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A linearized model of water exit

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A model of hydrodynamic loads acting on a rigid floating body during the lifting of the body from the liquid surface is presented. The liquid is of infinite depth, inviscid and incompressible. Initially the liquid is at rest. The body suddenly starts to move upwards from the liquid at a constant acceleration. Boundary conditions on the liquid surface are linearized and imposed on the equilibrium position of the liquid surface. The resulting boundary problem is solved by the methods of analytical functions. Negative pressures are allowed and the pressure is assumed continuous at the periphery of the wetted area. The unknown size of the wetted area is determined by the condition that the speed of the contact points is proportional to the local velocity of the flow. This condition provides a nonlinear Abel-type integral equation which is solved explicitly. Both two-dimensional and axisymmetric configurations are considered. Predicted hydrodynamic forces are compared with the computational fluid dynamics results by Piro & Maki (*11th International Conference on Fast Sea Transport. Honolulu, Hawaii, USA*, 2011) for both a rigid wedge and circular cylinder, which initially enter the water and then exit from it.

Key words: aerodynamics, flow–structure interactions, waves/free-surface flows

1. Introduction

The two-dimensional unsteady problem of a rigid body with small deadrise angle, which is initially in contact with an inviscid liquid at rest and then suddenly starts to move from the liquid at a constant acceleration, is considered (see figure 1). The hydrodynamic loads acting on the moving body are of primary interest. The liquid is assumed inviscid and incompressible, and its flow irrotational. The liquid is infinitely deep and initially at rest. Initially the body is in contact with the liquid over a finite interval, $-c_0 < x < c_0$ (figure 1). The intervals $y = 0$, $x > c_0$ and $x < -c_0$ correspond to the initial positions of the liquid free surface. The initial position of the body is described by the equation $y = f(x) - h_0$, where h_0 is the initial draft of the body. Only the initially wetted part of the body surface matters in the exit problem. The body is symmetric, $f(-x) = f(x)$, $f(0) = 0$ and $f'(x) \geqslant 0$, where $0 < x < c_0$. At the initial time instant, $t = 0$, the body starts to move upwards with a given acceleration $h''(t)$. The current position of the wetted part of the body is described by the equation $y = f(x) - h_0 + h(t)$, where $|x| < c(t)$, $h(t)$ is the body displacement, $h''(t) \geqslant 0$, $h(0) = 0$, $h'(0) = 0$ and the interval $-c(t) < x < c(t)$ corresponds to the wetted area of the moving surface.

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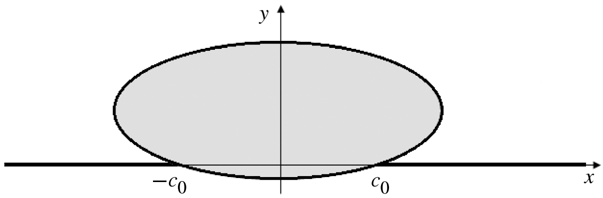

FIGURE 1. Initial position of a body on the surface of liquid of infinite depth at rest.

We shall determine the hydrodynamic force $F(t)$ acting on the moving body and the function $c(t)$, which describes the size of the wetted part of the body surface under the following assumptions: (i) the deadrise angle of the body is small, $0 < f'(x) \ll 1$, where $0 < x < c_0$; (ii) the acceleration of the body $h''(t)$ is much greater than the gravitational acceleration g; (iii) the wetted area is shrinking in time, $c'(t) < 0$ and $c(0) = c_0$; (iv) the hydrodynamic pressure in the wetted area can be below the atmospheric pressure p_{atm} but is higher than the cavitation pressure p_{cav} at which the liquid turns into a vapour. The atmospheric pressure p_{atm} is taken below as the reference pressure.

Note that if there is no surface tension, gravity, or air above the liquid, and hydrodynamic pressures in the contact region are not allowed to be below the atmospheric pressure, then the body surface separates from the liquid instantly at $t = 0$ without inducing any flow (see figure 2). This observation makes the present problem different from the water exit problem studied intensively in the past, where initially a body is submerged under the liquid free surface and then moves towards the liquid surface. Greenhow (1988) wrote: ‘For exit the calculations predict the lifting of the water above the cylinder and subsequent formation of thin layers; the draw-down and rush-up of the free surface beneath the cylinder, the rush-up terminating in localised breaking clearly seen in the experiments; the fluid motion after complete cylinder exit; and, perhaps most interestingly, the formation of large regions of strongly negative pressure on the cylinder surface during late stages of exit which explains the occurrence of the “spontaneous” breaking of the free surface seen in experiments.’ See also Greenhow & Moyo (1997) for more results on the water exit problem for horizontal circular cylinders.

The configuration shown in figure 1 is similar to the water entry problem studied within the Wagner theory (Wagner 1932; Korobkin & Pukhnachov 1988; Howison, Ockendon & Wilson 1991; Oliver 2002). Oliver (2002) wrote: ‘… the leading-order outer problem is linearly stable if and only if the turnover curve is advancing, i.e. the time reversal of the entry problem is linearly unstable. This suggests that modelling the water exit of a partially submerged hull by the time reversal of a water entry problem is illposed.’ This discussion by Oliver (2002) is related to our present problem, with the ‘turnover curve’ corresponding to the contact points $x = \pm c(t)$ in our notations. More results and explanations as to why the present exit problem cannot be treated as a reversed entry problem were published by Chapman *et al.* (1997) and Gillow (1998). We conclude that the formulation of the present problem is very different from the entry problem, the formulation of which is well understood at present. However, we expect that some ideas from the Wagner theory of water entry can be used in the exit problem as well. Namely, for small deadrise angles of the body, it is expected that the horizontal speeds of the contact points $x = \pm c(t)$ are greater than the vertical speed of the body, at least during the early stage. If so, the boundary conditions on both the free surface and the wetted portion of the body surface, which

(*a*)

(*b*)

(*c*)

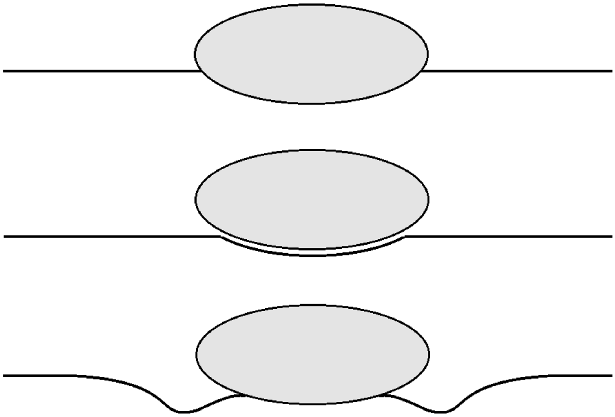

FIGURE 2. (*a*) Initial position of a body on the surface of liquid of infinite depth at rest. (*b*) Lifting of the body if there is no gravity, adhesion, or presence of air. (*c*) Expected shape of the liquid surface in a model where negative hydrodynamic pressures in the wetted area are allowed.

exits from the liquid, can be linearized and imposed on the equilibrium position of the liquid surface, similar to the linearization procedure adopted for entry problems during the initial stage. However, the motions of the contact points in the exit problem are governed by another mechanism, which is still unclear at present. In the entry problem, the motions of the corresponding contact points are described by the Wagner condition, which implies that the displacements of liquid particles are finite.

The problem under consideration in this paper is similar to that of the second stage of wave impact from below on a platform placed just above the water surface, so-called 'wetdeck slamming', studied by Baarholm (2001) and Faltinsen, Landrini & Greco (2004). During the first impact stage the wave hits the wet deck of the platform from below, and the wetted area of the platform increases in time. Then the wetted area starts to diminish, and the water falls down due to gravity. The problem of wetdeck slamming was studied numerically in two-dimensional formulation. Faltinsen *et al.* (2004) wrote: 'The water-exit phase lasted much longer than a von Kármán method would predict. The water-exit phase causes a negative force. The maximum absolute value of the negative force is comparable to the maximum force during the water-entry phase.' These findings were confirmed by Scolan, Remy & Thibault (2006), who performed three-dimensional experiments with a standing wave impact onto a horizontal transparent plate placed just above the water surface. The line of separation of the water from the plate during the exit stage was recorded and the wetted area as a function of time was plotted together with the measured hydrodynamic force. In the two-dimensional formulation of wetdeck slamming, Baarholm & Faltinsen (2004) suggested that, in addition to the Kutta condition, the speed of the contact point, which is $\mathrm{d}c/\mathrm{d}t$ in our notation, is equal to the velocity of the flow at this contact point (see equation (14) in their paper). They wrote that the latter condition 'provides a stable solution that makes it possible to simulate water exit until the deck becomes almost entirely dry.' In the present analysis this condition will also be employed but in a generalized form. The condition that the speed of the contact point is proportional to the local speed of the flow at the contact point will be used to close the formulation of the exit problem.

Two-dimensional problem of water entry and exit of both rigid and elastic wedges was investigated numerically by Piro & Maki (2011, 2012, 2013) with a Navier–Stokes solver from OpenFOAM library. The analysis was motivated by 'performance of vehicles such as seaplanes, planing craft, space craft that land in the ocean, and ships in general.' Piro & Maki performed computational fluid dynamics (CFD) computations, in particular, for a two-dimensional rigid wedge with deadrise angle of 10°. The wedge initially touches the flat water surface at a single point and suddenly starts to penetrate the liquid with a given speed and a constant deceleration. Conditions of the impact are selected in such a way that the speed of the wedge becomes zero before the wedge is completely wetted. Then the wedge moves upwards from the liquid and finally exits it. In particular, computations were performed by Piro & Maki (2011, 2013) for initial velocity 4 m s^{-1} and deceleration 92 m s^{-2}. Note that this deceleration is about ten times greater than the gravitational acceleration $g = 9.81$ m s^{-2}. It was found that the hydrodynamic force is initially positive and then becomes negative during the entry stage. The maximum magnitude of the negative force is achieved at the end of the entry stage. During the exit stage the force is negative and its magnitude decays in time. The force profile at the exit stage is rather particular; it is reproduced by the model of the present paper. Shapes of the free surface are shown by Piro & Maki (2013, figure 7) for different stages of the wedge motion. It is seen that the free surface comes to the separation point almost tangentially, which is in agreement with observations by Faltinsen *et al.* (2004) for wetdeck slamming. However, very close to the wedge surface the liquid motion is more complicated, with part of the liquid remaining on the body surface in the form of a thin film. This is due to viscous effects which were not included in the analysis by Faltinsen *et al.* (2004).

The CFD analysis by Piro & Maki does not permit cavitation in the fluid. However, they computed the pressure distribution in the wetted area of the parabolic section $y = x^2/(2R) - h(t)$, where $R = 1.4$ m, which enters water with the initial speed $h'(0) = 1$ m s^{-1} and constant deceleration $h''(t) = -19.5$ m s^{-2}, and concluded that the minimum pressure is well above the vapour pressure of water p_{cav}, which is 2.3 kPa at 20 °C (Tassin *et al.* 2013). In these computations, the maximum penetration depth was 2.5 cm and the minimum hydrodynamic pressure was computed just 7 kPa below the atmospheric pressure $p_{atm} \approx 101$ kPa, which gives the total pressure $\sim$94 kPa. The minimum pressure occurs at the centre of the wetted area when the speed of the parabolic section is zero.

Gravity and turbulence were neglected by Piro & Maki (2013) in the entry/exit problems with large decelerations, but gravity was included by Piro & Maki (2012) in the exit problems with constant speed. It was shown that for speeds of the body exit greater than $5\sqrt{gh_0}$, where h_0 is the initial-penetration depth of the wedge, the effect of gravity can be neglected. This result is for the wedge with deadrise angle 10°. The results by Piro & Maki (2011) were the first results on the exit problem with constant acceleration. The analysis of the present paper is based on the pioneering paper by Piro & Maki (2011).

The present study is also motivated by the '2D + t' approach applied to the analysis of aircraft landing on the surface of water (see Tassin, Korobkin & Cooker 2012). In this approach, the fuselage of the aircraft is considered as a slender structure entering the water with a horizontal speed much higher than the vertical speed of the aircraft. In a vertical plane of the global coordinate system, which is perpendicular to the direction of the aircraft motion, the liquid flow caused by impact is approximated as two-dimensional, independent of the longitudinal coordinate. The two-dimensional unsteady flow in such a control plane is governed by the time-dependent contour of the intersection between the fuselage and this vertical plane. Once the fuselage enters the plane, the contour starts from a single point, grows in all directions, enters the water, then decreases, exiting the water, and finally disappears when the fuselage leaves the plane. The two-dimensional problem of a body of time-varying shape, which enters the water and then exits from the water, was studied by Tassin *et al.* (2012) by using the modified Logvinovich model from Korobkin (2004) during the entry stage and the von Kármán model (see Kaplan 1987) during the exit stage. This combined MLM/vK model was applied to the vertical water entry and exit of the wedge studied by Piro & Maki (2011) using a fully nonlinear CFD method. It was found that during the entry stage these two methods agree very well. The agreement during the exit stage is good; however, the results by Piro & Maki (2011) predict a longer exit stage, which is not captured by the von Kármán method. The latter observation is consistent with the results by Faltinsen *et al.* (2004) in the problem of wetdeck slamming. Accurate prediction of the negative forces in the rear part of the fuselage during its landing on water is important in terms of the aircraft motion and the bending stresses in the fuselage caused by the highly non-uniform distribution of the loads along the fuselage.

The separation point, the motion of which was modelled by Faltinsen *et al.* (2004) by using a special condition on the speed of this point, also appears in the problems of oblique impact on the water surface of two-dimensional bodies. Three different conditions which may govern the motion of the separation point were proposed by Reinhard, Korobkin & Cooker (2012), with the Brillouin–Villat condition being one of them. It was shown that the choice of the separation condition significantly changes the predicted motion of the body. The flow before the separation starts was analysed by Moore *et al.* (2013) for the two-dimensional case, by Moore *et al.* (2012) for axisymmetric bodies and by Scolan & Korobkin (2012) for three-dimensional bodies. The flow with the separation point at the rear of the body during oblique impact was studied by Reinhard *et al.* (2012) for deep water and by Khabakhpasheva & Korobkin (2013) for shallow water. Skimming impacts and rebounds on shallow liquid layers were studied by Hicks & Smith (2011) with the separation point on the trailing sharp edge of a body. Khabakhpasheva & Korobkin (2013) wrote: ‘The separated part of the water surface and the subsequent motion of the separation point are determined by using the Brillouin–Villat condition, which requires the continuity of the pressure together with its tangential derivative at the separation point. This condition is not well established but reasonable.’ This condition was used, for example, by Tuck & Simakov (1999) and Semenov, Wu & Yoon (2012) in the steady problem of a body moving along the free surface of liquid of infinite depth. It might be expected that the local flow close to the separation point in the problem of water exit is similar to that in the problem of oblique impact with separation. However, the Brillouin–Villat condition at the moving separation point is not satisfied in the model of the present paper. The Brillouin–Villat condition was also used by Korobkin (2003) in a related problem of water entry of a decelerating body, where the liquid starts to separate from the moving body where the pressure on the body surface drops below the vapour pressure.

Reis *et al.* (2010) showed that the domestic cat laps by a subtle mechanism based on water adhesion to the dorsal side of the tongue. They wrote: ‘A combined experimental and theoretical analysis reveals that *Felis catus* exploits fluid inertia to defeat gravity and pull liquid into the mouth. At the lowest position of the tongue’s tip, its dorsal side rests on the liquid surface, without piercing it. When the cat lifts the tongue, liquid adhering to the dorsal side of the tip is drawn upward, forming a column. The tongue accelerates as it leaves the water surface, attains a remarkable maximum speed of 78 ± 2 cm s^{-1}, then decelerates as it enters the mouth.' The speed of $\sim$80 cm s^{-1} is achieved in 30 ms (see figure 2 in Reis *et al.* 2010). Therefore the tongue is lifted at an acceleration of $\sim$27 m s^{-2}, which is three times greater than the acceleration due to gravity, g. Experiments were performed by Reis *et al.* (2010) with a glass disc representing the tongue's tip. The disc was initially placed on the water surface and then pulled upwards vertically. The results of the experiments were summarized as: 'Estimation of the forces involved suggests that the fluid dynamics of lapping are governed by inertia and gravity, whereas viscous and capillary forces are negligible. Inertial entrainment draws liquid upward into a column, while gravity acts to collapse it. Ultimately, gravity prevails and the column pinches off.' This is a remarkable finding, which gives us the idea that during the early stage of exit and large acceleration of the body lifting, gravity can be neglected. The viscous effects and surface tension are important close to the body surface but can be neglected in terms of the global motion of the fluid.

We expect that the law of the body's motion during exit of the body from a fluid plays a most important role. Based on the results reviewed above we may assume that both gravity and inertia matter for gravity-driven flows such as wetdeck slamming (see Faltinsen *et al.* 2004) and motions of the body with acceleration of the order of the acceleration due to gravity g. When the body moves more rapidly, the gravity can be neglected during the initial stage of exit but has to be included at the later stages when the displacement of the body is not small. If a floating body suddenly starts to move from the liquid with non-zero speed, we expect the liquid to separate from the body surface instantly: see Norkin & Korobkin (2011) for the corresponding model of liquid separation. These arguments are additionally supported by Bugaenko (1973), who studied forces acting on an elliptic cylinder lifted from water surface; the paper by Bugaenko is reviewed in the book by (Korotkin 2009, chap. 5). Bugaenko assumed that the liquid free surface can be modelled as a flat rigid plate and the cylinder exits the liquid through this plate. This model is expected to be valid when the motion of the body is very slow and the gravity dominates. Theoretical predictions were compared with experimental results and good agreement was reported by Bugaenko (1973). A similar model of exit was published by Greenhow & Yanbao (1987).

The viscous effects and surface tension can be neglected everywhere except for a small vicinity of the body surface. We do not include the presence of air, and simplify the shape of the wetted part of the body using the so-called 'flat-plate approximation'. Only inertia forces are included in the exit model of this paper. The present model is concerned with the negative hydrodynamic force acting on a body lifted from the liquid surface during the early stage.

Formulation of the problem is given in § 2. The pressure distribution in the wetted area and the equation which relates the motion of the contact point and the body displacement are derived. The latter equation is solved by the power series method in § 3 in the case of constant acceleration of the body. The corresponding axisymmetric problem is solved in § 4. Comparisons with computational results by Piro & Maki (2011) and Tassin *et al.* (2013) for a wedge and parabolic contour are established in § 5. Conclusions are drawn and future work is discussed in § 6.

2. Formulation of the problem

A body with a smooth surface starts to move upwards from the position depicted in figure 1. The initial draft of the body h_0 is assumed to be much smaller than the initial horizontal size of the wetted area $2c_0$. The body displacement $h(t)$ is of the order of the initial draft h_0. The acceleration of the body $h''(t)$ is much greater than the gravitational acceleration g. The flow is assumed to be two-dimensional, potential and symmetric with respect to the centreline $x=0$. In the present analysis, we linearize the boundary conditions and impose them on the horizontal line $y=0$. The function $c(t)$, which describes the motion of the contact points, is unknown in advance and should be determined as part of the solution.

In order to justify this linearized model of water exit, we shall estimate the orders of the terms in the Navier–Stokes equation and in the boundary conditions governing the flow. We take c_0 as the length scale and the characteristic acceleration of the body a as the scale of the flow acceleration. We are concerned with the initial stage of exit, $0<t<T$, during which the vertical displacement of the body is of order $O(aT^2)$ and is much smaller than the characteristic length of the problem, $aT^2/c_0 \ll 1$. Then the velocity of the flow $\boldsymbol{v}(\boldsymbol{x}, t)$ is of the order of $O(aT)$. The viscous term $\nu\nabla^2\boldsymbol{v}$ in the Navier–Stokes equation, where ν is the kinematic viscosity of the liquid, can be neglected compared with the inertia term $\boldsymbol{v}_t$,

$$\frac{|\nu\nabla^2\boldsymbol{v}|}{|\boldsymbol{v}_t|} = O(\nu T/c_0^2), \tag{2.1}$$

during the early stage of exit when $\nu T/c_0^2 \ll 1$. Here $\nu \approx 1.004\times10^{-6}$ m^2 s^{-1} for water at 20 °C. The convective term $(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}$ can be neglected during the early stage,

$$\frac{|(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}|}{|\boldsymbol{v}_t|} = O(aT^2/c_0), \tag{2.2}$$

when the body displacement is small compared with the size of the wetted area. The estimates derived above are valid for the main flow region. They are not expected to be valid in close proximity to the moving rigid surface, where the body wettability, fluid viscosity and the contact line dynamics are important. The gravity can be neglected if $g/a \ll 1$. It is assumed below that the inequalities $g/a \ll 1$ and $aT^2/c_0 \ll 1$ are satisfied.

The derived estimates leave us with the linearized Euler equations

$$\boldsymbol{v}_t = -\frac{1}{\rho}\nabla p, \quad \nabla\cdot\boldsymbol{v} = 0, \tag{2.3}$$

where ρ is the liquid density, which dictate that the hydrodynamic pressure $p(\boldsymbol{x}, t)$ is of the order $O(\rho a c_0)$. The total pressure $p+p_{atm}$ in the wetted area of the body surface is below the atmospheric pressure during the body exit but above the vapour pressure p_{cav} when $p>p_{cav}-p_{atm}$. This inequality can be violated for large values of the product ac_0 and negative hydrodynamic pressure $p(\boldsymbol{x}, t)$. In the linearized exit model, which will be introduced below, the minimum hydrodynamic pressure is $-\rho a c_0$ (see (2.15)). Thus, cavitation in the wetted area is possible when $-\rho a c_0 < p_{cav}-p_{atm}$, which gives $ac_0 > 98$ m^2 s^{-2} for water at 20 °C.

It can be shown that the surface tension can be neglected in the main part of the free surface if $\sigma T^2/(\rho c_0^3) \ll 1$, where σ is the surface tension coefficient, $\sigma \approx 72\times10^{-3}$ N m^{-1} for water at 20 °C. The inequality $\sigma T^2/(\rho c_0^3) \ll 1$ can be written as

$$\left(\frac{\sigma}{\rho g c_0^2}\right)\left(\frac{g}{a}\right)\left(\frac{aT^2}{c_0}\right) \ll 1, \tag{2.4}$$

where $g/a \ll 1$ and $aT^2/c_0 \ll 1$. Therefore, in the exit problem, the surface tension must be taken into account only if $\sigma/(\rho g c_0^2) \gg 1$, which gives $c_0 \ll \sqrt{\sigma/(\rho g)}$. Correspondingly, both sides of the inequality $\nu T/c_0^2 \ll 1$ are convenient to square, and we write the result as

$$\left(\frac{\nu^2}{g c_0^3}\right)\left(\frac{g}{a}\right)\left(\frac{aT^2}{c_0}\right) \ll 1, \tag{2.5}$$

where $g/a \ll 1$ and $aT^2/c_0 \ll 1$. Therefore, the liquid viscosity must be taken into account only if $\nu^2/(g c_0^3) \gg 1$, which gives $c_0 \ll (\nu^2/g)^{1/3}$. For example, for water at 20 °C we calculate $(\nu^2/g)^{1/3} \approx 0.05$ mm and $\sqrt{\sigma/(\rho g)} \approx 3$ mm.

The obtained estimates provide that: (i) the boundary conditions can be linearized and imposed on the initial water level; (ii) gravity, surface tension and liquid viscosity can be neglected in the leading order; (iii) water does not cavitate in the wetted area during the early stage if

$$c_0 g \ll c_0 a < 98\ \mathrm{m}^2\ \mathrm{s}^{-2} \quad \text{and} \quad c_0 > 3\ \mathrm{mm}. \tag{2.6}$$

We conclude that the linearized model without gravity, surface tension, viscosity and cavitation can be used during the initial stage, $T \ll \sqrt{c_0/a}$, if $a \gg g$ and $3\ \mathrm{mm} < c_0 < (98/a)$ m.

Within the linearized exit model, the flow is described by the velocity potential $\varphi(x, y, t)$, which satisfies the Laplace equation in the lower half-plane $y < 0$ and decays at infinity, where $x^2 + y^2 \to \infty$. The linearized Bernoulli equation provides the hydrodynamic pressure

$$p(x, y, t) = -\rho \varphi_t(x, y, t) \quad (y \leqslant 0), \tag{2.7}$$

where ρ is the liquid density. The dynamic boundary condition on the free surface reads

$$\varphi_t(x, 0, t) = 0 \quad (y = 0, |x| > c(t)). \tag{2.8}$$

This condition implies that the horizontal velocity $\varphi_x(x, 0, t)$ on the free surface is zero, where $|x| > c_0$, but should be determined as part of the solution on the intervals $c(t) < |x| < c_0$, which are formed by liquid particles attached initially to the body surface. The linearized body boundary condition has the form

$$\varphi_y(x, 0, t) = h'(t) \quad (y = 0, |x| < c(t)). \tag{2.9}$$

We seek the solution of the formulated problem, which provides finite pressure in the contact region. This condition implies that both the pressure and velocity of the flow are continuous at the points $x = \pm c(t)$, $y = 0$. The function $c(t)$ is calculated by using the condition

$$\frac{\mathrm{d}c}{\mathrm{d}t} = \gamma \varphi_x(c(t), 0, t). \tag{2.10}$$

This condition can be considered as a generalized form of the condition used by Faltinsen *et al.* (2004) in numerical calculations of wetdeck slamming. The coefficient γ is undetermined in the present model and is chosen by using the numerical results by Piro & Maki (2011). It will be shown that the parameter γ can be included in the time scale of the problem.

Condition (2.10) comes from the assumption that the motion of the contact point is governed by the local flow, which is characterized by the tangential velocity along the body surface, $\varphi_x(c(t), 0, t)$. This component of the flow velocity increases with the body speed, which should lead to a higher speed of the contact point. In the present model, we assume that the relation between the speed of the contact point $c'(t)$ and the local tangential velocity of the flow is linear, with the coefficient γ dependent on the physical characteristics of both the liquid and the body surface, such as wettability of the body surface and viscosity of the liquid.

We shall solve the problem (2.7)–(2.10) and compare the obtained results with the numerical ones by Piro & Maki (2011) and Tassin *et al.* (2013) for the case of water exit of a wedge and circular cylinder with a constant acceleration. Note that in the present model, the shape of the body is assumed to provide a higher-order contribution to the hydrodynamic loads. It is expected that this contribution can be recovered by extending the model to higher-order effects.

It is convenient to introduce a complex acceleration potential

$$w(z, t) = \varphi_t(x, y, t) + \mathrm{i}\psi_t(x, y, t), \tag{2.11}$$

where $z = x+\mathrm{i}y$ and $\psi(x, y, t)$ is the stream function. The function $w(z, t)$ is analytic in the lower half-plane $y < 0$ and decays at infinity as $w(z, t) = O(z^{-1})$, where $|z| \to \infty$. Conditions (2.8), (2.9) and the symmetry of the flow give

$$\mathrm{Re}[w(x - \mathrm{i}0, t)] = 0 \quad (|x| > c(t)), \tag{2.12}$$

$$\mathrm{Im}[w(x - \mathrm{i}0, t)] = -h''(t)x \quad (|x| < c(t)). \tag{2.13}$$

Condition (2.13) is exact for any shape of the body if it is imposed on the actual position of the body surface. In (2.12), quadratic terms are missing and the condition is imposed on the horizontal line $y = 0$, but not on the actual position of the free surface.

The solution of the mixed boundary-value problem (2.12) and (2.13) has the form

$$w(z, t) = \mathrm{i}h''(t)\left(\sqrt{z^2 - c^2(t)} - z\right), \tag{2.14}$$

where the function $\sqrt{z^2 - c^2(t)}$ is defined on the complex plane z with the cut along the interval $y = 0$, $|x| < c(t)$, and such that it is real and positive on the interval $y = 0$, $x > a(t)$. The boundary values of this function on $y = -0$ (note that $y < 0$ in the flow region) are $\sqrt{x^2 - c^2(t)}$, where $y = -0$ and $x > c(t)$, $-\sqrt{x^2 - c^2(t)}$, where $y = -0$ and $x < -c(t)$, and $-\mathrm{i}\sqrt{c^2(t) - x^2}$, where $y = -0$ and $|x| < c(t)$. Note that this function tends to $+\mathrm{i}\sqrt{c^2(t) - x^2}$, where $y \to +0$ and $|x| < c(t)$.

The solution (2.14) and the linearized Bernoulli equation (2.7) provide the pressure distribution in the wetted area,

$$p(x, 0, t) = -\rho h''(t)\sqrt{c^2(t) - x^2} \quad (|x| < c(t)), \tag{2.15}$$

and the time derivative of the stream function on the free surface,

$$\psi_t(x, 0, t) = h''(t)(\sqrt{x^2 - c^2(t)} - x) \quad (x > c(t)). \tag{2.16}$$

The hydrodynamic force $F(t)$ acting on the moving body is obtained by integrating the pressure (2.15) along the contact region,

$$F(t) = \int_{-c(t)}^{c(t)} p(x, 0, t)\,\mathrm{d}x = -\rho h''(t)\int_{-c(t)}^{c(t)} \sqrt{c^2(t) - x^2}\,\mathrm{d}x = -m_a h''(t), \tag{2.17}$$

where $m_a = 0.5\pi\rho c^2(t)$ is the added mass of the equivalent flat plate. Therefore, the hydrodynamic force can be calculated if the function $c(t)$ is known.

The components of the flow velocity on the boundary $y=0$ are obtained by integrating (2.16) and the corresponding equations for $\varphi_t(x,0,t)$,

$$\varphi_t(x,0,t)=h''(t)\sqrt{c^2(t)-x^2}\quad (|x|<c(t)), \tag{2.18a}$$
$$\varphi_t(x,0,t)=0\quad (|x|>c(t)), \tag{2.18b}$$

in time, and subsequent differentiation of the results with respect to x along the boundary. The integration must be performed carefully. We introduce a function $t(x)$ such that $c[t(x)]\equiv x$ and $t(c_0)=0$. Then the velocity potential on the boundary is given by

$$\varphi(x,0,t)=\int_0^t h''(\tau)\sqrt{c^2(\tau)-x^2}\,\mathrm{d}\tau\quad (|x|<c(t)), \tag{2.19}$$
$$\varphi(x,0,t)=\int_0^{t(x)} h''(\tau)\sqrt{c^2(\tau)-x^2}\,\mathrm{d}\tau\quad (c(t)<|x|<c_0), \tag{2.20}$$
$$\varphi(x,0,t)=0\quad (|x|>c_0). \tag{2.21}$$

Equation (2.19) provides the velocity of the flow along the body surface:

$$\varphi_x(x,0,t)=-x\int_0^t \frac{h''(\tau)\,\mathrm{d}\tau}{\sqrt{c^2(\tau)-x^2}}. \tag{2.22}$$

A similar formula for the horizontal velocity of the liquid in the wake, $y=0$, $c(t)<x<c_0$, follows from (2.20):

$$\varphi_x(x,0,t)=-x\int_0^{t(x)} \frac{h''(\tau)\,\mathrm{d}\tau}{\sqrt{c^2(\tau)-x^2}}. \tag{2.23}$$

The horizontal velocity is continuous at the contact point $x=c(t)$, where

$$\varphi_x(c(t),0,t)=-c(t)\int_0^t \frac{h''(\tau)\,\mathrm{d}\tau}{\sqrt{c^2(\tau)-c^2(t)}}. \tag{2.24}$$

Equations (2.10) and (2.24) yield the following equation for the function $c(t)$:

$$\frac{\mathrm{d}c}{\mathrm{d}t}=-\gamma c(t)\int_0^t \frac{h''(\tau)\,\mathrm{d}\tau}{\sqrt{c^2(\tau)-c^2(t)}}. \tag{2.25}$$

Equation (2.25) will be studied in the next section.

The vertical velocity of the liquid boundary can be calculated by using (2.16) and the function $t(x)$. The result is

$$\varphi_y(x,0,t)=h'(t)-x\int_0^t \frac{h''(\tau)\,\mathrm{d}\tau}{\sqrt{x^2-c^2(\tau)}}\quad (x>c_0), \tag{2.26}$$
$$\varphi_y(x,0,t)=h'(t)-x\int_{t(x)}^t \frac{h''(\tau)\,\mathrm{d}\tau}{\sqrt{x^2-c^2(\tau)}}\quad (c(t)<x<c_0), \tag{2.27}$$
$$\varphi_y(x,0,t)=h'(t)\quad (0<x<c(t)). \tag{2.28}$$

Equations (2.26)–(2.28) show that the vertical velocity of the boundary is a continuous function of x.

3. Size of the contact region in the exit problem

To solve (2.25), it is convenient to introduce new non-dimensional variables α and σ such that

$$c^2(t) = c_0^2(1-\sigma), \quad c^2(\tau) = c_0^2(1-\alpha), \tag{3.1}$$

where α and σ are equal to zero when $t=0$ and $\tau=0$, correspondingly, and $\alpha=\sigma$ at $\tau=t$. A new unknown function $f(\sigma)$ is introduced by the equation

$$h''(t) = f(\sigma)\frac{\mathrm{d}c^2}{\mathrm{d}t}. \tag{3.2}$$

Equation (3.1) yields

$$\frac{\mathrm{d}c^2}{\mathrm{d}t} = -c_0^2\frac{\mathrm{d}\sigma}{\mathrm{d}t}, \tag{3.3}$$

which makes it possible to integrate (3.2) with the result

$$h'(t) = -c_0^2\int_0^\sigma f(\alpha)\,\mathrm{d}\alpha. \tag{3.4}$$

Substituting (3.1) and (3.2) in (2.25), we derive the nonlinear integral equation

$$h''(t) = 2\gamma c_0^3(1-\sigma)f(\sigma)\int_0^\sigma \frac{f(\alpha)\,\mathrm{d}\alpha}{\sqrt{\sigma-\alpha}}. \tag{3.5}$$

Equations (3.4) and (3.5) serve to evaluate the functions $f(\sigma)$ and $\sigma(t)$. The speed of the body $h'(t)$ and the body acceleration $h''(t)$ are assumed to be given functions of time in these equations. The equations are too complicated to be solved in a general case. Below we consider the exit problem for constant acceleration, $h''(t)=a$, of the body. Formally, the present model can only be used for small σ. However, in the following, we will use the derived equations up to $\sigma=1$ which corresponds to the end of the exit stage with $c=0$.

If the body acceleration is constant, then the body speed $h'(t)$ and the body displacement $h(t)$ are

$$h'(t) = at, \quad h(t) = \tfrac{1}{2}at^2, \tag{3.6}$$

and (3.4) and (3.5) can be significantly simplified. With respect to the non-dimensional unknown function

$$G(\sigma) = -f(\sigma)\left(\frac{a}{2\gamma c_0^3}\right)^{-1/2}, \tag{3.7}$$

equations (3.5) and (3.4) take the form

$$(1-\sigma)G(\sigma)\int_0^\sigma \frac{G(\alpha)\,\mathrm{d}\alpha}{\sqrt{\sigma-\alpha}} = 1, \tag{3.8}$$

$$t = \left(\frac{c_0}{2\gamma a}\right)^{1/2}\int_0^\sigma G(\alpha)\,\mathrm{d}\alpha. \tag{3.9}$$

It is seen that (3.8) is independent of any parameters of the problem and can be solved first with respect to the function $G(\sigma)$. Then we calculate $t(\sigma)$ using (3.9), which together with (3.1) provides $c(t)$ in parametric form.

The duration of the exit stage is given by (3.9) at $\sigma = 1$. The present linearized model is valid only for small displacements of the body, $h(t)/c_0 \ll 1$, that is, for small σ. However, we will use the obtained solution for $0 < \sigma < 1$ with the aim of comparing the asymptotic predictions with the results of numerical simulations. This comparison will give us ideas about the practical range of validity of the present model, which could be wider than that defined by the formal conditions derived in § 2. Note that the maximum displacement of the body during the exit stage is independent of the acceleration a and proportional to the initial size of the contact region,

$$h_{max} = \frac{c_0}{4\gamma}\left(\int_0^1 G(\alpha)\,\mathrm{d}\alpha\right)^2. \tag{3.10}$$

Equation (3.8) shows that the unknown function $G(\sigma)$ is singular at $\sigma = 0$, when the body starts to move, and at the end of the exit stage, $\sigma = 1$. For small σ, the solution is sought in the form

$$G(\sigma) = \tilde{q}_0\sigma^{-k} + \cdots. \tag{3.11}$$

Equations (3.8) and (3.11) provide in the leading order as $\sigma \to 0$

$$\tilde{q}_0^2\sigma^{-2k+(1/2)}\int_0^1 \frac{\xi^{-k}\,\mathrm{d}\xi}{\sqrt{1-\xi}} \sim 1, \tag{3.12}$$

which gives $k = 1/4$ and $\tilde{q}_0 = \pi^{-1/4}[\Gamma(5/4)/\Gamma(3/4)]^{1/2}$, where $\Gamma(x)$ is the gamma function. When $\sigma \to 1$, a similar analysis yields $G(\sigma) \sim D(1-\sigma)^{-3/4}$, where $D = \pi^{-1/4}[2\Gamma(5/8)/\Gamma(1/8)]^{1/2}$.

The solution of (3.8) is sought in the form

$$G(\sigma) = \tilde{q}_0\sigma^{-1/4}\sum_{n=0}^{\infty} q_n\sigma^n, \tag{3.13}$$

where $q_0 = 1$. The coefficients q_n in (3.13) are calculated by the recurrent relations

$$q_{n+1} = \left[q_n\left(1+\frac{3}{8}\beta_n\right) + \sum_{m=1}^{n-1} q_m\beta_m(q_{n-m} - q_{n-m+1})\right] \Big/ (1+\beta_{n+1}), \tag{3.14a}$$

$$\beta_{n+1} = \beta_n\left(1 - \frac{2}{4n+5}\right), \tag{3.14b}$$

$$\beta_0 = 1, \quad q_0 = 1, \quad q_1 = \frac{5}{8}, \quad q_2 = \frac{49\times 15}{22\times 64}. \tag{3.14c}$$

The results of the calculations by (2.17), (3.9), (3.13) and (3.14) are presented in figure 3 in terms of the non-dimensional size of the wetted area $c(t)/c_0 = \sqrt{1-\sigma}$, non-dimensional hydrodynamic force $F(t)/|F(0)| = \sigma - 1$, where $F(0) = -0.5\pi\rho a c_0^2$, and the body displacement $h(t) = 0.5at^2$, which is shown here to indicate the initial stage during which the non-dimensional displacement of the body is small and the present linearized model is expected to be valid. Note that the undefined parameter γ can be included in the time scale of the exit problem and does not affect the magnitude of the hydrodynamic force. At the end of the exit stage the force is small and the present model can be used even if the displacement of the body is not small.

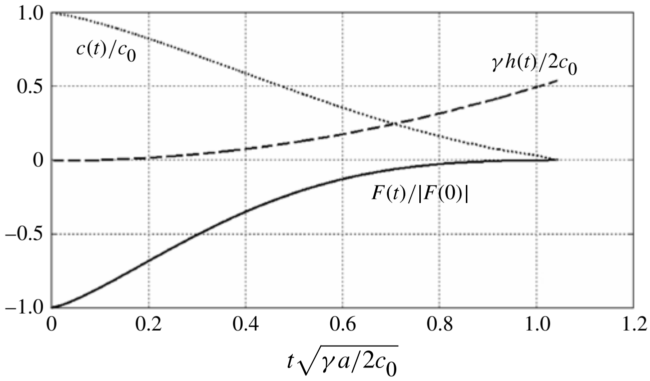

FIGURE 3. The non-dimensional size of the wetted area, hydrodynamic force and the vertical displacement of the body as functions of the non-dimensional time $t\sqrt{\gamma a/2c_0}$.

The obtained solution predicts that the speed of the contact point for small time behaves as

$$\frac{\mathrm{d}c}{\mathrm{d}t} \approx -\frac{1}{2}[3\gamma^2 c_0 a^2 t/\tilde{q}_0^4]^{1/3} \quad (t \to 0). \tag{3.15}$$

On the other hand, the small-time asymptotic analysis of the floating plate problem, where the plate is lifted upwards with a constant acceleration, can be performed in a same manner as that used by Iafrati & Korobkin (2008) in the problem of floating plate impact. The result is $c'(t) = O(t^{1/3})$, which is in agreement with the formula (3.15) predicted by the present model.

The shapes of the free surface at four time instants are shown in figure 4 for the parabolic contour, whose initial position is described by the equation $y = x^2/(2R) - h_0$, where $R = 1$ m and $h_0 = 2$ cm. Correspondingly, the initial coordinate of the intersection point is $c_0 = \sqrt{2Rh_0} = 20$ cm. Initially the liquid is at rest and the body starts to exit from the liquid at $t = 0$ with constant acceleration $a = 20$ m s^{-2}. The shape of the free surface, $y = \eta(x, t)$, is computed by integration of the linearized kinematic boundary condition $\partial\eta/\partial t = v(x, t)$ in time with the corresponding initial condition. Here $v(x, t) = \varphi_y(x, 0, t)$, where $|x| > c(t)$. The non-dimensional size of the wetted area, $c(t)/c_0$, is shown in figure 3 as a function of the non-dimensional time $\tau = t\sqrt{\gamma a/2c_0}$. In the present calculations, $\sqrt{\gamma a/2c_0} = 10$ s^{-1} for $\gamma = 2$. The differential equation for $v(x, t)$ follows from (2.16) and has the form

$$\frac{\partial v}{\partial t} = a\left(1 - \frac{x}{\sqrt{x^2 - c^2(t)}}\right) \quad (|x| > c(t)). \tag{3.16}$$

The differential equations for the elevation of the free surface $\eta(x, t)$ and the vertical velocity of the free surface $v(x, t)$ are integrated with respect to the non-dimensional time τ with time step 10^{-4} for values of the non-dimensional horizontal coordinate x/c_0 from zero to 0.44 with the step 0.0055. The function $c(t)$ in (3.16) was approximated by

$$c(\tau) = (1 - \mu_1\tau^{\mu_2} + \mu_3\tau^3)c_0, \tag{3.17}$$

where $\mu_1 = 1.46838$, $\mu_2 = 1.31525$, $\mu_3 = 0.509565$, in the interval $0 < \tau < 0.8$ (see figure 3) with the relative error 4×10^{-5}. Note that $\mu_2 \approx 4/3$, which is in agreement with the asymptotic formula (3.15). Figure 4 resembles the expected shape of the free

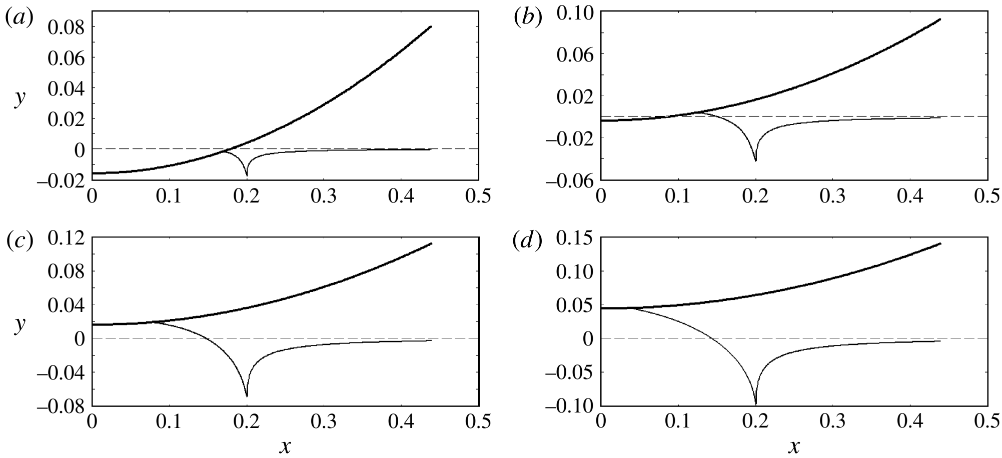

FIGURE 4. Shapes of the free surface at (*a*) $t = 0.02$ s, (*b*) $t = 0.04$ s, (*c*) $t = 0.06$ s and (*d*) $t = 0.08$ s are shown by thin lines. Thick lines show the positions of the rigid surface at the corresponding time instants. The horizontal and vertical axes are in metres. The initial position of the free surface is shown by the dashed line.

surface shown in figure 2(*c*) except for intervals around the initial position of the contact point $x = a_0$, where the slope of the free surface is significant and the surface tension is not negligible. To resolve this singularity, the surface tension has to be included in the analysis starting from the earliest stage. This can be done by using the asymptotic analysis similar to that employed by Korobkin & Iafrati (2005) in the problem of floating body impact by introducing inner regions close to the ends of the floating plate. The asymptotic analysis predicts that the size of the inner region is of the order of $t^{2/3}$ as $t \to 0$, the flow in the inner region is nonlinear and is governed by surface tension. The inner problem can only be solved by numerical methods. The inner flow close to the initial position of the contact point is not considered in this paper.

4. Axisymmetric problem

The results of § 3 are based on the formula (2.15) for the pressure distribution in the wetted area of the moving surface. In the corresponding axisymmetric problem, the pressure distribution in the contact region, $r < c(t)$, is given by

$$p(r, 0, t) = -\frac{2}{\pi}\rho h''(t)\sqrt{c^2(t) - r^2}, \tag{4.1}$$

where r and z are cylindrical coordinates. Equation (4.1) makes it possible to calculate the velocity potential in the wetted part of the circular wetted area and the radial velocity $\varphi_r(r, 0, t)$. The formula for the radial velocity is similar to (2.22) but with factor $2/\pi$. Correspondingly, in the equation (2.25) for the radius of the wetted area $c(t)$, γ should be changed to $2\gamma/\pi$. For constant acceleration of the body a, the analysis of § 3 provides

$$t\sqrt{\gamma a/2c_0} = \left(\frac{\pi}{8}\right)^{1/2}\int_0^{\sigma} G(\alpha)\,\mathrm{d}\alpha, \quad \frac{c(t)}{c_0} = (1-\sigma)^{1/2}, \tag{4.2}$$

$$F(t) = F(0)(1-\sigma)^{3/2}, \quad F(0) = -\frac{4}{3}\rho a c_0^3, \tag{4.3}$$

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where $G(\sigma)$ is given by (3.13) and (3.14). It is seen that the duration of the exit stage for a circular disc of diameter $2c_0$ is ∼40 % shorter than for a two-dimensional plate of the same width. The force $F(t)$ tends to zero more quickly at the end of the exit stage in the axisymmetric problem than in the two-dimensional problem.

5. Comparison with CFD results

Numerical calculations of the hydrodynamic force acting on a two-dimensional wedge of deadrise angle $\beta = 10^\circ$ were performed by Piro & Maki (2011) with a Navier–Stokes solver from OpenFOAM library. Both rigid and elastic wedges were studied. Only results for the rigid wedge are considered here. The wedge displacement is described by the equation

$$h(t) = Vt - \tfrac{1}{2}at^2, \tag{5.1}$$

where V is the initial speed of the wedge and a is the wedge deceleration. In computations, $V = 4\ \mathrm{m\ s^{-1}}$ and $a = 92\ \mathrm{m\ s^{-2}}$. Note that the body deceleration is about ten times higher than the gravitational acceleration g. The speed of the wedge is zero at $t = V/a$. The latter quantity is taken as the time scale, $t = V\tilde{t}/a$, where $\tilde{t}$ is the non-dimensional time. The time interval $0 < \tilde{t} < 1$ corresponds to the entry stage, during which the wedge penetrates the water, and $\tilde{t} > 1$ corresponds to the exit stage. The non-dimensional force $\tilde{F}(\tilde{t}) = F(t)/F_{sc}$ computed by Piro & Maki (2011) is shown in figure 5 by a thin line. The thick line in this figure corresponds to the hydrodynamic force calculated by the Wagner theory of water impact when $0 < \tilde{t} < 1$ and by the present linearized theory of water exit when $\tilde{t} > 1$. The parameter γ in the water exit model was set equal to 2, in order to fit the theoretical prediction to the computed force. Note that variation of the parameter γ stretches the force curve when $\tilde{t} > 1$ only in time but not in terms of the force magnitude. The force scale is taken as $\rho(V/2)^2B$, where $V/2$ is the mean speed of the body during the entry stage (see Piro & Maki 2011), ρ is the liquid density and $B = h(t_{sc})/\tan\beta$. The Wagner theory provides

$$\tilde{F}(\tilde{t}) = \frac{5\pi^3}{4\tan\beta}(\tilde{t}^2 - 2\tilde{t} + 4/5)(2 - \tilde{t})\tilde{t} \quad (0 < \tilde{t} < 1) \tag{5.2}$$

and, in particular, $\tilde{F}(1) \approx -43.9613$. It is seen that the theoretical force with $\gamma = 2$ reproduces well the numerical force during the exit stage. Moreover, the duration of the exit stage is well predicted. During the entry stage the force calculated by the Wagner model is higher than the force computed by Piro & Maki (2011). This is a well known feature of the Wagner model (see Wagner 1932). Prediction of the hydrodynamic force during the entry stage can be improved by using the modified Logvinovich model by Korobkin (2004): see figure 2 in Tassin *et al.* (2012).

Similar CFD computations were performed in Tassin *et al.* (2013) for parabolic contour $y = x^2/(2R)$, where $R = 1.37$ m, which motion is described by (5.1) with $V = 1\ \mathrm{m\ s^{-1}}$ and $a = 19.5376\ \mathrm{m\ s^{-2}}$. The speed of the contour is zero at $t = V/a$. This quantity is taken as the time scale, $t = V\tilde{t}/a$, where $\tilde{t}$ is the non-dimensional time. The time interval $0 < \tilde{t} < 1$ corresponds to the entry stage, and $\tilde{t} > 1$ to the exit stage. The non-dimensional force $\tilde{F}(\tilde{t}) = F(t)/F_{sc}$ computed in Tassin *et al.* (2013) is shown in figure 6 by a thin line. The thick line in this figure corresponds to the hydrodynamic force calculated by the Wagner theory of water impact when $0 < \tilde{t} < 1$ and by the present linearized theory of water exit when $\tilde{t} > 1$ with $\gamma = 2$. The force scale F_{sc} is taken as $\rho(V/2)^2B$, where $V/2$ is the mean speed of the body during the entry stage

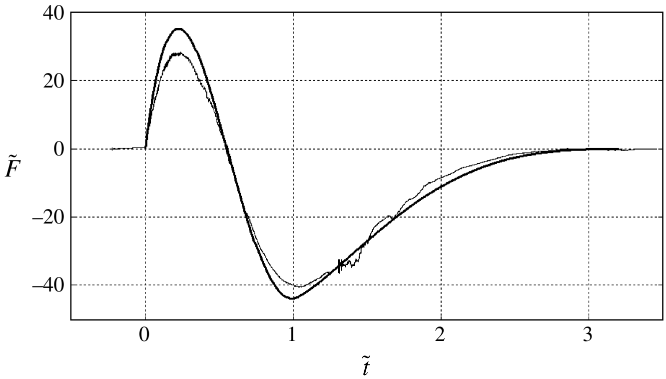

FIGURE 5. The non-dimensional hydrodynamic force $\tilde{F}$ acting on the wedge with deadrise angle of 10° entering water with the initial speed $V = 4\ \mathrm{m\ s^{-1}}$ and constant deceleration $a = 92\ \mathrm{m\ s^{-2}}$ as a function of the non-dimensional time $\tilde{t}$. The thin line corresponds to the numerical prediction by Piro & Maki (2011) and the thick line to the present model of water exit, where $\tilde{t} > 1$, and the Wagner model, where $0 < \tilde{t} < 1$.

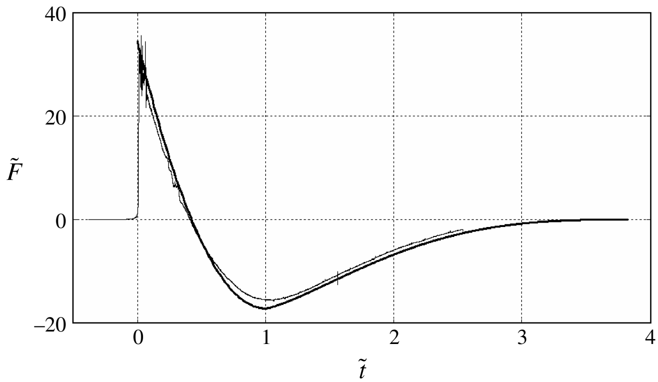

FIGURE 6. The non-dimensional hydrodynamic force $\tilde{F}$ acting on the parabolic contour entering water with the initial speed $V = 1\ \mathrm{m\ s^{-1}}$ and constant deceleration $a = 19.5376\ \mathrm{m\ s^{-2}}$ as a function of the non-dimensional time $\tilde{t}$. The thin line corresponds to the numerical prediction by Tassin *et al.* (2013) and the thick line to the present model of water exit, where $\tilde{t} > 1$, and the Wagner model, where $0 < \tilde{t} < 1$.

and $B = 1$ m in the CFD computations. The Wagner theory predicts

$$\tilde{F}(\tilde{t}) = 2\pi\rho R V^2(1 - 3\tilde{t} + 1.5\tilde{t}^2)/F_{sc} \quad (0 < \tilde{t} < 1). \tag{5.3}$$

Note that $\tilde{F}(1) = -0.5\tilde{F}(0)$. It is seen that both the entry and exit stages are well described by the corresponding theoretical models. The body returns to its initial position with $h = 0$ at $\tilde{t} = 2$, and the body is above the initial water level when $\tilde{t} > 2$.

6. Conclusion

A model of lifting of a body with small deadrise angle from a liquid surface has been presented. The model includes the liquid inertia only in a similar way to how it was done in the water entry problem by Wagner (1932). The shape of the body surface is not taken into account. The size of the wetted area of the body surface plays the most important role. The wetted area is diminishing in time. The speeds of the contact points are assumed to be proportional to the local speed of the flow at these points. It was shown that the coefficient in this condition can be included in the time scale of the problem and does not affect the magnitude of the hydrodynamic force during the exit stage. The problem was solved for constant acceleration of the body motion. The problem of a floating body lifting with a constant force can be solved in a similar way. More complicated motions require numerical solutions of (3.4) and (3.5). The problem of a body which exits from water and changes its shape in time is very challenging, but can also be treated within the approach presented. The solution of the latter problem is needed for evaluation of negative forces in the rear parts of high-speed vessels and the fuselage of a ditching aircraft within the $2D + t$ approximation.

Both two-dimensional and axisymmetric exit problems with constant accelerations were analysed in this paper. Three-dimensional problems can be solved within the present water exit model only numerically. Three-dimensional experimental results were published by Scolan *et al.* (2006) but for wave impact from beneath a horizontal plate when the gravity matters.

The hydrodynamic pressure (2.15) for constant acceleration a of the body exit is negative, that is, below the atmospheric pressure. Its magnitude is maximum at the centre of the wetted area, $x = 0$, at the initial instant, $t = 0$, when $c(0) = c_0$. Therefore, the minimum pressure in the contact area predicted by the linearized theory is equal to $-\rho a c_0$. In the conditions of the numerical analysis by Tassin *et al.* (2013), $\rho = 1000\ \mathrm{kg\ m^{-3}}$, $a = 19.5\ \mathrm{m\ s^{-2}}$ and $c_0 \approx 36$ cm, (2.15) gives $p(0, 0, +0) \approx -7.02$ kPa, which is close to the value computed numerically.

The shapes of the free surface during the exit stage are shown in figure 4. The free surface is tangential to the surface of the body at the contact points and drops down monotonically in the interval from $x = c(t)$ to $x = c_0$. Then the free surface rises to the initial liquid level where $x > c_0$. The tangent to the free surface is not continuous at $x = c_0$, which indicates that the surface tension is important close to this point. The initial stage of the body motion can be analysed using the approach by Korobkin & Iafrati (2005) developed for impact of a floating body. It can be shown that the flow close to the point $x = c_0$ depends on the surface tension and is self-similar in the inner variables $(x - c_0)/t^{2/3}$ and $y/t^{2/3}$. The inner solution can be obtained only numerically.

The energy conservation law is satisfied within the present exit model for any function $c(t)$. Note that simplified models of water entry do not satisfy this law, and additional effects should be included in the models, such as jetting or acoustic effects; see Korobkin & Peregrine (2000) for discussion of energy conservation in water impact problems.

The body wettability, contact line dynamics, viscous effects and surface tension are expected to play a major role close to the surface of the body. This local analysis has not yet been done. We expect that the local analysis will help us to justify (2.10) or to derive another condition governing the motion of the contact point within the inviscid model. In terms of the hydrodynamic force studied in § 5, condition (2.10) is fairly reasonable.

More dedicated numerical simulations by CFD and specially designed experiments are required to further develop theoretical models of water exit.

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