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CHARACTERIZATIONS OF STRICTLY SINGULAR AND STRICTLY COSINGULAR OPERATORS BY PERTURBATION CLASSES

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Abstract. We consider a class of operators that contains the strictly singular operators $\mathcal{SS}$ and it is contained in the perturbation class of the upper semi-Fredholm operators $P\Phi_+$. We show that this class is strictly contained in $P\Phi_+$, solving a question of Friedman. We obtain similar results for the strictly cosingular operators $\mathcal{SC}$ and the perturbation class of the lower semi-Fredholm operators $P\Phi_-$. We also characterize $\mathcal{SS}$ in terms of $P\Phi_+$ and $\mathcal{SC}$ in terms of $P\Phi_-$. As a consequence, we show that $\mathcal{SS}$ and $\mathcal{SC}$ are the biggest operator ideals contained in $P\Phi_+$ and $P\Phi_-$, respectively.

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1. Introduction. The strictly singular operators $\mathcal{SS}$ were introduced by Kato [**14**] and he showed that they are contained in the perturbation class of the upper semi-Fredholm operators $P\Phi_+$. It has been a long-standing open problem whether these two classes coincide, until a negative solution was given in [**9**]. In [**7**], Friedman considered the following condition (C) for an operator $K \in \mathcal{L}(X, Y)$:

$$\text{(C)} \quad S \in \mathcal{L}(X, Y), \dim(X^*/(R(K^*) + R(S^*))) < \infty \Rightarrow \dim(X^*/R(S^*)) < \infty,$$

where K^* is the conjugate operator of K. She showed that

$$K \in \mathcal{SS}(X, Y) \Rightarrow\ K \text{ satisfies (C) } \Rightarrow K \in P\Phi_+(X, Y),$$

proposing as a question whether condition (C) characterizes the operators in $P\Phi_+(X, Y)$. Moreover, she characterized the strictly singular operators in terms of $P\Phi_+$.

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Here, we give a negative answer to Friedman's question and two refinements of her characterization of $\mathcal{SS}$ in terms of $P\Phi_+$. As a consequence of the refinement, we show that $\mathcal{SS}$ is the biggest operator ideal contained in $P\Phi_+$.

The strictly cosingular operators $\mathcal{SC}$ were introduced by Pełczyński [**15**], and Vladimirskii [**19**] showed that they are contained in the perturbation class of the lower semi-Fredholm operators $P\Phi_-$. We consider the following condition (D) for an operator $K \in \mathcal{L}(X, Y)$:

$$\text{(D)} \quad S \in \mathcal{L}(X, Y), \dim(Y/(R(K) + R(S))) < \infty \Rightarrow \dim(Y/R(S)) < \infty.$$

We prove that $K \in \mathcal{SC}(X, Y) \Rightarrow K$ satisfies (D) $\Rightarrow K \in P\Phi_-(X, Y)$, and show that condition (D) does not characterize the operators in $P\Phi_-(X, Y)$. Moreover, we give two characterizations of the strictly cosingular operators in terms of $P\Phi_-$ and, as a consequence, we show that $\mathcal{SC}$ is the biggest operator ideal contained in $P\Phi_-$.

The questions whether conditions (C) and (D) characterize $\mathcal{SS}$ and $\mathcal{SC}$, respectively, remain open. We observe that to obtain negative answers we would need new counterexamples to the perturbation classes problem. Indeed, $\mathcal{SS}(X, Y) = P\Phi_+(X, Y)$ implies that condition (C) characterizes $\mathcal{SS}$ and $P\Phi_+$ for operators in $\mathcal{L}(X, Y)$, and the same happens with $\mathcal{SC}$, $P\Phi_-$ and condition (D). We refer to [**9**] and [**8**] for two counterexamples and to [**3**, **8**, **11**, **12**] for recent positive answers to the perturbation classes problem.

For X, Y Banach spaces, we denote by $\mathcal{L}(X, Y)$ the set of (continuous linear) operators from X into Y, and for $T \in \mathcal{L}(X, Y)$, $T^* \in \mathcal{L}(Y^*, X^*)$ is the conjugate operator of T. An operator $T \in \mathcal{L}(X, Y)$ is *upper semi-Fredholm* if its kernel $N(T)$ is finite dimensional and its range $R(T)$ is closed; and T is *lower semi-Fredholm* if its range is finite codimensional; hence, closed by [**18**, Theorem IV.5.10]. We denote respectively by Φ_+ and Φ_- the classes of all upper semi-Fredholm and lower semi-Fredholm operators. It follows from the basic duality relations for operators that $T \in \Phi_+$ if and only if $T^* \in \Phi_-$ and $T \in \Phi_-$ if and only if $T^* \in \Phi_+$. The class of *Fredholm operators* is $\Phi := \Phi_+ \cap \Phi_-$. Given a class $\mathcal{A}$ of operators, we denote

$$\mathcal{A}(X, Y) := \{T \in \mathcal{L}(X, Y) : T \in \mathcal{A}\}$$

and we write $\mathcal{A}(X)$ in the case $X = Y$.

Given a closed subspace M of X, let us denote by J_M the inclusion of M into X, and by Q_M the quotient map from X onto X/M. Recall that an operator $T \in \mathcal{L}(X, Y)$ is said to be *strictly singular* if a restriction TJ_E to a closed subspace E is an isomorphism only if E is finite dimensional; T is said to be *strictly cosingular* if for every closed subspace F of Y the composition $Q_F T$ is surjective only if F is finite codimensional; and T is said to be *inessential* if $I_X - ST \in \Phi(X)$ for every $S \in \mathcal{L}(Y, X)$. We denote respectively by $\mathcal{SS}$, $\mathcal{SC}$ and $\mathcal{I}n$ the classes of strictly singular, strictly cosingular and inessential operators. It is easy to show that $T^* \in \mathcal{SS} \Rightarrow T \in \mathcal{SC}$ and $T^* \in \mathcal{SC} \Rightarrow T \in \mathcal{SS}$, but the converse implications do not hold [**16**, Examples 1 and 2].

Let $\mathcal{A}$ be any of the classes Φ_+, Φ_- or Φ. The *perturbation class* of $\mathcal{A}$ is defined by its components:

$$P\mathcal{A}(X, Y) := \{K \in \mathcal{L}(X, Y) \colon K + T \in \mathcal{A}(X, Y) \text{ for all } T \in \mathcal{A}(X, Y)\},$$

when $\mathcal{A}(X, Y)$ is non-empty.

The components $P\mathcal{A}(X, Y)$ have been studied by many authors, but there are no good descriptions of them in general [**2**]. As we observed before, $\mathcal{SS}$ is contained in $P\Phi_+$, $\mathcal{SC}$ is contained in $P\Phi_-$ and both $P\Phi_+$ and $P\Phi_-$ are contained in the ideal $\mathcal{I}n$ of inessential operators, which coincides with $P\Phi$. We refer to [**1**] and [**10**] for an exposition of these facts.

In [**17**, 26.6.12], it was observed that the equality $P\Phi_+ = \mathcal{SS}$ (or $P\Phi_- = \mathcal{SC}$) holds if and only if the components $P\Phi_+(X)$ (or $P\Phi_-(X)$) determine an operator ideal. Therefore, the examples in [**9**] show that neither $P\Phi_+$ nor $P\Phi_-$ are operator ideals; i.e., they fail the compatibility conditions stated in [**17**, 1.1.3].

2. On Friedman's condition.

Recall that $K \in \mathcal{L}(X, Y)$ satisfies condition (C) if

$$S \in \mathcal{L}(X, Y), \dim(X^*/(R(K^*) + R(S^*))) < \infty \Rightarrow \dim(X^*/R(S^*)) < \infty.$$

Next we give a reformulation of this condition, which is easier to compare with the definitions of $\mathcal{SS}$ and $P\Phi_+$.

Given two operators $S, T \in \mathcal{L}(X, Y)$, we consider the operator $(S, T) \in \mathcal{L}(X, Y \times Y)$ defined by $(S, T)x := (Sx, Tx)$. Its conjugate operator $(S, T)^* \in \mathcal{L}(Y^* \times Y^*, X^*)$ is given by $(S, T)^*(g, h) = S^*g + T^*h$.

PROPOSITION 2.1. *An operator $K \in \mathcal{L}(X, Y)$ satisfies condition (C) if and only if for every $S \in \mathcal{L}(X, Y)$, $(S, K) \in \Phi_+$ implies $S \in \Phi_+$.*

Proof. Suppose that K satisfies condition (C) and $(S, K) \in \Phi_+(X, Y \times Y)$. Then $(S, K)^* \in \Phi_-$; hence, its range $R((S, K)^*) = R(S^*) + R(K^*)$ is finite codimensional. Thus, $R(S^*)$ is finite codimensional by condition (C), and [**18**, Theorem IV.5.10] implies that $R(S^*)$ is closed. Thus, $S^* \in \Phi_-$ and hence $S \in \Phi_+$.

Conversely, suppose that for every $S \in \mathcal{L}(X, Y)$, $(S, K) \in \Phi_+$ implies $S \in \Phi_+$. If $R(K^*) + R(S^*)$ is finite codimensional, then $(S, K)^* \in \Phi_-$, hence $(S, K) \in \Phi_+$. Our hypothesis implies $S \in \Phi_+$; hence $R(S^*)$ is finite codimensional. □

The previous characterization allows us to give an easier proof of the following result of Friedman.

PROPOSITION 2.2. *[7, Theorems 3 and 4] For an operator $K \in \mathcal{L}(X, Y)$,*

$$K \in \mathcal{SS} \Rightarrow K \text{ satisfies } (C) \Rightarrow K \in P\Phi_+.$$

Proof. For the first implication, note that $K \in \mathcal{SS}(X, Y)$ implies $(0, -K) \in \mathcal{SS}(X, Y \times Y)$, because $\mathcal{SS}$ is an operator ideal [**17**, 1.9.4 Theorem]. Thus, $(S, K) \in \Phi_+(X, Y \times Y)$ implies $(S, 0) = (S, K) + (0, -K) \in \Phi_+$; hence $S \in \Phi_+$.

For the second implication, suppose that K satisfies (C) and $S \in \Phi_+(X, Y)$. Then $S^* \in \Phi_-$; hence $R(S^* + K^*) + R(K^*)$ is finite codimensional. By condition (C), $R(S^* + K^*)$ is finite codimensional; hence $(S + K)^* \in \Phi_-$ and $S + K \in \Phi_+$. □

The following example shows that the answer to Friedman's question in [**7**, p. 350] is negative.

EXAMPLE 2.3. Let Z be the infinite dimensional, reflexive, hereditarily indecomposable Banach space constructed in [**13**] and let M be a closed subspace

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of Z with

$$\dim M = \dim Z/M = \infty.$$

Denoting by $J : M \to Z$ the inclusion, we consider the operator $K : M \times Z \to M \times Z$ defined by $K(m, z) := (0, Jm)$.

It was proved in [9] that $K \in P\Phi_+(M \times Z)$. Let us apply Proposition 2.1 to show that K does not satisfy condition (C). Indeed, if we consider the operator $S : M \times Z \to M \times Z$ defined by $S(m, z) := (0, z)$, then $S \notin \Phi_+$ but $(S, K) \in \Phi_+(M \times Z, (M \times Z) \times (M \times Z))$.

Next we give a dual version of the previous results for $\mathcal{SC}$ and $P\Phi_-$. Recall that $K \in \mathcal{L}(X, Y)$ satisfies condition (D) if

$$S \in \mathcal{L}(X, Y), \dim(Y/(R(K) + R(S))) < \infty \Rightarrow \dim(Y/R(S)) < \infty.$$

Given two operators $S, T \in \mathcal{L}(X, Y)$, we consider the operator $[S, T] \in \mathcal{L}(X \times X, Y)$ defined by $[S, T](x, z) := Sx + Tz$.

PROPOSITION 2.4. *An operator $K \in \mathcal{L}(X, Y)$ satisfies condition (D) if and only if for every $S \in \mathcal{L}(X, Y)$, $[S, K] \in \Phi_-$ implies $S \in \Phi_-$.*

Proof. Suppose that the operator K satisfies condition (D) and $[S, K] \in \Phi_-(X \times X, Y)$. Then $R([S, K]) = R(S) + R(K)$ is finite codimensional. Thus, $R(S)$ is finite codimensional by condition (D); hence $S \in \Phi_-$.

Conversely, suppose that for every $S \in \mathcal{L}(X, Y)$, $[S, K] \in \Phi_-$ implies $S \in \Phi_-$. If $R(K) + R(S)$ is finite codimensional, then $[S, K] \in \Phi_-$. Our hypothesis implies $S \in \Phi_-$; hence $R(S)$ is finite codimensional. □

The next result is a dual version of [**7**, Theorems 3 and 4].

PROPOSITION 2.5. *For an operator $K \in \mathcal{L}(X, Y)$,*

$$K \in \mathcal{SC} \Rightarrow \ K \text{ satisfies (D) } \Rightarrow K \in P\Phi_-.$$

Proof. For the first implication, note that $K \in \mathcal{SC}(X, Y)$ implies $[0, -K] \in \mathcal{SC}(X \times X, Y)$, because $\mathcal{SC}$ is an operator ideal [**17**, 1.10.4 Theorem]. Thus, $[S, K] \in \Phi_-(X \times X, Y)$ implies $[S, 0] = [S, K] + [0, -K] \in \Phi_-$; hence $S \in \Phi_-$.

For the second implication, suppose that K satisfies (D) and $S \in \Phi_-(X, Y)$. Then $R(S + K) + R(K)$ is finite codimensional. By condition (C), $R(S + K)$ is finite codimensional; hence $S + K \in \Phi_-$. □

The following example shows that the converse to the second implication in the previous result fails.

EXAMPLE 2.6. Let K be the operator in Example 2.3. Then $K^* \in P\Phi_-(M \times Z)$ but K^* does not satisfy condition (D). Indeed, if we consider the operator $S : M \times Z \to M \times Z$ in Example 2.3, then $S^* \notin \Phi_-$ but $(S, K)^* = [S^*, K^*] \in \Phi_-$. By Proposition 2.4, K^* does not satisfy condition (D).

The results of this section leave open the following questions.

QUESTION 1. Does condition (C) characterize the strictly singular operators?

QUESTION 2. Does condition (D) characterize the strictly cosingular operators?

Our impression is that the answer to both questions should be negative. However, we observe that in order to obtain such negative answers we would need new counterexamples to the perturbation classes problem different from those in [**8**] and [**9**].

3. On strictly singular and strictly cosingular operators. Here we give some characterizations of strictly singular operators and strictly cosingular operators in terms of perturbation classes for semi-Fredholm operators.

DEFINITION 3.1. Let X denote a Banach space. The *density character* $\text{dens}(X)$ of X is the least cardinal κ for which X has a dense subset of cardinality κ.

Given a non-empty set Γ, we denote by $|\Gamma|$ the cardinal of Γ and by $\ell_\infty(\Gamma)$ the space of all bounded scalar families $(a_i)_{i\in\Gamma}$ with index in Γ. Note that $\ell_\infty(\Gamma)$, endowed with the supremum norm $\|(a_i)_{i\in\Gamma}\|_\infty := \sup_{i\in\Gamma}|a_i|$, is a Banach space.

Let Γ be a set satisfying $|\Gamma| \geq \text{dens}(X)$. Then there is a natural isometric embedding of X into $\ell_\infty(\Gamma)$. Indeed, let $\{x_i : i \in \Gamma\}$ be a dense subset of X. For each $i \in \Gamma$ we select a norm-one $f_i \in X^*$ such that $f_i(x_i) = \|x_i\|$. Clearly,

$$J(x) := (f_i(x))_{i\in\Gamma} \quad \text{for all } x \in X,$$

defines an isometric embedding $J : X \longrightarrow \ell_\infty(\Gamma)$. In particular, $J \in \Phi_+(X, \ell_\infty(\Gamma))$.

The following result improves [**7**, Theorem 6].

THEOREM 3.2. *For $T \in \mathcal{L}(X, Y)$, the following statements are equivalent:*

(i) $T \in \mathcal{SS}(X, Y)$;

(ii) given Z for which $\Phi_+(X, Z) \neq \emptyset$, $ST \in P\Phi_+(X, Z)$ for every $S \in \mathcal{L}(Y, Z)$;

(iii) given a set Γ satisfying $|\Gamma| \geq \text{dens}(X)$, $ST \in P\Phi_+(X, \ell_\infty(\Gamma))$ for every $S \in \mathcal{L}(Y, \ell_\infty(\Gamma))$.

Proof. (i) $\Rightarrow$ (ii) Note that $T \in \mathcal{SS}(X, Y)$ and $S \in \mathcal{L}(Y, Z)$ imply $ST \in \mathcal{SS}(X, Z)$, and $\mathcal{SS}(X, Z) \subseteq P\Phi_+(X, Z)$.

(ii) $\Rightarrow$ (iii) Observe that $|\Gamma| \geq \text{dens}(X)$ implies $\Phi_+(X, \ell_\infty(\Gamma)) \neq \emptyset$.

(iii) $\Rightarrow$ (i) Suppose that $T \notin \mathcal{SS}$. Clearly, we can assume $\|T\| = 1$. We take an infinite dimensional closed subspace M of X such that TJ_M is an isomorphism. So there exists a constant $0 < C \leq 1$ such that $\|Tm\| \geq C\|m\|$ for each $m \in M$.

Let Γ be a set satisfying $|\Gamma| \geq \text{dens}(X)$. We consider two disjoint subsets Γ_1 and Γ_2 of Γ such that $|\Gamma_1| = |\Gamma_2| = |\Gamma|$ and $\Gamma = \Gamma_1 \cup \Gamma_2$. Clearly we can identify each $\ell_\infty(\Gamma_i)$ $(i = 1, 2)$ with a subspace of $\ell_\infty(\Gamma)$. In this way, we get a decomposition $\ell_\infty(\Gamma) = \ell_\infty(\Gamma_1) \oplus_\infty \ell_\infty(\Gamma_2)$. Since $\text{dens}(T(M)) \leq |\Gamma_1|$, we can take a dense subset $\{y_i : i \in \Gamma_1\}$ of $T(M)$ and norm-one elements $g_i \in Y^*$ $(i \in \Gamma_1)$ such that $g_i(y_i) = \|y_i\|$. We take $g_i = 0$ in Y^* for $i \in \Gamma_2$ and define

$$S(y) := (g_i(y))_{i\in\Gamma} \quad \text{for all } y \in Y.$$

Then $S \in \mathcal{L}(Y, \ell_\infty(\Gamma))$ is a norm-one operator with $R(S) \subset \ell_\infty(\Gamma_1)$ and $SJ_{T(M)}$ is an isometry; thus $\|STm\| \geq C\|m\|$ for each $m \in M$.

Similarly, since $\text{dens}(X/M) \leq |\Gamma_2|$, we can take a dense subset $\{z_i : i \in \Gamma_2\}$ of X/M and norm-one elements $h_i \in (X/M)^*$ $(i \in \Gamma_2)$ such that $h_i(z_i) = \|z_i\|$. We take $h_i = 0$ in

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$(X/M)^*$ for $i \in \Gamma_1$ and define

$$U(z) := (h_i(z))_{i\in\Gamma} \quad \text{for all } z \in X/M.$$

Then $U \in \mathcal{L}(X/M, \ell_\infty(\Gamma))$ is an isometric embedding and $R(U) \subset \ell_\infty(\Gamma_2)$.

Let us show that $V := UQ_M + ST$ is an isomorphic embedding. Let $x \in X$ with $\|x\| = 1$. Then either $\|Q_M x\| = \text{dist}(x, M) \geq C/3$ or $\|Q_M x\| < C/3$. In the latter case, we can write $x = m + z$ with $m \in M$ and $\|z\| < C/3$; hence $\|STx\| \geq \|STm\| - \|STz\| > C/3$. Therefore,

$$\|Vx\| = \max\{\|UQ_M x\|, \|STx\|\} \geq (C/3)\|x\| \quad \text{for every } x \in X.$$

Thus, $UQ_M + ST \in \Phi_+(X, \ell_\infty(\Gamma))$. Since $UQ_M \notin \Phi_+(X, \ell_\infty(\Gamma))$, we conclude $ST \notin P\Phi_+(X, \ell_\infty(\Gamma))$, and the proof is complete. □

COROLLARY 3.3. *Let X be a Banach space and let Γ be a set with $|\Gamma| \geq \text{dens}(X)$. Then $P\Phi_+(X, \ell_\infty(\Gamma)) = \mathcal{SS}(X, \ell_\infty(\Gamma))$.*

Proof. As we mentioned before, $|\Gamma| \geq \text{dens}(X)$ implies that $\Phi_+(X, \ell_\infty(\Gamma))$ is non-empty. Thus $P\Phi_+(X, \ell_\infty(\Gamma))$ is well defined, and the equality follows from the equivalence of (i) and (iii) in Theorem 3.2 for $Y = \ell_\infty(\Gamma)$. □

REMARK 3.4. We observe that the coincidence of $\mathcal{SS}(X, Y)$ and $P\Phi_+(X, Y)$ in some cases is related with the different positions in which a Banach space can be embedded as a subspace of another Banach space, as studied in [**6**].

Indeed, in the counterexample given in [**9**] an infinite dimensional, hereditarily indecomposable Banach space Z and a closed subspace M of Z with $\dim M = \dim Z/M = \infty$ were considered. A key fact for the construction is that, in the product space $M \times Z$, the position of M as $M \times 0$ is non-equivalent with the position as a subspace of $0 \times X$.

Conversely, it was proved in [**5**] that all the closed subspaces of $\ell_\infty(\Gamma)$ isomorphic to a Banach space X with $|\Gamma| \geq \text{dens}(X)$ are in equivalent positions, in the sense that given two of them, there exists an automorphism of $\ell_\infty(\Gamma)$ that takes one onto the other. This is the reason behind the result in Corollary 3.3.

The following maximality result follows from Theorem 3.2.

PROPOSITION 3.5. *The class of strictly singular operators $\mathcal{SS}$ is the biggest among the operator ideals $\mathcal{A}$ that satisfy $\mathcal{A}(X, Y) \subset P\Phi_+(X, Y)$ for every couple of Banach spaces X, Y for which $\Phi_+(X, Y)$ is non-empty.*

Proof. It is easy to check that the class $P\Phi_+$ is injective in the sense that given an operator $K \in \mathcal{L}(X, Y)$ and an isomorphic embedding $J \in \mathcal{L}(Y, Z)$, $JK \in P\Phi_+ \Rightarrow K \in P\Phi_+$. Therefore, if an operator ideal $\mathcal{A}$ is contained in $P\Phi_+$, the same happens with its injective hull $\mathcal{A}^{inj}$ [**17**, 4.6.1].

Suppose that $\mathcal{A}$ is an injective operator ideal contained in $P\Phi_+$ and $K \in \mathcal{A}(X, Y)$. Then part (iii) of Theorem 3.2 implies that K is strictly singular. □

We consider the space $\ell_1(\Gamma)$ of all absolutely summable scalar families $(a_i)_{i\in\Gamma}$ with index in Γ. Note that $\ell_1(\Gamma)$, endowed with the summing norm $\|(a_i)_{i\in\Gamma}\|_1 := \sum_{i\in\Gamma} |a_i|$, is a Banach space.

The unit vector basis of $\ell_1(\Gamma)$ is $\{e_j : j \in \Gamma\}$, where $e_j = (\delta_{ij})_{i\in\Gamma}$.

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Let Γ be a set satisfying $|\Gamma| \geq \text{dens}(X)$. Then there is a natural quotient map from $\ell_1(\Gamma)$ onto X. Indeed, if $\{x_i : i \in \Gamma\}$ is a dense subset of the unit ball B_X, then

$$Q((a_i)_{i\in\Gamma}) := \sum_{i\in\Gamma} a_i x_i \quad \text{for all } (a_i)_{i\in\Gamma} \in \ell_1(\Gamma)$$

defines a quotient map $Q : \ell_1(\Gamma) \longrightarrow X$. In particular, $Q \in \Phi_-(\ell_1(\Gamma), X)$.

THEOREM 3.6. For $T \in \mathcal{L}(X, Y)$, the following statements are equivalent:
(i) $T \in \mathcal{SC}(X, Y)$;
(ii) given Z for which $\Phi_-(Z, Y) \neq \emptyset$, $TS \in P\Phi_-(Z, Y)$ for every $S \in \mathcal{L}(Z, X)$;
(iii) given a set Γ satisfying $|\Gamma| \geq \text{dens}(Y)$, $TS \in P\Phi_-(\ell_1(\Gamma), Y)$ for every $S \in \mathcal{L}(\ell_1(\Gamma), X)$.

Proof. (i) $\Rightarrow$ (ii) Note that $T \in \mathcal{SC}(X, Y)$ and $S \in \mathcal{L}(Z, X)$ imply $TS \in \mathcal{SC}(Z, Y)$, and $\mathcal{SC}(Z, Y) \subseteq P\Phi_-(Z, Y)$.

(ii) $\Rightarrow$ (iii) Observe that $|\Gamma| \geq \text{dens}(Y)$ implies $\Phi_-(\ell_1(\Gamma), Y) \neq \emptyset$.

(iii) $\Rightarrow$ (i) Suppose that $T \notin \mathcal{SC}$. We take an infinite codimensional closed subspace N of Y such that $Q_N T$ is surjective. Thus, $Y = R(T) + N$.

Let Γ be a set satisfying $|\Gamma| \geq \text{dens}(Y)$. We consider two disjoint subsets Γ_1 and Γ_2 of Γ such that $|\Gamma_1| = |\Gamma_2| = |\Gamma|$ and $\Gamma = \Gamma_1 \cup \Gamma_2$. Clearly, we can identify each $\ell_1(\Gamma_i)$ $(i = 1, 2)$ with a subspace of $\ell_1(\Gamma)$. In this way, we get a decomposition $\ell_1(\Gamma) = \ell_1(\Gamma_1) \oplus_1 \ell_1(\Gamma_2)$.

Since $\text{dens}(Y/N) \leq |\Gamma_1|$, we can take a dense subset $\{z_i : i \in \Gamma_1\}$ in the unit ball of Y/N. Moreover, since $Q_N T$ is surjective, we can select a bounded family $\{x_i : i \in \Gamma_1\}$ in X such that $Q_N T x_i = z_i$ for each $i \in \Gamma_1$. We define $S \in \mathcal{L}(\ell_1(\Gamma), X)$ by $Se_i := x_i$ for $i \in \Gamma_1$ and $Se_i := 0$ for $i \in \Gamma_2$.

Similarly, since $\text{dens}(N) \leq |\Gamma_2|$, we can take a dense subset $\{y_i : i \in \Gamma_2\}$ in the unit ball of N and define $V \in \mathcal{L}(\ell_1(\Gamma), N)$ by $Ve_i := 0$ for $i \in \Gamma_1$ and $Ve_i := y_i$ for $i \in \Gamma_2$. Clearly, $R(V) = N$.

Now, since $Q_N TS$ is surjective, $R(TS) + N = Y$. Thus, the operator $U := J_N V + TS$ is surjective. Since $J_N V \notin \Phi_-(\ell_1(\Gamma), Y)$, we get $TS \notin P\Phi_-(\ell_1(\Gamma), Y)$, and the proof is complete. □

COROLLARY 3.7. *Let Y be a Banach space and let Γ be a set with $|\Gamma| \geq \text{dens}(Y)$. Then $P\Phi_-(\ell_1(\Gamma), Y) = \mathcal{SC}(\ell_1(\Gamma), Y)$.*

Proof. As we mentioned before, $|\Gamma| \geq \text{dens}(Y)$ implies that $\Phi_-(\ell_1(\Gamma), Y)$ is non-empty. Thus, $P\Phi_-(\ell_1(\Gamma), Y)$ is well defined, and the equality follows from the equivalence of (i) and (iii) in Theorem 3.6 for $X = \ell_1(\Gamma)$. □

Similar to Proposition 3.5, we can derive a maximality result from Theorem 3.6.

PROPOSITION 3.8. *The class of strictly cosingular operators $\mathcal{SC}$ is the biggest among the operator ideals $\mathcal{A}$ that satisfy $\mathcal{A}(X, Y) \subset P\Phi_-(X, Y)$ for every couple of Banach spaces X, Y for which $\Phi_-(X, Y)$ is non-empty.*

Proof. It is easy to check that the class $P\Phi_-$ is surjective in the sense that given an operator $K \in \mathcal{L}(X, Y)$ and a surjective operator $Q \in \mathcal{L}(Z, X)$, $KQ \in P\Phi_- \Rightarrow K \in P\Phi_-$. Therefore, if an operator ideal $\mathcal{A}$ is contained in $P\Phi_-$, the same happens with its surjective hull $\mathcal{A}^{sur}$ [**17**, 4.7.1].

Suppose that $\mathcal{A}$ is a surjective operator ideal contained in $P\Phi_-$ and $K \in \mathcal{A}(X, Y)$. Then part (iii) of Theorem 3.6 implies that K is strictly cosingular. □

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Given a semi-Fredholm operator $T \in \Phi_+(X, Y) \cup \Phi_-(X, Y)$, the *index ind*$(T)$ *of* T is defined by

$$\mathrm{ind}(T) := \dim N(T) - \dim \frac{Y}{R(T)} \in \mathbb{Z} \cup \{\pm\infty\}.$$

It is well known that the index is constant on the connected components of the semi-Fredholm operators [**1**]. Therefore, the following weakenings of condition (ii) in Theorems 3.2 and 3.6 imply that $T \in \mathcal{L}(X, Y)$ is inessential:

(a) for every $S \in \mathcal{L}(Y, X)$, $ST \in P\Phi_+(X)$;
(b) for every $S \in \mathcal{L}(Y, X)$, $TS \in P\Phi_-(Y)$.

Indeed, if T satisfies (a), then $I_X - tST \in \Phi_+(X)$ for every $t > 0$. Since I_X is a Fredholm operator with index equal to 0, we conclude that for every $S \in \mathcal{L}(Y, X)$, $I_X - ST$ is a Fredholm operator; hence $T \in \mathcal{I}n$. For (b) we can give a similar argument.

However, we will give examples below showing that neither of the two conditions is a characterization of the inessential operators. Observe that it is enough to show a Banach space Z for which $\mathcal{I}n(Z) \neq P\Phi_+(Z)$ and $\mathcal{I}n(Z^*) \neq P\Phi_-(Z^*)$.

Recall that $T \in \mathcal{L}(X, Y)$ is *weakly compact* if it sends bounded sets into relatively weakly compact subsets, and it is *completely continuous* if it sends weakly compact sets into norm-compact sets. A Banach space X has the *Dunford–Pettis property* (in short, X has the *DPP*) if every weakly compact operator $T \in \mathcal{L}(X, Y)$ is completely continuous. The $C(K)$ spaces, the $L_1(\mu)$ spaces and their dual spaces have the *DPP*. For these results and additional information, we refer to [**4**].

The following result is essentially known. We give a proof for completeness.

PROPOSITION 3.9. *(a) Suppose that X has the DPP. Then every weakly compact operator $U : X \to Y$ is strictly singular.*

(b) Suppose that Y^* has the DPP. Then every weakly compact operator $U : X \to Y$ is strictly cosingular.

Proof. (a) Suppose that $U : X \to Y$ is weakly compact, M is a closed subspace of X and UJ_M is an isomorphism. Since UJ_M is weakly compact, M is reflexive. Therefore, J_M is weakly compact and U is completely continuous, which implies that UJ_M is compact; hence M is finite dimensional; thus $U \in \mathcal{SS}$.

(b) Suppose that $U : X \to Y$ is weakly compact. Then $U^* : Y^* \to X^*$ is weakly compact. By part (a), U^* is strictly singular; hence $U \in \mathcal{SC}$. □

It follows from Proposition 3.9 that the inclusion $i_1 : L_2[0, 1] \to L_1[0, 1]$ is strictly cosingular because it is weakly compact and $L_1[0, 1]^*$ has the *DPP*. However, it is not strictly singular because it is an isomorphism on the subspace generated by the Rademacher functions (r_n) [**4**, Theorem 6.2.3]. Note that (r_n) is an orthonormal sequence in $L_2[0, 1]$.

The inclusion $j_\infty : L_\infty[0, 1] \to L_2[0, 1]$ is strictly singular, but not strictly cosingular. Indeed, since j_∞ is weakly compact and $L_\infty[0, 1]$ has the DPP, $j_\infty \in \mathcal{SS}$ (see Proposition 3.9). Moreover, j_∞ is the conjugate operator of i_1. Hence, $i_1 \notin \mathcal{SS} \Rightarrow j_\infty \notin \mathcal{SC}$.

Examples 3.10. There exists a Banach space Z and an operator $T \in \mathcal{L}(Z)$ such that $T \in P\Phi_+(Z) \setminus \mathcal{I}n(Z)$ and $T^* \in P\Phi_-(Z^*) \setminus \mathcal{I}n(Z^*)$.

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Let $S : L_2[0, 1] \to L_2[0, 1]$ denote an isomorphism from $L_2[0, 1]$ onto the subspace generated by the Rademacher functions (r_n).

We take $Z = L_2[0, 1] \times L_1[0, 1] \times L_1[0, 1]$ and define $T \in \mathcal{L}(Z)$ by

$$T(f, g, h) := (0, i_1 Sf, 0).$$

Note that $Z^* = L_2[0, 1] \times L_\infty[0, 1] \times L_\infty[0, 1]$ and the conjugate operator $T^* \in \mathcal{L}(Z^*)$ is given by $T^*(f, g, h) := (S^* j_\infty g, 0, 0)$.

Since $\mathcal{I}n$ is an operator ideal, $T \in \mathcal{I}n(Z)$ and $T^* \in \mathcal{I}n(Z^*)$. Let us see that $T = I_Z T \notin P\Phi_+(Z)$ and $T^* = T^* I_{Z^*} \notin P\Phi_-(Z^*)$.

We consider the operator $U \in \mathcal{L}(Z)$ defined by $U(f, g, h) := (0, -i_1 Sf, \phi)$, where $\phi(t) := g(2t)$ for $0 \leq t \leq 1/2$ and $\phi(t) := h(2t - 1)$ for $1/2 < t \leq 1$.

Clearly, $U \in \Phi_+(Z)$ and $U + T \notin \Phi_+(Z)$; hence $T \notin P\Phi_+(Z)$. Similarly, $U^* \in \Phi_-(Z^*)$ and $U^* + T^* \notin \Phi_-(Z^*)$; hence $T^* \notin P\Phi_+(Z^*)$.

Observe that

- $P\Phi_+(L_2[0, 1], L_1[0, 1]) = \mathcal{SS}(L_2[0, 1], L_1[0, 1])$ [**12**, Theorem 15];
- $P\Phi_-(L_\infty[0, 1], L_2[0, 1]) = \mathcal{SC}(L_\infty[0, 1], L_2[0, 1])$ [**12**, Corollary 17].

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