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COMPUTING THE FUNDAMENTAL GROUP OF A HIGHER-RANK GRAPH

SOORAN KANG[1], DAVID PASK[2] AND SAMUEL B.G. WEBSTER[3]

[1] *College of General Education, Chung-Ang University, Seoul 06974, Republic of Korea* (sooran09@cau.ac.kr)

[2] *School of Mathematics and Applied Statistics, The University of Wollongong, Wollongong, NSW 2522, Australia* (dpask@uow.edu.au)

[3] *Independent Hospital Pricing Authority, Level 6, 1 Oxford Street, Sydney, NSW 2000, Australia* (sbgwebster@gmail.com)

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Abstract We compute a presentation of the fundamental group of a higher-rank graph using a coloured graph description of higher-rank graphs developed by the third author. We compute the fundamental groups of several examples from the literature. Our results fit naturally into the suite of known geometrical results about higher-rank graphs when we show that the abelianization of the fundamental group is the homology group. We end with a calculation which gives a non-standard presentation of the fundamental group of the Klein bottle to the one normally found in the literature.

Keywords: Higher-rank graph; fundamental group

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Secondary: 18D99

1. Introduction

Higher-rank graphs, or k-graphs, are countable categories Λ equipped with a degree functor $d : \Lambda \to \mathbb{N}^k$ which satisfies a certain factorization property. They were introduced in [11] as a graphical approach to the higher-rank Cuntz–Krieger algebras introduced by Robertson and Steger in [21]. Since then k-graphs have been studied by many authors from several points of view (see [1–5,7,10,16,17,20,23], for example). The motivation for this paper comes from the recent developments of [9,12–14,18], where the geometric nature of k-graphs has been investigated and then used to construct new families of twisted k-graph C^*-algebras.

In fact, the geometric aspects of a k-graph have been of interest in their own right. In [18], the notion of a fundamental group was introduced for a connected k-graph, where connectivity is defined in the categorical sense (see Definition 5.1). In [9, §4], a connected k-graph is viewed as a connected k-dimensional CW-complex (and so has a 1-skeleton).

It was then shown that a k-graph may be realized as a topological space in a way which preserves homotopy type. It is therefore of interest to provide a facility to compute the fundamental group of a k-graph in terms of generators and relations. This is the main purpose of this paper.

To clarify the nature of the fundamental group we wish to compute, we briefly describe the approach of [9,18] here. We adapt the description of the fundamental group of a graph as described in [15, Chapter 6] and [22, §2] to our situation. The 1-skeleton Sk_Λ of a k-graph Λ is a directed graph, which we then view as a graph Sk_Λ^+ (the augmented graph). We may then quotient out the finite path space $\mathcal{P}(\mathrm{Sk}_\Lambda^+)$ by trivial paths, and identifications made by the factorization rules in Λ to form the fundamental groupoid and fundamental group of Λ, see [18, §3] and [9, §4]. To establish notation, we review this construction in slightly more detail in § 2.

Following [22], the fundamental group of a graph is easy to compute: Given a maximal spanning tree of the graph, the generators of the fundamental group are indexed by the edges of the graph which are not in the tree. Furthermore, the group generated is a free group (see [22, §2.1.8]). We adapt this process to compute the fundamental group of a k-graph: Take a maximal spanning tree of Sk_Λ^+ and quotient out the generators of the fundamental group of Sk_Λ by the relations coming from the factorization rules in Λ. Hence, the fundamental group of a k-graph is usually not a free group (see Theorem 5.4 below).

Key to our analysis is the frequent use of coloured graphs to visualize and describe the structure of a k-graph (for a complete description of how this works, see [8]). Briefly, the 1-skeleton Sk_Λ of a k-graph Λ is a directed graph, which together with a colouring $c : \mathrm{Sk}_\Lambda^1 \to \{c_1, \ldots, c_k\}$ of the edges forms a k-coloured graph (Sk_Λ, c), called the skeleton. As shown in [8], a k-coloured graph does not define, or completely determine a k-graph. To make a satisfactory correspondence between k-graphs and k-coloured graphs, we need some additional combinatorial data $\mathcal{C}$, which encodes the natural quotient structure of Λ. To establish notation, we review the relationship between k-graphs and k-coloured graphs in slightly more detail in § 3 and § 4.

In § 5, we implement the method described in the third paragraph above to give a presentation of the fundamental group of a k-graph in terms of the fundamental group of its 1-skeleton (see Theorem 5.4). To illustrate the efficacy of our result, we give several computations in Examples 5.5. Finally, in § 6, we show that the abelianization of the fundamental group of a k-graph agrees with its homology as defined in [12]. Then, in Example 6.3, we compute the fundamental group of a 2-graph from the Klein bottle example in [9, Example 3.13] which reveals a non-standard presentation of this group.

2. Conventions

For $k \geq 1$, let $\mathbb{N}^k$ denote the monoid of k-tuples of natural numbers under addition and denote the canonical generators by $e_1, e_2, \ldots, e_k$. For $m \in \mathbb{N}^k$, we write $m = \sum_{i=1}^k m_i e_i$ then for $m, n \in \mathbb{N}^k$ we say that $m \leq n$ if and only if $m_i \leq n_i$ for $i = 1, \ldots, k$.

A *directed graph* E is a quadruple (E^0, E^1, r_E, s_E), where E^0 is the set of *vertices*, E^1 is the set of *edges*, and $r_E, s_E : E^1 \to E^0$ are *range* and *source* maps, giving a direction to each edge (if there is no chance of confusion we will drop the subscripts). We follow the conventions of [19] which are suited to the categorical setting we wish to pursue:

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a *path of length* n is a sequence $\mu = \mu_1\mu_2\cdots\mu_n$ of edges such that $s(\mu_i) = r(\mu_{i+1})$ for $1 \le i \le n-1$. We denote by E^n the set of all paths of length n, and define $E^* = \bigcup_{n\in\mathbb{N}} E^n$. We extend r and s to E^* by setting $r(\mu) = r(\mu_1)$ and $s(\mu) = s(\mu_n)$.

To align with the established literature on the fundamental groupoid of a k-graph, the following definitions are taken from [18, Definition 5.1]: Let E be a directed graph. For each $e \in E^1$, we introduce an *inverse* edge e^{-1} with $s(e^{-1}) = r(e)$ and $r(e^{-1}) = s(e)$. Let $E^{-1} = \{e^{-1} : e \in E^1\}$ and $E^u = E^1 \cup E^{-1}$, then $E^+ = (E^0, E^u, r, s)$ is a directed graph called the *augmented graph of* E. Let $\mathcal{P}(E^+)$ be the path category of E^+. If we set $(e^{-1})^{-1} = e$ then $e^{-1} \in \mathcal{P}(E^+)$ for any $e \in E^u$ and by extension $\lambda^{-1} = \lambda_n^{-1}\cdots\lambda_1^{-1} \in \mathcal{P}(E^+)$ for any $\lambda = \lambda_1\cdots\lambda_n \in \mathcal{P}(E^+)$. Elements of $\mathcal{P}(E^+)$ are called *undirected paths in* E, and elements of $\mathcal{P}(E^+)$ which do not contain ee^{-1} for any $e \in E^u$ are called *reduced* undirected paths in E (vertices are reduced paths).

Let E be a directed graph then E is *connected* if for every $u, v \in E^0$ there is $\alpha \in \mathcal{P}(E^+)$ with $u = s(\alpha)$ and $v = r(\alpha)$. A tree T is a connected directed graph such that the only reduced $\alpha \in \mathcal{P}(T^+)$ with the same source and range are vertices. Let E be a directed graph with subgraph T which is a tree, then T is a *maximal spanning tree* if $T^0 = E^0$. Every connected directed graph has a (not necessarily unique) maximal spanning tree (see [22, §2.1.5]).

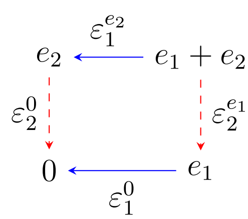
Remark 2.1. Fix a maximal spanning tree T of a connected directed graph E and $v \in E^0$. For each $w \in E^0$, there is a unique reduced path η_w in the augmented graph T^+ from v to w, which is an element of the fundamental groupoid $\mathcal{G}(E)$. For $\lambda \in \mathcal{P}(E^+)$ define $\zeta_\lambda = \eta_{r(\lambda)}^{-1}\lambda\eta_{s(\lambda)} \in v\mathcal{P}(E^+)v$. Note that $\zeta_\lambda^{-1} = \zeta_{\lambda^{-1}}$.

3. Coloured graphs

For $k \ge 1$, a *k-coloured graph* E is a directed graph along with a *colour map* $c_E : E^1 \to \{c_1, \ldots, c_k\}$. By considering $\{c_1, \ldots, c_k\}$ as generators of the free group $\mathbb{F}_k$, we may extend $c_E : E^* \setminus E^0 \to \mathbb{F}_k^+$ by $c_E(\mu_1\cdots\mu_n) = c_E(\mu_1)c_E(\mu_2)\cdots c_E(\mu_n)$. We will drop the subscript from c_E if there is no risk of confusion. For k-coloured directed graphs E and F, a *coloured-graph morphism* $\phi : F \to E$ is a graph morphism satisfying $c_E \circ \phi^1 = c_F$.

For 2-coloured graphs, the convention is to draw edges with colour c_1 blue (or solid) and edges with colour c_2 red (or dashed).

Example 3.1. For $k \ge 1$ and $m \in \mathbb{N}^k$, the k-coloured graph $E_{k,m}$ is defined by $E^0_{k,m} = \{n \in \mathbb{N}^k : 0 \le n \le m\}$, $E^1_{k,m} = \{\varepsilon_i^n : n, n + e_i \in E^0_{k,m}\}$, with $r(\varepsilon_i^n) = n$, $s(\varepsilon_i^n) = n + e_i$ and $c_E(\varepsilon_i^n) = c_i$. The 2-coloured graph E_{2,e_1+e_2} is used often and is depicted below.



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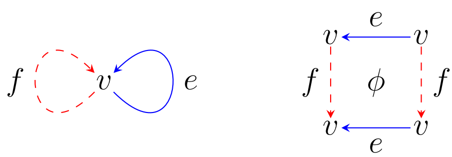
Let E be a k-coloured graph and $i \neq j \leq k$. A coloured graph morphism $\phi : E_{k,e_i+e_j} \to E$ is called a *square* in E. One may represent a square ϕ as a labelled version of E_{k,e_i+e_j}. For instance, the 2-coloured graph below on the left has only one square ϕ, shown to its right, given by $\phi(n) = v$ for all $n \in E^0_{2,e_1+e_2}$, $\phi(\varepsilon^0_1) = \phi(\varepsilon^{e_2}_1) = e$ and $\phi(\varepsilon^0_2) = \phi(\varepsilon^{e_1}_2) = f$.



$\mathcal{C}_E = \{\phi : E_{k,e_i+e_j} \to E : 1 \leq i \neq j \leq k\}$ denotes the set of squares in a k-coloured graph E.

A collection of squares $\mathcal{C}$ in a k-coloured graph E is called *complete* if for each $i \neq j \leq k$ and c_ic_j-coloured path $fg \in E^2$, there exists a unique $\phi \in \mathcal{C}$ such that $\phi(\varepsilon^0_i) = f$ and $\phi(\varepsilon^{e_i}_j) = g$. In this case, uniqueness of ϕ gives a unique c_jc_i-coloured path $g'f'$ with $g' = \phi(\varepsilon^0_j)$ and $f' = \phi(\varepsilon^{e_j}_i)$. We will write $fg \sim_{\mathcal{C}} g'f'$ and refer to elements $(fg, g'f')$ of this relation as *commuting squares.*

Example 3.2. For $n \geq 1$ define $\underline{n} = \{1, \ldots, n\}$. For $m, n \geq 1$, let $\theta : \underline{m} \times \underline{n} \to \underline{m} \times \underline{n}$ be a bijection. Let E_θ be the 2-coloured graph with $E^0_\theta = \{v\}$, $E^1_\theta = \{f_1, \ldots, f_m, g_1, \ldots, g_n\}$, and colouring map $c : E^1_\theta \to \{c_1, c_2\}$ by $c(f_i) = c_1$ for $i \in \underline{m}$ and $c(g_j) = c_2$ for $j \in \underline{n}$. For each $(i, j) \in \underline{m} \times \underline{n}$, define $\phi_{(i,j)} : E_{2,e_1+e_2} \to E_\theta$ by

$$\phi_{(i,j)}(\varepsilon^0_1) = f_i, \quad \phi_{(i,j)}(\varepsilon^{e_2}_1) = f_{i'}, \ \phi_{(i,j)}(\varepsilon^0_2) = g_{j'}, \text{ and}$$
$$\phi_{(i,j)}(\varepsilon^{e_1}_2) = g_j, \text{ where } \theta(i, j) = (i', j').$$

As θ is a bijection $\mathcal{C}_{E_\theta} = \{\phi_{(i,j)} : (i, j) \in \underline{m} \times \underline{n}\}$ is a complete collection of squares.

4. Higher-rank graphs

A *k-graph* (or a *higher-rank graph*) is a countable category Λ with a *degree* functor $d : \Lambda \to \mathbb{N}^k$ satisfying the *factorization property*: if $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ are such that $d(\lambda) = m + n$, then there are unique $\mu, \nu \in \Lambda$ with $d(\mu) = m$, $d(\nu) = n$ and $\lambda = \mu\nu$.

Given $m \in \mathbb{N}^k$ we define $\Lambda^m := d^{-1}(m)$. Given $v, w \in \Lambda^0$ and $F \subseteq \Lambda$ define $vF := r^{-1}(v) \cap F$, $Fw := s^{-1}(w) \cap F$, and $vFw := vF \cap Fw$. The factorization property allows us to identify Λ^0 with $\mathrm{Obj}(\Lambda)$, and we call its elements *vertices.*

By the factorization property, for each $\lambda \in \Lambda$ and $m \leq n \leq d(\lambda)$, we may write $\lambda = \lambda'\lambda''\lambda'''$, where $d(\lambda') = m$, $d(\lambda') = n - m$ and $d(\lambda'') = d(\lambda) - n$; then $\lambda(m, n) := \lambda''$. For more information about k-graphs, see [8,11,20] for example.

Examples 4.1. (a) Let E be a directed graph. The collection E^* of finite paths in E forms a category, called the *path category* of E, denoted by $\mathcal{P}(E)$. The map $d : \mathcal{P}(E) \to \mathbb{N}$ defined by $d(\mu) = n$ if and only if $\mu \in E^n$ is a functor which satisfies the factorization property, hence $(\mathcal{P}(E), d)$ is a 1-graph. It turns out that every 1-graph arises in this way (see [11, Example 1.3]).

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(b) For $k \geq 1$; let $\Delta_k = \{(m, n) : m, n \in \mathbb{Z}^k : m \leq n\}$. With structure maps $r(m, n) = m$, $s(m, n) = n$, so that $(\ell, n) = (\ell, m)(m, n)$ then Δ_k is a category. Set $d(m, n) = n - m$, then d is a functor and (Δ_k, d) is a k-graph. The vertices $\Delta_k^0 = \{(m, m) : m \in \mathbb{Z}^k\}$ may be identified with $\mathbb{Z}^k$.

(c) Resuming the notation of Example 3.2 let $\theta : \underline{m} \times \underline{n} \to \underline{m} \times \underline{n}$ be a bijection. Let $\mathbb{F}_\theta^2$ be the semigroup with generators $\{f_1, \ldots, f_m, g_1, \ldots g_n\}$ and relations $f_i g_j = g_{j'} f_{i'}$ where $\theta(i, j) = (i', j')$ for $(i, j) \in \underline{m} \times \underline{n}$. Let $d(f_i) = e_1$ for $i = 1, \ldots, m$ and $d(g_j) = e_2$ for $j = 1, \ldots, n$ then d extends to a functor from $\mathbb{F}_\theta^2$ to $\mathbb{N}^2$ with the factorization property, and so $\mathbb{F}_\theta^2$ is a 2-graph (see [23, §2]).

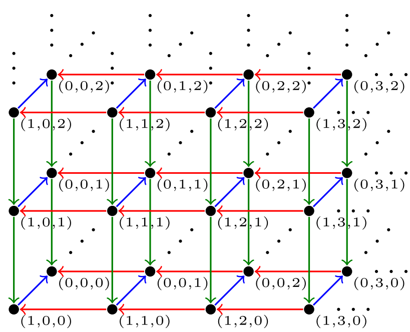
(d) Recall from [6] that if Λ is a k-graph and α is an automorphism of Λ, then there is a $(k+1)$-graph $\Lambda \times_\alpha \mathbb{Z}$ with morphisms $\Lambda \times \mathbb{N}$, range and source maps given by $r(\lambda, n) = (r(\lambda), 0)$, $s(\lambda, n) = (\alpha^{-n}(s(\lambda)), 0)$, degree map given by $d(\lambda, n) = (d(\lambda), n)$ and composition given by $(\lambda, m)(\mu, n) := (\lambda\alpha^m(\mu), m + n)$. In particular $(\Lambda \times_\alpha \mathbb{Z})^0 = \Lambda^0 \times \{0\}$.

We define the *skeleton* of a k-graph Λ to be a k-coloured graph. It consists of the 1-skeleton Sk_Λ of Λ, which is a directed graph given by $\mathrm{Sk}_\Lambda^0 = \mathrm{Obj}(\Lambda)$, $\mathrm{Sk}_\Lambda^1 = \bigcup_{i \leq k} \Lambda^{e_i}$, with range and source as in Λ. There is a natural colouring map $c : \mathrm{Sk}_\Lambda^1 \to \{c_1, \ldots, c_k\}$ given by $c(f) = c_i$ if and only if $f \in \Lambda^{e_i}$. The skeleton (Sk_Λ, c) comes with a canonical set of bi-coloured squares $\mathcal{C}_\Lambda := \{\phi_\lambda : \lambda \in \Lambda^{e_i + e_j} : i \neq j \leq k\}$, where the colour-preserving graph morphism $\phi_\lambda : E_{k, e_i + e_j} \to \mathrm{Sk}_\Lambda$ is given by $\phi_\lambda(\varepsilon_\ell^n) = \lambda(n, n + e_\ell)$ for each $n \leq e_i + e_j$ and $\ell = i, j$. The collection $\mathcal{C}_\Lambda$ is complete by [8, Lemma 4.2].

Conversely, in [8, Theorem 4.4, Theorem 4.5], it is shown that for a k-coloured graph E with a complete, associative* collection of squares $\mathcal{C}_E$ determines a unique k-graph $\Lambda_{E, \mathcal{C}_E}$.

Examples 4.2. (a) The 2-graph $\Lambda_{E_\theta, \mathcal{C}_{E_\theta}}$, determined by the 2-coloured graph (E_θ, c) with squares $\mathcal{C}_{E_\theta}$ described in Example 3.2 is isomorphic to $\mathbb{F}_\theta^2$ defined in Examples 4.1 (c).

(b) Recall the k-graph Δ_k described in Examples 4.1. Part of the skeleton of Δ_3, as seen from the first octant is shown below:



* The associative condition which only applies if $k \geq 3$ is quite complicated, and we will not deal with it here. For more details, see [8, §3]

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It is straightforward to see that $\mathrm{Sk}^0_{\Delta_3} = \mathbb{Z}^3$, $\mathrm{Sk}^1_{\Delta_3} = \{(m,\, m+e_j) : m \in \mathbb{Z}^3,\, 1 \le j \le 3\}$, $r(m,\, m+e_j) = m$ and $s(m,\, m+e_j) = m+e_j$. The commuting squares are

$$\begin{aligned}\mathcal{C} = \{(m, m+e_i)&(m+e_i, m+e_i+e_j)\\ &= (m, m+e_j)(m+e_i, m+e_i+e_j) : m \in \mathbb{Z}^3, 1 \le i \ne j \le 3\}.\end{aligned}$$

One checks that this collection of squares is complete and associative.

5. Computing the fundamental group of a k-graph

In this section, we define and provide a presentation of the fundamental group of a k-graph. Kaliszewski, Kumjian, Quigg and Sims show in [9, Corollary 4.2] that the fundamental group of a k-graph may be realized as a quotient of the fundamental group of its skeleton. We provide an alternative proof of this in Theorem 5.4 which yields a natural presentation of the group. We demonstrate the practical use of our result in Examples 5.5.

Definition 5.1 (Kumjian et al. [12, Definition 2.8]). We say that the k-graph Λ is *connected* if the equivalence relation on Λ^0 generated by the relation $u \sim v$ iff $u\Lambda v \ne \emptyset$ is $\Lambda^0 \times \Lambda^0$.

We review the construction of the fundamental groupoid of a connected k-graph from [18]. First, we describe the fundamental groupoid $\mathcal{G}(E)$ of a directed graph E.

Following [18, p. 197], let E be a directed graph, then a *relation* for E is a pair $(\alpha,\, \beta)$ of paths in $\mathcal{P}(E)$ such that $s(\alpha) = s(\beta)$ and $r(\alpha) = r(\beta)$. If K is a set of relations for E, then $\mathcal{P}(E)/K$ is the quotient of $\mathcal{P}(E)$ by the equivalence relation generated by K, for more details, see [18, §2].

As in [18, Definition 5.2], let $C = \{(e^{-1}e,\, s(e)) : e \in E^u\}$ and call C the set of *cancellation relations* for E^+. The quotient $\mathcal{P}(E^+)/C$ is then the fundamental groupoid, $\mathcal{G}(E)$ of E. We denote the quotient functor $\mathcal{P}(E^+) \to \mathcal{G}(E)$ by q_C. Elements of $\mathcal{G}(E)$ are reduced undirected paths in E with composition given by concatenation followed by cancellation.

Now we turn to defining the fundamental groupoid of a k-graph Λ. First, apply the above construction to form $\mathcal{G}(E)$ where $E = \mathrm{Sk}_\Lambda$. As in [8,18], let S be the equivalence relation on $\mathcal{P}(E)$ generated by $\mathcal{C}_\Lambda$, the commuting squares in E determined by Λ. That is, the transitive closure in $\mathcal{P}(E) \times \mathcal{P}(E)$ of

$$\begin{aligned}&\textstyle\bigcup_{n\ge 2}\{(\mu,\nu) \in E^n \times E^n :\ \text{there exists } i < n \text{ such that}\\ &\qquad \mu_j = \nu_j \text{ whenever } j \notin \{i, i+1\} \text{ and } \mu_i\mu_{i+1} \sim_{\mathcal{C}_\Lambda} \nu_i\nu_{i+1}\}.\end{aligned}$$

As in [18, Observation 5.3], this relation may be extended uniquely to a relation S^+ on $\mathcal{P}(E^+)$ by adding the relation $(f^{-1}e^{-1},\, h^{-1}g^{-1})$ whenever $(ef,\, gh) \in S$. This induces a relation, also called S^+, on $\mathcal{G}(E)$.

Definition 5.2. Let Λ be a connected k-graph. Then the *fundamental groupoid*, $\mathcal{G}(\Lambda)$ is

$$\mathcal{G}(\Lambda) := \mathcal{G}(\mathrm{Sk}_\Lambda)/S^+ = (\mathcal{P}(\mathrm{Sk}^+_\Lambda)/C)/S^+ = \mathcal{P}(\mathrm{Sk}^+_\Lambda)/(C \cup S^+).$$

For $v \in \Lambda^0$ the *fundamental group based at* $v \in \Lambda^0$ is the isotropy group $\pi_1(\Lambda,\, v) := v\mathcal{G}(\Lambda)v$.

The above definition of the fundamental groupoid of a k-graph is consistent with the one given in [18, Definition 5.6] (see also the accompanying discussion).

Our goal is to obtain a practical way of giving a presentation of $\pi_1(\Lambda, v)$. First recall that for a connected directed graph E, the quotient functor $q_C : \mathcal{P}(E^+) \to \mathcal{P}(E^+)/C = \mathcal{G}(E)$ restricts to $v\mathcal{P}(E^+)v$ and the image is the isotropy group $v\mathcal{G}(E)v$, which is by definition the fundamental group of E at v, denoted $\pi_1(E, v)$.

The following result is well known (see [22] for instance).

Lemma 5.3. *Let E be a connected directed graph, $v \in E^0$, and T be a maximal spanning tree of E. Then $\pi_1(E, v) \cong \langle E^1 \mid T^1 \rangle := \langle e \in E^1 \mid e = 1 \text{ if } e \in T^1 \rangle$.*

Proof. Suppose $e \notin T^u = T^1 \sqcup T^{-1}$. With notation as in Remark 2.1, all the edges of $\eta_{r(e)}, \eta_{s(e)}$ are in T^u, so ζ_e is reduced undirected path in E and hence $q_C(\zeta_e) = \zeta_e$. Suppose that $e \in T^u$ then ζ_e is an undirected path in T from v to v and so its reduced form must be v, hence $q_C(\zeta_e) = v$.

To complete the proof, it suffices to show that $\{\zeta_e : e \in E^1 \backslash T^1\}$ freely generate $\pi_1(E, v)$. This is a standard result, see [22, §2.1.7, §2.1.8] for example. □ □

Since Lemma 5.3 holds for any choice of maximal spanning tree, it follows that $\pi_1(E, v)$ does not depend on the choice of basepoint v. We denote the fundamental group of a graph E by $\pi_1(E)$. Now we turn our attention to computing the fundamental group of a connected k-graph Λ. Since $\Lambda \cong \mathcal{P}(\mathrm{Sk}_\Lambda)/S$, we expect the relation S to appear in the description of $\pi_1(\Lambda)$.

Theorem 5.4. *Let Λ be a connected k-graph, $v \in \Lambda^0$ and let T be a maximal spanning tree for Sk_Λ. Then $\pi_1(\Lambda, v) \cong \langle \mathrm{Sk}_\Lambda^1 \mid t = 1 \text{ if } t \in T^1,\ ef = gh \text{ if } (ef, gh) \in S^+ \rangle$.*

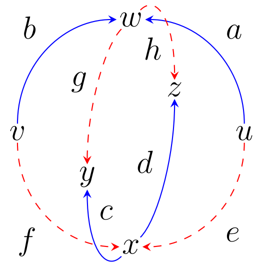
Proof. Denote by $q_S : \mathcal{G}(\mathrm{Sk}_\Lambda) \to \mathcal{G}(\mathrm{Sk}_\Lambda)/S^+ = \mathcal{G}(\Lambda)$ the quotient map. With notation as in Remark 2.1, observe that $\zeta_e\zeta_f = \zeta_{ef}$ and $\zeta_g\zeta_h = \zeta_{gh}$ in $\mathcal{G}(\mathrm{Sk}_\Lambda)$. Hence $q_S(\zeta_e\zeta_f) = q_S(\zeta_g\zeta_h)$ if and only if $(ef, gh) \in S^+$. Since taking quotients preserves objects, $\pi_1(\Lambda, v) = \pi_1(\mathrm{Sk}_\Lambda)/S^+$. Then Lemma 5.3 implies that $\pi_1(\Lambda, v) \cong \langle \mathrm{Sk}_\Lambda^1 \mid t = 1 \text{ if } t \in T^1,\ ef = gh \text{ if } (ef, gh) \in S^+ \rangle$. □ □

Since Theorem 5.4 holds for every choice of T, the group $\pi_1(\Lambda, v)$ does not depend on T. We henceforth denote by $\pi_1(\Lambda)$ the fundamental group of Λ. Theorem 5.4 gives us an explicit presentation of $\pi_1(\Lambda)$, as seen in the following examples.

Examples 5.5. 1 Let Σ be the 2-graph, which is completely determined by its skeleton, shown below. In [9, Example 3.10], it was shown that the topological realization of Σ is the 2-sphere S^2. Let T be the maximal spanning tree for Sk_Σ consisting of edges $T^1 = \{a, b, c, d, e\}$. The commuting squares in Sk_Σ are (ga, ce), (gb, cf), (de, ha) and (df, hb), thus Theorem 5.4 gives

$$\pi_1(\Sigma) \cong \langle \mathrm{Sk}_\Sigma^1 \mid t = 1 \text{ if } t \in T^1, ga = ce, gb = cf, de = ha, df = hb \rangle.$$

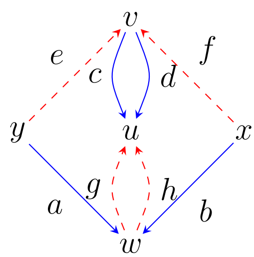
The first relation forces $g = 1$, the second $g = f$, the third $h = 1$ and the fourth $f = h$. Hence all the generators of $\pi_1(\Sigma)$ are equal to 1. Therefore $\pi_1(\Sigma)$ is trivial.



2 Consider the 2-graph Π with skeleton Sk_Π shown below, with commuting squares (ga, ce), (gb, df), (hb, cf) and (ha, de). In [9, Example 3.12], it was shown that the topological realization of Π is the projective plane. Choose spanning tree T of Sk_Π with $T^1 = \{a, b, c, f\}$. Then Theorem 5.4 gives

$$\pi_1(\Pi) = \langle \mathrm{Sk}^1_\Pi \mid t = 1 \text{ if } t \in T^1, ga = ce, gb = df, hb = cf, ha = de\rangle.$$

The relations become $g = e$, $g = d$, $h = 1$ and $de = 1$. So the fundamental group of $\pi_1(\Pi) \cong \langle e \mid e^2 = 1\rangle \cong \mathbb{Z}/2\mathbb{Z}$, the fundamental group of the projective plane.



3 Recall the skeleton of the 2-graph $\mathbb{F}^2_\theta$ described in Example 3.2. Since $\mathbb{F}^2_\theta$ has a single vertex, v, the maximal spanning tree for its 1-skeleton $\mathrm{Sk}_{\mathbb{F}^2_\theta}$ is v. The commuting squares of $\mathrm{Sk}_{\mathbb{F}^2_\theta}$ are $(f_i g_j,\, g_{j'} f_{i'})$ where $\theta(i, j) = (i', j')$ for $(i, j) \in \underline{m} \times \underline{n}$. Hence by Theorem 5.4 the fundamental group of $\mathbb{F}^2_\theta$ is

$$\langle f_i, g_j \mid f_i g_j = g_{j'} f_{i'} \text{ where } \theta(i, j) = (i', j')\rangle.$$

For different choices of θ, we get quite different fundamental groups:

(a) If $m = n = 2$ and $\theta : \underline{2} \times \underline{2} \to \underline{2} \times \underline{2}$ is the identity map then

$$\pi_1(\mathbb{F}^2_\theta) \cong \langle f_1, f_2\rangle \times \langle g_1, g_2\rangle \cong (\mathbb{Z} * \mathbb{Z}) \times (\mathbb{Z} * \mathbb{Z}) = \pi_1(S^1 \vee S^1) \times \pi_1(S^1 \vee S^1),$$

where $*$ is the free product, and $\vee$ is the wedge sum.

(b) If $m = n = 2$ and θ is given by $\theta(i, j) = (j, i)$ then by Theorem 5.4, we have

$$\pi_1(\mathbb{F}^2_\theta) \cong \langle f_1, f_2, g_1, g_2 \mid f_1g_1 = g_1f_1, f_1g_2 = g_1f_2, f_2g_1 = g_2f_1, f_2g_2 = g_2f_2\rangle.$$

The first and fourth relations give $f_i^{-1} g_i = g_i f_i^{-1}$ for $i = 1, 2$, then using the second relation, we have $f_1^{-1} g_1 = g_2 f_2^{-1}$. Putting these together gives $g_1 f_1^{-1} =$

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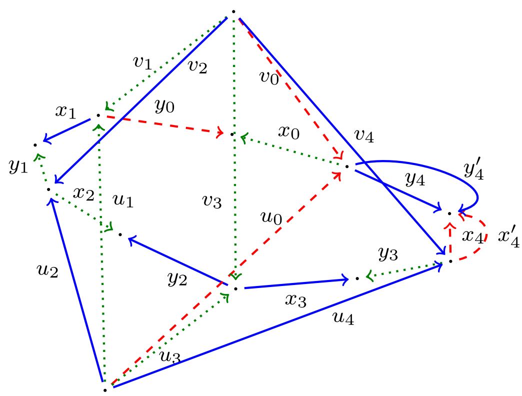
$f_2^{-1}g_2$, and hence $f_2g_1 = g_2f_1$. So the third relation is redundant and

$$\pi_1(\mathbb{F}_\theta^2) \cong (\mathbb{Z}^2 * \mathbb{Z}^2)/\langle g_1f_1^{-1} = g_2f_2^{-1}\rangle = \mathbb{Z}^2 *_{\mathbb{Z}} \mathbb{Z}^2,$$

where $\{f_1, g_1\}$ generate the first copy of $\mathbb{Z}^2$ and $\{f_2, g_2\}$ generate the second copy, and the amalgamation over $\mathbb{Z}$ is with respect to the identifications of $\mathbb{Z}$ in $\mathbb{Z}^2$ given by $1 \mapsto g_if_i^{-1}$ for $i = 1, 2$. So $\mathbb{F}_\theta^2$ has the same fundamental group as the two-holed torus.

4 It would be nice to include more higher-dimensional examples with a significant geometric content, such as those in [16,17]. However, the computation becomes difficult to work with as the spanning tree only uses relatively few edges of the 1-skeleton. The following example takes a 3-graph whose geometric realization is a sphere and adds two extra edges x_4', y_4' to make its fundamental group non-trivial (cf. Example (i) above).



(2)

Corresponding to the relations

$$\begin{array}{ll} y_0u_1 = x_0u_0 & y_0v_1 = x_0v_0 \\ x_1u_1 = y_1u_2 & x_1v_1 = y_1v_2 \\ y_2u_3 = x_2u_2 & y_2v_3 = x_2v_2 \\ x_3u_3 = y_3u_4 & x_3v_3 = y_3v_4 \\ y_4'u_0 = x_4u_4 & y_4v_0 = x_4'v_4 \\ y_4u_0 = x_4'u_4 & y_4'v_0 = x_4v_4 \end{array}$$

Choose spanning tree with edges y_0, x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4, u_0, v_0. Then by Theorem 5.4, the fundamental group of the 3-graph shown above is

$$\langle x_i, y_i, u_i, v_i, i = 0, \ldots, 4, x_4', y_4' : x_j = 1, j = 1, \ldots 4, \\ y_i = 1, i = 0, \ldots, 4,\ u_0 = 1, v_0 = 1\rangle$$

Applying the relations in the first column, we get $x_0 = u_1 = u_2 = u_3 = u_4 = y_4'$, $x_4'u_4 = 1$, and from the second column, we have $x_0 = v_1 = v_2 = v_3 = v_4 = y_4'$, $x_4'v_4 = 1$. Hence, the fundamental group is $\langle x_0, x_4' : x_4'x_0 = 1\rangle \cong \mathbb{Z}$ (since the generating set is redundant, $x_4' = x_0^{-1}$).

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6. Relationship with first homology group

Full versions of the following definitions may be found in [12, §3]. Let X be a set. We write $\mathbb{Z}X$ for the free abelian group generated by X. For a k-graph Λ, set $C_0(\Lambda) = \mathbb{Z}\Lambda^0$, $C_1(\Lambda) = \mathbb{Z}\Lambda^{e_1} \oplus \cdots \oplus \mathbb{Z}\Lambda^{e_k}$ and $C_2(\Lambda) = \oplus_{1\le i<j\le k}\mathbb{Z}\Lambda^{e_i+e_j}$.

Let $\partial_1^\Lambda : C_1(\Lambda) \to C_0(\Lambda)$ be the homomorphism determined by $\partial_1^\Lambda(\lambda) = s(\lambda) - r(\lambda)$. Define $\partial_2^\Lambda : C_2(\Lambda) \to C_1(\Lambda)$ as follows. Suppose $\lambda \in \Lambda^{e_i+e_j}$ where $1 \le i < j \le k$. Factorize $\lambda = f_1 g_1 = g_2 f_2$ where $f_r \in \Lambda^{e_i}$ and $g_r \in \Lambda^{e_j}$ for $r = 1, 2$, then set $\partial_2^\Lambda(\lambda) = f_1 + g_1 - f_2 - g_2$ and extend to a homomorphism from $C_2(\Lambda)$ to $C_1(\Lambda)$. Then $\partial_2^\Lambda \circ \partial_1^\Lambda = 0$, and $H_0(\Lambda) = \mathbb{Z}\Lambda^0/\operatorname{Im}\partial_1^\Lambda$, $H_1(\Lambda) = \ker\partial_1^\Lambda/\operatorname{Im}\partial_2^\Lambda$.

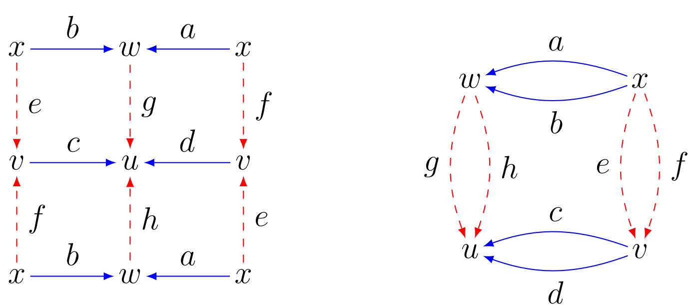
Recall the following definitions from [12, Definition 2.7, Definition 3.10].

Definition 6.1. Given $h = h_1^{m_1}\cdots h_n^{m_n} \in \mathcal{G}(\Lambda)$, where $m_i = \pm 1$, define $t : \mathcal{G}(\Lambda) \to C_1(\Lambda)$ by $t(h) = \sum_{i=1}^n m_i h_i \in C_1(\Lambda)$; then $t(h)$ is called a *trail*. If h is a circuit (that is $r(h) = s(h)$) then $t(h)$ is called a *closed trail*. If h is also simple (that is $s(h_i^{m_i}) \ne s(h_j^{m_j})$ for $i \ne j$),then $t(h)$ is called a *simple closed trail*.

Proposition 6.2. *Let Λ be a connected k-graph. Then the map t defined in Definition 13 induces an isomorphism* $\operatorname{Ab}\pi_1(\Lambda) \cong H_1(\Lambda)$.

Proof. Fix $v \in \Lambda^0$, then $\pi_1(\Lambda) \cong \pi_1(\mathrm{Sk}_\Lambda, v)/S^+$ by [9]. Fix a maximal spanning tree $T \subset \mathrm{Sk}_\Lambda$, then $\pi_1(\mathrm{Sk}_\Lambda, v) = \langle \zeta_e \mid e \in E^1 \setminus T^1\rangle$ (see [22] for example). Then $t : \pi_1(\mathrm{Sk}_\Lambda, v) \to C_1(\Lambda)$ is a homomorphism. Since t sends simple reduced circuits to simple closed trails, [12, Proposition 3.15] implies that $\ker(\partial_1^\Lambda) = t(\pi_1(\mathrm{Sk}_\Lambda, v))$. As $t(\zeta_e\zeta_f) - t(\zeta_g\zeta_h) = e + f - g - h \in \operatorname{Im}\partial_2^\Lambda$ whenever if $(ef, gh) \in S^+$, t descends to a homomorphism $t' : \pi_1(\Lambda) \to H_1(\Lambda)$ which maps $[a]$ to $[t(a)]$ for $a \in \pi_1(\mathrm{Sk}_\Lambda, v)$. Routine calculation then shows that $\ker t'$ is the commutator subgroup of $\pi_1(\Lambda)$, so t is an isomorphism from $\operatorname{Ab}\pi_1(\Lambda)$, the abelianization of $\pi_1(\Lambda)$ to $H_1(\Lambda)$. □

Example 6.3. Recall the 2-graph Λ shown below on the right with commuting squares shown on the on the left from [12, Example 5.7]



which has the same homology as the Klein bottle. However, as we shall see, it does have the same fundamental group, but with a quite different presentation to the one given in [9, Example 3.13]. To see this, choose spanning tree T with $T^1 = \{a, c, g\}$. By Theorem

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5.4, the fundamental group is generated by $\Lambda^{e_1} \cup \Lambda^{e_2}$ subject to the relations

$$a = c = g = 1, \quad gb = ce,\ ga = df,\ hb = cf,\ ha = de,$$

which simplify to $b = e,\ \ 1 = df,\ \ hb = f,\ \ h = de$. Eliminating b and simplifying further, we have

$$\pi_1(\Lambda) = \langle e, f, h : fh = e, he = f\rangle = \langle e, f : f^2 = e^2\rangle, \tag{6.1}$$

is equal to the fundamental group of the Klein bottle, $\langle a,\ b : aba = b\rangle$. To see this, set $e = ab$ and $f = b$, then

$$e^2 = (ab)(ab) = (aba)b = (b)(b) = b^2 = f^2.$$

A slightly easier calculation shows that in the case $n = 2$, the 2-graph in [12, Example 5.1] has the same fundamental group (6.1) as Λ, which is not a surprise as it has the same topological realization as Λ (see [12, Remark 5.9]). The presentation (6.1) in abelian form is $\langle e,\ f : 2(f - e) = 0\rangle$.

One sees that the abelianization of $\pi_1(\Lambda)$ is $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, the homology group of the Klein bottle, as stated in [12, Example 5.7].

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