslant f(t,x),$$

or for all $t \in I \setminus N$ *and all* $x \in \mathbb{R}$ *we have*

$$f(t,x) \leqslant \max\left\{\liminf_{y\to x^-} f(t,y), \liminf_{y\to x^+} f(t,y)\right\}.$$

(3) *There exist mappings* $j_n : I_n \subset I \to \mathbb{R}$, $n \in \mathbb{N}$, *such that, for each* $t \in I \setminus N$, *the set of discontinuity points of* $f(t,\cdot)$ *is exactly* $\bigcup_{n/t\in I_n}\{j_n(t)\}$. *Moreover, the mappings* j_n *and* $f(\cdot, j_n(\cdot))$ *are measurable.*

Then the mapping $t \in I \mapsto f(t, x(t))$ *is measurable for each* $x \in \mathcal{C}(I)$.

Proof. Assume that the first alternative in (2) holds, let $x \in \mathcal{C}(I)$ be fixed and let $J = \{t \in I \setminus N : x(t) = j_n(t)$ for some $n \in \mathbb{N}\}$ and $J_n = \{t \in J : x(t) = j_n(t)\}$, $n \in \mathbb{N}$. For all $t \in I$, we have that

$$f(t,x(t))\chi_J(t) = \sum_{n=1}^{\infty} f(t, j_n(t))\chi_{\tilde{J}_n}(t),$$

where $\tilde{J}_1 = J_1$, $\tilde{J}_n = J_n \setminus (J_1 \cup J_2 \cup \cdots \cup J_{n-1})$, $n \geqslant 2$, and χ_A stands for the characteristic function of the set A. Therefore, $f(\cdot, x(\cdot))\chi_J$ is measurable.

Now we consider $(I \setminus N) \setminus J = \{t \in I \setminus N : f(t,\cdot)$ is continuous at $x(t)\} =: I_{\mathrm{c}}$. Then, for all $t \in I \setminus N$,

$$\begin{aligned} f(t,x(t)) &= \liminf_{y\to(x(t))^+} f(t,y)\chi_{I_{\mathrm{c}}}(t) + f(t,x(t))\chi_J(t) \\ &= \lim_{n\to\infty}\Big[\inf_{y\in(x(t),x(t)+1/n)} f(t,y)\chi_{I_{\mathrm{c}}}(t)\Big] + f(t,x(t))\chi_J(t), \end{aligned}$$

which implies that $f(\cdot, x(\cdot))$ is measurable by virtue of lemma 3.1.

To establish the result using the second alternative in (2), it suffices to replace inf by sup to express $f(\cdot, x(\cdot))$ as a limit of a sequence of measurable functions. □

3.2. Existence results

It is the aim of this part to prove an analogous result to theorem 2.4 for $m = 1$ in order to cover the case of nonlinear $f : I \times \mathbb{R} \to \mathbb{R}$ which, for a given null-measure set $N \subset I$, satisfies (i) and the following conditions.

(ii)′ $f(\cdot, v(\cdot))$ is measurable on I whenever $v \in AC(I)$.

(iii)′ For all $t \in I \setminus N$, we have

$$\limsup_{y\to x^-} f(t,y) \leqslant f(t,x) \leqslant \liminf_{y\to x^+} f(t,y) \quad \text{for all } x \in \mathbb{R}\setminus K(t),$$

$$\liminf_{y\to x^-} f(t,y) \geqslant f(t,x) \geqslant \limsup_{y\to x^+} f(t,y) \quad \text{for all } x \in K(t),$$

where $K(t)=\bigcup_{n=1}^{\infty}K_n(t)$ and, for each $n\in\mathbb{N}$ and $x\in K_n(t)$, we have

$$\bigcap_{\varepsilon>0}\overline{\operatorname{co}}\, f(t,x+\varepsilon B)\cap DK_n(t,x)(1)\subset\{f(t,x)\}.$$

REMARK 3.3. In this case, there is no hope to have $\mathcal{K}=\mathcal{C}$, since $\mathcal{C}$ need not be closed nor connected in $\mathcal{C}(I,\mathbb{R}^m)$, even though $K(t)=\emptyset$ for all $t\in I$. To see this, it suffices to consider the problem $x'=f(t,x)$ for a.a. $t\in[0,1]$, $x(0)=0$, for

$$f(t,x)=\begin{cases}2 & \text{if } x\geqslant t,\\ 1-1/n & \text{if } (1-1/n)t\leqslant x<[1-1/(n+1)]t,\\ 0 & \text{if } x<0.\end{cases}$$

To work with this new type of nonlinearity, we follow lemma 1 in [3] and we define $h:I\times\mathbb{R}^2\to\mathbb{R}$ as follows:

$$h(t,\alpha,\beta)=\begin{cases}\sup\{f(t,\delta):\alpha\leqslant\delta\leqslant\beta\} & \text{if } \alpha\leqslant\beta,\\ \inf\{f(t,\delta):\beta\leqslant\delta\leqslant\alpha\} & \text{if } \alpha\geqslant\beta.\end{cases}\tag{3.2}$$

Furthermore, we shall need the following multi-valued extension of h: we define $H:I\times\mathbb{R}^2\to\mathcal{P}(\mathbb{R})$ as

$$H(t,\alpha,\beta)=\bigcap_{\varepsilon>0}\overline{\operatorname{co}}\, h(t,\alpha+\varepsilon B,\beta).\tag{3.3}$$

The following statement and its proof are nothing but immediate adaptations of those of [3, lemma 1]. However, some minor differences arise due to our weaker assumptions.

LEMMA 3.4. *Assume that (i), (ii)′ and (iii)′ hold. Then the function h defined in (3.2) satisfies the following conditions.*

(a) $h(t,x,x)=f(t,x)$.

(b) *For almost all t and each x, $h(t,x,\cdot)$ is non-decreasing.*

(c) *For each absolutely continuous $v:I\to\mathbb{R}$, the function*

$$(t,x)\mapsto h(t,x,v(t))$$

satisfies (i) and (ii). Moreover, for each $t\in I\setminus N$, $h(t,\cdot,v(t))$ is continuous on $\mathbb{R}\setminus K(t)$.

Proof. Parts (a) and (b) are immediate. To prove that, for a.a. $t\in I$, $h(t,\cdot,v(t))$ is continuous on $\mathbb{R}\setminus K(t)$, it suffices to note that $h(t,\cdot,\beta)$ is non-increasing for all $t\in I\setminus N$ and all β and to show that

$$\lim_{y\to\alpha^-}h(t,y,\beta)\leqslant h(t,\alpha,\beta)\leqslant\lim_{y\to\alpha^+}h(t,y,\beta)\quad\text{for each } \alpha\in\mathbb{R}\setminus K(t).\tag{3.4}$$

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To see that, let $t \in I \setminus N$ be fixed and assume that $\alpha \leqslant \beta$ is such that $\alpha \notin K(t)$. Then we have

$$\begin{aligned}\lim_{y\to\alpha^-} h(t,y,\beta) &= \lim_{y\to\alpha^-} \sup\{f(t,\delta) : y \leqslant \delta \leqslant \beta\} \\ &= \lim_{y\to\alpha^-} \sup\{\sup\{f(t,\delta) : y \leqslant \delta < \alpha\}, \sup\{f(t,\delta) : \alpha \leqslant \delta \leqslant \beta\}\} \\ &\qquad\qquad \Big(\text{by condition (iii)}', \limsup_{y\to\alpha^-} f(t,y) \leqslant f(t,\alpha)\Big) \\ &\leqslant \sup\{f(t,\alpha), \sup\{f(t,\delta) : \alpha \leqslant \delta \leqslant \beta\}\} \\ &= h(t,\alpha,\beta),\end{aligned}$$

and if $\alpha > \beta$, $\alpha \notin K(t)$, we have

$$\begin{aligned}\lim_{y\to\alpha^-} h(t,y,\beta) &= \lim_{y\to\alpha^-} \inf\{f(t,\delta) : \beta \leqslant \delta \leqslant y\} \\ &= \inf\{f(t,\delta) : \beta \leqslant \delta < \alpha\} \\ &\qquad \Big(\text{by condition (iii)}', \liminf_{y\to\alpha^-} f(t,y) \leqslant f(t,\alpha)\Big) \\ &= \inf\{f(t,\delta) : \beta \leqslant \delta \leqslant \alpha\} \\ &= h(t,\alpha,\beta).\end{aligned}$$

We note that the previous limits exist because the mappings involved are monotone. The proof of the other half of (3.4) is similar.

Now we have to prove that $h(\cdot, x, v(\cdot))$ is measurable for each $v \in AC(I)$ and each $x \in \mathbb{R}$, but this follows directly from the assumptions, lemma 3.1 and

$$\begin{aligned}&h(t,x,v(t)) \\ &\quad = \max\{f(t,x), f(t,v(t)), \sup\{f(t,\delta) : x < \delta < v(t)\}\}\chi_{I_1}(t) \\ &\qquad + \min\{f(t,x), f(t,v(t)), \inf\{f(t,\delta) : v(t) < \delta < x\}\}\chi_{I_2}(t) \\ &\qquad\qquad + f(t,x)\chi_{I_3}(t), \quad t \in I,\end{aligned}$$

where $I_1 = \{t \in I : x < v(t)\}$, $I_2 = \{t \in I : x > v(t)\}$ and $I_3 = I \setminus (I_1 \cup I_2)$.

Finally, the mapping $(t,x) \mapsto h(t,x,v(t))$ satisfies (i) with $\psi(t)$ replaced by, for instance, $\bar{\psi}(t) = \psi(t)(1 + |v(t)|)$. □

Now we can proceed to establish some properties of H.

LEMMA 3.5. *Assume that for a null-measure set $N \subset I$, $f : I \times \mathbb{R} \to \mathbb{R}$ satisfies (i), (ii)′ and (iii)′, and consider the mappings h and H defined in (3.2) and (3.3), respectively. Then, for each $t \in I \setminus N$ and all $x \in \mathbb{R}$, we have the following.*

(a) $H(t,x,x) \subset F(t,x) := \bigcap_{\varepsilon>0} \overline{\text{co}}\, f(t, x + \varepsilon B)$.

(b) $H(t,x,\cdot)$ *is non-decreasing in the following sense:*

$$y_1 \leqslant y_2 \quad\Rightarrow\quad H(t,x,y_1) \subset H(t,x,y_2) - \mathbb{R}_+ \quad \textit{and} \quad H(t,x,y_2) \subset H(t,x,y_1) + \mathbb{R}_+.$$

Proof. Note that, for each $(t, x) \in I \times \mathbb{R}$, we have

$$F(t,x) = \Big[\min\Big\{f(t,x), \liminf_{y\to x} f(t,y)\Big\}, \max\Big\{f(t,x), \limsup_{y\to x} f(t,y)\Big\}\Big],$$

and for each $\varepsilon > 0$,

$$\begin{aligned}&\overline{\mathrm{co}}\, h(t, x+\varepsilon B, x)\\ &\quad = [\inf\{h(t,y,x) : x-\varepsilon \leqslant y \leqslant x+\varepsilon\}, \sup\{h(t,y,x) : x-\varepsilon \leqslant y \leqslant x+\varepsilon\}]. \quad (3.5)\end{aligned}$$

Now we take into account the fact that

$$\begin{aligned}&\inf\{h(t,y,x) : x-\varepsilon \leqslant y \leqslant x+\varepsilon\}\\ &\quad = \min\{h(t,x,x), \inf\{h(t,y,x) : x-\varepsilon \leqslant y < x\}, \inf\{h(t,y,x) : x < y \leqslant x+\varepsilon\}\},\end{aligned}$$

and we compute

$$\begin{aligned}\inf\{h(t,y,x) &: x-\varepsilon \leqslant y < x\}\\ &= \inf\{\sup\{f(t,\delta) : y \leqslant \delta \leqslant x\} : x-\varepsilon \leqslant y < x\}\\ &= \inf\{\max\{f(t,x), \sup\{f(t,\delta) : y \leqslant \delta < x\}\} : x-\varepsilon \leqslant y < x\}\\ &\geqslant \inf\{\sup\{f(t,\delta) : y \leqslant \delta < x\} : x-\varepsilon \leqslant y < x\}\\ &= \limsup_{y\to x^-} f(t,y) \geqslant \liminf_{y\to x^-} f(t,y)\end{aligned}$$

and

$$\begin{aligned}\inf\{h(t,y,x) &: x < y \leqslant x+\varepsilon\}\\ &= \inf\{\inf\{f(t,\delta) : x \leqslant \delta \leqslant y\} : x < y \leqslant x+\varepsilon\}\\ &= \inf\{f(t,\delta) : x \leqslant \delta \leqslant x+\varepsilon\}\\ &= \min\{f(t,x), \inf\{f(t,\delta) : x < \delta \leqslant x+\varepsilon\}\}.\end{aligned}$$

Symmetric arguments with the right end of the interval (3.5) show that, for each $\varepsilon > 0$, we have

$$\begin{aligned}\overline{\mathrm{co}}\, h(t, x+\varepsilon B, x) \subset \Big[&\min\Big\{f(t,x), \liminf_{y\to x^-} f(t,y), \inf\{f(t,y) : x < y \leqslant x+\varepsilon\}\Big\},\\ &\max\Big\{f(t,x), \limsup_{y\to x^+} f(t,y), \sup\{f(t,y) : x-\varepsilon \leqslant y < x\}\Big\}\Big],\end{aligned}$$

and, since these intervals decrease with ε, we can go to the limit when ε tends to 0^+ to obtain the desired estimate,

$$\begin{aligned}H(t,x,x) &= \bigcap_{\varepsilon>0} \overline{\mathrm{co}}\, h(t, x+\varepsilon B, x)\\ &\subset \Big[\min\Big\{f(t,x), \liminf_{y\to x^-} f(t,y), \liminf_{y\to x^+} f(t,y)\Big\},\\ &\qquad \max\Big\{f(t,x), \limsup_{y\to x^+} f(t,y), \limsup_{y\to x^-} f(t,y)\Big\}\Big]\\ &= F(t,x).\end{aligned}$$

To establish part (b), it suffices to show that both endpoints of the interval $H(t,x,y_1)$ are smaller than the corresponding ones of $H(t,x,y_2)$ when $y_1 \leqslant y_2$. We shall only prove the result for the left extremes, as the arguments to prove it for the right ones are similar. Since $h(t,x,\cdot)$ is non-decreasing, for each $\varepsilon > 0$, we have

$$\begin{aligned}\inf \overline{\mathrm{co}}\, h(t,x+\varepsilon B,y_1) &= \inf\{h(t,y,y_1) : x-\varepsilon \leqslant y \leqslant x+\varepsilon\}\\ &\leqslant \inf\{h(t,y,y_2) : x-\varepsilon \leqslant y \leqslant x+\varepsilon\}\\ &= \inf \overline{\mathrm{co}}\, h(t,x+\varepsilon B,y_2),\end{aligned}$$

and then $\inf H(t,x,y_1) = \sup\{\inf \overline{\mathrm{co}}\, h(t,x+\varepsilon B,y_1) : \varepsilon > 0\} \leqslant \inf H(t,x,y_2)$. □

Finally, we establish this section's main result. Its proof is based on the theory of generalized iterative techniques for finding fixed points of discontinuous operators, described by Heikkilä and Lakshmikantham in [15]. It will be divided in several steps for the sake of clearness.

Theorem 3.6. *If conditions (i), (ii)′ and (iii)′ hold, then problem (1.1) has the minimal solution x_* and the maximal one x^*.*

Moreover, for each $t \in I$, we have

$$x^*(t) = \max\{v(t) : v \in AC(I),\ v'(s) \leqslant f(s,v(s)) \text{ a.e.},\ v(t_0) \leqslant x_0\}, \tag{3.6}$$

$$x_*(t) = \min\{v(t) : v \in AC(I),\ v'(s) \geqslant f(s,v(s)) \text{ a.e.},\ v(t_0) \geqslant x_0\}. \tag{3.7}$$

Proof. We start by defining an operator $G : AC(I) \to AC(I)$ as follows. For each $v \in AC(I)$, Gv is the minimal Krasovskij solution of $x' = h(t,x,v(t))$, $x(t_0) = x_0$, or, equivalently, the minimal solution of the multi-valued problem

$$x'(t) \in H(t,x(t),v(t)) \quad \text{for a.a. } t \in I, \quad x(t_0) = x_0. \tag{3.8}$$

Claim 1. *Gv is well defined.*

By lemma 3.4 (c), the mapping $(t,x) \mapsto h(t,x,v(t))$ satisfies (i) and (ii). Hence it follows from proposition 2.1 that problem (3.8) has extremal solutions, and, in particular, the minimal solution exists.

Claim 2. *$G : AC(I) \to AC(I)$ is non-decreasing.*

Let $v_i \in AC(I)$, $i = 1,2$, be such that $v_1 \leqslant v_2$ on I and put $y_i = Gv_i$, $i = 1,2$. By part (b) of lemma 3.5, we have, for a.a. $t \in I$, that

$$y_2'(t) \in H(t,y_2(t),v_2(t)) \subset H(t,y_2(t),v_1(t)) + \mathbb{R}_+,$$

which implies that $y_1 \leqslant y_2$ by virtue of (2.2) and the definition of y_1.

A priori bounds on the solutions. As a consequence of (i) and Gronwall's inequality, we have that each solution v of (1.1) satisfies

$$|v(t)| \leqslant (1+|x_0|)\exp\left(\int_{t_0}^{t} \psi(s)\,\mathrm{d}s\right) - 1 =: b(t) \quad \text{for all } t \in I.$$

Claim 3. *$Gb \leqslant b$.*

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Indeed, from (i) and the definition of b, we have that

$$h(t, b(t), b(t)) = f(t, b(t)) \leqslant \psi(t)(1 + b(t)) = b'(t) \quad \text{for a.a. } t \in I.$$

Then $b'(t) \in H(t, b(t), b(t)) + \mathbb{R}_+$ for a.a. $t \in I$ and, moreover, $b(t_0) = |x_0| \geqslant x_0$. Therefore, by (2.2), we deduce that $Gb \leqslant b$.

Claim 4. *There exists $a \in AC(I)$, $a \leqslant b$, such that $Gv \geqslant a$ for all $v \leqslant b$ (in particular, $Ga \geqslant a$). Moreover, if $v \in AC(I)$ is a solution of (1.1), then*

$$v \in [a, b] := \{z \in AC(I) : a(t) \leqslant z(t) \leqslant b(t) \text{ for all } t \in I\}.$$

By the definition of h and (i), we have, for each $v \in AC(I)$ with $v \leqslant b$, that

$$|h(t, x, v(t))| \leqslant \psi(t)(1 + b(t))(1 + |x|) \quad \text{for a.a. } t \in I \text{ and for all } x \in \mathbb{R}.$$

Since the right-hand side of the above inequality is independent of v, there exists $\bar{\psi} \in L^1(I)$ such that, for each $v \in AC(I)$ with $v \leqslant b$, we have

$$|(Gv)'(t)| \leqslant \bar{\psi}(t) \quad \text{for a.a. } t \in I. \tag{3.9}$$

Let us define

$$a(t) = \min\left\{-b(t), x_0 - \int_{t_0}^{t} \bar{\psi}(s)\, \mathrm{d}s\right\} \quad \text{for all } t \in I.$$

By (3.9), for all $v \in AC(I)$ such that $v \leqslant b$, we have that $a \leqslant Gv$. Since $a \leqslant b$ in particular, it holds that $a \leqslant Ga$. Moreover, for any solution v of (1.1), we have that $|v(t)| \leqslant b(t)$ for all $t \in I$, and by the definition of a, we also have that $v \in [a, b]$.

Claim 5. *G has the minimal fixed point in the functional interval $[a, b]$.*

By claims 2, 3 and 4, we have that $a \leqslant Ga$, $Gb \leqslant b$ and G is non-decreasing. Moreover, equation (3.9) holds for each $v \in [a, b]$. Then, by [15, proposition 1.4.4], there exists x_*, the minimal fixed point of G in $[a, b]$, which satisfies

$$x_* = \min\{x \in [a, b] : Gx \leqslant x\}. \tag{3.10}$$

Claim 6. *x_* is the minimal solution of problem (1.1).*

Since $Gx_* = x_*$, we have that $x_*(t_0) = x_0$ and $x_*'(t) \in H(t, x_*(t), x_*(t))$ for a.a. $t \in I$. Therefore, part (a) of lemma 3.5 guarantees that $x_*'(t) \in F(t, x_*(t))$ for a.a. $t \in I$.

We define $A = \{t \in I : x_*(t) \in K(t)\}$ and $B = I \setminus A$. By (iii)$'$ and lemma 2.3 (a), we have that $x_*'(t) = f(t, x_*(t))$ for a.a. $t \in A$. On the other hand, $h(t, \cdot, x_*(t))$ is continuous on $\mathbb{R} \setminus K(t)$ for a.a. $t \in I$ (see lemma 3.4 (c)), and then

$$H(t, x_*(t), x_*(t)) = \{h(t, x_*(t), x_*(t))\} = \{f(t, x_*(t))\} \quad \text{for a.a. } t \in B.$$

Therefore, we also have $x_*'(t) = f(t, x_*(t))$ for a.a. $t \in B$, and thus x_* is a (Carathéodory) solution of (1.1).

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To see that x_* is the minimal solution of (1.1), we have to take an arbitrary solution of (1.1), say, x, and show that $x_* \leqslant x$ on I. We have that $x(t_0) = x_0$, $x \in [a, b]$ by claim 4, and

$$x'(t) = f(t, x(t)) = h(t, x(t), x(t)) \in H(t, x(t), x(t)) + \mathbb{R}_+ \quad \text{for a.a. } t \in I.$$

Therefore, by (2.2) and the definition of G, we deduce that $Gx \leqslant x$. Now it follows from (3.10) that $x_* \leqslant x$.

Claim 7. *x_* satisfies (3.7).*

Suppose that $v \in AC(I)$ and that

$$v'(t) \geqslant f(t, v(t)) \quad \text{for a.a. } t \in I, v(t_0) \geqslant x_0.$$

The mapping $y(t) = \min\{v(t), b(t)\}$, $t \in I$, belongs to $AC(I)$. Moreover,

$$y'(t) \geqslant f(t, y(t)) = h(t, y(t), y(t)) \quad \text{for a.a. } t \in I, y(t_0) = x_0,$$

which implies that $y'(t) \in H(t, y(t), y(t)) + \mathbb{R}_+$ for a.a. $t \in I$. Then, by (2.2), we have that $Gy \leqslant y$. Since $y \leqslant b$, it follows from claim 4 that $a \leqslant Gy$, and therefore $a \leqslant Gy \leqslant y \leqslant b$. Hence we deduce from (3.10) that $x_* \leqslant y$. Therefore, $x_* \leqslant v$ and (3.7) is proved.

The arguments to prove that (1.1) has a maximal solution are dual. □

3.3. Particular cases

In this section we give two corollaries of theorem 3.6 in order to more easily obtain applicable results. Both results cover the case in which the discontinuity set graph(K) consists of a countable union of possibly intersecting 'curves' in the (t, x)-plane and improve theorem 3.1 of [20] in some aspects.

Corollary 3.7. *Assume that, for $f : I \times \mathbb{R} \to \mathbb{R}$, there exists a null-measure set $N \subset I$ such that (i), (ii)′ and the following condition holds.*

(iii)″ *There exist curves $\gamma_n : I_n \subset I \to \mathbb{R}$, $n \in \mathbb{N}$, which are right-differentiable a.e. on the interval I_n, such that, for all $t \in I \setminus N$, we have*

$$\limsup_{y \to x^-} f(t, y) \leqslant f(t, x) \leqslant \liminf_{y \to x^+} f(t, y) \quad \text{for } x \in \mathbb{R} \setminus \bigcup_{n=1}^{\infty} \{\gamma_n(t)\},$$

$$\liminf_{y \to x^-} f(t, y) \geqslant f(t, x) \geqslant \limsup_{y \to x^+} f(t, y) \quad \text{for all } x \in \bigcup_{n=1}^{\infty} \{\gamma_n(t)\}.$$

Moreover, for each $n \in \mathbb{N}$ and a.a. $t \in I_n$, the relation

$$\min\left\{f(t, \gamma_n(t)), \liminf_{y \to \gamma_n(t)} f(t, y)\right\} \leqslant (\gamma_n)'_+(t) \leqslant \max\left\{f(t, \gamma_n(t)), \limsup_{y \to \gamma_n(t)} f(t, y)\right\}$$

implies $(\gamma_n)'_+(t) = f(t, \gamma_n(t))$.

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Then problem (1.1) has extremal solutions, which satisfy (3.6) and (3.7).

Proof. We may assume that γ_n is right-differentiable on $I_n \setminus N$. For each $n \in \mathbb{N}$, we define $K_n(t) = \{\gamma_n(t)\}$ for $t \in I_n$ and $K_n(t) = \emptyset$ otherwise. By lemma 1.2 (a), we have, for each $t \in I_n$, $t \notin N$, that

$$DK_n(t, \gamma_n(t))(1) = \{(\gamma_n)'_+(t)\},$$

and, following our convention, $DK(t, \gamma_n(t))(1) = \emptyset$ for $t \notin I_n$. On the other hand, for each $n \in \mathbb{N}$ and $t \in I_n \setminus N$, we have that

$$\bigcap_{\varepsilon>0} \overline{\mathrm{co}}\, f(t, \gamma_n(t) + \varepsilon B) = \Big[\min\Big\{f(t, \gamma_n(t)), \liminf_{y\to\gamma_n(t)} f(t,y)\Big\}, \max\Big\{f(t, \gamma_n(t)), \limsup_{y\to\gamma_n(t)} f(t,y)\Big\}\Big],$$

and the result follows from theorem 3.6. □

Now we state another consequence of theorem 3.6 and lemma 1.2.

COROLLARY 3.8. *Assume that, for $f : I \times \mathbb{R} \to \mathbb{R}$, there exists a null-measure set $N \subset I$ such that (i), (ii)′ and the following condition holds.*

(iii)‴ *There exist curves $\gamma_n : I_n \subset I \to \mathbb{R}$, $n \in \mathbb{N}$, such that, for all $t \in I \setminus N$, we have*

$$\limsup_{y\to x^-} f(t,y) \leqslant f(t,x) \leqslant \liminf_{y\to x^+} f(t,y) \quad \textit{for } x \in \mathbb{R} \setminus \bigcup_{n=1}^{\infty}\{\gamma_n(t)\},$$

$$\liminf_{y\to x^-} f(t,y) \geqslant f(t,x) \geqslant \limsup_{y\to x^+} f(t,y) \quad \textit{for all } x \in \bigcup_{n=1}^{\infty}\{\gamma_n(t)\}.$$

Moreover, for each $n \in \mathbb{N}$ and a.a. $t \in I_n$, we have that either

$$D^+\gamma_n(t) < \min\Big\{f(t, \gamma_n(t)), \liminf_{y\to\gamma_n(t)} f(t,y)\Big\}$$

or

$$D_+\gamma_n(t) > \max\Big\{f(t, \gamma_n(t)), \limsup_{y\to\gamma_n(t)} f(t,y)\Big\}.$$

Then problem (1.1) has extremal solutions, which satisfy (3.6) and (3.7).

We illustrate the applicability of corollaries 3.7 and 3.8 in the following examples. As far as the authors are aware, there is no previous existence result which can be applied to study these examples.

EXAMPLE 3.9. Let $\{q_n\}_{n=1}^{\infty}$ be an enumeration of all rational numbers in $(-\infty, 0)$ and define

$$\varphi(x) = \sum_{q_n<x} 2^{-n} \quad \text{for all } x \in \mathbb{R}.$$

Note that φ is non-decreasing and, in particular, Borel measurable, discontinuous exactly on $\mathbb{Q} \cap (-\infty, 0)$, $0 < \varphi(x) \leqslant 1$ for all $x \in \mathbb{R}$ and $\varphi(x) = 1$ for all $x \geqslant 0$.

Define now $\psi : [0,1] \times \mathbb{R} \to \mathbb{R}$ as

$$\psi(t,x) = \begin{cases} 2 & \text{if } x > 0, \\ 0 & \text{if } x < -t, \\ \displaystyle\sum_{-t/n \leqslant x} 2^{-n} & \text{elsewhere,} \end{cases}$$

and note that ψ is non-decreasing to both of its variables. Finally, we define $f(t,x) = \varphi(x)(1-\psi(t,x))$ for all $(t,x) \in [0,1] \times \mathbb{R}$.

It is easy to check using the above-mentioned properties about φ and ψ that the conditions of proposition 3.2 are satisfied with $j_0(t) = 0$, $j_n(t) = -t/n$ for all $t \in [0,1]$ and all $n \in \mathbb{N}$. Therefore, f satisfies condition (ii)$'$. Condition (i) is immediately verified and thus it only remains to check condition (iii)$'$ in order to be in a position to apply corollary 3.7. To this end, we define $\gamma_n = j_n$ for $n = 0, 1, 2, \dots$ and we observe that, for $t \in [0,1] \setminus \mathbb{Q}$, we have that φ is continuous at $-t/n$, and hence

$$\lim_{y \to x^-} f(t,y) \leqslant f(t,x) \leqslant \lim_{y \to x^+} f(t,y) \quad \text{if } x \neq -\frac{t}{n}$$

and

$$\lim_{y \to x^-} f(t,y) \geqslant f(t,x) \geqslant \lim_{y \to x^+} f(t,y) \quad \text{if } x = -\frac{t}{n} \text{ or } x = 0.$$

Moreover, for each $n \in \mathbb{N}$ and all $t \in [0,1]$, we have that

$$\gamma_n'(t) = -1/n < 0 < \min\Big\{f(t,\gamma_n(t)), \liminf_{y \to \gamma_n(t)} f(t,y)\Big\}.$$

On the other hand, for $n = 0$ and all $t \in [0,1]$, we have

$$-1 = \min\Big\{f(t,0), \liminf_{y \to 0} f(t,y)\Big\} \leqslant \gamma_0'(t) = 0 = \max\Big\{f(t,0), \limsup_{y \to 0} f(t,y)\Big\},$$

and also $\gamma_0'(t) = 0 = f(t,\gamma_0(t))$.

Then the problem $x'(t) = f(t,x(t))$, $x(0) = x_0$, has extremal solutions on $[0,1]$ for each $x_0 \in \mathbb{R}$.

Example 3.10. Let $\{q_n\}_{n=1}^{\infty}$ be an enumeration of all rational numbers and consider the mapping

$$\phi(x) = \sum_{q_n < x} 2^{-n} \quad \text{for all } x \in \mathbb{R}.$$

Note that ϕ is non-decreasing, left-continuous everywhere, discontinuous exactly on $\mathbb{Q}$ and $0 < \phi(x) < 1$ for all $x \in \mathbb{R}$.

Let $f(t,x) = \phi(t-x) + \phi(x) - 1$ for all $(t,x) \in [0,1] \times \mathbb{R}$.

Since ϕ is Borel measurable, condition (ii)$'$ holds. The remaining conditions in corollary 3.8 can be easily checked with $\gamma_n(t) = t - q_n$ for all $t \in [0,1]$ and $n \in \mathbb{N}$. Notice that, for all $n \in \mathbb{N}$ and all $t \in [0,1]$, we have

$$\gamma_n'(t) = 1 > \phi(q_n^+) + \phi((t-q_n)^+) - 1 > \max\Big\{f(t,\gamma_n(t)), \limsup_{y \to \gamma_n(t)} f(t,y)\Big\}.$$

Therefore, the initial-value problem $x'(t) = f(t, x(t))$, $x(0) = x_0$, has extremal solutions on $[0,1]$ for each $x_0 \in \mathbb{R}$.

REMARK 3.11. We note that the previous corollaries and theorem 3.1 of [20] are not really comparable: conditions (iii)$''$ and (iii)$'''$ are clearly milder than condition (II) in [20, theorem 3.1]. However, condition (ii)$'$ is stronger than (I) in [20], which only requires that $f(\cdot, x)$ be measurable for each x.

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