Bull. Austral. Math. Soc. **77** (2008), 261–276
doi:10.1017/S0004972708000257

AUTOMORPHISMS OF METABELIAN PRIME POWER ORDER GROUPS OF MAXIMAL CLASS

S. FOULADI ✉ and R. ORFI

(Received 2 July 2007)

Abstract

Let G be a p-group of maximal class of order p^n. It is shown that the order of the group of all automorphisms of G centralizing the Frattini quotient takes the maximum value p^{2n-4} if and only if G is metabelian. A structure theorem is proved for the Sylow p-subgroup, $\mathrm{Aut}_p(G)$, of the automorphism group of G when G is metabelian. For $p = 2$, $\mathrm{Aut}_2(G)$ is the full automorphism group of G. For $p = 3$, we prove a structure theorem for the full automorphism group of G.

2000 *Mathematics subject classification*: primary 20D45, 20D15.

Keywords and phrases: finite p-group, maximal class, metabelian group, automorphism group.

1. Introduction

There have been a number of studies of the automorphism groups of p-groups of maximal class (see for example, Baartmans and Woeppel [2], Juhász [8], Malinowska [10], Caranti and Mattarei [4], Caranti and Scoppola [5]). These concentrate mostly on small automorphism groups. In this paper we consider large automorphism groups.

Let G be a p-group of maximal class of order p^n and let $\Phi = \Phi(G)$ be the Frattini subgroup of G. It is well known [7, Satz III.3.17] that the order of $\mathrm{Aut}^{\Phi}(G)$, the group of all automorphisms of G centralizing G/Φ, divides p^{2n-4}. Moreover, the order of $\mathrm{Aut}_p(G)$, the Sylow p-subgroup of the automorphism group of G, divides p^{2n-3}. In Section 4 we prove that $|\mathrm{Aut}^{\Phi}(G)|$, the order of $\mathrm{Aut}^{\Phi}(G)$, is p^{2n-4} if and only if G is metabelian (Theorem 4.3).

Juhász [8, Theorem 2.3] proved that if G is a p-group of maximal class then $\mathrm{Aut}^{\Phi}(G)$ is a split extension of $\mathrm{Inn}(G)$, the inner automorphism group of G. In Section 3 we prove that when G is metabelian, a complement of $\mathrm{Inn}(G)$ in $\mathrm{Aut}^{\Phi}(G)$ is *almost homocyclic of rank* $p-1$; that is, it is a direct product of exactly $p-1$ cyclic groups of order p^r or p^{r+1} for some nonnegative integer r (Theorem 3.3). Also we give conditions on G for $|\mathrm{Aut}_p(G)| = p^{2n-3}$ (Corollary 3.8). In this case $\mathrm{Aut}_p(G)$

is a split extension of $\mathrm{Aut}^{\Phi}(G)$ by a cyclic group of order p (Theorem 3.10). For $p = 2$, the automorphism group is a 2-group. In Section 5 we give a simple proof for the structure theorem in this case (Theorem 5.9). It is straightforward to see that when p is odd, the (full) automorphism group $\mathrm{Aut}(G)$ of G is a split extension of $\mathrm{Aut}_p(G)$ by a subgroup of the direct product of two cyclic groups of order $p - 1$, see [2, Section 1]. By using this result we prove a structure theorem for $\mathrm{Aut}(G)$ when $p = 3$ (Theorem 5.8).

Throughout this paper the following notation is used. The terms of the lower and the upper central series of G are denoted by $\gamma_i(G)$ and $\zeta_i(G)$, respectively. The centre of G is denoted by $Z = Z(G)$. The nilpotency class of a group G is denoted by $\mathrm{cl}(G)$. If α is an automorphism of G and x is an element of G, we write x^{α} for the image of x under α. The inner automorphism induced by the element g is denoted by σ_g. For a normal subgroup N of G, we let $\mathrm{Aut}^N(G)$ denote the group of all automorphisms of G centralizing G/N. Also C_n denotes the cyclic group of order n. All unexplained notation is standard and follows that of [9].

2. Some basic results

In this section we give some basic results needed for the main results of the paper.

Let G be a p-group of maximal class and order p^n $(n \geq 4)$, where p is a prime. Following [9], we define the 2-step centralizer K_i in G to be the centralizer in G of $\gamma_i(G)/\gamma_{i+2}(G)$ for $2 \leq i \leq n-2$ and define $P_i = P_i(G)$ by $P_0 = G$, $P_1 = K_2$, $P_i = \gamma_i(G)$ for $2 \leq i \leq n$. The degree of commutativity $l = l(G)$ of G is defined to be the maximum integer such that $[P_i, P_j] \leq P_{i+j+l}$ for all $i, j \geq 1$ if P_1 is not Abelian and $l = n - 3$ if P_1 is Abelian.

Take $s \in G - \bigcup_{i=2}^{n-2} K_i$, $s_1 \in P_1 - P_2$ and $s_i = [s_{i-1}, s]$ for $2 \leq i \leq n-1$. It is easily seen that $\{s, s_1\}$ is a generating set for G and $P_i(G) = \langle s_i, \ldots, s_{n-1}\rangle$ for $1 \leq i \leq n-1$.

For the rest of the section we fix the above notation and assume that $n \geq 4$.

Lemma 2.1 [9, Corollary 3.2.7]. *Let G be a p-group of maximal class. The degree of commutativity of G is positive if and only if the 2-step centralizers of G are all equal.*

Lemma 2.2 [7, Hilfssatz III. 14.13]. *If G is a p-group of maximal class of order p^n and $s \notin K_i$ for $2 \leq i \leq n-2$, then $C_G(s) = \langle s\rangle P_{n-1}(G)$ and $s^p \in P_{n-1}$.*

Lemma 2.3. *Let G be a p-group of maximal class of order p^n.*

(i) *If G has positive degree of commutativity, then $s_i^p s_{i+p-1} \in P_{i+p}$ for $i > 1$.*

(ii) *If $n > p + 1$ then $s_1^p s_p \in P_{p+1}$.*

(iii) *If $y \in P_2$ then $(sy)^p = s^p$.*

(iv) *If G is metabelian then G has positive degree of commutativity.*

(v) *If G is metabelian then $s_i^p s_{i+1}^{\binom{p}{2}} \cdots s_{i+p-1} = 1$ for $2 \leq i \leq n-1$.*

https://doi.org/10.1017/S0004972708000257 Published online by Cambridge University Press

PROOF. Conclusions (i)–(iv) follow by [9, Propositions 3.3.8, 3.3.3, Lemma 3.3.7] and [3, Corollary p. 59]. Conclusion (v) is obvious since $(ss_i)^p = s^p s_i^p s_{i+1}^{\binom{p}{2}} \cdots s_{i+p-1}$. □

LEMMA 2.4 [3, Theorem 3.10]. *If G is a metabelian p-group of maximal class of order p^n $(n \geq p+1)$, then $[P_1(G), P_i(G)] \leq P_{n-p+i}(G)$.*

LEMMA 2.5. *Let G be a metabelian p-group of maximal class of order p^n.*

(i) $s_i^{p^t} = s_{i+(p-1)t}^{(-1)^t} x$, *where* $x \in P_{i+(p-1)t+1}$ *for* $i \geq 2$; *so if* $s_i^{p^t} \neq 1$ *then* $s_i^{p^t} \in P_{i+(p-1)t} - P_{i+(p-1)t+1}$.

(ii) $P_i^{p^t} \leq P_{i+(p-1)t}$ *for* $i \geq 2$.

PROOF.

(i) We use induction on t and Lemma 2.3(i).

(ii) This follows from (i). □

LEMMA 2.6. *Let G be a metabelian p-group of maximal class of order p^n and m be a positive integer.*

(i) $[s_1, s^m] = s_2^m s_3^{\binom{m}{2}} \cdots s_{m+1}^{\binom{m}{m}}$.

(ii) $[s_i^m, s_1] = [s_i, s_1]^m$ *for* $i \geq 2$.

(iii) $[s_i^m, s] = [s_i, s]^m$ *for* $i \geq 2$.

(iv) $|s_2| \geq |s_3| \geq \cdots \geq |s_{n-1}|$ *and so* $\exp(P_i) = |s_i|$ *for* $i \geq 2$.

(v) $(s_1 s_i)^m = s_1^m s_i^m [s_i, s_1] [s_i, s_1^2] \cdots [s_i, s_1^{m-1}]$ *for* $i \geq 2$.

LEMMA 2.7. *Let G be a metabelian p-group of maximal class of order p^n. Then*

$$|s_i| = \begin{cases} p^{((n-i)/(p-1))}, & p-1 \mid n-i, \\ p^{[(n-i)/(p-1)]+1}, & p-1 \nmid n-i, \end{cases}$$

for $i \geq 2$.

PROOF. It is enough to prove that $|s_{n-i(p-1)-j}| = p^{i+1}$ for all $i \geq 0$ and all $1 \leq j \leq p-1$. We use induction on i. It is easy to show that $|s_{n-j}| = p$ for $1 \leq j \leq p-1$, by Lemma 2.3(i). Now the result follows from Lemmas 2.3(i), 2.6(iv) and 2.5. □

LEMMA 2.8. *Let G be a metabelian p-group of maximal class of order p^n. Then* $s_{ri-(r-1)}^{\binom{p^t}{r}} \in P_{i+2+t(p-1)}$ *when* $t \geq 0$, $r \geq 2$ *and* $i \geq 3$.

PROOF. Suppose that $r = p^w m$, where $(m, p) = 1$ and $w \geq 0$. So $\binom{p^t}{r} = p^{t-w}k$, where $(k, p) = 1$. Therefore $s_{ri-(r-1)}^{\binom{p^t}{r}} \in P_{ri-(r-1)+(t-w)(p-1)}$ by Lemma 2.5(i). We have the equality $p^w > w(p-1)$ for $w \geq 0$ and $p \geq 3$. Hence, $(i-1)(r-1) \geq w(i-1)(p-1)$. Moreover, $w(p-1)(i-1) \geq 2w(p-1) \geq w(p-1)+2$ when $w > 0$, and $(i-1)(r-1) \geq 2$ when $w = 0$, completing the proof. □

https://doi.org/10.1017/S0004972708000257 Published online by Cambridge University Press

LEMMA 2.9. *Let G be a metabelian p-group of maximal class of order p^n.*

(i) *Suppose that $i \geq 2$ and $1 \leq k \leq p-2$. If $s_i^{p^t} = s_{i+1}^{p^{m_1}u_1} \cdots s_{i+k}^{p^{m_k}u_k}$, where $(u_j, p) = 1$ and $m_j \geq t-1$ for $1 \leq j \leq k$, then $s_i^{p^t} = 1$.*

(ii) *$\langle s_i \rangle \cap \langle s_{i+1}, \ldots, s_{i+k} \rangle = 1$ for $i \geq 2$ and $1 \leq k \leq p-2$.*

(iii) *$P_i = \langle s_i \rangle \times \langle s_{i+1} \rangle \times \cdots \times \langle s_{i+p-2} \rangle$ for $i \geq 2$.*

(iv) *Suppose that $n - r = (p-1)k + j$ for $0 \leq j \leq p-2$, $r \geq 2$. If $j \neq 1$, then $|Zs_r| = |s_r|$ and $|Zs_r| = |s_r|/p$ if $j = 1$.*

(v) *$P_i/Z \cong \langle Zs_i \rangle \times \cdots \times \langle Zs_{i+p-2} \rangle$ for $i \geq 2$.*

PROOF. (i) We first note that if $m_j \geq t$ for all j, $1 \leq j \leq k$, then by Lemma 2.5(i) $s_{i+j}^{p^{m_j}u_j} \in P_{i+t(p-1)+1}$ and so $s_i^{p^t} = 1$. Now suppose that j $(1 \leq j \leq k)$ is the least positive integer such that $m_j = t-1$, so $m_1, m_2, \ldots, m_{j-1} \geq t$. We claim that $s_{i+j}^{p^{m_j}u_j} = 1$. Suppose that this is false; then by Lemma 2.5(i) $s_{i+j}^{p^{m_j}u_j} \in P_{i+j+(t-1)(p-1)} - P_{i+j+(t-1)(p-1)+1}$. On the other hand, we see that $s_{r+i}^{p^{m_r}u_r} \in P_{i+j+(t-1)(p-1)+1}$ for $1 \leq r \leq k$, $r \neq j$. Since $j \leq p-2$, we have $s_i^{p^t} \in P_{i+j+(t-1)(p-1)+1}$, which is impossible. Therefore, by the above note, the proof is established.

(ii) We use induction on k. By Lemma 2.7, $|s_{i+j}| \geq |s_i|/p$ for $1 \leq j \leq p-2$. For $k = 1$, we suppose that $s_i^{p^t} \in \langle s_i \rangle \cap \langle s_{i+1} \rangle$. We may write $s_i^{p^t} = s_{i+1}^{p^m u}$, where $(u, p) = 1$. By considering the order of both sides we conclude that $m \geq t-1$ so that $s_i^{p^t} = 1$ by (i). Now suppose that $i \geq 2$ and the equality holds for all positive integers less than k. If $\langle s_i \rangle \cap \langle s_{i+1}, \ldots, s_{i+k} \rangle \neq 1$ then we may write $s_i^{p^t} = s_{i+1}^{p^{m_1}u_1} \cdots s_{i+k}^{p^{m_k}u_k}$, where $(u_j, p) = 1$ for $1 \leq j \leq k$. Again by considering the order of both sides and using the induction hypothesis we deduce that $|s_i^{p^t}| = \max\{|s_{i+1}|/p^{m_1}, \ldots, |s_{i+k}|/p^{m_k}\}$. Hence $m_j \geq t-1$ for $1 \leq j \leq k$ and therefore the proof is completed by applying (i).

(iii) On setting $H_i = \langle s_i, \ldots, s_{i+p-2} \rangle$ we see that $H_i \leq P_i$; also by (ii), $H_i \cong \langle s_i \rangle \times \cdots \times \langle s_{i+p-2} \rangle$. Now by Lemma 2.7 we deduce that $|H_i| = p^{n-i}$ and hence $H_i = P_i$, as required.

(iv) This is proved by Lemma 2.5(i).

(v) We may proceed as in (iii) above. □

COROLLARY 2.10. *P_i is an almost homocyclic p-group of rank $p-1$ for $i \geq 2$.*

PROOF. This follows from Lemma 2.9(iii) and the fact that $|P_i| = p^{n-i}$. Also we note that elementary Abelian groups of order $p, p^2, \ldots, p^{p-1}$ are almost homocyclic of rank $p-1$ with $r = 0$, by our definition in the introduction and for each n, d there is exactly one almost homocyclic group of order p^n and rank d. □

LEMMA 2.11. *Let G be a group and x, y be elements of G. If $[[x, y], x^{-1}] = 1$ then $[x, y] = [y, x^{-1}]$.*

LEMMA 2.12. *Let G be a metabelian p-group of maximal class of order p^n. If $n \geq 2p-3$, $p \geq 3$ and m is a positive integer, then:*

(i) $[[s_2, s_1], s_1] = z \in [P_{p-1}, P_1]$ *and* $[[s_i, s_1], s_1] = 1$ *for* $i \geq 3$, *moreover* $z = 1$ *if* $n \geq 2p-2$;

(ii) $[[s_i, s_1]^m, s] = [s_{i+1}, s_1]^m$ *for* $i \geq 2$;

(iii) $(s_1 s_2)^m = s_1{}^m s_2{}^m [s_2, s_1]^{\binom{m}{2}} z^{\binom{m}{3}}$;

(iv) $[s_1^m, s] = s_2^m [s_2, s_1]^{\binom{m}{2}} z^{\binom{m}{3}}$;

(v) $[s_i, s_1]^p = 1$ *for* $i \geq 2$.

PROOF. (i) This follows from Lemma 2.4.

(ii) We use induction on m, the Witt identity, Lemma 2.11 and (i).

(iii)–(iv) We use induction on m, (i) and Lemma 2.6(ii).

(v) We have $[s_i, s_1] \in P_{n-p+i}$ by Lemma 2.4 and $\exp(P_{n-p+i}) | p$ by Lemma 2.5(ii). □

LEMMA 2.13. *We have $\sum_{k=u}^{m} k\binom{k}{u} = m\binom{m+1}{u+1} - \binom{m+1}{u+2}$ for all positive integers u and m.*

LEMMA 2.14. *Let G be a metabelian p-group of maximal class of order p^n. If $n \geq 2p-3$ and $p \geq 3$, then $(ss_1)^p = s^p s_1^{\binom{p}{1}} \cdots s_p^{\binom{p}{p}} [s_1, s_{p-1}]$.*

PROOF. By using induction on m, Lemmas 2.13 and 2.12 we see that

$$(ss_1)^m = s^m s_1^{\binom{m}{1}} \cdots s_m^{\binom{m}{m}} [s_2, s_1]^{b_2} \cdots [s_i, s_1]^{b_i} \cdots [s_m, s_1]^{b_m} z^{b_3},$$

where $b_i = \sum_{k=i-1}^{m-1} k\binom{k}{i-1}$ for $i \geq 2$. Now we take $m = p$ and observe that $[s_p, s_1] = 1$ by Lemma 2.4. We have $b_i = (p-1)\binom{p}{i} - \binom{p}{i+1}$ by Lemma 2.13, so $p | b_i$ when $i < p-1$ and $b_{p-1} = p(p-1) - 1$. Therefore $[s_i, s_1]^{b_i} = 1$ for $i < p-1$ and $[s_{p-1}, s_1]^{b_{p-1}} = [s_1, s_{p-1}]$ by Lemma 2.12(v). Also if $p = 3$, then by Lemma 2.12(i), $z = 1$ since $n \geq 4$; and if $p > 3$, then $p | b_3$. Hence, $z^{b_3} = 1$ since $z \in Z(G)$. □

In what follows we give a presentation for a metabelian p-group G of maximal class of order p^n $(n \geq 2p-3)$. Suppose that k is the largest positive integer such that $[P_1, P_2] = P_k$, so $k \geq n-p+2$ by Lemma 2.4. Therefore we may write $[s_1, s_2] = s_k^{a_1} s_{k+1}^{a_2} \cdots s_{n-1}^{a_{n-k}}$, where $a_1 \neq 0$ and $o \leq a_i < p$. Also, by Lemma 2.2, $s^p = s_{n-1}^w$ $(0 \leq w < p)$, and, by Lemma 2.3(v), $s_i^p s_{i+1}^{\binom{p}{2}} \cdots s_{i+p-1} = 1$ for $i \geq 2$. Now by Lemmas 2.14 and 2.4 we see that $s_1^{\binom{p}{1}} \cdots s_p^{\binom{p}{p}} = s_{n-1}^z$ $(0 \leq z < p)$. So we have proved the following theorem.

https://doi.org/10.1017/S0004972708000257 Published online by Cambridge University Press

THEOREM 2.15. *Let G be a metabelian p-group of maximal class of order p^n, $p \geq 3$ and $n \geq 2p-3$. Then*

$$G \cong \langle s, s_1, \ldots, s_{n-1} \mid s_i = [s_{i-1}, s], [s_{n-1}, s] = 1, [s_1, s_2] = s_k^{a_1} s_{k+1}^{a_2} \cdots s_{n-1}^{a_{n-k}}$$
$$[s_i, s_j] = 1, s^p = s_{n-1}^w, s_1^{\binom{p}{1}} \cdots s_p^{\binom{p}{p}} = s_{n-1}^z, s_i^p s_{i+1}^{\binom{p}{2}} \cdots s_{i+p-1} = 1\rangle,$$

where $2 \leq i \leq j \leq n-1$, $a_1 \neq 0$, $0 \leq a_1, \ldots, a_{n-k} < p$, $0 \leq z < p$ and $0 \leq w < p$.

3. $\mathrm{Aut}^{P_i}(G)$ and $\mathrm{Aut}_p(G)$

In this section we prove a structure theorem for $\mathrm{Aut}^{P_i}(G)$ $(i \geq 2)$ and $\mathrm{Aut}_p(G)$ when G is a metabelian p-group of maximal class of order p^n. We note that if $n \leq 3$ and G is not cyclic then $\mathrm{Aut}^{\Phi}(G) = \mathrm{Inn}(G)$ and $\mathrm{Aut}_p(G) \cong \mathrm{Aut}^{\Phi}(G) \rtimes C_p$. Moreover, when G is cyclic then $\mathrm{Aut}_p(G) = \mathrm{Aut}^{\Phi}(G) \cong C_p$. Therefore, in the rest of this section we assume that $n \geq 4$.

THEOREM 3.1 [6, Theorem 3.2]. *Let $G = \langle a, b\rangle$ be a two-generated metabelian group. Then the following are equivalent:*
(i) *for all $u, v \in G'$ there is an automorphism of G that maps a to au and b to bv;*
(ii) *G is nilpotent.*

By the above theorem we see that if G is a noncyclic metabelian p-group of maximal class of order p^n, then for any elements $x, y \in G'$ there is an automorphism that maps s to sx and s_1 to s_1y, so $|\mathrm{Aut}^{\Phi}(G)| = p^{2n-4}$. Now we define α_i, $2 \leq i \leq n-1$, by $s^{\alpha_i} = s$ and $s_1^{\alpha_i} = s_1 s_i$. Clearly, $[\alpha_i, \alpha_j] = 1$. Also $\alpha_2 = \sigma_s$ has order p.

LEMMA 3.2. *Let G be a metabelian p-group of maximal class of order p^n. Then $|\alpha_i| = |s_i|$ for $i \geq 3$.*

PROOF. We observe that $(s_1)^{\alpha_i^m} = s_1 s_i^m s_{2i-1}^{\binom{m}{2}} \cdots s_{ri-(r-1)}^{\binom{m}{r}} \cdots s_{mi-(m-1)}^{\binom{m}{m}}$ for every positive integer m. On setting $m = |s_i| = p^t$, we see that $t \geq ((n-i)/(p-1))$ by Lemma 2.7, so $\alpha_i^m = 1$ by Lemma 2.8. Also $\alpha_i^{p^k} \neq 1$ for $p^k < |s_i|$; otherwise $s_i^{p^k} s_{2i-1}^{\binom{p^k}{2}} \cdots s_{ri-(r-1)}^{\binom{p^k}{r}} \cdots = 1$. However, $s_i^{p^k} \in P_{i+k(p-1)} - P_{i+k(p-1)+1}$ by Lemma 2.5(i) and $s_{ri-(r-1)}^{\binom{p^k}{r}} \in P_{i+k(p-1)+2}$ for $2 \leq r \leq m$ by Lemma 2.8, which is impossible. □

THEOREM 3.3. *Let G be a metabelian p-group of maximal class of order p^n. We set $A_i = \langle \alpha_i, \ldots, \alpha_{n-1}\rangle$ and $I_i = \{\sigma_g | g \in P_{i-1}\}$ for $2 \leq i \leq n-1$.*
(i) $\mathrm{Aut}^{\Phi}(G) = \mathrm{Inn}(G) \rtimes A_3$, *where A_3 is an almost homocyclic p-group of rank $p-1$.*
(ii) $\mathrm{Aut}^{P_i}(G) = I_i \rtimes A_i$ *for $2 \leq i \leq n-1$.*
(iii) *$A_i \cong I_i \cong P_i$ is an almost homocyclic p-group of rank $p-1$, having order p^{n-i} for $i \geq 3$ and $A_2 \ncong P_2$ when $n > p+1$.*

https://doi.org/10.1017/S0004972708000257 Published online by Cambridge University Press

PROOF. (i) By [8, Theorems 2.3, 4.3], $\mathrm{Aut}^{\Phi}(G) = \mathrm{Inn}(G) \rtimes A_3$ and $A_3 \cong P_3$, so A_3 is almost homocyclic by Corollary 2.10.

(ii) We have $|I_i| = p^{n-i}$ for $2 \le i \le n-1$ and, by Theorem 3.1, $|\mathrm{Aut}^{P_i}(G)| = p^{2(n-i)}$ for $2 \le i \le n-1$. Also $A_{n-1} < \cdots < A_3 < A_2$ implies that $|A_i| \ge p^{n-i}$. Therefore by (i), $|A_i| = p^{n-i}$ and $I_i \cap A_i = 1$ for $3 \le i \le n-1$. So it remains to prove that $I_2 \cap A_2 = 1$. Otherwise there exists an element $g \in P_1 - Z(G)$ such that $\sigma_g \in A_2$ and so $[s, g] = 1$. Hence by Lemma 2.2, $g = s^i z$ for some $0 < i < p$ and some $z \in Z(G)$. Thus $[s_1, g] = [s_1, s^i] = s_2^i s_3^{\binom{i}{2}} \cdots s_{i+1}^{\binom{i}{i}}$, by Lemma 2.6(i). It follows that $[s_1, g] \in P_2 - P_3$. Since $g \in P_1$ and G has positive degree of commutativity, we find that $[s_1, g] \in P_3$, which is a contradiction. Therefore, $I_2 \cap A_2 = 1$.

(iii) We have $|A_i| = p^{n-i}$ by (ii). By [8, Theorem 4.3], $A_i \cong P_i$ for $i \ge 3$ and so, by Corollary 2.10, A_i is almost homocyclic. Now we prove that $I_i \cong P_i$ for $i \ge 3$. To see this we note that $I_i \cong P_{i-1}/Z$. Also $P_{i-1}/Z \cong P_i$ by Lemma 2.9(v), (iv) and Corollary 2.10 we have $I_i \cong P_i \cong A_i$ for $i \ge 3$. Finally, $A_2 = \langle \alpha_2 \rangle A_3$ and $\alpha_2 = \sigma_s$. Therefore $A_2 = \langle \alpha_2 \rangle \times A_3$ since σ_s has order p. So for $n > p + 1$ the minimal number of generators of P_2 and A_2 are different. □

LEMMA 3.4. *With the notation and assumption of Theorem 3.3, the following inequalities hold for all positive integers m.*

(i) $[\sigma_{s_t}, \alpha_k] = \sigma_{s_{t+k-1}}$ *for* $2 \le k \le n-1$ *and* $1 \le t \le n-1$.

(ii) $[A_k, I_t] = I_{t+k-1}$ *for* $2 \le k, t \le n-1$.

(iii) $[\sigma_{s_1^m}, \alpha_k] = \sigma_y$, *where* $y = s_k^m x$ *and* $x \in P_{k+1}$ *for* $k \ge 2$. *Furthermore,* $[\sigma_{s_i^m}, \alpha_k] = [\sigma_{s_i}, \alpha_k]^m$ *for* $i, k \ge 2$.

(iv) $[\sigma_{s_i}, \alpha_j^m] = \sigma_y$, *where* $y = s_{i+j-1}^m x$ *and* $x \in P_{i+j}$ *for* $i \ge 1$ *and* $j \ge 2$.

(v) *If* $i, j \ge 2$, $(p, m_{j-1}) = 1$, $m_{j-1} \ne 0$ *and* $[\sigma_{s_{j-1}^{m_{j-1}} s_j^{m_j} \cdots}, \alpha_i] = 1$, *then* $j \ge n - (i-1)$.

(vi) *If* $i, j \ge 2$, $m_j \ne 0$, $(p, m_j) = 1$ *and* $[\alpha_{n-1}^{m_{n-1}} \alpha_{n-2}^{m_{n-2}} \cdots \alpha_j^{m_j}, \sigma_{s_{i-1}}] = 1$, *then* $j \ge n - (i-1)$.

PROOF.

(i) This is clear by $\sigma_{s_t}^{\alpha_k} = \sigma_{s_t^{\alpha_k}}$.

(ii) This is obvious by (i).

(iii) We have $[\sigma_{s_1^m}, \alpha_k] = \sigma_{s_1^{-m}(s_1 s_k)^m}$. The proof is completed by Lemma 2.6(v).

(iv) We use induction on m and (i).

(v) By (iii), (i) and the fact that G' is Abelian, $s_{i+j-2}^{m_{j-1}} s_{i+j-1}^{m_j} \cdots \in Z(G)$ for $j \ge 3$, so $i + j - 2 \ge n - 1$. If $j = 2$ then $s_i^{m_1} x s_{i+1}^{m_2} s_{i+2}^{m_3} \cdots \in Z(G)$, where $x \in P_{i+1}$ therefore $i \ge n-1$, completing the proof.

(vi) We use (iv) and then proceed as in (v). □

THEOREM 3.5. *With the notation and assumption of Theorem 3.3, the following results hold:*

(i) $\gamma_j(\mathrm{Aut}^{P_i}(G)) = I_{(i-1)j+1}$ *for* $i, j \ge 2$;

(ii) $\mathrm{cl}(\mathrm{Aut}^{P_2}(G)) = n - 2$;

https://doi.org/10.1017/S0004972708000257 Published online by Cambridge University Press

(iii) *if* $i \geq 3$ *and* $n = (i-1)c + r$ $(0 \leq r \leq i-2)$, *then*

$$\mathrm{cl}(\mathrm{Aut}^{P_i}(G)) = \begin{cases} c-1, & 0 \leq r \leq 1, \\ c, & 2 \leq r \leq i-2; \end{cases}$$

(iv) $\mathrm{Aut}^{P_i}(G)$ *is Abelian if and only if* $i \geq ((n+1)/2)$;
(v) $Z(\mathrm{Aut}^{P_i}(G)) = \mathrm{Aut}^{P_{n-(i-1)}}(G)$ *for* $2 \leq i \leq ((n+1)/2)$;
(vi) $\zeta_j(\mathrm{Aut}^{P_i}(G)) = \mathrm{Aut}^{P_{n-(i-1)j}}(G)$ *for* $2 \leq i \leq ((n+1)/2)$ *and* $1 \leq j \leq ((n-i)/(i-1))$.

PROOF.
(i) We see that $I_i = \langle \sigma_{s_{i-1}}, \ldots, \sigma_{s_{n-2}} \rangle$ and $[I_i, \sigma_{s_1}] \leq I_{i+1}$ for $i \geq 2$. Then Lemma 3.4(ii) implies that $[A_i, I_i] = I_{2i-1}$. Also by Theorem 3.3(ii), we deduce that $\gamma_2(\mathrm{Aut}^{P_i}(G)) = I_{2(i-1)+1}$. Now by using induction on j and Lemma 3.4(ii) the result is proved.
(ii) We have $\gamma_j(\mathrm{Aut}^{P_2}(G)) = I_{j+1}$ by (i). The result is immediate since $I_{n-1} \neq 1$ and $I_n = 1$.
(iii) This is evident from (i).
(iv) This is easily proved by considering (iii) and (i).
(v) It is obvious that $\mathrm{Aut}^{P_{n-(i-1)}}(G) \leq Z(\mathrm{Aut}^{P_i}(G))$ by Theorem 3.3(ii) and 3.4(ii). If $\alpha\sigma_g \in Z(\mathrm{Aut}^{P_i}(G))$ for $\alpha \in A_i$ and $g \in P_{i-1}$, then we may write $g = s_{j-1}^{m_{j-1}} \cdots s_{n-1}^{m_{n-1}}$, where $j \geq i$, $m_{j-1} \neq 0$ and $(m_{j-1}, p) = 1$. Also since $|A_r : A_{r+1}| = p$ for $2 \leq r \leq n-2$, we may write $\alpha = \alpha_r^{m_r} \cdots \alpha_{n-1}^{m_{n-1}}$, where $r \geq i$, $m_r \neq 0$ and $(m_r, p) = 1$. Consequently $[\alpha\sigma_g, \alpha_i] = 1$ so $[\alpha_i, \sigma_g] = 1$. Hence, $j \geq n-(i-1)$ by Lemma 3.4(v). Therefore, $\sigma_g \in Z(\mathrm{Aut}^{P_i}(G))$, which implies that $[\alpha, \sigma_{s_{i-1}}] = 1$. This shows that $r \geq n-(i-1)$ by Lemma 3.4(vi), completing the proof.
(vi) By (v) and Theorem 3.3(iii), $|Z(\mathrm{Aut}^{P_i}(G))| = p^{2i-2}$ for any metabelian p-group G of maximal class and order p^n with $i \leq ((n+1)/2)$. In what follows we prove, by induction on j, that $\zeta_j(\mathrm{Aut}^{P_i}(G)) = \mathrm{Aut}^{P_{n-(i-1)j}}(G)$. Suppose that the equality holds for all positive integers less than j. We set $H = P_{n-(i-1)\,(j-1)}$ and observe that

$$\zeta_j(\mathrm{Aut}^{P_i}(G))/\zeta_{j-1}(\mathrm{Aut}^{P_i}(G)) = Z(\mathrm{Aut}^{P_i}(G)/\mathrm{Aut}^{H}(G)).$$

Also $\mathrm{Aut}^{P_i}(G)/\mathrm{Aut}^{H}(G) \hookrightarrow \mathrm{Aut}^{P_i/H}(G/H)$. Since G/H is a metabelian p-group of maximal class and $P_i(G/H) = P_i(G)/H$, we find that $|Z(\mathrm{Aut}^{P_i/H}(G/H))| = p^{2i-2}$. Then $|\mathrm{Aut}^{P_i/H}(G/H)| = |\mathrm{Aut}^{P_i}(G)/\mathrm{Aut}^{H}(G)|$. Hence $|\zeta_j(\mathrm{Aut}^{P_i}(G))| = p^{2(i-1)j}$ and $\mathrm{Aut}^{P_{n-(i-1)j}}(G) \leq \zeta_j(\mathrm{Aut}^{P_i}(G))$, completing the proof. □

In the rest of this section, we find a necessary and sufficient condition on G for $|\mathrm{Aut}_p(G) : \mathrm{Aut}^{\Phi}(G)| = p$. We also give a structure theorem for $\mathrm{Aut}_p(G)$.

THEOREM 3.6. *Suppose that G is a metabelian p-group of maximal class of order p^n, where $p \geq 3$ and $n \geq 2p-3$. Define the map γ by $s^\gamma = ss_1$, $s_1^\gamma = s_1$ and $s_i^\gamma = [s_{i-1}^\gamma, s^\gamma]$. Then γ extends to an automorphism of G if and only if $s^p = (ss_1)^p$ and $[P_1, P_{p-1}] = 1$.*

PROOF. This is obvious when $(p, n) = (3, 4)$ so for the rest of the proof suppose that $(p, n) \neq (3, 4)$. We first note that if $n \geq 2p-2$, then $s_i^\gamma = s_i[s_i, s_1]^{i-1}[s_{i-1}, s_1]^{i-2}$ $(2 \leq i \leq p-1)$, $s_p^\gamma = s_p[s_{p-1}, s_1]^{p-2}$ and $s_j^\gamma = s_j$ for $j \geq p+1$, by Lemmas 2.12 and 2.4. Now suppose that γ is an automorphism. According to Lemma 2.3(ii), we have $s_1^p s_p \in P_{p+1}$ so that $[s_{p-1}, s_1] = 1$ since $(s_1^p s_p)^\gamma = s_1^p s_p$. Therefore, $[P_1, P_{p-1}] = 1$. On the other hand, $s^p = s_{n-1}^w$ by Theorem 2.15 and hence $(s^p)^\gamma = (s_{n-1}^w)^\gamma$, which implies that $s^p = (ss_1)^p$. Now suppose that $s^p = (ss_1)^p$ and $[P_1, P_{p-1}] = 1$. By using induction on i we may see that $[P_1, P_{p-i}] \leq P_{n-(i-1)}$ $(1 \leq i \leq p-2)$. So by considering the presentation of G given in Theorem 2.15, $k \geq n-p+3$ or equivalently $k \geq p+1$. Finally, we see that γ is an automorphism of G by Lemma 2.12(i), (v). Now if $n = 2p-3$, then clearly $p \geq 5$ and so $s_i^\gamma = s_i[s_i, s_1]^{i-1}[s_{i-1}, s_1]^{i-2}z$, where $z = [[s_2, s_1], s_1]$ and $i \in \{3, 4\}$. The value of γ on s_i for $i \neq 3, 4$ is the same as above. Hence, by the same argument we may conclude the result. □

THEOREM 3.7. *Let G be a metabelian p-group of maximal class of order p^n. Then $|\mathrm{Aut}_p(G) : \mathrm{Aut}^\Phi(G)| = p$ if and only if there exists an automorphism of G that maps s to ss_1 and s_1 to s_1.*

PROOF. If there exists an automorphism of G that maps s to ss_1 and s_1 to s_1, then $|\mathrm{Aut}_p(G) : \mathrm{Aut}^\Phi(G)| = p$. Assume that $|\mathrm{Aut}_p(G) : \mathrm{Aut}^\Phi(G)| = p$, so there exists an automorphism α such that $\alpha \notin \mathrm{Aut}^\Phi(G)$. We have $\alpha \in \mathrm{Aut}_p^{P_1}(G)$ since $\mathrm{Aut}(G)/\mathrm{Aut}^{P_1}(G) \hookrightarrow \mathrm{Aut}(G/P_1)$. Hence we may write $s^\alpha = ss_1^i x$ and $s_1^\alpha = s_1^{j+1} y$, where $0 \leq i, j < p$ and $x, y \in \Phi(G)$. We choose u and w in $\Phi(G)$ such that $u^\alpha = x$ and $w^\alpha = y$. Then by Theorem 3.1 the map β defined by $s^\beta = su^{-1}$, $s_1^\beta = s_1 w^{-1}$ is an automorphism of G lying in $\mathrm{Aut}^\Phi(G)$. On setting $\delta := \beta\alpha$, we have $\delta \in \mathrm{Aut}_p(G) - \mathrm{Aut}^\Phi(G)$ and $s^\delta = ss_1^i$, $s_1^\delta = s_1^{j+1}$. Now by considering the order of s_1 and s_1^δ, we see that $1 \leq j+1 < p$ which implies that $(s)^{\delta^{p-1}} = ss_1^t$, $(s_1)^{\delta^{p-1}} = s_1 x_1$, where $t \neq 0$ and $(t, p) = 1$, since $\delta^{p-1} \notin \mathrm{Aut}^\Phi(G)$ and $x_1 \in \Phi(G)$. Now by the same argument as above we obtain an automorphism τ such that $s^\tau = ss_1^t$, $s_1^\tau = s_1$ and $\tau \in \mathrm{Aut}_p(G) - \mathrm{Aut}^\Phi(G)$. So $(s)^{\tau^m} = ss_1$ and $(s_1)^{\tau^m} = s_1$, where m is a positive integer satisfying $tm \equiv 1 \pmod{|s_1|}$. □

COROLLARY 3.8. *Let G be a metabelian p-group of maximal class of order p^n, where $p \geq 3$ and $n \geq 2p-3$. Then $|\mathrm{Aut}_p(G) : \mathrm{Aut}^\Phi(G)| = p$ if and only if $[P_1, P_{p-1}] = 1$ and $s^p = (ss_1)^p$.*

https://doi.org/10.1017/S0004972708000257 Published online by Cambridge University Press

LEMMA 3.9. *Let G be a metabelian p-group of maximal class of order p^n, where $p \geq 3$ and $n \geq 2p-3$. If there exists an automorphism γ that maps s to ss_1 and s_1 to s_1, then $\gamma^p \in \mathrm{Inn}(G)$ and $\gamma \notin \mathrm{Inn}(G)$.*

PROOF. We have $[P_1, P_{p-1}] = 1$ by Theorem 3.6 so $z = [[s_2, s_1], s_1] = 1$, by Lemma 2.12(i). On setting $g = s_1^{\binom{p}{2}} s_2^{\binom{p}{3}} \cdots s_{p-1}^{\binom{p}{p}}$, we have $[g, s] = s_2^{\binom{p}{2}} [s_2, s_1]^{\binom{m}{2}} s_3^{\binom{p}{3}} \cdots s_p^{\binom{p}{p}}$, where $m = \binom{p}{2}$ since G' is Abelian, and by Lemmas 2.12(iv) and 2.6(iii). Also $[s_2, s_1]^{\binom{m}{2}} = 1$ by Lemma 2.12(v). According to Lemma 2.14 and Theorem 3.6, $s_1^{\binom{p}{1}} \cdots s_p^{\binom{p}{p}} = 1$ and hence $[s, g] = s_1^p$. Moreover, $[g, s_1] = 1$ by Lemmas 2.6(ii), 2.12(v) and Theorem 3.6. Therefore, $\gamma^p = \sigma_g$ and obviously γ is not an inner automorphism. □

THEOREM 3.10. *Let G be a metabelian p-group of maximal class of order p^n, where $p \geq 3$ and $n \geq 2p-3$. If $|\mathrm{Aut}_p(G) : \mathrm{Aut}^{\Phi}(G)| = p$ then $\mathrm{Aut}_p(G) = \mathrm{Aut}^{\Phi}(G) \rtimes C_p$.*

PROOF. By Theorem 3.7, γ is an automorphism; and by Lemma 3.9, $\gamma^p \in \mathrm{Inn}(G)$. Now following the proof of [8, Proposition 4.1], G can be embedded in a p-group H of maximal class of order p^{n+1} and $P_i(H) = P_{i-1}(G)$ for $3 \leq i \leq n$. We now choose $t \in H$ such that $t \notin P_1(H) \cup G$. But H has positive degree of commutativity by [9, Theorem 3.3.5] and so we deduce that $t^p \in P_n(H)$, by Lemmas 2.1 and 2.2. Let α be the restriction of σ_t to G; then α has order p since $Z(G) = P_n(H)$. We have $\alpha \notin \mathrm{Aut}^{\Phi}(G)$, for otherwise $[G, t] \leq \Phi(G) = P_2(G) = P_3(H)$. However, $H = \langle G, t\rangle$ would imply that $[H, t] \leq P_3(H)$ or equivalently $t \in P_2(H)$, which is impossible. Therefore $\mathrm{Aut}_p(G) = \mathrm{Aut}^{\Phi}(G) \rtimes \langle\alpha\rangle$, which yields the proof. □

REMARK. Let G be a p-group of maximal class having order p^n. In [8, Theorem 4.2] Juhász proves that if G can be embedded in a p-group of maximal class, then $|\mathrm{Aut}_p(G) : \mathrm{Aut}^{\Phi}(G)| = p$. We note that the converse of this result is also true when G is metabelian with $n \geq 2p-3$. Consider the map γ defined by $s^\gamma = ss_1$ and $s_1^\gamma = s_1$. Then γ is an automorphism with $\gamma^p \in \mathrm{Inn}(G)$ by Theorem 3.7 and Lemma 3.9. Now since γ satisfies the conditions of [8, Proposition 4.1], G can be embedded in a p-group of maximal class. Notice that our Corollary 3.8 gives another necessary and sufficient condition on G for $|\mathrm{Aut}_p(G) : \mathrm{Aut}^{\Phi}(G)| = p$. Juhász also proves that if $G/P_{p+1}(G)$ cannot be embedded in a p-group of maximal class and G has positive degree of commutativity, then $\mathrm{Aut}_p(G) = \mathrm{Aut}^{\Phi}(G)$. In particular, if G is metabelian, then $|\mathrm{Aut}_p(G)| = p^{2n-4}$. Finally, we note that the above embedding conditions given by Juhász do not cover the whole class of p-groups of maximal class. The following example provides an infinite class of metabelian p-groups of maximal class which do not satisfy these conditions.

https://doi.org/10.1017/S0004972708000257 Published online by Cambridge University Press

EXAMPLE. Suppose that

$$G \cong \langle s, s_1, \ldots, s_{n-1} \mid s_i = [s_{i-1}, s], [s_{n-1}, s] = 1, [s_1, s_2] = 1, [s_i, s_j] = 1$$
$$s^p = 1, s_1^{\binom{p}{1}} \cdots s_p^{\binom{p}{p}} = s_{n-1}, s_i^p s_{i+1}^{\binom{p}{2}} \cdots s_{i+p-1} = 1\rangle,$$

where $2 \leq i \leq j \leq n-1$, $n \geq 2p-3$ and $p \geq 3$. Then G is a p-group of maximal class and order p^n, which cannot be embedded in a p-group of maximal class. However, $G/P_{p+1}(G)$ can be embedded in a p-group of maximal class.

To prove this, we see that any element g of G may be written as $g = s^r s_1^{r_1} \cdots s_{n-1}^{r_{n-1}}$, where $0 \leq r, r_1, \ldots, r_{n-1} < p$. Also $\gamma_i(G) = \langle s_i, \ldots, s_{n-1}\rangle$, $2 \leq i \leq n-1$, implies that $\mathrm{cl}(G) = n-1$ and $|G| = p^n$. By Corollary 3.8, $|\mathrm{Aut}_p(G) : \mathrm{Aut}^{\Phi}(G)| \neq p$ since $s^p \neq (ss_1)^p$. Therefore, by [8, Theorem 4.2], G cannot be embedded in a p-group of maximal class. Now by adding the relations $s_{p+1} = \cdots = s_{n-1} = 1$ to those of G, we find a presentation for $G/P_{p+1}(G)$ in which the relation $s_1^{\binom{p}{1}} \cdots s_p^{\binom{p}{p}} = 1$ holds. On setting $H = G/P_{p+1}(G)$, we see that the map γ defined by $s^{\gamma} = ss_1$ and $s_1^{\gamma} = s_1$ is an automorphism of H fixing s_i, $2 \leq i \leq p$. By considering the presentation of H, it is easily seen that $\gamma^p = \sigma_g$, where $g = s_1^{\binom{p}{2}} \cdots s_{p-1}^{\binom{p}{p}}$. Hence, by [8, Proposition 4.1], H can be embedded in a p-group of maximal class.

4. The order of $\mathrm{Aut}^{\Phi}(G)$

In this section we prove that for a noncyclic p-group G of maximal class of order p^n, $|\mathrm{Aut}^{\Phi}(G)| = p^{2n-4}$ if and only if G is metabelian. This is evident for the case where $p = 2$. Therefore, for the rest of the section we assume that p is an odd prime and $n \geq 4$.

We first give some elementary lemmas.

LEMMA 4.1. *Let G be a p-group of maximal class and order p^n. If $[P_2(G), P_2(G)] \leq Z(G)$, then $[s_i^{-1}, s] = s_{i+1}^{-1}[s_{i+1}, s_i]$ $(i \geq 2)$.*

PROOF. This is immediate by $[s_i s_i^{-1}, s] = 1$. □

LEMMA 4.2. *Let G be a p-group of maximal class and order p^n, $[P_{r+1}, P_{r+1}] = 1$ and $[s_r, s_{r+1}] = z \in Z(G)$ for some $r \geq 2$. Then:*

(i) $[s_r, s_i] = 1$ *for* $i \geq r+2$;

(ii) $[s_{r+1}, s_{r-1}] = [s_{r-1}, s_{r+1}^{-1}]$;

(iii) $[[s_{r-1}, s_r], s] = [s, [s_r, s_{r-1}]]$;

(iv) *if* $[s_{r-1}, s_r] = s_k^{a_k} s_{k+1}^{a_{k+1}} \cdots s_{n-1}^{a_{n-1}}$ *for* $k \geq r+2$ *then* $[s_{r-1}, s_{r+1}] = s_{k+1}^{a_k} \cdots s_{n-1}^{a_{n-2}} z^{-1}$.

PROOF.

(i) We use induction on i $(i \geq r+2)$ and the Witt identity.

(ii) This follows from Lemma 2.11.

https://doi.org/10.1017/S0004972708000257 Published online by Cambridge University Press

(iii) This is easily seen from Lemma 2.11 and the fact that $[s_r, s_{r-1}] \in P_{r+1}$.

(iv) We have $[s_{r-1}, s_{r+1}]s_{r+1}^{-1} = s_{r+1}^{-s_{r-1}} = [s, s_r]^{s_{r-1}} = [s^{s_{r-1}}, s_r^{s_{r-1}}] = [s[s, s_{r-1}], s_r[s_r, s_{r-1}]] = [ss_r^{-1}, s_r[s_r, s_{r-1}]] = [s, s_r[s_r, s_{r-1}]]^{s_r^{-1}}$ by (i) and the fact that $[s_r, s_{r-1}] \in P_{r+2}$. So $[s_{r-1}, s_{r+1}]s_{r+1}^{-1} = [s, [s_r, s_{r-1}]]\,[s, s_r]^{s_r^{-1}}$. Now by (iii), $[s_{r-1}, s_{r+1}] = [[s_{r-1}, s_r], s]z^{-1}$ since $[s_r, s_{r+1}] = z$, which completes the proof. □

THEOREM 4.3. *Let G be a p-group of maximal class of order p^n, where n is a positive integer. Then $|\mathrm{Aut}^{\Phi}(G)| = p^{2n-4}$ if and only if G is metabelian.*

PROOF. If G is metabelian, then by Theorem 3.1, $|\mathrm{Aut}^{\Phi}(G)| = p^{2n-4}$. Now suppose that $|\mathrm{Aut}^{\Phi}(G)| = p^{2n-4}$. By induction on $|G|$ we prove that G is metabelian. If $|G| = p^4$ then $[P_2(G), P_2(G)] \le P_4(G) = 1$ and obviously G is metabelian. Suppose that $|G| \ge p^5$ and the theorem is true for each p-group of order less than $|G|$. We have $|\mathrm{Aut}^{Z}(G)| = p^2$ by [1, Theorem 1] and the fact that G has no nontrivial Abelian direct factor. Also we have

$$\mathrm{Aut}^{\Phi}(G)/\mathrm{Aut}^{Z}(G) \hookrightarrow \mathrm{Aut}^{\Phi(G/Z)}(G/Z).$$

It follows that $|\mathrm{Aut}^{\Phi(G/Z)}(G/Z)| = p^{2(n-1)-4}$. Now the group G/Z is metabelian by the induction hypothesis, so $[P_2(G), P_2(G)] \le Z(G)$. By the way of contradiction suppose that $P_2(G)$ is not Abelian. Let r be the largest positive integer such that $[P_r(G), P_r(G)] \ne 1$, so $[P_{r+1}(G), P_{r+1}(G)] = 1$. We may write $[s_r, s_{r+1}] = z \in Z(G)$ and so, by Lemma 4.2(i), $z \ne 1$. Now the map α defined by $s^{\alpha} = s$ and $s_1^{\alpha} = s_1 s_2^{-1}$ is an automorphism of G since $|\mathrm{Aut}^{\Phi}(G)| = p^{2n-4}$. By Lemma 4.1 and using induction on i, we deduce that $s_i^{\alpha} = s_i s_{i+1}^{-1}[s_{i+1}, s_i]$ $(i \ge 2)$. Therefore, $s_r^{\alpha} = s_r s_{r+1}^{-1} z^{-1}$ and $s_k^{\alpha} = s_k s_{k+1}^{-1}$ $(k \ge r+1)$. Also $[P_r(G), P_{r-1}(G)] \le P_{2r}(G)$ when $2r \le n-1$, since G/Z has positive degree of commutativity. However, $2r \le n-1$ always holds, for otherwise $[P_r(G), P_{r+1}(G)] \le P_{2r+1}(G) = 1$ would imply that $[s_r, s_{r+1}] = 1$, a contradiction. Hence, $[P_r(G), P_{r-1}(G)] \le P_{r+2}(G)$. Therefore, we may write $[s_{r-1}, s_r] = s_k^{a_k} s_{k+1}^{a_{k+1}} \cdots s_{n-1}^{a_{n-1}}$ when $(k \ge r+2)$. Now since α is an automorphism of G, we have $[s_{r-1}^{\alpha}, s_r^{\alpha}] = (s_k^{\alpha})^{a_k} \cdots (s_{n-1}^{\alpha})^{a_{n-1}}$. Moreover, $[s_{r-1}^{\alpha}, s_r^{\alpha}] = [s_{r+1}, s_{r-1}]\,[s_{r-1}, s_r]z$ by Lemma 4.2(i) and (ii). Also

$$(s_k^{\alpha})^{a_k} \cdots (s_{n-1}^{\alpha})^{a_{n-1}} = [s_{r-1}, s_r]([s_{r-1}, s_{r+1}]z)^{-1} = [s_{r-1}, s_r]\,[s_{r+1}, s_{r-1}]z^{-1},$$

by Lemma 4.2(iv), from which we conclude that $z^2 = 1$, which is impossible. □

5. A structure theorem for Aut(G) when $p = 2, 3$

Let G be a 3-group of maximal class of order 3^n. As in Section 3, it is an easy matter to find $\mathrm{Aut}(G)$ when $n \le 3$. Therefore, for the rest of this section we assume that $n \ge 4$. When $n = 4$, G is metabelian; and for $n \ge 5$, G has degree of commutativity $n-4$ by [3, Theorem 3.13] and so is metabelian.

https://doi.org/10.1017/S0004972708000257 Published online by Cambridge University Press

We deduce the following theorem from Blackburn's observation [3, p. 88] which gives us a presentation for G.

THEOREM 5.1. *If G is a 3-group of maximal class of order 3^n, then*

$$G \cong \langle s, s_1, \ldots, s_{n-1} \mid s_i = [s_{i-1}, s], [s_{n-1}, s] = 1, [s_1, s_2] = s_{n-1}^a \\ s^3 = s_{n-1}^b, s_1^3 s_2^3 s_3 = s_{n-1}^c, s_i^3 s_{i+1}^3 s_{i+2} = 1\rangle,$$

where $a, b, c \in \{0, 1, 2\}$ and $2 \le i \le n-1$. For $n > 4$ there exist three groups which possess no Abelian maximal subgroup, given by $c = 0$, $a = 1$ and $b = 0, 1, 2$. If n is even and $n \ge 4$, there exist four groups with an abelian maximal subgroup, given by $a = b = 0$, $c = 1, 2$ or $a = c = 0$, $b = 0, 1$. If n is odd and $n > 4$ then there exist three groups with an Abelian maximal subgroup, given by $a = b = 0$, $c = 1$ or $a = c = 0$, $b = 0, 1$.

By Theorem 3.3 and Corollary 2.10 we obtain the following corollary.

COROLLARY 5.2. *If G is a 3-group of maximal of order 3^n, then $\mathrm{Aut}^{\Phi}(G) = \mathrm{Inn}(G) \rtimes A$, where A is an Abelian subgroup of $\mathrm{Aut}(G)$. Moreover, $A \cong C_{3^m} \times C_{3^m}$ when $n = 2m + 3$ $(m \ge 1)$ and $A \cong C_{3^m} \times C_{3^{m+1}}$ when $n = 2m + 4$ $(m \ge 0)$.*

COROLLARY 5.3. *Let G be a 3-group of maximal class of order 3^n. Then $|\mathrm{Aut}_3(G) : \mathrm{Aut}^{\Phi}(G)| = 3$ if and only if P_1 is Abelian and $(ss_1)^3 = s^3$; in this case $\mathrm{Aut}_3(G) \cong \mathrm{Aut}^{\Phi}(G) \rtimes C_3$.*

PROOF. This follows from Corollary 3.8 and Theorem 3.10. □

Now our aim is to find a structure theorem for $\mathrm{Aut}_2(G)$, the Sylow 2-subgroup of $\mathrm{Aut}(G)$. Since $P_1(G)$ and $P_2(G)$ are characteristic subgroups of G, G/P_2 and P_1/P_2 are invariant under $\mathrm{Aut}_2(G)$. So by Maschke's theorem there exists $s \in G - P_1$ such that $G/P_2 = P_1/P_2 \oplus \langle P_2, s\rangle/P_2$ and $\langle P_2, s\rangle/P_2$ is invariant under $\mathrm{Aut}_2(G)$. In the rest of this section s will be as above. Therefore, if $\alpha \in \mathrm{Aut}_2(G)$ then $s^{\alpha} = s^i x$ and $s_1^{\alpha} = s_1^j y$, where $x, y \in P_2$ and $i, j \in \{1, -1\}$.

The next lemma follows at once from Theorem 5.1.

LEMMA 5.4. *Let G be a 3-group of maximal class of order 3^n. By considering the presentation of G we define the maps β_j, $j \in \{1, 2, 3\}$, by $s^{\beta_1} = s$, $s_1^{\beta_1} = s_1^{-1}$, $s^{\beta_2} = s^{-1}$, $s_1^{\beta_2} = s_1$, $s^{\beta_3} = s^{-1}$, $s_1^{\beta_3} = s_1^{-1}$. Then:*

(i) *β_1 is an automorphism of G if and only if P_1 is Abelian and $s^3 = 1$;*
(ii) *β_2 is an automorphism of G if and only if either n is odd and $s_1^3 s_2^3 s_3 = 1$, or n is even, P_1 is Abelian and $s^3 = 1$;*
(iii) *β_3 is an automorphism of G if n is even.*

LEMMA 5.5. *Let G be a p-group of maximal class of order p^n having positive degree of commutativity. If P_1 is not Abelian then neither is any maximal subgroup of G.*

https://doi.org/10.1017/S0004972708000257 Published online by Cambridge University Press

PROOF. We have $n > 4$ by Theorem 5.1. By way of contradiction, let M be an Abelian maximal subgroup of G. Then $M = \langle \Phi(G), y \rangle$, where $y \in G - P_1$. So $\mathcal{C}_G(y) = \langle y \rangle P_{n-1}$ and $y^p \in P_{n-1}$ by Lemmas 2.2 and 2.1. However, $\Phi(G) \leq \mathcal{C}_G(y)$ implies that $|\mathcal{C}_G(y)| \geq p^3$, which is impossible. □

Now by the fact that $\mathrm{Aut}(G) \cong \mathrm{Aut}_3(G) \rtimes H$, where $H \leq C_2 \times C_2$, we prove the following theorem.

THEOREM 5.6. *Let G be a 3-group of maximal class of order 3^n.*
(i) *If P_1 is not Abelian, then $\mathrm{Aut}_2(G) \cong C_2$.*
(ii) *If P_1 is Abelian and $(ss_1)^3 = s^3$, then $\mathrm{Aut}_2(G) \cong C_2 \times C_2$ when $|s| = 3$ and $\mathrm{Aut}_2(G) \cong C_2$ when $|s| = 9$.*
(iii) *If P_1 is Abelian and $(ss_1)^3 \neq s^3$, then $\mathrm{Aut}_2(G) \cong C_2 \times C_2$ when n is even and $\mathrm{Aut}_2(G) \cong C_2$ when n is odd.*

PROOF. (i) According to Theorem 5.1 and Lemmas 5.4 and 5.5, $\mathrm{Aut}_2(G) = \langle \beta_2 \rangle$ if n is odd and $\mathrm{Aut}_2(G) = \langle \beta_3 \rangle$ if n is even.

(ii) By Lemma 5.4, we have $\mathrm{Aut}_2(G) = \langle \beta_1 \rangle \times \langle \beta_2 \rangle$ when $|s| = 3$. Now if $|s| = 9$, by Lemma 5.4, we have $\mathrm{Aut}_2(G) = \langle \beta_2 \rangle$ or $\mathrm{Aut}_2(G) = \langle \beta_3 \rangle$ depending on the parity of n.

(iii) By considering Theorem 5.1, we see that $b = 0$ and so $\mathrm{Aut}_2(G) = \langle \beta_1 \rangle \times \langle \beta_2 \rangle$ by Lemma 5.4, when n is even. Now if n is odd, by Theorem 5.1 and Lemma 5.4, we have $\mathrm{Aut}_2(G) = \langle \beta_1 \rangle$. □

LEMMA 5.7. *Let G be a 3-group of maximal class of order 3^n. Then every element out of P_1 has order 3 or 9. Furthermore, when P_1 is Abelian, all elements out of P_1 have the same order if and only if $(ss_1)^3 = s^3$.*

PROOF. According to Lemma 2.2, every element out of P_1 has order 3 or 9. We have $(ss_1)^3 = s^3 s_1^3 s_2^3 s_3$ when P_1 is Abelian. If $(ss_1)^3 = s^3$ then $c = 0$ by Theorem 5.1. Also any element out of P_1 has the form $s^t s_1^{t_1} \cdots s_{n-1}^{t_{n-1}}$, where $0 < t < 3$ and $0 \leq t_i < 3$. Therefore by [3, Equation 40] $(s^t s_1^{t_1} \cdots s_{n-1}^{t_{n-1}})^3 = s^{3t}$, completing the proof. Now suppose that all elements out of P_1 have the same order. If $c = 0$ then $(ss_1)^3 = s^3$, and if $c \neq 0$ then $b = 0$ or equivalently $s^3 = 1$ by Theorem 5.1. Hence $(ss_1)^3 = 1$, as desired. □

THEOREM 5.8. *Let G be a 3-group of maximal class of order 3^n. If G has no Abelian maximal subgroup, then $\mathrm{Aut}(G) \cong \mathrm{Aut}^{\Phi}(G) \rtimes C_2$. If G has an Abelian maximal subgroup, then P_1 is Abelian and every element out of P_1 has order 3 or 9.*
(i) *If all elements out of P_1 have order 3 then*

$$\mathrm{Aut}(G) \cong (\mathrm{Aut}^{\Phi}(G) \rtimes C_3) \rtimes (C_2 \times C_2),$$

and if all elements out of P_1 have order 9 then

$$\mathrm{Aut}(G) \cong (\mathrm{Aut}^{\Phi}(G) \rtimes C_3) \rtimes C_2.$$

https://doi.org/10.1017/S0004972708000257 Published online by Cambridge University Press

(ii) *Suppose that elements out of P_1 do not have the same order. If n is even then* $\mathrm{Aut}(G) \cong \mathrm{Aut}^{\Phi}(G) \rtimes (C_2 \times C_2)$*, and if n is odd then*

$$\mathrm{Aut}(G) \cong \mathrm{Aut}^{\Phi}(G) \rtimes C_2.$$

PROOF. This is a straightforward consequence of the lemmas above. □

Now if G is a 2-group of maximal class, we may deduce some parts of the following theorem from the result of Section 3; however, there is also a simple proof as shown below.

THEOREM 5.9. *Let G be a 2-group of maximal class of order 2^n $(n \geq 3)$. If G is the dihedral group of order 2^n or the quaternion group of order 2^n, then* $\mathrm{Aut}^{\Phi}(G) \cong \mathrm{Inn}(G) \rtimes C_{2^{n-3}}$ *and* $\mathrm{Aut}(G) \cong \mathrm{Aut}^{\Phi}(G) \rtimes C_2$*. If G is the semi-dihedral group of order 2^n, then* $\mathrm{Aut}(G) = \mathrm{Aut}^{\Phi}(G) \cong \mathrm{Inn}(G) \rtimes C_{2^{n-3}}$.

PROOF. We know that

$$\begin{aligned}
D_{2^n} &\cong \langle x, y \mid x^{2^{n-1}} = y^2 = (xy)^2 = 1\rangle,\\
Q_{2^n} &\cong \langle x, y \mid x^{2^{n-2}} = y^2,\ y^{-1}xy = x^{-1},\ x^{2^{n-1}} = 1\rangle \quad \text{and}\\
SD_{2^n} &\cong \langle x, y \mid x^{2^{n-1}} = y^2 = 1,\ y^{-1}xy = x^{-1+2^{n-2}}\rangle.
\end{aligned}$$

If G is D_{2^n} or Q_{2^n}, then there are automorphisms α, β, γ and δ defined by $x^{\alpha} = x^{-1}$, $y^{\alpha} = y$, $x^{\beta} = x$, $y^{\beta} = x^2 y$, $x^{\gamma} = x^5$, $y^{\gamma} = y$ and $x^{\delta} = x^{-1}$, $y^{\delta} = x^{-1}y$. It is then easy to check that $\mathrm{Inn}(G) = \langle \alpha, \beta\rangle$, $|\gamma| = 2^{n-3}$, $\delta \notin \mathrm{Aut}^{\Phi}(G)$ and $|\delta| = 2$. If $G = SD_{2^n}$, there are automorphisms α, β and γ defined by $x^{\alpha} = x^{-1+2^{n-2}}$, $y^{\alpha} = y$, $x^{\beta} = x$, $y^{\beta} = x^{-2+2^{n-2}} y$ and $x^{\gamma} = x^5$, $y^{\gamma} = y$. Hence $\mathrm{Inn}(G) = \langle \alpha, \beta\rangle$ and $|\gamma| = 2^{n-3}$ and the rest is clear. □

Acknowledgement

This work was carried out under the supervision of Professor M. F. Newman during our visit to the Australian National University. We are deeply indebted to Professor Newman for setting up the problem, showing us how to find helpful examples, reading and correcting the first version of the manuscript and supervising our work kindly.

References

[1] J. E. Adney and T. Yen, 'Automorphisms of a p-group', *Illinois J. Math.* **9** (1965), 137–143.
[2] A. H. Baartmans and J. J. Woeppel, 'The automorphism group of a p-group of maximal class with an Abelian maximal subgroup', *Fund. Math.* **93**(1) (1976), 41–46.
[3] N. Blackburn, 'On a special class of p-groups', *Acta Math.* **100** (1958), 45–92.
[4] A. Caranti and S. Mattarei, 'Automorphisms of p-groups of maximal class', *Rend. Sem. Mat. Univ. Padova* **115** (2006), 189–198.
[5] A. Caranti and C. M. Scoppola, 'A remark on the orders of p-groups that are automorphism groups', *Boll. Unione Mat. Ital. A* (7) **4**(2) (1990), 201–207.

[6] A. Caranti and C. M. Scoppola, 'Endomorphisms of two-generated metabelian groups that induce the identity modulo the derived subgroup', *Arch. Math.* **56**(3) (1991), 218–227.
[7] B. Huppert, *Endliche gruppen*, Vol. 1 (Springer, Berlin, 1967).
[8] A. Juhász, 'The group of automorphisms of a class of finite p-groups', *Trans. Amer. Math. Soc.* **270**(2) (1982), 469–481.
[9] C. R. Leedham-Green and S. McKay, *The structure of groups of prime power order* (Oxford University Press, Oxford, 2002).
[10] I. Malinowska, 'Finite p-groups with few p-automorphisms', *J. Group Theory* **4** (2001), 395–400.

S. FOULADI, Faculty of Mathematics, University for Teacher Education,
599 Taleghani Avenue, Tehran 15618, Iran
e-mail: s_fouladi@tmu.ac.ir

R. ORFI, Faculty of Mathematics, University for Teacher Education,
599 Taleghani Avenue, Tehran 15618, Iran
e-mail: r_orfi@tmu.ac.ir

https://doi.org/10.1017/S0004972708000257 Published online by Cambridge University Press