INTEGRALS OF MOTION

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Abstract. The properties of conservative dynamical systems of two or more degrees of freedom are reviewed. The transition from integrable to ergodic systems is described. Nonintegrability is due to the interaction of two, or more, resonances. Then one sees, on a surface of section, infinite types of islands of various orders, while the asymptotic curves from unstable invariant points intersect each other along homoclinic and heteroclinic points producing an apparent 'dissolution' of the invariant curves. A threshold energy is defined separating near integrable systems from near ergodic ones. The possibility of real ergodicity for large enough energies is discussed. In the case of many degrees of freedom we also distinguish between integrable, ergodic, and intermediate cases. Among the latter are systems of particles interacting with Lennard-Jones interparticle potential. A threshold energy was derived, which is proportional to the number of particles. Finally some recent results about the general three-body problem are described. One can extend the families of periodic orbits of the restricted problem to the general three-body problem. Many of these orbits are stable. An empirical study of orbits near the stable periodic orbits indicates the existence of 2 integrals of motion besides the energy.

1. Introduction

Ten years have passed since the first international Meeting in Stellar Dynamics, that took place in Thessaloniki, in 1964. During these ten years much progress has been made in several fields of Stellar Dynamics, which is reflected in the program of the present Meeting.

In the area of the Integrals of Motion we have now a fairly complete understanding of the behavior of 2-dimensional conservative dynamical systems, e.g. the orbits of stars in the meridian plane of an axisymmetric galaxy, or in the plane of symmetry of a spiral galaxy.

However we are still in the first steps towards an understanding of systems of many degrees of freedom.

Today I will review shortly this general area, mentioning also some recent developments and applications.

2. Two-Dimensional Systems

The most simple nontrivial two-dimensional system is that of two coupled oscillators. E.g., the motion of a star in the meridian plane of an axisymmetric galaxy can be described by the Hamiltonian

$$H = \tfrac{1}{2}(\dot{x}^2 + \omega_1^2 x^2 + \dot{y}^2 + \omega_2^2 y^2) + \text{higher order terms}, \tag{1}$$

where the y-axis is parallel to the axis of symmetry of the galaxy, while the negative x-axis intersects it perpendicularly.

Hayli (ed.), Dynamics of Stellar Systems, 209–225.

Suppose now that there is another analytic integral

$$\Phi = \Phi(x, y, \dot{x}, \dot{y}) = \text{const}, \tag{2}$$

besides H. Such a system is called 'integrable'.

If we solve Equation (1) for $\dot{y}$ and insert this value in Equation (2) we have the equation of a surface

$$\Phi[x, y, \dot{x}, \dot{y}(x, y, \dot{x})] = \text{const} \tag{3}$$

in the three-dimensional space $(x, \dot{x}, y)$. This surface is a torus, on which lies every orbit whose initial point is on it.

Let us consider orbits with the same value of the Hamiltonian. By varying the constant (3) we find a family of tori, one inside the other. The innermost torus is reduced to a periodic orbit.

Most of the properties of these tori are found if we take their intersections by a 'surface of section', e.g. the plane $y=0$. Each orbit is represented by an 'invariant curve' (Figure 1) on this surface of section, which contains the successive points of intersection $(x, \dot{x})$ of the orbit by the plane $y=0$. A periodic orbit is represented by a finite number of invariant points. In particular the point C in Figure 1 represents the 'central' periodic orbit.

Thus the existence of a second integral of motion implies the existence of invariant curves on the surface of section.

If, on the other hand, we have numerical evidence that the successive points of intersection of many orbits lie on closed smooth curves, this is an indication (but not proof) of the existence of another integral of motion besides the Hamiltonian.

One convenient way to study the Hamiltonian system (1) is by using the actions

$$I_1 = \frac{1}{2\omega_1}(\dot{x}^2 + \omega_1^2 x^2), \qquad I_2 = \frac{1}{2\omega_2}(\dot{y}^2 + \omega_2^2 y^2) \tag{4}$$

of the unperturbed problem (i.e. the quadratic part of the Hamiltonian (1)), and the corresponding angles. Then H is written

$$H = \omega_1 I_1 + \omega_2 I_2 + \sum I_1^{n_1/2} I_2^{n_2/2} \{c_{(m_1 m_2)}^{(n_1 n_2)} \cos(m_1 \vartheta_1 + m_2 \vartheta_2) + \\ + s_{(m_1 m_2)}^{(n_1 n_2)} \sin(m_1 \vartheta_1 + m_2 \vartheta_2)\}, \tag{5}$$

where

$$n_1 + n_2 \geqslant 3, \quad n_1 \geqslant |m_1|, \quad n_2 \geqslant |m_2| \quad \text{and} \quad n_1 - m_1 = \text{even}, \\ n_2 - m_2 = \text{even}.$$

The most simple integrable case is one in which the angles are missing from the Hamiltonian (5); thus

$$H = \omega_1 I_1 + \omega_2 I_2 + f_1(I_1, I_2), \tag{6}$$

where f_1 is of degree three, or larger, in $\sqrt{I_i}$. This is what we call a 'normal form' of a Hamiltonian.

In such a case I_1 and I_2 are integrals of motion.

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For a given value of H each value of I_1 defines a corresponding torus. In particular the value $I_1 = 0$ defines a periodic orbit. The angle along a torus is ϑ_2, while the angle around the torus is ϑ_1. These angles vary linearly in time with frequencies

$$\varpi_1 = \frac{\partial H}{\partial I_1}, \; \varpi_2 = \frac{\partial H}{\partial I_2}, \tag{7}$$

therefore the orbits are quasi-periodic.

The ratio of the frequencies (7) is called the rotation number

$$\mathrm{Rot} = \frac{\varpi_1}{\varpi_2} = \frac{\partial H/\partial I_1}{\partial H/\partial I_2}. \tag{8}$$

This is found empirically as the average angle between the successive intersections of an orbit, as seen from C, in units of 2π. Namely (Figure 1)

$$\mathrm{Rot} = \lim_{n\to\infty} \frac{\widehat{1C2} + \widehat{2C3} + \cdots}{n}. \tag{9}$$

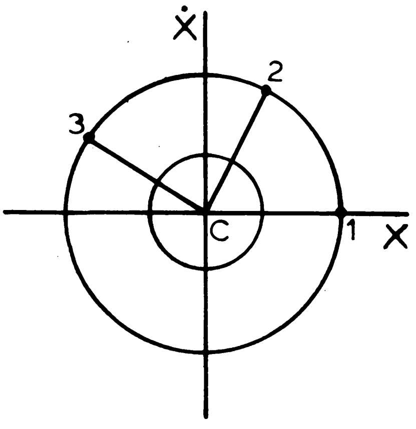

Fig. 1. Invariant curves (schematically) around an invariant point, C, representing a periodic orbit. The successive points of intersection 1, 2, 3,... define empirically a 'rotation number' (see text).

The rotation number is a function of H and I_1. If the value of H is fixed it can be expressed as a function of x, along the x-axis (Figure 2). All orbits on the same torus have the same rotation number.

If the rotation number is rational, say $\frac{2}{3}$, the orbit closes after three revolutions, therefore it is represented by three invariant points on the surface of section. The corresponding torus is filled with periodic orbits of the same type. Therefore all the points of the invariant curve are starting points of periodic orbits.

A more complicated case is a Hamiltonian containing one trigonometric term, besides the terms with I_1, I_2; e.g.

$$H = \omega_1 I_1 + \omega_2 I_2 + f_1(I_1, I_2) + f_{23}(I_1, I_2)\cos(3\vartheta_1 - 2\vartheta_2). \tag{10}$$

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This is also an integrable case. In fact the combination

$$J_{23} = 2I_1 + 3I_2 \tag{11}$$

is an integral of motion. However the actions I_1, I_2 are no more integrals of motion.

In this case we find only two periodic orbits with rotation number $\varpi_1/\varpi_2 = \frac{2}{3}$, one

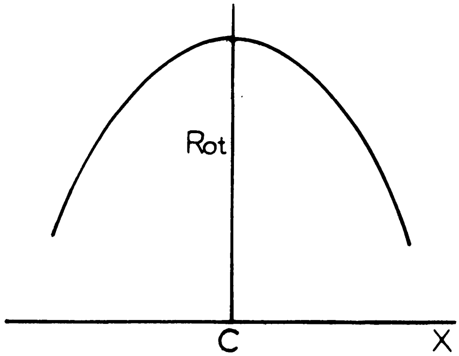

Fig. 2. The rotation number as a function of x (x is the intersection of an invariant curve by the x-axis) (schematically).

stable and one unstable (Figure 3). The stable periodic orbit is represented by three invariant points on the surface of section, which are surrounded by sets of islands. The orbits represented by islands are called tube orbits. In phase space they lie on tori surrounding the stable resonant periodic orbits, i.e. these tori are like tubes closing after three revolutions.

The outermost islands in Figure 3 go through the three invariant points, which

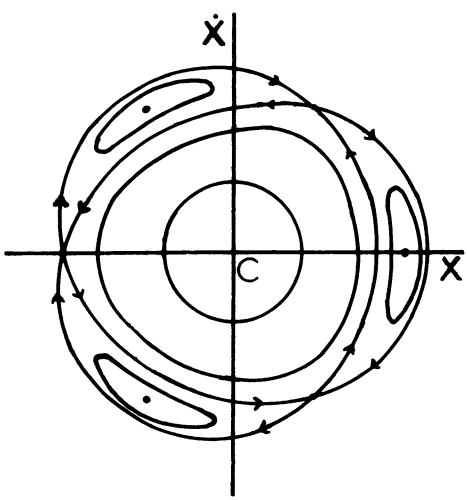

Fig. 3. Regular invariant curves (closing around the central invariant point C) and islands. In the integrable case the outermost islands (separatrices, or asymptotic curves) join the unstable invariant points (schematically).

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represent an unstable periodic orbit. These curves are called 'separatrices' as they separate the sets of islands from the regular invariant curves, which close around the central point C. They are also called 'asymptotic curves', because they represent orbits approaching asymptotically the unstable periodic orbits as $t \to \infty$, or $t \to -\infty$.

The corresponding points of intersection of every asymptotic orbit also approach asymptotically the three unstable invariant points, as indicated by the arrows in Figure 3.

The positions of the stable and unstable invariant points can be found approximately (for small energies) from Equation (8) applied to the 'unperturbed' Hamiltonian (6). We must notice here that Equation (8) may not have (real) solutions for I_1. For a given ratio of the unperturbed frequencies ω_1/ω_2 (different from $\frac{2}{3}$) we do not have real solutions for small enough H.

If we have a similar Hamiltonian

$$H = \omega_1 I_1 + \omega_2 I_2 + f_1(I_1, I_2) + f_{25}(I_1, I_2) \cos(5\vartheta_1 - 2\vartheta_2), \tag{12}$$

this is also integrable, but the integral is now

$$J_{25} = 2I_1 + 5I_2, \tag{13}$$

therefore quite different from the above integral (11). In the present case $\mathrm{Rot} = \frac{2}{5}$, therefore we have five sets of islands and the corresponding asymptotic curves.

Now let us introduce a more general Hamiltonian with two, or more, trigonometric terms of different type, e.g.

$$\begin{aligned} H = \omega_1 I_1 + \omega_2 I_2 + f_1(I_1, I_2) + f_{23}(I_1, I_2) \cos(3\vartheta_1 - 2\vartheta_2) + \\ + f_{25}(I_1, I_2) \cos(5\vartheta_1 - 2\vartheta_2). \end{aligned} \tag{14}$$

This Hamiltonian has some common characteristics with both Hamiltonians (10) and (12).

If ω_1/ω_2 is different from $\frac{2}{3}$ and $\frac{2}{5}$, for small enough energies no resonant periodic orbits appear at all. For larger energies both types of periodic orbits appear (3-periodic and 5-periodic), one type first and the other later (for somewhat larger H) depending on the value of ω_1/ω_2. The corresponding sets of islands are then well separated by 'regular' invariant curves (Figure 4). These sets of islands are approximately described by the Hamiltonians (10) and (12) respectively.

However, for even larger energies, the islands of the Hamiltonians (10) and (12) overlap. Then some invariant curves of the Hamiltonian (14) are not closed any more, but form very complicated patterns. The successive points of intersection seem scattered at random (Figure 5). In such cases we speak about 'dissolution' of invariant curves.

This 'interaction of resonances' appears even for energies much smaller than that needed for overlapping of the original resonances. Its effects are the following:

(a) Whenever the rotation number of the unperturbed Hamiltonian (6) is rational there is no more an infinity of periodic orbits, but only two periodic orbits, one stable and one unstable. The stable orbit is represented by a number of invariant points surrounded by islands (see, e.g., the 7 islands of Figure 4).

(b) We can define a new rotation number around each stable invariant point. When this new rotation number is rational we have second order islands around the main islands (Figure 4). In the same way we have third order islands, etc.

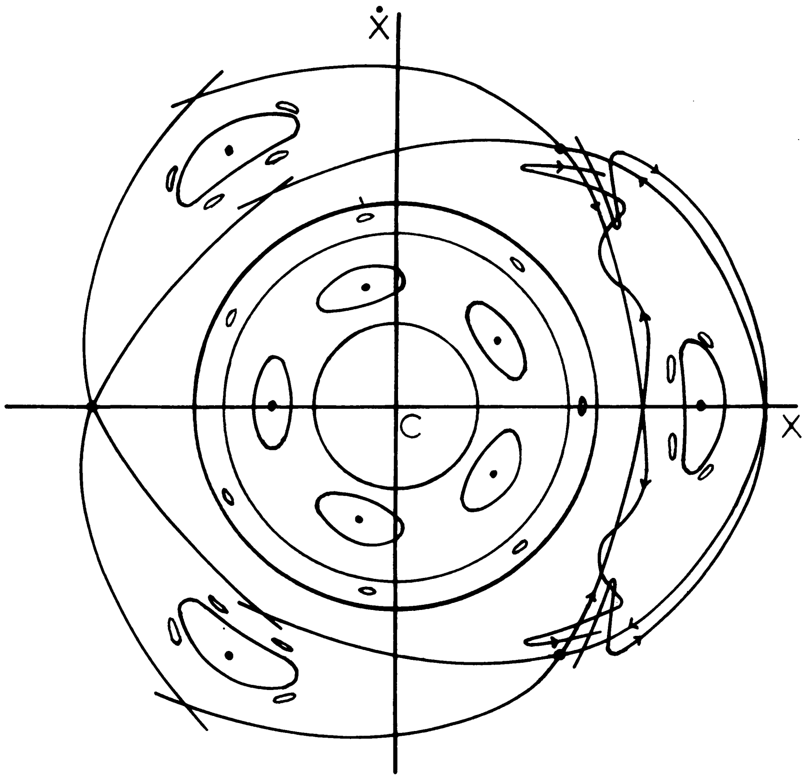

Fig. 4. Interaction of two resonances (schematically). Although the sets of 3 and 5 islands are separated by invariant curves closing around the central invariant point C, secondary islands are formed around the origin (e.g. 7 islands), and around the 3 main islands (e.g. 4 islands). Furthermore the asymptotic curves intersect each other an infinite number of times at homoclinic points.

(c) The asymptotic curves emanating from the invariant points of the same unstable periodic orbit do not join, any more, the successive invariant points, but intersect each other along an infinity of points, which are called 'homoclinic points' (Figure 4) (Poincaré, 1899). Such points generate doubly asymptotic orbits, approaching the same periodic orbit asymptotically, as $t \to \infty$, and $t \to -\infty$.

(d) At the same time the asymptotic curves from one invariant point intersect the asymptotic curves from invariant points of different multiplicities. Such points of intersection are called 'heteroclinic points' (Poincaré, 1899). The corresponding orbits are asymptotic to two different periodic orbits as $t \to \infty$ and $t \to -\infty$.

Heteroclinic points in our Galaxy were found by Martinet (1974).

An indication that the appearance of heteroclinic points is quite general is the fact that the rotation curves calculated empirically show discontinuities near the unstable periodic orbits (Figure 6) (Contopoulos, 1967). The rotation number changes abruptly in a small region near an unstable point and at the boundaries of the regions of islands. In these regions there is an infinity of resonances, which interact with each other.

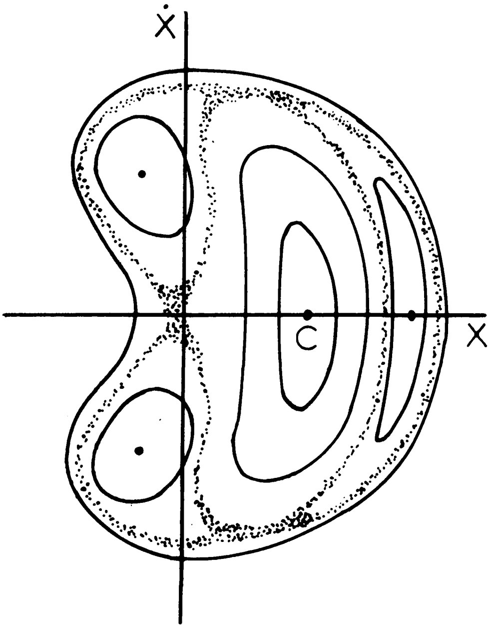

Fig. 5. The 'dissolution' of invariant curves appears first near the unstable invariant points, where the asymptotic curves form an intricate net. For larger energies the region of dissolution increases.

A proof that an infinity of heteroclinic orbits follows the appearance of homoclinic points was given recently by Moser (1973).

The above four effects are the basic characteristics of nonintegrability. The most conspicuous, however, is the fourth characteristic, which shows clearly the interaction of resonances. It is obvious that in any region that we have interaction of resonances we cannot define closed invariant curves.

On the other hand even in nonintegrable systems there are closed invariant curves. Moser (1962) and Arnold (1963) proved rigorously the existence of invariant surfaces in phase space for small enough energies (or, what is equivalent, small enough perturbations). Their intersections by a surface of section are closed invariant curves around the stable invariant points. If the energy is small enough, the set of regular invariant curves has almost the totality of measure, although between any two invariant curves there are regions containing islands and nets of intersecting asymptotic curves.

In practical applications we find empirically that good invariant curves exist in most problems of actual interest, even for large enough energies.

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Whenever the Hamiltonian is given in the form (1), or (5), the higher order terms are small for small energies. Thus the resonance effects are small, except if the ratio ω_1/ω_2 is exactly rational. One can perform canonical transformations of variables that bring the Hamiltonian in a normal form (Whittaker, 1904; Birkhoff, 1927) i.e. H is expressed

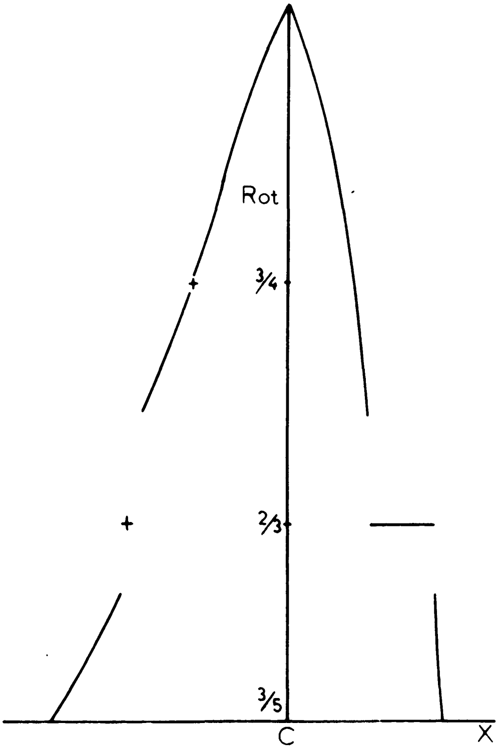

Fig. 6. The discontinuities of the rotation curve near the unstable invariant points (+) indicate that we have interaction of several neighboring resonances. The corresponding asymptotic curves intersect each other along heteroclinic points.

as a function of the new actions $\tilde{I}_1$, $\tilde{I}_2$ only. Thus we have an integral of the form

$$\tilde{I}_1 = I_1 + \text{higher order terms}, \tag{15}$$

besides the Hamiltonian, where the higher order terms are series in the original variables. A similar integral was developed in Stellar Dynamics, where it is known as a 'third' integral (Contopoulos, 1960).

The series (15) in general diverges; however, it represents asymptotically the regular invariant surfaces, whose existence was proved by Moser and Arnold.

On the other hand such a series is not applicable near resonances, because it contains an infinity of terms with small divisors of the form $(m\omega_1 - n\omega_2)$. These small divisors are responsible for the appearance of islands on the surface of section.

Near each resonance one can construct a different form of the 'third' integral. E.g., if the rotation number is near $\frac{2}{3}$, we can find an integral

$$J_{23} = 2J_1 + 3J_2 + \text{higher order terms}. \tag{16}$$

Such a form is applicable also away from all resonances. Near a different resonance a different integral is needed. A computer program, developed a few years ago (Contopoulos, 1966), calculates the algebraic form of the third integral near every resonance of interest, or away from all resonances, for any Hamiltonian of the form (1), or (5).

If the Hamiltonian contains more than one type of resonant terms (i.e. terms with more than one value of n/m) the formal integral (15) contains an infinity of small divisor terms, that do not appear in the Hamiltonian (5). This explains why in Figure 4 we see 7 islands (and, in fact, one finds infinite types of resonant islands) although the Hamiltonian contains only terms of multiplicity 3 and 5.

The resonance effects become larger as the value of the Hamiltonian increases. It was found (Contopoulos, 1967) that if the ratio ω_1/ω_2 is near the integer n/m (i.e., if the difference $|(\omega_1/\omega_2)-(n/m)|$ is smaller than a quantity of $O(H)$ we find m islands. The area covered by these islands is of the order of $H^{(n+m-4)/4}$. At the same time Equation (8) gives approximately the positions of the various islands. Therefore the theory of the 'third' integral allows the calculation of the value of H for which two nearby resonances interact, and produce a 'dissolution' of invariant curves.

This interaction, and the corresponding dissolution, become much stronger for larger H. This is more marked if the interacting resonances correspond to relatively large m and n. Therefore there is a kind of threshold energy, H_d, above which the dissolution is very conspicuous. In specific problems we can plot the proportion of the

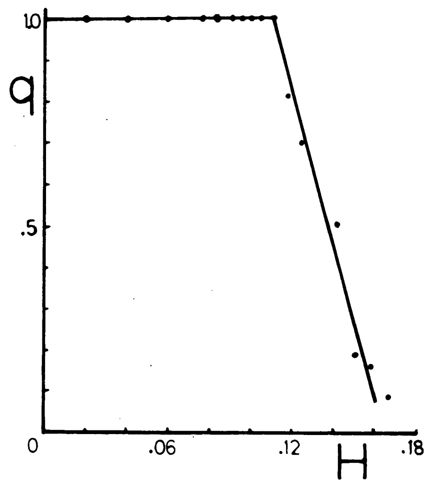

Fig. 7. The proportion, q, of the space covered by good invariant curves vs. the energy, H, in a case studied by Hénon and Heiles (1964). The transition from near integrability to near ergodicity is rather abrupt.

surface of section covered with good invariant curves versus the energy. Thus we find a curve like the one of Figure 7. We see that the transition from cases with very small dissolution to cases where the dissolution is practically complete is rather abrupt.

Thus we can state the following. For energies H much smaller than the threshold energy, H_d, the system behaves like an integrable one. In such a case we have practically only regular invariant curves (closing around the central invariant point) and, possibly, one main set of islands, corresponding to one particular resonant term in the Hamiltonian, if ω_1/ω_2 is near a rational number n/m.

The islands corresponding to other resonances are very small and in many problems they may be disregarded. In fact ω_1/ω_2 can be approximated by an infinity of rationals n/m, but for most of them n and m are large, therefore the corresponding islands are extremely small.

On the other hand for energies much larger than H_d there are practically no closed invariant curves at all and the system is almost ergodic. This means that most orbits approach arbitrarily closely the greatest majority of points on the surface of constant energy. The points of intersection are scattered in a random way on the surface of section.

The remaining problem is whether real ergodicity is ever reached, or whether there remain always small regions covered with closed invariant curves.

In particular we can check if all periodic orbits are unstable, because otherwise the system cannot be ergodic.

The study of highly perturbed dynamical systems (for large H) indicates the following (Contopoulos, 1970). If we calculate the stability index (the trace of the monodromy matrix) of various families of periodic orbits as a function of H we find curves like those of Figure 8. Every particular family becomes eventually unstable,

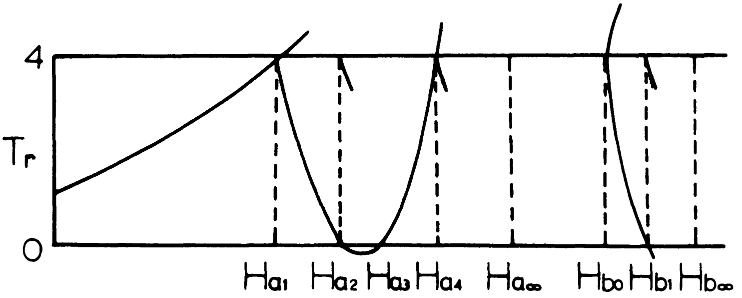

Fig. 8. Stability diagram (trace vs. energy) of some families of periodic orbits (schematically). If the trace is between 0 and 4 the orbits are stable, otherwise unstable. The orbits of the same group of families (a, or b) can be transformed to each other continuously by varying the energy H. However two different groups are quite independent.

but a new stable family starts whenever the characteristic curve crosses the boundary of the stable region at an angle (not perpendicularly), e.g. at the points $H_{a1}, H_{a2}, \ldots$. However the range of values of energy for which these new families remain stable is smaller and smaller. Thus it seems that the total range of stability of the group of families generated from the original one (the one that appears also for $H=0$) is finite. Beyond the limit $H_{a\infty}$ all the families of this type become unstable.

However there are new families independent of the above, which appear only beyond a certain minimum value of $H=H_{b0}$. The corresponding characteristic curve intersects the boundary of the stability region in Figure 8 perpendicularly, generating two families of periodic orbits, one stable and one unstable. The stable family becomes again unstable at H_{b1}, generating a new stable family, etc.

There is again a limit, $H_{b\infty}$, beyond which all the families of this type are unstable. The situation reminds the spectrum of a star, with groups of lines, like the Lyman and Balmer series.

The question remains open whether there is an upper limit of the values $H_{a\infty}, H_{b\infty}, \ldots$ beyond which all periodic orbits are unstable. In such a case systems with large enough energy would be probably genuinely ergodic. However, for large enough H usually most orbits are escaping.

If, on the other hand, there exist always stable families, for all H, there are always small islands of stability and there is never real ergodicity. In such a case the regions of stability would be extremely small for large enough H, therefore we may speak of 'practical ergodicity' in the sense that every orbit can approach any point on the energy surface to a distance $\leqslant \delta$, where δ is a small, but not arbitrarily small number.

There are further problems of interest concerning systems that are ergodic or 'practically ergodic'. One question is how long does it take for an initial distribution to get randomized. It is known that orbits starting along an asymptotic curve from an unstable periodic orbit deviate exponentially, like $e^{\lambda t}$. However the parameter λ differs from one orbit to the other. In many cases we define an average value $\bar{\lambda}$ empirically, by calculating several orbits. However a similar system with different energy may have a smaller or large $\bar{\lambda}$, therefore it may reach randomness slower or faster than the first, although both systems are ergodic.

These examples indicate that we have to be particularly careful in applying the methods of statistical mechanics in particular systems. Because (a) the system may be integrable or approximately integrable, in which case the assumption of equal a priori probability has to be applied to a space of fewer dimensions than the energy surface, and (b) even if the system is ergodic, or very nearly ergodic, the time needed for a particular quantity to get randomized may be long, or it may depend on the initial distribution.

Such problems are of even greater interest in cases of many degrees of freedom.

Further reviews of the problems connected with the transition from integrable to ergodic systems are given by Walker and Ford (1969), Chirikov (1971), Galgani and Scotti (1972), Ford (1973), and Contopoulos (1973).

3. Many Degrees of Freedom

In the case of many degrees of freedom we can also divide the systems into integrable, ergodic, and intermediate.

3.1. Integrable systems

A trivial example is a set of n uncoupled oscillators. A nontrivial case is the Toda lattice. This is a set of particles, x_i, on a string, attracted by its neighbours according to the law

$$f_i = \exp[-(x_i - x_{i-1})].$$

Therefore we have

$$\ddot{x}_i = f_i - f_{i+1} = \exp[-(x_i - x_{i-1})] - \exp[-(x_{i+1} - x_i)].$$

This example is extremely interesting because it seems of a quite general type. One might think that such a force law is so different from the linear law that characterizes the uncoupled oscillators, that one should expect ergodicity. However, Hénon (1974a) proved rigorously that this system is integrable.

3.2. Ergodic systems

Sinai (1970) proved rigorously that the hard sphere gas is ergodic. In this case the orbits are straight lines until a collision occurs, when we have perfect reflection. This result of Sinai is particularly important and it provides a rigorous foundation of statistical mechanics, at least for the hard sphere gas. It looks probable that it is also valid in cases of purely repulsive forces. However it is doubtful whether it applies in cases where we have attractive forces, or if the forces are attractive for large distances and repulsive for small distances.

3.3. Intermediate

A well known intermediate case is the celebrated Fermi-Pasta-Ulam problem (1955). This is the case of N particles on a string attracted with non linear forces

$$f_i = -k^2(x_i - x_{i-1}) - \alpha(x_i - x_{i-1})^2.$$

This problem can be reduced, by a linear transformation of variables, to a set of N coupled oscillators. Fermi, Pasta and Ulam expected that, if one mode is excited initially, the energy would be shared by all modes because of the non-linearity. However, it was found, by numerical experiments, that only a few modes shared part of the energy. The exchange of energy between the various modes took place in an almost periodic way. Therefore the system was definitely not ergodic.

A great literature has developed following this unexpected result. Among other developments I should mention the representation of a continuous string (i.e. the case when the number N tends to infinity) by a Kortewert-de Vries equation by Kruskal and Zabusky, which led to the idea of solitons (Zabusky, 1967). One important result in this area was the discovery of an infinity of conserved quantities for this problem (Miura, Gardner and Kruskal 1968). Recently Zakharov and Faddeev (1972) proved that the Kortewert-de Vries equation can be considered as an integrable problem of infinite degrees of freedom and this explains the conservation laws of Kruskal and his associates.

Many numerical experiments were made in the case of N finite with different laws of attraction, and different energies. In general, it was found that there is a threshold of energy, above which the behavior of the system is ergodic.

In particular many numerical experiments were made with a Lennard-Jones inter-particle potential

$$V_{i,\,i-1} = 4\varepsilon\left[\left(\frac{1}{x}\right)^{12} - \left(\frac{1}{x}\right)^{6}\right],$$

where $x = x_i - x_{i-1}$, and ε is the depth of the potential well, i.e. the escape energy from the minimum of potential.

The force is attractive for large distances, while it is repulsive for small enough distances. This potential is assumed to govern the interaction of molecules in a fluid or in a solid, therefore it is of particular interest in physics. Stoddard and Ford (1973) studied this problem numerically in cases of rather large energies and found practically complete stochasticity. However, Galgani and Scotti (1972) found that if the energy is small enough there is no stochasticity at all, but the problem is governed by N integrals of motion.

Their study consists of two parts. First they calculated formal integrals of motion for systems of N-degrees of freedom and checked numerically how well they are conserved if they are truncated after the terms of a given degree. For this purpose they extended to many degrees of freedom the method of Contopoulos (1966) to calculate the 'third integral' in a two-dimensional system, by performing algebraic manipulations with the help of a computer. Their first results derived for systems of 4 degrees of freedom are very encouraging. In fact for small enough energies the N integrals are conserved better and better as higher and higher order terms are added in their expressions. On the other hand for large energies the numerical values of the truncated formulas vary considerably, and do not improve by adding higher order terms. This indicates that the system is then ergodic rather than integrable.

The second method is purely numerical. They find the deviations of two systems which are very close to each other initially. This deviation is linear in time if the energy is small enough, while it is exponential if we go beyond a certain threshold. The transition is rather abrupt, and allows to define a threshold energy, H_d.

The most important result of Galgani and Scotti is the dependence of the threshold energy on the number of degrees of freedom, N. The function H_d/N runs as shown in Figure 9.

For $N = 2$ it seems that H_d is infinite. The system behaves as integrable, even for large energies. For $N = 4$ we have $H_d/N \simeq 4$ but as N increases H_d/N tends to 1 in the particular units used. It is remarkable that the value H_d/N remains near 1 even if N becomes as larger as 500.

If this result is valid also when $N \to \infty$ (and, in my opinion, the fact that $H_d/N \simeq 1$ from $N = 8$ to $N = 500$ is a very good indication for that) then we can derive many interesting conclusions. In particular one can define a zero-point energy, which is usually considered as a pure quantum-mechanical effect. This allows one to derive a classical analogue of Planck's law, as it was first done by Einstein and Stern (1913). Galgani has drawn some far reaching, but very interesting conclusions concerning a classical foundation of quantum mechanics from these studies.

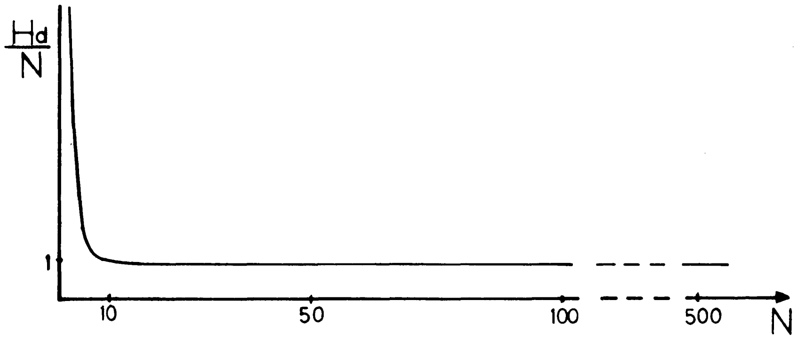

Fig. 9. The ratio H_d/N (transition energy divided by the number of particles) as a function N for a Lennard-Jones interparticle potential (after Galgani and Scotti).

Another problem of more than two degrees of freedom, where new important results have been found quite recently, is the general three body problem. I will report here on some recent work by Hadjidemetriou in Thessaloniki. Hadjidemetriou uses a rotating frame of reference, whose origin is the center of mass of the bodies m_1 and m_2, while the x-axis contains always these two bodies; thus the planar three-body problem is reduced to a system of three degrees of freedom, with coordinates x_1, x_3, y_3. Then it is proved that all families of periodic orbits of the restricted three-body problem can be extended to the general three-body problem.

The theoretical proof was implemented by numerical calculations for increasing values of the mass of the third body. Eventually families of periodic orbits were found for three equal masses. A study of the stability of these orbits indicates that there may be some rather open triple systems, which are stable. Similar results were found by Hénon (1974b).

Further, Hadjidemetriou studied the behavior of non periodic orbits in the vicinity of stable periodic orbits. At every intersection of an orbit by the plane $y_3=0$ the values of $x_1, \dot{x}_1, x_3, \dot{x}_3, \dot{y}_3$ were calculated. If there are two more integrals of motion besides the energy, one can eliminate two of these quantities and find the equation of an invariant surface in a reduced three-dimensional space, say $(x_1, \dot{x}_1, x_3)$. This corresponds to an invariant curve on a surface of section in the 2-dimensional case.

The successive points on the invariant surface can be joined by a continuous curve. The projection of such a curve on the plane $(x_1, \dot{x}_1)$ is given in Figure 10. One can then define three mean periods. The first, P_1, is the period between two successive intersections with the plane $y_3=0$. The second, $P_2 \sim 50\, P_1$, is the period of one loop in Figure 10, while the third, P_3, of the order of some $100\, P_2$, is the period needed for the loops to complete a libration, or rotation. It is remarkable that the 3 periods above are of quite different orders of magnitude.

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The loops of Figure 10 were drawn empirically by joining the projections of the successive points of intersection of an orbit by the surface $y_3 = 0$. The fact that these loops are smooth curves indicates the existence of two integrals of motion besides the energy in the general three body problem.

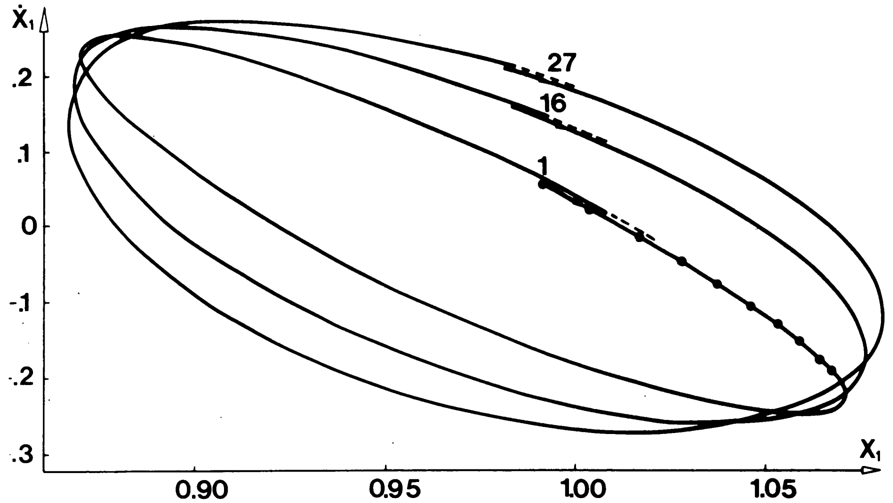

Fig. 10. Projection on the plane $(x_1, \dot{x}_1)$ of the curve joining the successive intersections of an orbit by the surface $y_3 = 0$. Only the 1st, 16th and 27th loops are marked, and the 10 first points on the 1st loop (after Hadjidemetriou).

In one particular case it was found that the curve on the plane $(x_1, \dot{x}_1)$ is closed, i.e. successive loops coincide. This phenomenon is not well understood. It means either that the libration period, P_3, is extremely large, or that there are four integrals of motion in this special problem, which is rather improbable.

The above phenomena appear whenever the third body is far from the two other bodies. If the distance of the third body is small the successive points of intersection are no more on smooth loops. However in many cases the orbits do not escape to infinity, although the energy integral does not provide such a restriction. On the other hand orbits far away from periodic orbits eventually escape.

The fact that no escape appears for a large set of orbits and for long times indicates that no Arnold diffusion (Arnold and Avez, 1967) is operative in these systems, or that its time scale is very large.

An extension of this study to systems of more than three bodies has already started in Thessaloniki. This research opens important new possibilities for the general N-body problem. By studying the stable families of periodic orbits one will be able to separate the regions of stability in each case. In such regions one has to take into account the effects of 3 N integrals of motion, instead of the 10 classical ones. This will allow a better statistical treatment of N-body systems.

I cannot make many predictions about the future developments in these very interesting areas of many degrees of freedom. But I can be sure of one thing. That they will keep us busy for the whole next decade.

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DISCUSSION

Miller: In these solutions of the three-body problem, if you somehow bound the relevant part of the phase space, what fraction of that part is filled with these 'non-ergodic' orbits?

Contopoulos: It is still too early to give numerical results. My impression is that the set of bounded orbits is small. However as this set is not infinitesimal one cannot be sure what will be the final evolution of a particular triple system by taking initial conditions at random. This is why Dr Hadjidemetriou started a systematic exploration of the regions of stability.

Lecar: Is there an analog to the triangular points in the problem with three equal masses?

Contopoulos: Yes. Motions near the triangular equilibrium points in the planar general three body problem have been studied already (see e.g., Siegel, C. L. and Moser, J. K.: 1971, *Lectures on Celestial Mechanics*, Springer, Berlin, p. 113).

Lynden-Bell: With such widely different periods involved as 1, 50 and 500 are not the apparent integrals directly related to the adiabatic invariants?

Contopoulos: Yes. I believe that the new integrals are, in fact, to be considered as adiabatic invariants.

Froeschle: I ask further explanations about the appearance of 'wild behaviour' in the model used by Galgani and Scotti.

Scotti: The threshold turns out to be one only because of the particular units employed.

Galgani: I think the question of Dr Froeschlé will be answered by the following clarification. The figure discussed here refers to a class of computations with a fixed kind of initial conditions, for example equipartition of energy among the normal modes and phases zero. Then the unique remaining parameter is the value of the energy, which just appears in Figure 7 as abscissa (what is reported in ordinate has already been said).

I would like to add now some considerations. Obviously a graph as the one referred to above should be given for all classes of initial conditions, which is a formidable problem, actually that of exploring a surface of $2N-1$ dimensions if N is the number of particles. In this connection I can only say that many classes of initial conditions have been considered and in all of them the threshold was well defined and did not apparently vary from case to case. This, I believe, gives meaning and support to the statement that there is something as a threshold energy which is proportional to N. Naturally we expect that the situation be much more complicated and even we attach great importance, from the point of view of principle, to the circumstance, that the statement given above should not be taken literally. Indeed the presence of nonstochastic regions on every energy surface is of fundamental importance in determining the statistics, i.e. the dynamically correct invariant measure, on the stochastic region. But this, I hope, is work for the near future, while the rough statement given above was directed to disprove the often expressed opinion that the Kolmogorov-Arnold-Moser theorem guaranteeing the existence of invariant tori, or, as we say, of ordered motions, should be of no interest for physical systems of many degrees of freedom.