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Robust backstepping control of an underactuated one-legged hopping robot in stance phase

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SUMMARY
Exponentially stabilizing a non-Spring Loaded Inverted Pendulum (SLIP) model-based one-legged hopping robot in stance phase is studied. Differing from the SLIP model systems, the hopping robot with non-SLIP model considered in this paper does not restrict the center of mass of the robot coinciding to the hip joint. A specific underactuated one-legged hopping robot with two actuated arms are selected to investigate the dynamics and control problem. It is shown that the system holds the essential nonlinear prosperities of general systems and belongs to a class of second-order nonholonomic mechanical systems, which cannot be stabilized by any smooth time-invariant state feedback. By using a coordinates transform based on the so-called normalized momentum, a robust backstepping control method is presented for the specific hopping robot system. Both theoretical analysis and numerical simulations show that the robust backstepping controller can stabilize the underactuated one-legged hopping robot to its balance configuration as well as a periodic motion trajectory near to the balance configuration. These results are significative for designing a new non-SLIP model based hopping robot systems with more biological characteristics.

KEYWORDS: Hopping robots; Underactuation; Nonholonomy; Nonlinear control.

1. Introduction

The dynamically stable legged robots are an important class of biological robot systems, which have been investigated by many scholars theoretically or practically in the past three decades. The one-legged hopping robots,[1,2] which are the minimal system for the multi-legged running robots, hold special importance for developing the dynamic legged robots. The legged hopping robots that had been fabricated in the past are mostly concentrated on the Spring Loaded Inverted Pendulum (SLIP) model-based systems.[1–8] The SLIP model is a point mass attached on a massless spring that is free to rotate around its point of contacting with the ground, thus the rotary motion of the body is decoupled from the telescopic motion of the leg, and the nonlinear dynamics coupling between the swing motion of the leg and the rotary motion of the body can be linearized if the swing angle of the leg is sufficiently small. In order to satisfy the decoupling conditions of the dynamics, the leg of the hopping robot should be sufficiently light compared to the body, and the center of mass (CM) of the body should be coincident with the hip joint as possible as ref. [9]. The simple dynamics of a fully actuated (except for the ankle) SLIP model-based hopping robot does not require much control effort, as shown by refs. [4–8], even a linear time-invariant feedback can stabilize the SLIP robot systems and yield quite natural running gaits. Nevertheless, the SLIP model-based hopping robot is far from a biological robot system because that the CM of animals' body in the nature is set off the hip joint (see ref. [9] and the references therein), and even the position of the CM varies relative to the hip joint when the animals are running. For instance, swinging arms will result in the variation of the position of CM of human being, but the variation of the position of CM can help balance in walking or running and augments the stride in our intuition. In the track-and-field sports such as high jump, long jump, diving, discus, and javelin etc, it is well known that a proper movement of swinging the arms also effectively affects the result of the athlete. Therefore, from the point of view of a biomechanical system, the legged hopping or running robots should be designed to imitate the skeletal structure of the animals in the natural, and more importantly, to understand the principle of locomotion of them, such that a biological robot designed holds both the best energy-efficiency and mobility. Nevertheless, the popularity of the SLIP model systems in dynamic legged robots is just due to the easiness in designing the control. With considering the unnatural simple dynamics shown by the SLIP model-based legged robots, one can conclude that the balance principle and the requirements for stabilizing the biological legged robot system cannot be understood adequately through the SLIP model based robot system.

Of course, a biologically legged robot is anything but merely showing a variable CM. Nevertheless, this "small" advancement for the legged robots, as to be shown in this paper, considerably changes the dynamics and control problem of the SLIP model-based legged robot systems. For investigating the feasibility of designing a hopping robot with more complex nonlinear dynamics than the SLIP

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model-based system has, by a specific hopping robot with a non-SLIP model that releases the restriction of invariant CM, the dynamics and the control problem are studied in this paper. It will be shown that, an important benefit from the non-SLIP model system is that under certain conditions this class system can be controlled even the leg of the robot has more than one passive joint, such that the leg has mass less. Reducing the mass of the leg can effectively reduce the energy loss for the hopping/running robots.

So far, there are few of non-SLIP model-based robot prototype had been fabricated except for the "Kenken"[9] and the "Uniroo,"[10] and these prototypes are driven by hydraulic or pneumatic actuators which are not fit for a long-distance movement. Besides the power problem, another difficulty for the non-SLIP model-based hopping robots, comes from the intractable nonlinear dynamics of the systems. Since the ankle of the hopping robots has no actuator, in general the hopping robots are second-order nonholonomic underactuated systems, and the nonholonomic constraints differential equations of the non-SLIP model systems are far complex than that of the SLIP model system. The control problem of the second-order nonholonomic nonlinear system is generally a difficult task. The essential progresses of nonlinear control theory in the past decades mainly come from the developments of geometric control theory.[11,12] The geometric control theory developed in recent decades shows that some systematic design methodologies can be applied to control a nonlinear system if the nonlinear system can be transformed into some special normal form, such as upper/lower triangular form ref. [11], for the nonholonomic systems such as the chained form ref. [12]. Many benchmark nonlinear systems[13–16] can be stabilized depends on that a method of transforming the original system into the upper/lower triangular form has been discovered. It is unfortunately that there does not exist a general result for getting the transformation for a specific nonlinear system. Therefore, the approximate normal form transformation combining with a robust control maybe a valuable way for resolving the control problem of a general nonlinear system, such as the nilpotent approximation combined with iterate steering scheme suggested by Oriolo *et al.*[17]

For controlling the underactuated mechanical systems, Olfati-Saber[13] discovers that the so-called normalized momentum can be used to transform the system to a special Byrnes–Isidori normal form, and combining with some additional conditions, the normal forms can be transformed into some special triangular forms, such as the strict feedback forms or feedforward forms. The importance of the strict feedback normal forms for the nonlinear system is confirmed by the existence of standard backstepping procedure.[11,13,19,20] The backstepping control technique has been developed to be a systemic nonlinear control methodology, which can be used to broad classes of nonlinear systems including some first-order or second-order nonholonomic systems. Based on the Olfati-Saber transformation and backstepping procedure, the control problem of the non-SLIP model-based hopping robot is investigated in this paper. The main contribution of the paper is that a robust backstepping controller is proposed for a class of second-order nonholonomic underactuated mechanical systems with nonconstant potential energy. It is shown that the proposed controller can stabilize the periodic motion system such as the underactuated hopping/running robots with a non-SLIP model even the dynamics of the system has no accurate strict feedback normal form. The method presented in this paper maybe the first try to invent a systematic method for designing a control for the non-SLIP model-based legged robots, which shows more biological properties than the SLIP model systems in the sense of the nonlinear dynamics shown by the animalized robot systems. This result releases the restrictions of designing a SLIP model legged robot system such that many optimization methods can be utilized to improve the energy-efficiency of the legged robots at the stage of mechanical design by redistributing the mass as well as the inertia of the bodies of the robot system. Encouraged by the result of this paper, the energy-efficiency design for the non-SLIP model-based legged robots is obviously an interesting and important problem that will be studied in our future.

The paper is organized as follow. In Section 2, the robot model is introduced and the dynamics of the robot is analyzed. In Section 3, the approximately strict feedback normal form with perturbation terms is presented based on the Olfati-Saber transformation. Since the approximations introduce unavoidable uncertainties, a robust backstepping controller is proposed in Section 4. Section 5 presents some numerical simulation results that verify the scheme suggested in the former two sections. Finally, we conclude the paper in Section 6.

2. Dynamics of the Hopping Robot

Figure 1 shows the model of the hopping robot considered in this paper, of which the single telescopic leg consists of two segments. One segment of the leg is nonzero mass and another is massless (this can be effectively approximated by a careful mechanical design, see ref. [1]). The length of the segment with nonzero mass is l_1 with mass m_1, and the CM of it lies in the middle of the link. The massless segment that has length l_2 is serially connected to the former by a linear

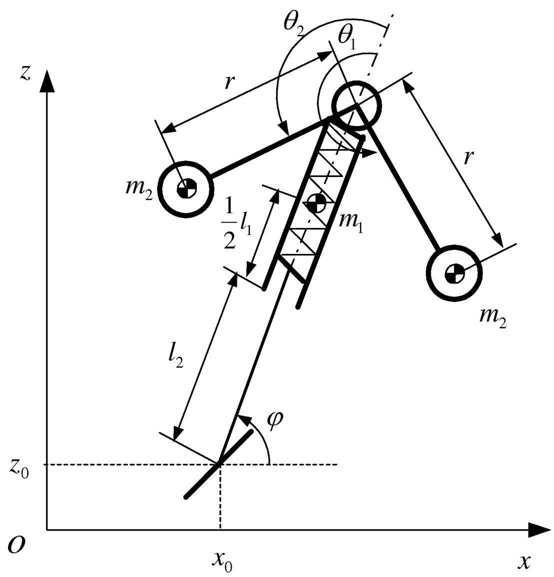

Fig. 1. The mechanism model of the underactuated one-legged hopping robot.

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spring with the same axis. The stiffness of the linear spring is k. The physical parameters of the two arms are identical with length r and mass m_2, respectively. The CM of the arm lies in the end of it. The two arms are hinged to the top of the nonzero mass segment of the leg. Define the generalized coordinates of the model to be $(x_0, z_0, l_2, \varphi, \theta_1, \theta_2)$, of which (x_0, z_0) is the position of the foot toe in the vertical plane, l_2 is the length of the massless leg ($l_2 = l_0$ when the spring is free, $l_2 = l_{20}$ when the leg is vertical with static balance.), φ is the angle between leg's axis line and horizontal plane, θ_1 and θ_2 are angular variables of the two arms relative to the leg, respectively. Positive direction of all angles is defined to be anticlockwise. Since the CM of the robot varies with respect to the hip joint, it is obvious that the model of the robot cannot be approximated to a SLIP model system.

The one-legged hopping robot shown in Fig. 1 is designed to be an underactuated mechanical system, which is four degrees of freedom (DOF) system in stance phase while actuated by two arms. Thus the telescopic and swing motions of the leg have to be actuated by the dynamics coupling from the two arms indirectly. To understand the characteristics of the robot, we analyze the dynamics of the robot by Lagrangian mechanics. The Lagrangian of the robot system has form

$$L(\boldsymbol{q}, \dot{\boldsymbol{q}}) = K(\boldsymbol{q}, \dot{\boldsymbol{q}}) - V(\boldsymbol{q}) = \frac{1}{2}\dot{\boldsymbol{q}}^{\mathrm{T}}\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}} - V(\boldsymbol{q}), \quad (1)$$

where $K(\boldsymbol{q}, \dot{\boldsymbol{q}})$ denotes the kinetic energy and $V(\boldsymbol{q})$ denotes the potential energy. With considering the coordinates (x_0, z_0) are constants and do not appear in the dynamics, the generalized coordinates in stance phase are reduced and can be denoted by $\boldsymbol{q} = [l_2 \quad \varphi \quad \theta_1 \quad \theta_2]^{\mathrm{T}}$. More specifically, the Lagrangian function (1) can be written as

$$L(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{1}{2}\begin{bmatrix}\dot{\boldsymbol{q}}_{\mathrm{p}}\\ \dot{\boldsymbol{q}}_{\mathrm{a}}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\boldsymbol{m}_{\mathrm{pp}} & \boldsymbol{m}_{\mathrm{pa}}\\ \boldsymbol{m}_{\mathrm{ap}} & \boldsymbol{m}_{\mathrm{aa}}\end{bmatrix}\begin{bmatrix}\dot{\boldsymbol{q}}_{\mathrm{p}}\\ \dot{\boldsymbol{q}}_{\mathrm{a}}\end{bmatrix} - V(\boldsymbol{q}), \quad (2)$$

where $\boldsymbol{q}_{\mathrm{p}} = [l_2 \quad \varphi]^{\mathrm{T}}$ is the passive generalized coordinates part, while $\boldsymbol{q}_{\mathrm{a}} = [\theta_1 \quad \theta_2]^{\mathrm{T}}$ is the actuated generalized coordinates part. Refer to the Appendix A, one can find the inertia matrix

$$\boldsymbol{M}(\boldsymbol{q}_{\mathrm{p}}, \boldsymbol{q}_{\mathrm{a}}) = \begin{bmatrix}\boldsymbol{m}_{\mathrm{pp}} & \boldsymbol{m}_{\mathrm{pa}}\\ \boldsymbol{m}_{\mathrm{ap}} & \boldsymbol{m}_{\mathrm{aa}}\end{bmatrix} \quad (3)$$

is a matrix of functions about variables $(l_2, \theta_1, \theta_2)$ merely, thus the kinetic energy $K(\boldsymbol{q}, \dot{\boldsymbol{q}})$ is independent of the variable φ. According to Lagrangian mechanics, if the Lagrangian function $L(\boldsymbol{q}, \dot{\boldsymbol{q}})$ is independent of a generalized coordinates q_i, then we say the Lagrangian is symmetric with respect to the generalized coordinates q_i, and q_i is said to be a cyclic coordinates. Lagrangian symmetry gives an identical equation as

$$\partial L(\boldsymbol{q}, \dot{\boldsymbol{q}})/\partial q_i = 0. \quad (4)$$

For a pure mechanics system (without any controls), the Lagrangian dynamics has an expression

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L(\boldsymbol{q}, \dot{\boldsymbol{q}})}{\partial \dot{q}_i} - \frac{\partial L(\boldsymbol{q}, \dot{\boldsymbol{q}})}{\partial q_i} = 0. \quad (5)$$

If Eq. (4) holds, then Eq. (5) indicates

$$p_i = \partial L(\boldsymbol{q}, \dot{\boldsymbol{q}})/\partial \dot{q}_i = \text{constant}, \quad (6)$$

this means that p_i is a conserved quantity. For the underactuated hopping robot shown in Fig. 1, one can verify that the potential energy of the robot system is a function of all the generalized coordinates such that there is no symmetry in classical sense (Lagrangian symmetry). Nevertheless, as shown above, the kinetic energy $K(\boldsymbol{q}, \dot{\boldsymbol{q}})$ is symmetric with respect to variable φ, thus one has

$$\partial K(\boldsymbol{q}, \dot{\boldsymbol{q}})/\partial \varphi = 0. \quad (7)$$

Since the robot system considered here has a nonconstant potential energy, the existence of kinetic symmetry in presence of nonconstant potential field does not lead to the existence of conserved quantities. In fact, from the point view of controlling an underactuated mechanical system, the existence of the conserved quantities always results in losing the controllability of the system. For instance, the Acrobot is a benchmark underactuated mechanical system with two DOF. The controllability of the Acrobot is similar to our hopping robot if the leg of our robot has no the telescopic motion. In gravitational field the Acrobot system is controllable, whereas in zero gravitational field the Acrobot system is uncontrollable since there exists a conserved quantity with respect to the unactuated joint.[13] This important problem is also discussed in the Remark 6 of the next section.

With considering the kinetic symmetry (7), the Lagrangian dynamics for the robot shown in Fig. 1 can be expressed by

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial K}{\partial \dot{l}_2} - \frac{\partial L}{\partial l_2} = 0, \quad (8a)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial K}{\partial \dot{\varphi}} + \frac{\partial V}{\partial \varphi} = 0, \quad (8b)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial K}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1} = \tau_1, \quad (8c)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial K}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2} = \tau_2, \quad (8d)$$

where $\tau_i, i = 1, 2$, in the Eqs. (8c) and (8d), are the joint torques of the two arms. The former two Eqs. (8a) and (8b) with zero right side, can be regarded as differential constraints of the actuated part. As $\partial L/\partial l_2 \neq 0$ and $\partial V/\partial \varphi \neq 0$, the second-order differential Eqs. (8a) and (8b) cannot be integrated to the first-order differential equations or an algebraic equations, therefore the system (8) is second-order nonholonomic system. However, this terminology is somewhat misleading since the Lagrangian of an underactuated mechanical system satisfies Euler–Lagrange equations without any external differential-algebraic constraints that require use of Lagrangian multiplier in a variational setting as in ref. [21] for first-order nonholonomic constraints systems. During the past decade, some researches on the underactuated mechanical systems, such as the underactuated ships[22] and the underactuated manipulators,[23] also confirmed that the underactuated mechanical systems were second-order nonholonomic systems generally. In fact, a rigorous proof for a nonholonomic constraint can be done by the Frobenius theorem[24] though this is a difficult task for a multi-DOF mechanical system generally. The benefit from regarding the underactuated mechanical system as a second-order nonholonomic system is that the motion planning and control problems for the underactuated system could be dealt with based on the famous Brockett's theorem.[25]

For the purpose of explicitness, we concluded the properties of the robot model as:

(a) The robot is a non-SLIP model system since the body of the robot cannot be approximated to a point mass;
(b) The robot is an underactuated mechanical system since the four DOF mechanical system has only two actuators;
(c) The robot is a second-order nonholonomic system because of the nonconstant potential field.
(d) The robot has no Lagrangian symmetry but has kinetic symmetry. This plays a vital role in controllability and stabilization of the underactuated robot system, as to be shown in the following sections.

3. Normal Form of the Dynamics

An underactuated mechanical system generally cannot be globally linearized by static or dynamic feedbacks[26] except for the differentially flat systems,[27] therefore the problems of motion planning or stabilization of the underactuated mechanical systems are nonlinear in nature. The main progresses in recent years for the nonlinear control theory highly depends on the nonlinear systems holding some special geometric or algebraic structures, such as the strict feedback normal form refs. [11, 13] or feedforward normal form refs. [19, 28], the former can be stabilized by the backstepping procedure and the latter can be stabilized by state feedback in explicit form as nested saturations.[28] For nonholonomic systems, if the control distribution of the systems has nilpotent property,[14] the systems can be transformed into a triangular normal form. In some special cases, the nilpotent nonholonomic systems can be further transformed into the so-called chained form refs. [14–16], which can be exponentially stabilized. As to the nonlinear systems without a special normal form, stabilizing the system is generally an open problem.[13] Attracted by the benefits of the normal forms, in this section, we search a reasonable way to transform the underactuated system (8) into a normal form such that the control problem can be resolved effectively.

Specifically, the dynamics (8) can be expressed by

$$\begin{aligned} \boldsymbol{m}_{\mathrm{pp}}\ddot{\boldsymbol{q}}_{\mathrm{p}} + \boldsymbol{m}_{\mathrm{pa}}\ddot{\boldsymbol{q}}_{\mathrm{a}} + \boldsymbol{c}_{\mathrm{p}}(\boldsymbol{q}, \dot{\boldsymbol{q}}) &= 0 \\ \boldsymbol{m}_{\mathrm{pa}}^{\mathrm{T}}\ddot{\boldsymbol{q}}_{\mathrm{p}} + \boldsymbol{m}_{\mathrm{aa}}\ddot{\boldsymbol{q}}_{\mathrm{a}} + \boldsymbol{c}_{\mathrm{a}}(\boldsymbol{q}, \dot{\boldsymbol{q}}) &= \boldsymbol{\tau} \end{aligned}, \tag{9}$$

where the terms $\boldsymbol{c}_{\mathrm{p}}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ and $\boldsymbol{c}_{\mathrm{a}}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ include the centrifugal, Coriolis, gravitational, and frictional forces. For the first step, let's consider the partial feedback linearization that is due to Spong,[18] by a change of control

$$\boldsymbol{\tau} = \left(\boldsymbol{m}_{\mathrm{aa}} - \boldsymbol{m}_{\mathrm{pa}}^{\mathrm{T}}\boldsymbol{m}_{\mathrm{pp}}^{-1}\boldsymbol{m}_{\mathrm{pa}}\right)\ddot{\boldsymbol{q}}_{\mathrm{a}} + \left(\boldsymbol{c}_{\mathrm{a}} - \boldsymbol{m}_{\mathrm{pa}}^{\mathrm{T}}\boldsymbol{m}_{\mathrm{pp}}^{-1}\boldsymbol{c}_{\mathrm{p}}\right) \tag{10}$$

the system (9) can be transformed into a partial linearization form

$$\begin{aligned} \dot{\boldsymbol{q}}_{\mathrm{p}} &= \boldsymbol{y}_{\mathrm{p}} \\ \dot{\boldsymbol{y}}_{\mathrm{p}} &= -\boldsymbol{m}_{\mathrm{pp}}^{-1}\boldsymbol{c}_{\mathrm{p}} - \boldsymbol{m}_{\mathrm{pp}}^{-1}\boldsymbol{m}_{\mathrm{pa}}\boldsymbol{u} \\ \dot{\boldsymbol{q}}_{\mathrm{a}} &= \boldsymbol{y}_{\mathrm{a}} \\ \dot{\boldsymbol{y}}_{\mathrm{a}} &= \boldsymbol{u} \end{aligned}, \tag{11}$$

where $\boldsymbol{u} = \ddot{\boldsymbol{q}}_{\mathrm{a}}$ is defined to be a new input. Obviously, the subsystem $(\boldsymbol{q}_{\mathrm{a}}, \boldsymbol{y}_{\mathrm{a}})$ is linearized whereas the subsystem $(\boldsymbol{q}_{\mathrm{p}}, \boldsymbol{y}_{\mathrm{p}})$ is still highly nonlinear, and the new control $\boldsymbol{u}$ appears in the dynamics of both subsystems. This is one of the main sources of the complexity of control design for underactuated systems.

Furthermore, we consider the Olfati-Saber transformation,[13] which is a state change based on the normalized momentum. The main advantage of the Olfati-Saber transformation is that, under kinetic symmetry properties of the underactuated system, it is possible to change the partial linearization form (11) into a special case of the famous Byrnes–Isidori normal form[11] with a double integrator, such that the control input does not appear in the unactuated subsystem. This simplifies the control design for the underactuated system by reducing the control of the original higher order system into the control of its lower order nonlinear unactuated subsystem. With different additional conditions, the normal form can be changed to strict feedback form or feedforward form. We restate the related notions and the main theorem provided by Olfati-Saber as following due to applications in this paper.

Definition 1: (external variables and shape variables) The variables that appear in the kinetic energy of the mechanical system with Lagrangian (1) are called shape variables. A configuration variable is called an external variable, if it does not appears in the kinetic energy, i.e., $\partial K(q, \dot{q})/\partial q_i = 0$.

Definition 2: (normalized momentum) Consider the underactuated system (9), we define

$$\pi_{\mathrm{p}} = \boldsymbol{m}_{\mathrm{pp}}^{-1}\frac{\partial L}{\partial \dot{q}_{\mathrm{p}}} = \dot{\boldsymbol{q}}_{\mathrm{p}} + \boldsymbol{m}_{\mathrm{pp}}^{-1}\boldsymbol{m}_{\mathrm{pa}}\dot{\boldsymbol{q}}_{\mathrm{a}},$$

$$\pi_{\mathrm{a}} = \boldsymbol{m}_{\mathrm{pa}}^{-1}\frac{\partial L}{\partial \dot{q}_{\mathrm{a}}} = \dot{\boldsymbol{q}}_{\mathrm{p}} + \boldsymbol{m}_{\mathrm{pa}}^{-1}\boldsymbol{m}_{\mathrm{aa}}\dot{\boldsymbol{q}}_{\mathrm{a}},$$

to be the normalized momentum with respect to the generalized coordinates $\boldsymbol{q}_{\mathrm{p}}, \boldsymbol{q}_{\mathrm{a}}$, respectively.

Definition 3: (strict feedback form) A nonlinear system is said to be in strict feedback form, if it has the following triangular structure

$$\begin{aligned} \dot{z} &= \boldsymbol{f}(z, \boldsymbol{\xi}_1) \\ \dot{\boldsymbol{\xi}}_1 &= \boldsymbol{\xi}_2 \\ &\cdots \\ \dot{\boldsymbol{\xi}}_m &= \boldsymbol{u} \end{aligned}.$$

Now for the underactuated system (9), we set a proposition that can be found in the work of Olfati-Saber.[13] The usefulness of the proposition is that it proves a class of underactuated mechanical system can be transformed into

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the strict feedback normal form, such that the system can be stabilized by the backstepping procedure.[19]

Proposition 1. (strict feedback form transformation) Assume that the unactuated coordinates $\boldsymbol{q}_{\mathrm{p}}$ are external variables and that the actuated coordinates $\boldsymbol{q}_{\mathrm{a}}$ are shape variables, if the normalized momentum π_{p} is integrable, and the part $\boldsymbol{w} = [\boldsymbol{m}_{\mathrm{pp}}^{-1}(\boldsymbol{q}_{\mathrm{a}})\boldsymbol{m}_{\mathrm{pa}}(\boldsymbol{q}_{\mathrm{a}})]\,\mathrm{d}\boldsymbol{q}_{\mathrm{a}}$ of π_{p} has the form $\boldsymbol{w} = \mathrm{d}\boldsymbol{\gamma}(\boldsymbol{q}_{\mathrm{a}})$, then it can be obtained from the Lagrangian of the system that there exists a global change of coordinates with the following form

$$\begin{aligned} \boldsymbol{q}_{\mathrm{r}} &= \boldsymbol{q}_{\mathrm{p}} + \boldsymbol{\gamma}(\boldsymbol{q}_{\mathrm{a}}) \\ \boldsymbol{p}_{\mathrm{r}} &= \partial L/\partial \dot{\boldsymbol{q}}_{\mathrm{p}} = \boldsymbol{m}_{\mathrm{pp}}(\boldsymbol{q}_{\mathrm{a}})\dot{\boldsymbol{q}}_{\mathrm{p}} + \boldsymbol{m}_{\mathrm{pa}}(\boldsymbol{q}_{\mathrm{a}})\dot{\boldsymbol{q}}_{\mathrm{a}} \end{aligned} \tag{12}$$

by associating with the control change (10), it transforms the dynamics of the system (9) into a cascade nonlinear system in a strict feedback form

$$\begin{aligned} \dot{\boldsymbol{q}}_{\mathrm{r}} &= \boldsymbol{m}_{\mathrm{pp}}^{-1}(\boldsymbol{q}_{\mathrm{a}})\boldsymbol{p}_{\mathrm{r}} \\ \dot{\boldsymbol{p}}_{\mathrm{r}} &= -\frac{\partial V(\boldsymbol{q}_{\mathrm{r}} - \gamma(\boldsymbol{q}_{\mathrm{a}}), \boldsymbol{q}_{\mathrm{a}})}{\partial \boldsymbol{q}_{\mathrm{r}}}, \\ \dot{\boldsymbol{q}}_{\mathrm{a}} &= \boldsymbol{p}_{\mathrm{a}} \\ \dot{\boldsymbol{p}}_{\mathrm{a}} &= \boldsymbol{u} \end{aligned} \tag{13}$$

where $\boldsymbol{u} = \ddot{\boldsymbol{q}}_{\mathrm{a}}$ is the new input.

Remark 1: The proof of Proposition 1 is intuitional as long as substituting Eq. (12) into Eq. (13). As shown by Olfati-Saber,[13] the finding of the integral of the normalized momentum π_{p} is generally not guaranteed in the multi-DOF underactuated mechanical systems, except for some simple two DOF systems. Therefore the searching of a reasonable approximation for the strict feedback form (13) is crucial for general underactuated systems, and this is the original intention of the paper.

Remark 2: The robot system considered in this paper does not satisfy the conditions of Proposition 1 since the passive coordinate l_2 is not a kinetic symmetrical coordinate. Whereas, as to be shown by the following two propositions, the system (8) can be approximated to satisfy the conditions of Proposition 1.

Proposition 2: (approximate momentum integral) Consider the dynamics of the underactuated hopping robot system (8), if the kinetic energy $K(l_2, \theta_1, \theta_2)$ is approximated by $\tilde{K}(l_{20}, \theta_1, \theta_2)$, viz. let the unactuated coordinate $l_2 \approx l_{20}$. Then it follows that:

(a) The approximate kinetic energy $\tilde{K}(l_{20}, \theta_1, \theta_2)$ is symmetric about the passive coordinates $\boldsymbol{q}_p = [l_2 \quad \varphi]^{\mathrm{T}}$, and that
(b) If the matrix $\tilde{\boldsymbol{m}}_{\mathrm{pp}}(l_{20}, \theta_1, \theta_2)$ is approximated by $\tilde{\boldsymbol{m}}_{\mathrm{pp}}(l_{20}, \theta_1^*, \theta_2^*)$, of which θ_1^*, θ_2^* are given positions of the two arms, respectively, then the approximate momentum part $\tilde{\boldsymbol{w}} = [\tilde{\boldsymbol{m}}_{\mathrm{pp}}^{-1}(l_{20}, \theta_1^*, \theta_2^*)\tilde{\boldsymbol{m}}_{\mathrm{pa}}(l_{20}, \theta_1, \theta_2)]\,\mathrm{d}\boldsymbol{q}_{\mathrm{a}}$ is integrable.

Proof:

(a) Referring to the Appendix A, by letting $l_2 \approx l_{20}$, the approximate kinetic energy $\tilde{K}(l_{20}, \theta_1, \theta_2)$ is independent of the passive coordinates $\boldsymbol{q}_{\mathrm{p}} = [l_2 \quad \varphi]^{\mathrm{T}}$, thus $\tilde{K}(l_{20}, \theta_1, \theta_2)$ is symmetric about the passive coordinates $\boldsymbol{q}_{\mathrm{p}}$.
(b) Let $\tilde{\boldsymbol{m}}_{\mathrm{pp}}(l_{20}, \theta_1, \theta_2) \approx \tilde{\boldsymbol{m}}_{\mathrm{pp}}(l_{20}, \theta_1^*, \theta_2^*)$, referring to the Appendix A, the approximate momentum part can be written as

$$\begin{aligned} \tilde{\boldsymbol{w}} &= \left[\tilde{\boldsymbol{m}}_{\mathrm{pp}}^{-1}(l_{20}, \theta_1^*, \theta_2^*)\tilde{\boldsymbol{m}}_{\mathrm{pa}}(l_{20}, \theta_1, \theta_2)\right] \mathrm{d}\boldsymbol{q}_{\mathrm{a}} \\ &= \tilde{\boldsymbol{m}}_{\mathrm{pp}}^{-1}(l_{20}, \theta_1^*, \theta_2^*) \begin{bmatrix} m_{13}(\theta_1)\mathrm{d}\theta_1 + m_{14}(\theta_2)\mathrm{d}\theta_2 \\ m_{23}(\theta_1)\mathrm{d}\theta_1 + m_{24}(\theta_2)\mathrm{d}\theta_2 \end{bmatrix}, \end{aligned}$$

this is exact one-forms and can be denoted by $\tilde{\boldsymbol{w}} = \mathrm{d}\boldsymbol{\gamma}(l_{20}, \theta_1, \theta_2)$. Then it follows that

$$\begin{aligned} \boldsymbol{\gamma}(l_{20}, \theta_1, \theta_2) &= \tilde{\boldsymbol{m}}_{\mathrm{pp}}^{-1}(l_{20}, \theta_1^*, \theta_2^*) \\ &\times \begin{bmatrix} m_2 r(\cos\theta_1 + \cos\theta_2) \\ m_2 r^2(\theta_1 + \theta_2) + m_2(l_1 + l_{20})r(\sin\theta_1 + \sin\theta_2) \end{bmatrix}, \end{aligned} \tag{14}$$

this complete the proof. □

Remark 3: The proposition 2 shows that the underactuated hopping robot system (8) can be approximate to satisfy the conditions of Proposition 1. This is feasible for periodic motion systems such as the hopping robot system.

Remark 4: When one uses the Propositions 1 and 2 to transform an underactuated system into the strict feedback form, the resulted errors of the dynamics model should be estimated. Consider the Lagrangian (1), since the potential energy $V(\boldsymbol{q})$ is just a function of generalized coordinates, viz. $\partial V(\boldsymbol{q})/\partial \dot{\boldsymbol{q}} = 0$, the generalized momenta can be written as

$$\frac{\partial L(\boldsymbol{q}, \dot{\boldsymbol{q}})}{\partial \dot{\boldsymbol{q}}} = \frac{\partial K(\boldsymbol{q}, \dot{\boldsymbol{q}})}{\partial \dot{\boldsymbol{q}}} - \frac{\partial V(\boldsymbol{q})}{\partial \dot{\boldsymbol{q}}} = \frac{\partial K(\boldsymbol{q}, \dot{\boldsymbol{q}})}{\partial \dot{\boldsymbol{q}}} = \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}. \tag{15}$$

We are interest in the momenta part that is relative to the passive coordinates, i.e.,

$$\boldsymbol{p}_{\mathrm{r}} = \partial L/\partial \dot{\boldsymbol{q}}_{\mathrm{p}} = \boldsymbol{m}_{\mathrm{pp}}(\boldsymbol{q})\dot{\boldsymbol{q}}_{\mathrm{p}} + \boldsymbol{m}_{\mathrm{pa}}(\boldsymbol{q})\dot{\boldsymbol{q}}_{\mathrm{a}}. \tag{16}$$

Considering the approximations $l_2 \approx l_{20}$ and $\boldsymbol{m}_{\mathrm{pp}}(\boldsymbol{q}) = \boldsymbol{m}_{\mathrm{pp}}(l_2, \theta_1, \theta_2) \approx \tilde{\boldsymbol{m}}_{\mathrm{pp}}(l_{20}, \theta_1^*, \theta_2^*)$, then the approximate momenta of Eq. (16) can be expressed by

$$\tilde{\boldsymbol{p}}_{\mathrm{r}} = \tilde{\boldsymbol{m}}_{\mathrm{pp}}(l_{20}, \theta_1^*, \theta_2^*)\dot{\boldsymbol{q}}_{\mathrm{p}} + \tilde{\boldsymbol{m}}_{\mathrm{pa}}(l_{20}, \theta_1, \theta_2)\dot{\boldsymbol{q}}_{\mathrm{a}}. \tag{17}$$

The normalized momenta corresponding to Eqs. (16) and (17) are given by

$$\boldsymbol{\psi}_{\mathrm{r}} = \dot{\boldsymbol{q}}_{\mathrm{p}} + \boldsymbol{m}_{\mathrm{pp}}^{-1}(l_2, \theta_1, \theta_2)\boldsymbol{m}_{\mathrm{pa}}(l_2, \theta_1, \theta_2)\dot{\boldsymbol{q}}_{\mathrm{a}}, \tag{18}$$

$$\tilde{\boldsymbol{\psi}}_{\mathrm{r}} = \dot{\boldsymbol{q}}_{\mathrm{p}} + \tilde{\boldsymbol{m}}_{\mathrm{pp}}^{-1}(l_{20}, \theta_1^*, \theta_2^*)\tilde{\boldsymbol{m}}_{\mathrm{pa}}(l_{20}, \theta_1, \theta_2)\dot{\boldsymbol{q}}_{a}. \tag{19}$$

The errors of the normalized momenta have a form

$$\begin{aligned} \boldsymbol{\psi}_{\mathrm{r}}^{\mathrm{e}} &= \boldsymbol{\psi}_{\mathrm{r}} - \tilde{\boldsymbol{\psi}}_{\mathrm{r}} \\ &= \big[\boldsymbol{m}_{\mathrm{pp}}^{-1}(l_2, \theta_1, \theta_2)\boldsymbol{m}_{\mathrm{pa}}(l_2, \theta_1, \theta_2) - \tilde{\boldsymbol{m}}_{\mathrm{pp}}^{-1} \\ &\quad \times (l_{20}, \theta_1^*, \theta_2^*)\tilde{\boldsymbol{m}}_{\mathrm{pa}}(l_{20}, \theta_1, \theta_2)\big]\dot{\boldsymbol{q}}_{\mathrm{a}}. \end{aligned} \tag{20}$$

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By the errors analysis, we present the following Proposition, which shows that the underactuated system (9) can be transformed into a strict feedback normal form with perturbation terms such that the nonlinear dynamic system (9) can be controlled by a robust backstepping procedure.

Proposition 3: (strict feedback form with perturbation terms) Consider the underactuated hopping robot system (9), by combining the partial feedback linearization input change (10) and the following coordinates changes

$$\begin{aligned} \boldsymbol{q}_{\mathrm{r}} &= \boldsymbol{q}_{\mathrm{p}} + \boldsymbol{\gamma}(l_{20}, \theta_1, \theta_2) \\ \boldsymbol{p}_{\mathrm{r}} &= \partial L/\partial \dot{\boldsymbol{q}}_{\mathrm{p}} = \boldsymbol{m}_{\mathrm{pp}}(\boldsymbol{q}_{\mathrm{a}})\dot{\boldsymbol{q}}_{\mathrm{p}} + \boldsymbol{m}_{\mathrm{pa}}(\boldsymbol{q}_{\mathrm{a}})\dot{\boldsymbol{q}}_{\mathrm{a}}, \end{aligned} \tag{21}$$

the system (9) can be transformed into a strict feedback form with perturbation terms

$$\begin{aligned} \dot{\boldsymbol{q}}_{\mathrm{r}} &= \tilde{\boldsymbol{m}}_{\mathrm{pp}}^{-1}(l_{20}, \theta_1^*, \theta_2^*)\boldsymbol{p}_{\mathrm{r}} + \boldsymbol{\varepsilon}_1 \\ \dot{\boldsymbol{p}}_{\mathrm{r}} &= \boldsymbol{h}_{\mathrm{r}}(\boldsymbol{q}_{\mathrm{r}}, \boldsymbol{q}_{\mathrm{a}}) + \boldsymbol{\varepsilon}_2 \\ \dot{\boldsymbol{q}}_{\mathrm{a}} &= \boldsymbol{p}_{\mathrm{a}} \\ \dot{\boldsymbol{p}}_{\mathrm{a}} &= \boldsymbol{u} \end{aligned}, \tag{22}$$

where

$$\boldsymbol{h}_{\mathrm{r}}(\boldsymbol{q}_{\mathrm{r}}, \boldsymbol{q}_{\mathrm{a}}) = -\frac{\partial V(\boldsymbol{q}_{\mathrm{r}} - \boldsymbol{\gamma}(l_{20}, \theta_1, \theta_2), \boldsymbol{q}_{\mathrm{a}})}{\partial \boldsymbol{q}_{\mathrm{r}}},$$

and the perturbation terms have form

$$\boldsymbol{\varepsilon}_1 = \tilde{\boldsymbol{p}}_{\mathrm{r}} - \boldsymbol{p}_{\mathrm{r}}, \boldsymbol{\varepsilon}_2 = \begin{bmatrix} \partial K/\partial l_2 \\ 0 \end{bmatrix}.$$

Proof:

By the input change (10), the underactuated system (9) is transformed to

$$\begin{aligned} \boldsymbol{m}_{\mathrm{pp}}\ddot{\boldsymbol{q}} + \boldsymbol{m}_{\mathrm{pa}}\ddot{\boldsymbol{q}} + \boldsymbol{c}_{\mathrm{p}}(\boldsymbol{q}, \dot{\boldsymbol{q}}) &= 0 \\ \dot{\boldsymbol{q}}_{\mathrm{a}} &= \boldsymbol{p}_{\mathrm{a}} \\ \dot{\boldsymbol{p}}_{\mathrm{a}} &= \boldsymbol{u} \end{aligned}. \tag{23}$$

Thus the last two equations of Eq. (22) are verified.

To proof the first two equations of (22), consider the Eqs. (8a) and (8b), one has

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \partial K/\partial \dot{l}_2 \\ \partial K/\partial \dot{\varphi} \end{bmatrix} - \begin{bmatrix} \partial L/\partial l_2 \\ -\partial V/\partial \varphi \end{bmatrix} \\ &\quad = \boldsymbol{m}_{\mathrm{pp}}\ddot{\boldsymbol{q}}_{\mathrm{p}} + \boldsymbol{m}_{\mathrm{pa}}\ddot{\boldsymbol{q}}_{\mathrm{a}} + \boldsymbol{c}_{\mathrm{p}}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = 0. \end{aligned} \tag{24}$$

Since $\partial V/\partial \dot{\boldsymbol{q}}_{\mathrm{p}} = 0$, the left side of Eq. (24) has relationship

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}_{\mathrm{p}}}\right) = \begin{bmatrix} \partial K/\partial l_2 \\ 0 \end{bmatrix} - \frac{\partial V}{\partial \boldsymbol{q}_{\mathrm{p}}}. \tag{25}$$

Referring to Eqs. (16) and (25) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}_{\mathrm{p}}}\right) = \dot{\boldsymbol{p}}_{\mathrm{r}} = -\frac{\partial V}{\partial \boldsymbol{q}_{\mathrm{p}}} + \begin{bmatrix} \partial K/\partial l_2 \\ 0 \end{bmatrix}. \tag{26}$$

Let

$$\varepsilon_2 = \begin{bmatrix} \partial K/\partial l_2 \\ 0 \end{bmatrix}. \tag{27}$$

From Eqs. (25) and (26), one has that

$$\dot{\boldsymbol{p}}_{\mathrm{r}} = -\partial V/\partial \boldsymbol{q}_{\mathrm{p}} + \boldsymbol{\varepsilon}_2. \tag{28}$$

By the first expression of the given coordinates changes (21), it gives that

$$\begin{aligned} -\frac{\partial V}{\partial \boldsymbol{q}_{\mathrm{p}}} &= -\frac{\partial V(\boldsymbol{q}_{\mathrm{p}}, \boldsymbol{q}_{\mathrm{a}})}{\partial \boldsymbol{q}_{\mathrm{p}}} \\ &= -\frac{\partial V(\boldsymbol{q}_{\mathrm{r}} - \boldsymbol{\gamma}(l_{20}, \theta_1, \theta_2), \boldsymbol{q}_{\mathrm{a}})}{\partial \boldsymbol{q}_{\mathrm{r}}}\frac{\partial \boldsymbol{q}_{\mathrm{r}}}{\partial \boldsymbol{q}_{\mathrm{p}}} \\ &= \boldsymbol{h}_{\mathrm{r}}(\boldsymbol{q}_{\mathrm{r}}, \boldsymbol{q}_{\mathrm{a}}) \times \boldsymbol{I} \\ &= \boldsymbol{h}_{\mathrm{r}}(\boldsymbol{q}_{\mathrm{r}}, \boldsymbol{q}_{\mathrm{a}}), \end{aligned} \tag{29}$$

where $\boldsymbol{I}$ is an identity matrix. By Eqs. (28) and (29), one has

$$\dot{\boldsymbol{p}}_{\mathrm{r}} = \boldsymbol{h}_{\mathrm{r}}(\boldsymbol{q}_{\mathrm{r}}, \boldsymbol{q}_{\mathrm{a}}) + \boldsymbol{\varepsilon}_2. \tag{30}$$

Then the second equation of Eq. (22) is verified.

The first equation of Eq. (22) can be verified directly by the given coordinates changes (21), that is,

$$\begin{aligned} \dot{\boldsymbol{q}}_{\mathrm{r}} &= \dot{\boldsymbol{q}}_{\mathrm{p}} + \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\gamma}(l_{20}, \theta_1, \theta_2) \\ &= \dot{\boldsymbol{q}}_{\mathrm{p}} + \tilde{\boldsymbol{m}}_{\mathrm{pp}}^{-1}(l_{20}, \theta_1^*, \theta_2^*)\tilde{\boldsymbol{m}}_{\mathrm{pa}}(l_{20}, \theta_1, \theta_2)\dot{\boldsymbol{q}}_{\mathrm{a}} \\ &= \tilde{\boldsymbol{m}}_{\mathrm{pp}}(l_{20}, \theta_1^*, \theta_2^*)\tilde{\boldsymbol{p}}_{\mathrm{r}} \\ &= \tilde{\boldsymbol{m}}_{\mathrm{pp}}(l_{20}, \theta_1^*, \theta_2^*)(\boldsymbol{p}_{\mathrm{r}} + \boldsymbol{\varepsilon}_1), \end{aligned}$$

where

$$\boldsymbol{\varepsilon}_1 = \tilde{\boldsymbol{p}}_{\mathrm{r}} - \boldsymbol{p}_{\mathrm{r}}. \tag{31}$$

This completes the proof. □

Remark 5: Let $(z_1, z_2) = (\boldsymbol{q}_{\mathrm{r}}, \boldsymbol{p}_{\mathrm{r}})$ and $(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2) = (\boldsymbol{q}_{\mathrm{a}}, \boldsymbol{p}_{\mathrm{a}})$, Eq. (22) can be rewritten as a more familiar form

$$\begin{aligned} \dot{z} &= \boldsymbol{f}(z, \boldsymbol{\xi}_1, \boldsymbol{\varepsilon}) \\ \dot{\boldsymbol{\xi}}_1 &= \boldsymbol{\xi}_2 \\ \dot{\boldsymbol{\xi}}_2 &= \boldsymbol{u} \end{aligned}. \tag{32}$$

The normal form (32) with nonlinear perturbation $\boldsymbol{\varepsilon}$ indicates that a robust controller is expected since the standard backstepping procedure cannot be used directly.

Remark 6: Referring to Eq. (28), if the passive coordinates are external variables i.e. $\partial K/\partial \boldsymbol{q}_{\mathrm{p}} = 0$, and the generalized momenta (16) are integrable, then the perturbation $\varepsilon = 0$. Under this case, that the potential force terms satisfy $\partial V/\partial \boldsymbol{q}_{\mathrm{p}} \neq 0$ is a necessary condition that ensures the controllability of the system (9). On the contrary, if $\partial V/\partial \boldsymbol{q}_{\mathrm{p}} = 0$, then the generalized momenta has a conserved quantity $\boldsymbol{p}_{\mathrm{r}}$ = constant. Obviously, the existence of the conserved quantity indicates that the underactuated system (22) as well as its original system (9) cannot be stabilized from any initial states if $\boldsymbol{p}_{\mathrm{r}}(t_0) \neq 0$. This confirms the conclusion (d) of Section 2.

4. Robust Backstepping Control

The standard backstepping procedure is an effective control technique especially fitting for the class of nonlinear systems that are or can be transformed to the strict feedback form. The standard backstepping control methods and some improved methods can be found in references.[19,20] Some scholars, for example, Kristić *et al.*,[19] Isidori,[11] and Freeman *et al.*[20] have made significant contributions to the development of this theory. In this section, we will present a robust backstepping controller for the system (22). In order that the proposed controller is fit for both the set-point regulation and trajectory tracking task, define the transformations $z_1 = \boldsymbol{q}_{\mathrm{r}}^{\mathrm{d}} - \boldsymbol{q}_{\mathrm{r}}$, $z_2 = \boldsymbol{p}_{\mathrm{r}}^{\mathrm{d}} - \boldsymbol{p}_{\mathrm{r}}$, $\boldsymbol{\xi}_1 = \boldsymbol{q}_{\mathrm{a}}^{\mathrm{d}} - \boldsymbol{q}_{\mathrm{a}}$, $\boldsymbol{\xi}_2 = \boldsymbol{p}_{\mathrm{a}}^{\mathrm{d}} - \boldsymbol{p}_{\mathrm{a}}$, where the superscript "d" denotes the desired trajectory of corresponding variables. Then, the system (22) can be transformed into a form

$$\begin{aligned} \dot{z}_1 &= \dot{\boldsymbol{q}}_{\mathrm{r}}^{\mathrm{d}} - \tilde{\boldsymbol{m}}_{\mathrm{pp}}^{-1}(l_{20}, \theta_1^*, \theta_2^*)\left(\boldsymbol{p}_{\mathrm{r}}^{\mathrm{d}} - z_2\right) - \boldsymbol{\varepsilon}_1 \\ \dot{z}_2 &= \dot{\boldsymbol{p}}_{\mathrm{r}}^{\mathrm{d}} - \boldsymbol{h}_{\mathrm{r}}\left(\boldsymbol{q}_{\mathrm{r}}^{\mathrm{d}} - z_1, \boldsymbol{q}_{\mathrm{a}}^{\mathrm{d}} - \boldsymbol{\xi}_1\right) - \boldsymbol{\varepsilon}_2 \\ \dot{\boldsymbol{\xi}}_1 &= \boldsymbol{\xi}_2 \\ \dot{\boldsymbol{\xi}}_2 &= \dot{\boldsymbol{p}}_{\mathrm{a}}^{\mathrm{d}} - \boldsymbol{u} \end{aligned} . \tag{33}$$

The corresponding unperturbed system of Eq. (33) has form

$$\begin{aligned} \dot{z}_1 &= \dot{\boldsymbol{q}}_{\mathrm{r}}^{\mathrm{d}} - \tilde{\boldsymbol{m}}_{\mathrm{pp}}^{-1}(l_{20}, \theta_1^*, \theta_2^*)(\boldsymbol{p}_{\mathrm{r}}^{\mathrm{d}} - z_2) \\ \dot{z}_2 &= \dot{\boldsymbol{p}}_{\mathrm{r}}^{\mathrm{d}} - \boldsymbol{h}_{\mathrm{r}}(\boldsymbol{q}_{\mathrm{r}}^{\mathrm{d}} - z_1, \boldsymbol{q}_{\mathrm{a}}^{\mathrm{d}} - \boldsymbol{\xi}_1) \\ \dot{\boldsymbol{\xi}}_1 &= \boldsymbol{\xi}_2 \\ \dot{\boldsymbol{\xi}}_2 &= \dot{\boldsymbol{p}}_{\mathrm{a}}^{\mathrm{d}} - \boldsymbol{u} \end{aligned} . \tag{34}$$

In the z subsystem of the system (34), function $\boldsymbol{h}_{\mathrm{r}}$ is not affine with respect to the input variable $\boldsymbol{\xi}_1$. Though a backstepping theorem was presented for the nonaffine system in the literature,[11] the theorem contains a necessary condition that assumes the Lyapunov function $H(z)$ about the z subsystem can be found. Whereas such knowledge about $H(z)$ is not always available. Now let's consider the affine approximation of the function $\boldsymbol{h}_{\mathrm{r}}$ for the specific system (22). Referring to Eq. (29) and the Appendix A, we have

$$\boldsymbol{h}_{\mathrm{r}}(\boldsymbol{q}_{\mathrm{r}}, \boldsymbol{q}_{\mathrm{a}}) = -\partial V(\boldsymbol{q}_{\mathrm{p}}, \boldsymbol{q}_{\mathrm{a}})/\partial \boldsymbol{q}_{\mathrm{p}} = [h_1 \quad h_2]^{\mathrm{T}}, \tag{35}$$

where

$$\begin{aligned} h_1 &= -(m_1 + 2m_2)g \sin\varphi - k(l_2 - l_0), \\ h_2 &= -\left[m_1(0.5l_1 + l_2) + 2m_2(l_1 + l_2)\right] g \cos\varphi \\ &\quad - m_2 gr\left[\cos(\varphi + \theta_1) + \cos(\varphi + \theta_2)\right]. \end{aligned}$$

Obviously, the functions h_1 and h_2 are not affine in variables $\boldsymbol{\xi}_1 = \boldsymbol{q}_{\mathrm{a}} = [\theta_1 \quad \theta_2]^{\mathrm{T}}$. We can always consider the swing angle of the leg of the hopping robots in steady motion to be small[29] since the limit of the actuators must be considered. Thus the motion of the leg is near to the vertical position $\varphi \approx 0.5\pi$, *viz.* $\sin\varphi \approx 1$, $\cos\varphi \approx 0.5\pi - \varphi$, and let $l_2 \approx l_{20}$ in h_2, then h_1 and h_2 can be approximated by

$$\begin{aligned} \tilde{h}_1 &= -(m_1 + 2m_2)g - k(l_2 - l_0), \\ \tilde{h}_2 &= H_0 g(\varphi - 0.5\pi) + m_2 gr(\sin\theta_1 + \sin\theta_2), \end{aligned} \tag{36}$$

where $H_0 = m_1(0.5l_1 + l_{20}) + 2m_2(l_1 + l_{20})$. From Eq. (14) and the first equation of Eq. (21), we have

$$\boldsymbol{q}_1 = \begin{bmatrix} l_2 \\ \varphi \end{bmatrix} + \tilde{\boldsymbol{m}}_{\mathrm{pp}}^{-1}(l_{20}, \theta_1^*, \theta_2^*) \begin{bmatrix} m_2 r(\cos\theta_1 + \cos\theta_2) \\ m_2 r^2(\theta_1 + \theta_2) + m_2(l_1 + l_{20})r(\sin\theta_1 + \sin\theta_2) \end{bmatrix}. \tag{37}$$

By the Eqs. (36) and (37) with considering the first-order approximation of $\cos\theta_i$, $\sin\theta_i$, $i = 1, 2$ at the point (θ_1^*, θ_2^*), then the affine approximation of function $\boldsymbol{h}_r$ can be expressed by

$$\tilde{\boldsymbol{h}}_r = \boldsymbol{h}_0(\boldsymbol{q}_{\mathrm{r}}, l_{20}, \theta_1^*, \theta_2^*) + \boldsymbol{h}_1(l_{20}, \theta_1^*, \theta_2^*)(\boldsymbol{q}_{\mathrm{a}}^{\mathrm{d}} - \boldsymbol{\xi}_1). \tag{38}$$

where

$$\begin{aligned} &\boldsymbol{h}_0(\boldsymbol{q}_{\mathrm{r}}, l_{20}, \theta_1^*, \theta_2^*) = \boldsymbol{A} + \boldsymbol{B}\left(\boldsymbol{q}_{\mathrm{r}} - \tilde{\boldsymbol{m}}_{\mathrm{pp}}^{-1}\boldsymbol{C}\right) \\ &\boldsymbol{h}_1(l_{20}, \theta_1^*, \theta_2^*) = \boldsymbol{D} - \boldsymbol{B}\tilde{\boldsymbol{m}}_{\mathrm{pp}}^{-1}\boldsymbol{E} \\ &\boldsymbol{A} = \begin{bmatrix} -(m_1 + 2m_2)g + kl_{20} \\ -0.5\pi H_0 g + m_2 gr(\sin\theta_1^* + \theta_1^*\cos\theta_1^* + \sin\theta_2^* \\ +\theta_2^*\cos\theta_2^*) \end{bmatrix} \\ &\boldsymbol{B} = \begin{bmatrix} -k & 0 \\ 0 & H_0 g \end{bmatrix} \\ &\boldsymbol{C} = \begin{bmatrix} m_2 r(\cos\theta_1^* - \theta_1^*\sin\theta_1^* + \cos\theta_2^* - \theta_2^*\sin\theta_2^*) \\ m_2(l_1 + l_{20})r(\sin\theta_1^* + \theta_1^*\cos\theta_1^* + \sin\theta_2^* + \theta_2^*\cos\theta_2^*) \end{bmatrix} \\ &\boldsymbol{D} = \begin{bmatrix} 0 & 0 \\ -m_2 gr\cos\theta_1^* & -m_2 gr\cos\theta_2^* \end{bmatrix} \\ &\boldsymbol{E} = \begin{bmatrix} m_2 r\sin\theta_1^* & m_2 r\sin\theta_2^* \\ m_2 r^2 - m_2(l_1 + l_{20})r\cos\theta_1^* & m_2 r^2 - m_2(l_1 + l_{20})r\cos\theta_2^* \end{bmatrix} \end{aligned}.$$

With considering the expression (38), the system (33) can be rewritten as an affine system in strict feedback form with perturbation terms

$$\begin{aligned} \dot{z}_1 &= \dot{\boldsymbol{q}}_{\mathrm{r}}^{\mathrm{d}} - \tilde{\boldsymbol{m}}_{\mathrm{pp}}^{-1}(l_{20}, \theta_1^*, \theta_2^*)(\boldsymbol{p}_{\mathrm{r}}^{\mathrm{d}} - z_2) - \boldsymbol{\varepsilon}_1 \\ \dot{z}_2 &= \dot{\boldsymbol{p}}_{\mathrm{r}}^{\mathrm{d}} - \boldsymbol{h}_0(\boldsymbol{q}_{\mathrm{r}}, l_{20}, \theta_1^*, \theta_2^*) - \boldsymbol{h}_1(l_{20}, \theta_1^*, \theta_2^*) \\ &\quad \times (\boldsymbol{q}_{\mathrm{a}}^{\mathrm{d}} - \boldsymbol{\xi}_1) - \boldsymbol{\varepsilon}_3 \\ \dot{\boldsymbol{\xi}}_1 &= \boldsymbol{\xi}_2 \\ \dot{\boldsymbol{\xi}}_2 &= \dot{\boldsymbol{p}}_{\mathrm{a}}^{\mathrm{d}} - \boldsymbol{u} \end{aligned} , \tag{39}$$

where $\boldsymbol{\varepsilon}_3 = \boldsymbol{\varepsilon}_2 + \boldsymbol{h}_{\mathrm{r}} - \tilde{\boldsymbol{h}}_{\mathrm{r}}$. Despite that the system (39) introduces an additional model error $\boldsymbol{h}_{\mathrm{r}} - \tilde{\boldsymbol{h}}_{\mathrm{r}}$, the following theorem is readily to be proved for ensuring the perturbations are bounded.

Theorem 1: (The perturbation terms are bounded) For the hopping robot illustrated in Fig. 1, the perturbations terms

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$\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_3$ in the dynamics (39) of the robot are bounded, viz. there exist positive constants $\Gamma_i > 0, i = 1, 2$ such that $\|\boldsymbol{\varepsilon}_1\| \le \Gamma_1$ and $\|\boldsymbol{\varepsilon}_3\| \le \Gamma_2$ are satisfied.

Proof:
The hopping robot system is a periodic motion system, thus the generalized coordinates $\boldsymbol{q} = [l_2\ \varphi\ \theta_1\ \theta_2]^{\mathrm{T}}$, generalized velocities $\dot{\boldsymbol{q}} = [\dot{l}_2\ \dot{\varphi}\ \dot{\theta}_1\ \dot{\theta}_2]^{\mathrm{T}}$, generalized accelerations $\ddot{\boldsymbol{q}} = [\ddot{l}_2\ \ddot{\varphi}\ \ddot{\theta}_1\ \ddot{\theta}_2]^{\mathrm{T}}$, and the generalized momenta $\boldsymbol{p}_{\mathrm{r}}$ defined by Eq. (16) are all bounded. Then, any perturbation terms of the system (39) are bounded, and one has $\Gamma_1 = \max(\|\varepsilon_1\|)$, $\Gamma_2 = \max(\|\varepsilon_3\|)$. □

For the purpose of clarity, the equations of Eq. (39) can be rewritten as a more compact form

$$\begin{aligned} \dot{z}_1 &= \boldsymbol{f}_1 + \boldsymbol{g}_1 z_2 - \boldsymbol{\varepsilon}_1 \\ \dot{z}_2 &= \boldsymbol{f}_2 + \boldsymbol{g}_2 \boldsymbol{\xi}_1 - \boldsymbol{\varepsilon}_3 \\ \dot{\boldsymbol{\xi}}_1 &= \boldsymbol{f}_3 + \boldsymbol{g}_3 \boldsymbol{\xi}_2 \\ \dot{\boldsymbol{\xi}}_2 &= \boldsymbol{f}_4 + \boldsymbol{g}_4 \boldsymbol{u} \end{aligned}, \tag{40}$$

where

$$\begin{aligned} &\boldsymbol{f}_1 = \dot{\boldsymbol{q}}_{\mathrm{r}}^{\mathrm{d}} - \tilde{\boldsymbol{m}}_{\mathrm{pp}}^{-1} \boldsymbol{p}_{\mathrm{r}}^{\mathrm{d}} && \boldsymbol{g}_1 = \tilde{\boldsymbol{m}}_{\mathrm{pp}}^{-1} \\ &\boldsymbol{f}_2 = \dot{\boldsymbol{p}}_{\mathrm{r}}^{\mathrm{d}} - \boldsymbol{h}_0 - \boldsymbol{h}_1 \boldsymbol{q}_{\mathrm{a}}^{\mathrm{d}} && \boldsymbol{g}_2 = \boldsymbol{h}_1 \\ &\boldsymbol{f}_3 = 0 && \boldsymbol{g}_3 = \boldsymbol{I} \\ &\boldsymbol{f}_4 = \dot{\boldsymbol{p}}_{\mathrm{a}}^{\mathrm{d}} && \boldsymbol{g}_4 = -\boldsymbol{I} \end{aligned}.$$

$\boldsymbol{I}$ is the identical matrix.

Then, based on the Theorem 1, the following theorem can be proved. This theorem is the main result of the paper, which presents a method for stabilizing the non-SLIP model-based hopping system in stance phase.

Theorem 2: (Robust backstepping) Consider the system (40), if let $k_i > 0, i = 1, \ldots, 4$, $\eta_i > 0, i = 1, 2$, $\Gamma_1 = \max(\|\varepsilon_1\|)$, $\Gamma_2 = \max(\|\varepsilon_3\|)$, and define some positive definite function as following

$$\begin{aligned} &H_1(z_1) = \frac{1}{2} z_1^{\mathrm{T}} z_1,\ H_2(z_1, z_2) = H_1(z_1) + \frac{1}{2}(z_2 - \boldsymbol{\alpha}_1)^{\mathrm{T}}(z_2 - \boldsymbol{\alpha}_1) \\ &H_3(z_1, z_2, \boldsymbol{\xi}_1) = H_2(z_1, z_2) + \frac{1}{2}(\boldsymbol{\xi}_1 - \boldsymbol{\alpha}_2)^{\mathrm{T}}(\boldsymbol{\xi}_1 - \boldsymbol{\alpha}_2) \\ &H_4(z_1, z_2, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2) = H_3(z_1, z_2, \boldsymbol{\xi}_1) + \frac{1}{2}(\boldsymbol{\xi}_2 - \boldsymbol{\alpha}_3)^{\mathrm{T}}(\boldsymbol{\xi}_2 - \boldsymbol{\alpha}_3) \end{aligned}.$$

Then, the following smooth state feedback

$$\begin{aligned} \boldsymbol{u} = \boldsymbol{g}_4^{-1} \Bigg[& -k_4(\boldsymbol{\xi}_2 - \boldsymbol{\alpha}_3) - \left(\frac{\partial H_3}{\partial \boldsymbol{\xi}_1} \boldsymbol{g}_3 \right)^{\mathrm{T}} - \boldsymbol{f}_4 \\ & + \frac{\partial \boldsymbol{\alpha}_3}{\partial \boldsymbol{\xi}_1}(\boldsymbol{f}_3 + \boldsymbol{g}_3 \boldsymbol{\xi}_2) + \frac{\partial \boldsymbol{\alpha}_3}{\partial z_2}(\boldsymbol{f}_2 + \boldsymbol{g}_2 \boldsymbol{\xi}_1) \\ & + \frac{\partial \boldsymbol{\alpha}_3}{\partial z_1}(\boldsymbol{f}_1 + \boldsymbol{g}_1 z_2) \Bigg], \end{aligned} \tag{41}$$

where

$$\begin{aligned} \boldsymbol{\alpha}_3(z_1, z_2, \boldsymbol{\xi}_1) = \boldsymbol{g}_3^{-1} \Bigg[& -k_3(\boldsymbol{\xi}_1 - \boldsymbol{\alpha}_2) - \left(\frac{\partial H_2}{\partial z_2} \boldsymbol{g}_2 \right)^{\mathrm{T}} \\ & - \boldsymbol{f}_3 + \frac{\partial \boldsymbol{\alpha}_2}{\partial z_2}(\boldsymbol{f}_2 + \boldsymbol{g}_2 \boldsymbol{\xi}_1) + \frac{\partial \boldsymbol{\alpha}_2}{\partial z_1}(\boldsymbol{f}_1 + \boldsymbol{g}_1 z_2) \Bigg], \\ \boldsymbol{\alpha}_2(z_1, z_2) = \boldsymbol{g}_2^{-1} \Bigg[& -(k_2 + \eta_2 \Gamma_2^2)(z_2 - \boldsymbol{\alpha}_1) \\ & - \left(\frac{\partial H_1}{\partial z_1} \boldsymbol{g}_1 \right)^{\mathrm{T}} - \boldsymbol{f}_2 + \frac{\partial \boldsymbol{\alpha}_1}{\partial z_1}(\boldsymbol{f}_1 + \boldsymbol{g}_1 z_2) \Bigg], \\ \boldsymbol{\alpha}_1(z_1) = \boldsymbol{g}_1^{-1} & \left[-(k_1 + \eta_1 \Gamma_1^2) z_1 - \boldsymbol{f}_1 \right], \end{aligned}$$

renders $(z, \boldsymbol{\xi})$ uniformly bounded for the system (40) and, furthermore, converges to a compact residual set

$$\Omega = \left\{ (z_1, z_2, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2) : H(z, \boldsymbol{\xi}) \le \sigma \right\},$$

where $H(z, \boldsymbol{\xi}) = H_4(z_1, z_2, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2)$, and σ is an arbitrary small positive constant.

Proof:
Consider the z_1 subsystem of Eq. (40), select $H_1(z_1)$ as the candidate Lyapunov function and assume z_2 be the virtual input for z_1 subsystem, then given $\forall \eta_1 > 0$, by the Young's Inequality $2ab \le a^2 + b^2$ and Cauchy–Schwarz Inequality $|\boldsymbol{x}^{\mathrm{T}} \boldsymbol{y}| \le \|\boldsymbol{x}\| \cdot \|\boldsymbol{y}\|$, one has

$$\begin{aligned} \dot{H}_1(z_1) &= z_1^{\mathrm{T}} \left[\boldsymbol{f}_1 + \boldsymbol{g}_1 z_2 - \boldsymbol{\varepsilon}_1 \right] \\ &\le z_1^{\mathrm{T}} \left[\boldsymbol{f}_1 + \boldsymbol{g}_1 z_2 \right] + \eta_1 \left\| z_1^{\mathrm{T}} \right\|^2 \|\boldsymbol{\varepsilon}_1\|^2 + 1/(4\eta_1) \\ &\le z_1^{\mathrm{T}} \left[\boldsymbol{f}_1 + \boldsymbol{g}_1 z_2 + \eta_1 \Gamma_1^2 z_1 \right] + 1/(4\eta_1). \end{aligned}$$

Let $z_2 = \boldsymbol{\alpha}_1(z_1) = -\boldsymbol{g}_1^{-1}(k_1 + \eta_1 \Gamma_1^2) z_1 - \boldsymbol{g}_1^{-1} \boldsymbol{f}_1$, where $\forall k_1 > 0$, then we obtain

$$\dot{H}_1(z_1) \le -k_1 z_1^{\mathrm{T}} z_1 + 1/(4\eta_1). \tag{42}$$

Furthermore, select $H_2(z_1, z_2)$ as the candidate Lyapunov function for composite subsystems z_1, z_2, assume that $\boldsymbol{\xi}_1$ is the virtual input for the same subsystems, and let $\boldsymbol{e}_{z_2} = z_2 - \boldsymbol{\alpha}_1(z_1)$, then one has

$$\begin{aligned} \dot{H}_2(z_1, z_2) &= \dot{H}_1(z_1) + \boldsymbol{e}_{z_2}^{\mathrm{T}}(\dot{z}_2 - \dot{\boldsymbol{\alpha}}_1(z_1)) \\ &= \frac{\partial H_1}{\partial z_1} \left[\boldsymbol{f}_1 + \boldsymbol{g}_1(\boldsymbol{e}_{z_2} + \boldsymbol{\alpha}_1) - \boldsymbol{\varepsilon}_1 \right] \\ &\quad + \boldsymbol{e}_{z_2}^{\mathrm{T}} \left[\boldsymbol{f}_2 + \boldsymbol{g}_2 \boldsymbol{\xi}_1 - \boldsymbol{\varepsilon}_3 - \frac{\partial \boldsymbol{\alpha}_1}{\partial z_1}(\boldsymbol{f}_1 + \boldsymbol{g}_1 z_2) \right] \\ &= \frac{\partial H_1}{\partial z_1} \left[\boldsymbol{f}_1 + \boldsymbol{g}_1 \boldsymbol{\alpha}_1 - \boldsymbol{\varepsilon}_1 \right] \\ &\quad + \boldsymbol{e}_{z_2}^{\mathrm{T}} \Bigg[\left(\frac{\partial H_1}{\partial z_1} \boldsymbol{g}_1 \right)^{\mathrm{T}} + \boldsymbol{f}_2 + \boldsymbol{g}_2 \boldsymbol{\xi}_1 - \boldsymbol{\varepsilon}_3 \\ &\quad - \frac{\partial \boldsymbol{\alpha}_1}{\partial z_1}(\boldsymbol{f}_1 + \boldsymbol{g}_1 z_2) \Bigg]. \end{aligned} \tag{43}$$

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With considering the inequality (42), the Eq. (43) follows that

$$\begin{aligned}\dot{H}_2(z_1,z_2) &\leq -k_1 z_1^{\mathrm{T}} z_1 + \boldsymbol{e}_{z_2}^{\mathrm{T}}\left[\left(\frac{\partial H_1}{\partial z_1}\boldsymbol{g}_1\right)^{\mathrm{T}} + \boldsymbol{f}_2 + \boldsymbol{g}_2\boldsymbol{\xi}_1 - \boldsymbol{\varepsilon}_3 \right.\\ &\quad \left. - \frac{\partial \boldsymbol{\alpha}_1}{\partial z_1}(\boldsymbol{f}_1 + \boldsymbol{g}_1 z_2)\right] + \frac{1}{4\eta_1}\\ &\leq -k_1 z_1^{\mathrm{T}} z_1 + \boldsymbol{e}_{z_2}^{\mathrm{T}}\left[\left(\frac{\partial H_1}{\partial z_1}\boldsymbol{g}_1\right)^{\mathrm{T}} + \boldsymbol{f}_2 + \boldsymbol{g}_2\boldsymbol{\alpha}_2 \right.\\ &\quad \left. + \boldsymbol{g}_2(\boldsymbol{\xi}_1 - \boldsymbol{\alpha}_2) - \frac{\partial \boldsymbol{\alpha}_1}{\partial z_1}(\boldsymbol{f}_1 + \boldsymbol{g}_1 z_2) + \eta_2\Gamma_2^2 \boldsymbol{e}_{z_2}\right] + \delta,\end{aligned} \tag{44}$$

where $\delta = 1/(4\eta_1) + 1/(4\eta_2)$. Let

$$\begin{aligned}\boldsymbol{\alpha}_2(z_1,z_2) &= -\boldsymbol{g}_2^{-1}\left(k_2 + \eta_2\Gamma_2^2\right)\boldsymbol{e}_{z_2} - \boldsymbol{g}_2^{-1}\left[\left(\frac{\partial H_1}{\partial z_1}\boldsymbol{g}_1\right)^{\mathrm{T}}\right.\\ &\quad \left. + \boldsymbol{f}_2 - \frac{\partial \boldsymbol{\alpha}_1}{\partial z_1}(\boldsymbol{f}_1 + \boldsymbol{g}_1 z_2)\right],\end{aligned}$$

where $\forall k_2 > 0$. From the inequality (44), we have

$$\dot{H}_2(z_1,z_2) \leq -k_1 z_1^{\mathrm{T}} z_1 - k_2 \boldsymbol{e}_{z_2}^{\mathrm{T}}\boldsymbol{e}_{z_2} + \frac{\partial H_2}{\partial z_2}\boldsymbol{g}_2(\boldsymbol{\xi}_1 - \boldsymbol{\alpha}_2) + \delta. \tag{45}$$

Furthermore, select $H_3(z_1, z_2, \boldsymbol{\xi}_1)$ as the candidate Lyapunov function for composite subsystems $(z_1, z_2, \boldsymbol{\xi}_1)$, assume that $\boldsymbol{\xi}_2$ is a new virtual input for the subsystems $(z_1, z_2, \boldsymbol{\xi}_1)$, and let $\boldsymbol{e}_{\xi_1} = \boldsymbol{\xi}_1 - \boldsymbol{\alpha}_2$, then we have

$$\begin{aligned}\dot{H}_3(z_1,z_2,\boldsymbol{\xi}_1) &= \dot{H}_2(z_1,z_2) + \boldsymbol{e}_{\xi_1}^{\mathrm{T}}(\dot{\boldsymbol{\xi}}_1 - \dot{\boldsymbol{\alpha}}_2)\\ &\leq = -k_1 z_1^{\mathrm{T}} z_1 - k_2 \boldsymbol{e}_{z_2}^{\mathrm{T}}\boldsymbol{e}_{z_2} + \boldsymbol{e}_{\xi_1}^{\mathrm{T}}\left[\left(\frac{\partial H_2}{\partial z_2}\boldsymbol{g}_2\right)^{\mathrm{T}}\right.\\ &\quad + \boldsymbol{f}_3 + \boldsymbol{g}_3\boldsymbol{\alpha}_3 + \boldsymbol{g}_3(\boldsymbol{\xi}_2 - \boldsymbol{\alpha}_3)\\ &\quad \left. - \frac{\partial \boldsymbol{\alpha}_2}{\partial z_2}(\boldsymbol{f}_2 + \boldsymbol{g}_2\boldsymbol{\xi}_1) - \frac{\partial \boldsymbol{\alpha}_2}{\partial z_1}(\boldsymbol{f}_1 + \boldsymbol{g}_1 z_2)\right] + \delta.\end{aligned} \tag{46}$$

Let

$$\begin{aligned}\boldsymbol{\alpha}_3(z_1,z_2,\boldsymbol{\xi}_1) &= -\boldsymbol{g}_3^{-1}\left[k_3\boldsymbol{e}_{\xi_1} + \left(\frac{\partial H_2}{\partial z_2}\boldsymbol{g}_2\right)^{\mathrm{T}} + \boldsymbol{f}_3\right.\\ &\quad \left. - \frac{\partial \boldsymbol{\alpha}_2}{\partial z_2}(\boldsymbol{f}_2 + \boldsymbol{g}_2\boldsymbol{\xi}_1) - \frac{\partial \boldsymbol{\alpha}_2}{\partial z_1}(\boldsymbol{f}_1 + \boldsymbol{g}_1 z_2)\right],\end{aligned}$$

where $\forall k_3 > 0$. The expression (46) has the form

$$\begin{aligned}\dot{H}_3(z_1,z_2,\boldsymbol{\xi}_1) &\leq \dot{H}_2(z_1,z_2) - k_3\boldsymbol{\xi}_1^{\mathrm{T}}\boldsymbol{\xi}_1 + \boldsymbol{e}_{\xi_1}^{\mathrm{T}}\boldsymbol{g}_3(\boldsymbol{\xi}_2 - \boldsymbol{\alpha}_3) + \delta\\ &= -k_1 z_1^{\mathrm{T}} z_1 - k_2 \boldsymbol{e}_{z_2}^{\mathrm{T}}\boldsymbol{e}_{z_2} - k_3\boldsymbol{e}_{\xi_1}^{\mathrm{T}}\boldsymbol{e}_{\xi_1}\\ &\quad + \frac{\partial H_3}{\partial \boldsymbol{\xi}_1}\boldsymbol{g}_3(\boldsymbol{\xi}_2 - \boldsymbol{\alpha}_3) + \delta.\end{aligned} \tag{47}$$

Finally, select $H_4(z_1, z_2, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2)$ as the candidate Lyapunov function for the system (40), and let $e_{\xi_2} = \boldsymbol{\xi}_2 - \boldsymbol{\alpha}_3$, then it follows that

$$\begin{aligned}\dot{H}_4(z_1,z_2,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2) &= \dot{H}_3(z_1,z_2,\boldsymbol{\xi}_1) + e_{\xi_2}^{\mathrm{T}}(\dot{\boldsymbol{\xi}}_2 - \dot{\boldsymbol{\alpha}}_3)\\ &\leq -k_1 z_1^{\mathrm{T}} z_1 - k_2 \boldsymbol{e}_{z_2}^{\mathrm{T}}\boldsymbol{e}_{z_2} - k_3\boldsymbol{e}_{\xi_1}^{\mathrm{T}}\boldsymbol{e}_{\xi_1}\\ &\quad + \boldsymbol{e}_{\xi_2}^{\mathrm{T}}\left[\left(\frac{\partial H_3}{\partial \boldsymbol{\xi}_1}\boldsymbol{g}_3\right)^{\mathrm{T}} + \boldsymbol{f}_4 + \boldsymbol{g}_4\boldsymbol{u}\right.\\ &\quad - \frac{\partial \boldsymbol{\alpha}_3}{\partial \boldsymbol{\xi}_1}(\boldsymbol{f}_3 + \boldsymbol{g}_3\boldsymbol{\xi}_2) - \frac{\partial \boldsymbol{\alpha}_3}{\partial z_2}(\boldsymbol{f}_2 + \boldsymbol{g}_2\boldsymbol{\xi}_1)\\ &\quad \left. - \frac{\partial \boldsymbol{\alpha}_3}{\partial z_1}(\boldsymbol{f}_1 + \boldsymbol{g}_1 z_2)\right] + \delta.\end{aligned} \tag{48}$$

Let the actual input for the system (40) be

$$\begin{aligned}\boldsymbol{u} &= -\boldsymbol{g}_4^{-1}\left[k_4\boldsymbol{e}_{\xi_2} + \left(\frac{\partial H_3}{\partial \boldsymbol{\xi}_1}\boldsymbol{g}_3\right)^{\mathrm{T}} + \boldsymbol{f}_4 - \frac{\partial \boldsymbol{\alpha}_3}{\partial \boldsymbol{\xi}_1}(\boldsymbol{f}_3 + \boldsymbol{g}_3\boldsymbol{\xi}_2)\right.\\ &\quad \left. - \frac{\partial \boldsymbol{\alpha}_3}{\partial z_2}(\boldsymbol{f}_2 + \boldsymbol{g}_2\boldsymbol{\xi}_1) - \frac{\partial \boldsymbol{\alpha}_3}{\partial z_1}(\boldsymbol{f}_1 + \boldsymbol{g}_1 z_2)\right] + \delta,\end{aligned} \tag{49}$$

where $\forall k_4 > 0$. From Eq. (48), it follows that

$$\begin{aligned}&\dot{H}_4(z_1,z_2,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2) \leq -k_1 z_1^{\mathrm{T}} z_1 - k_2 \boldsymbol{e}_{z_2}^{\mathrm{T}}\boldsymbol{e}_{z_2} - k_3\boldsymbol{e}_{\xi_1}^{\mathrm{T}}\boldsymbol{e}_{\xi_1}\\ &- k_4\boldsymbol{e}_{\xi_2}^{\mathrm{T}}\boldsymbol{e}_{\xi_2} + \delta.\end{aligned} \tag{50}$$

If let $k_1 = k_2 = k_3 = k_4 = 0.5\lambda > 0$, the expression (50) can be rewritten as

$$\dot{H}_4 \leq -\lambda H_4 + \delta. \tag{51}$$

Thus

$$\begin{aligned}H_4(t) &\leq H_4(t_0)e^{-\lambda(t-t_0)} + (\delta/\lambda)(1 - e^{-\lambda(t-t_0)})\\ &\leq H_4(t_0)e^{-\lambda(t-t_0)} + \delta/\lambda\end{aligned}. \tag{52}$$

Since $H_4(t) \geq \max\{\frac{1}{2}z_1^{\mathrm{T}}z_1, \frac{1}{2}\boldsymbol{e}_{z_2}^{\mathrm{T}}\boldsymbol{e}_{z_2}, \frac{1}{2}\boldsymbol{e}_{\xi_1}^{\mathrm{T}}\boldsymbol{e}_{\xi_1}, \frac{1}{2}\boldsymbol{e}_{\xi_2}^{\mathrm{T}}\boldsymbol{e}_{\xi_2}\}$, and $\lim_{t\to\infty} H_4(t_0)e^{-\lambda(t-t_0)} = 0$, for all $\delta_0 > 0$, there exists a constant $T > 0$ such that

$$H_4(t_0)e^{-\lambda(t-t_0)} < \delta_0,\ t > T.$$

Therefore we have

$$\left\|\boldsymbol{\rho}_i\right\| < 2(\delta_0 + \delta/\lambda), \quad i = 1,2,3,4, \tag{53}$$

where $\boldsymbol{\rho}_1 = z_1$, $\boldsymbol{\rho}_2 = \boldsymbol{e}_{z_2}$, $\boldsymbol{\rho}_3 = \boldsymbol{e}_{\xi_1}$, and $\boldsymbol{\rho}_4 = \boldsymbol{e}_{\xi_2}$ are considered. By selecting the independent parameters λ, η_1, η_2, an arbitrary small positive number σ can always be found such that

$$H_4(z_1,z_2,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2) \leq \sigma \leq \delta/\lambda.$$

This finishes the proof. □

Remark 7: The robust backstepping control (41) depends on the bounded perturbation terms of the system (40).

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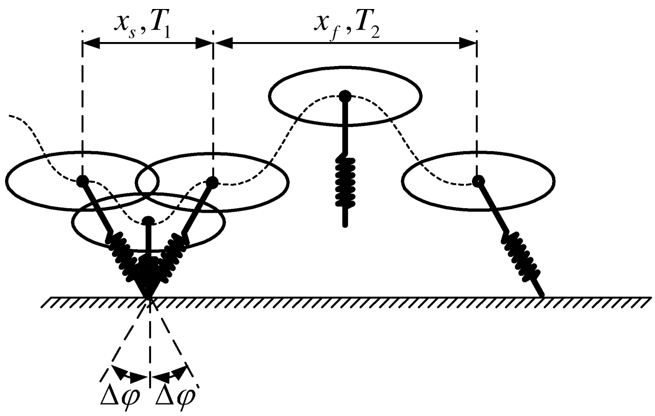

Fig. 2. The sketch of once hop for the one-legged robot.

Thanks to the periodic motion manner of the hopping robots, the perturbations in system (40) are bounded in principle. Furthermore, the hopping robot system works near to its vertical equilibrium configuration, then the bounds of the perturbation terms are limited to a small level.

5. Numerical Simulations

Figure 2 shows a sketch for understanding the steady motion of the hopping robot, where x_s and T_1 denote the moving distance and the duration in stance phase, respectively, x_f and T_2 correspond to the moving distance and the duration in flight phase, respectively. $\Delta\varphi$ is the swing angle deviating from the vertical position of the leg (assume that the swing angle is symmetric about the vertical position in stance phase).

Here we just give a brief description of the target motion of the hopping robot in stance phase, for more detailed contents one can refer to the reference.[30] Due to the periodic motion manner, the telescopic motion of the leg should be the resonance

$$l_2^{\mathrm{d}}(t) = l_{20} + A(t)\sin(\omega_n t + \beta_{l_2}), \tag{54}$$

where l_{20} is the static balance length of the telescopic leg segment in gravity. $A(t)$ is the desired amplitude of the telescopic motion of the passive leg. $\omega_n = \sqrt{k/m}$ is the inherent angular frequency of the robot, and β_{l_2} is the phase angle, which has a formulation[30]

$$\beta_{l_2} = -\arcsin(\Delta l_2/A(t)), \tag{55}$$

where $0 < \Delta l_2 = l_2(t_0) - l_{20}$ (since $l_2(t_0) \geq l_{20}$ is satisfied when the robot is free). Similarly, the swing of the leg should satisfy a formulation given by

$$\varphi^{\mathrm{d}}(t) = \varphi_0 + \Delta\varphi(t)\sin\left(\frac{1}{2}\omega_n t + \beta_\varphi\right), \tag{56}$$

where φ_0 $(= 0.5\pi)$ is the vertical position of the leg. $\Delta\varphi(t)$ is the swing amplitude, and β_φ is the phase angle. As to the desired motion of the two arms, one cannot give a motion plan intuitionally except for their equilibrium position

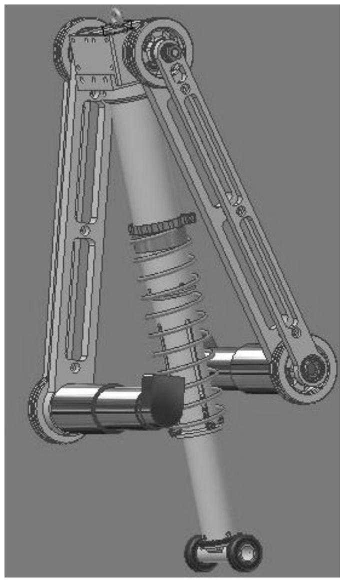

Fig. 3. The 3D model of the robot prototype.

$$\theta_i^* = \frac{1}{2}\left(\theta_i^{\min} + \theta_i^{\max}\right) = 0.5\pi,\ i = 1, 2. \tag{58}$$

The parameters of the robot model are listed in Appendix B. A 3D model of the robot prototype also is shown in Fig. 3 for helping to understand the structure of the robot.

By the robust controller (41), the first simulation results are plotted in Fig. 4, which corresponds to a control task for stabilizing the robot from an initial static configuration $[l_2\ \varphi\ \theta_1\ \theta_2]^{\mathrm{T}}(t_0) = [l_{20} + 0.01\,\mathrm{m}\ 100^\circ\ 130^\circ\ -110^\circ]^{\mathrm{T}}$ to the target configuration $[l_2^d\ \varphi^d\ \theta_1^d\ \theta_2^d]^{\mathrm{T}} = [l_{20}\ 90^\circ\ 90^\circ\ -90^\circ]^{\mathrm{T}}$ that is the stance balance configuration.

The second simulation results are illustrated in Fig. 5. The corresponding control task is tracking the trajectory given by Eqs. (54–58) with the same initial conditions as that in Fig. 4 while the amplitudes of telescopic motion and swing motion of the leg are set to $A(t) = 0.04\,\mathrm{m}$ and $\Delta\varphi(t) = 10^\circ$, respectively. The former value is selected such that the leg does not leave the ground, while the later value is selected as a typical small angle of the leg deviating from the vertical position such that the robot can be stabilized with bounded inputs. For the purpose of looking into the stabilization of the closed-loop system, we plot the phase trajectory of the coordinates l_2, φ and θ_1 by Figs. 6–8, respectively, of which the data are taken from the last 10 s of Fig. 5. The phase trajectory of coordinate θ_2 is omitted since it is very similar to that of θ_1. Figure 9 shows the first two s torque of Fig. 5 (d) for clearly, while Fig. 10 redraws the torque of steady motion of Fig. 5 during two periods of swing of the leg.

It is worthy to be mentioned that motion planning for the hopping robot is far from an intuitional task.[30,31] For obtaining an acceptable motion with small control inputs, the initial state and target state/trajectory of the system must be selected carefully. For instance, the swing angle of the leg should be limited to the neighborhood of the vertical position at start, the vibration amplitude of the spring should not

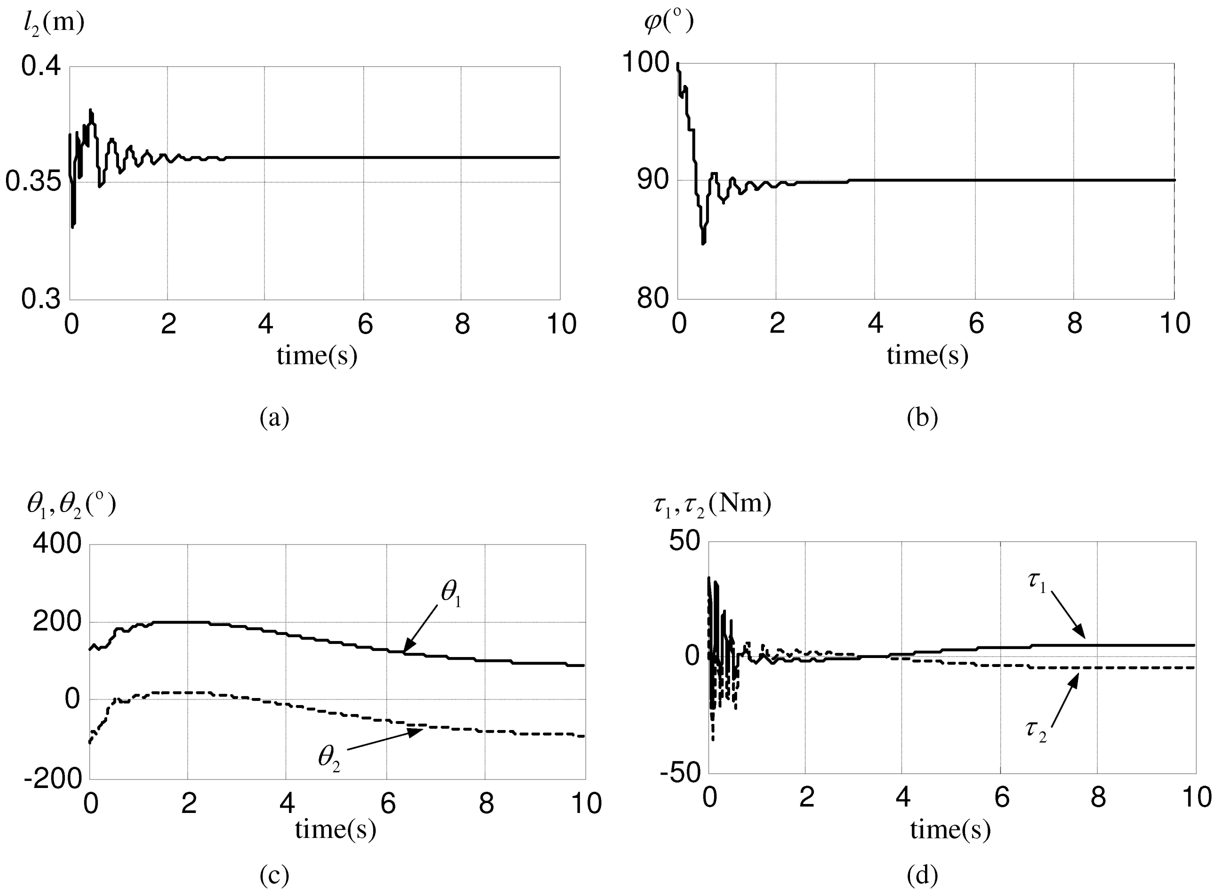

Fig. 4. Simulation results of controlling the hopping robot to stance balance.

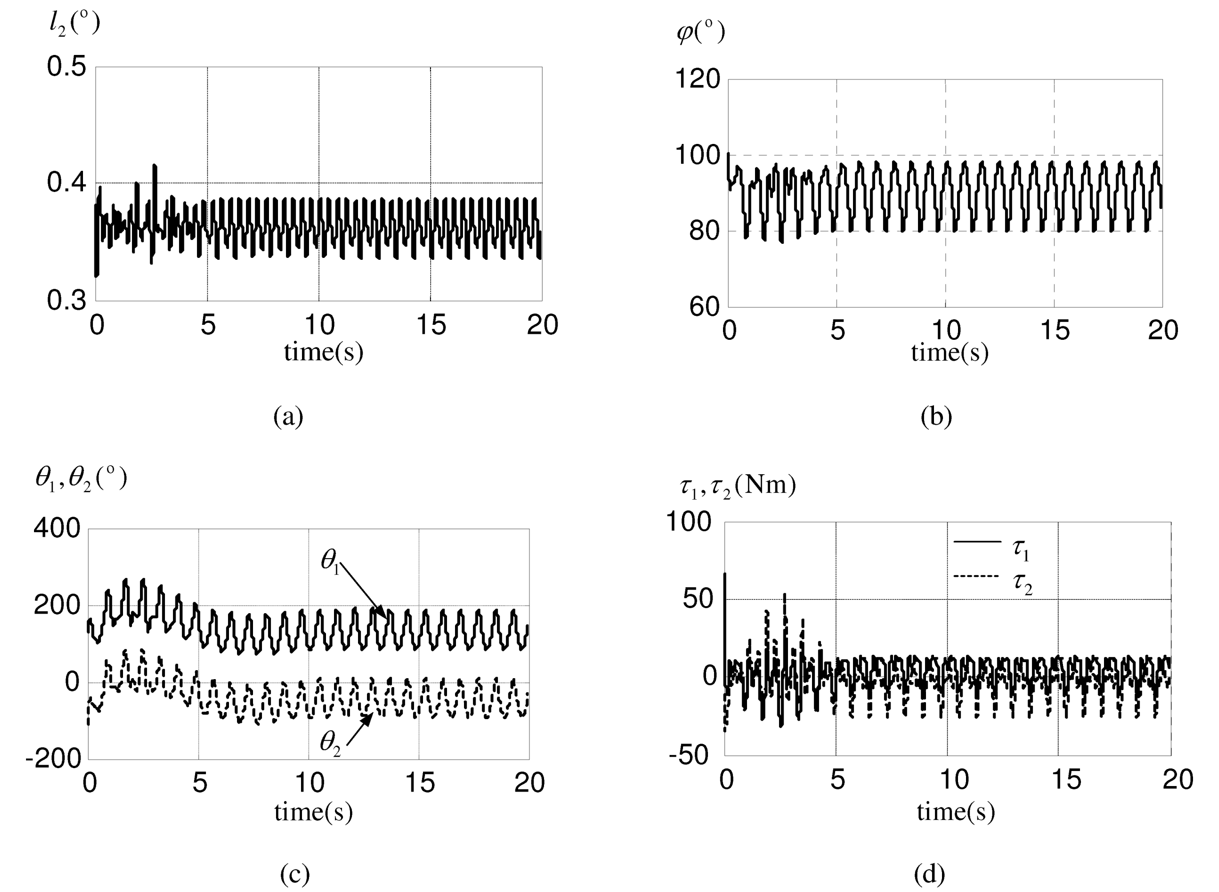

Fig. 5. Simulation results of controlling the hopping robot to a small-amplitude periodic motion in stance phase.

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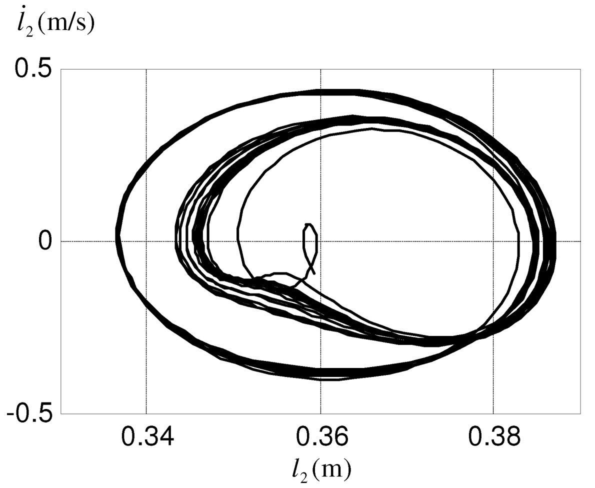

Fig. 6. The phase trajectory of variable l_2 with the last 10 s data of Fig. 5(a).

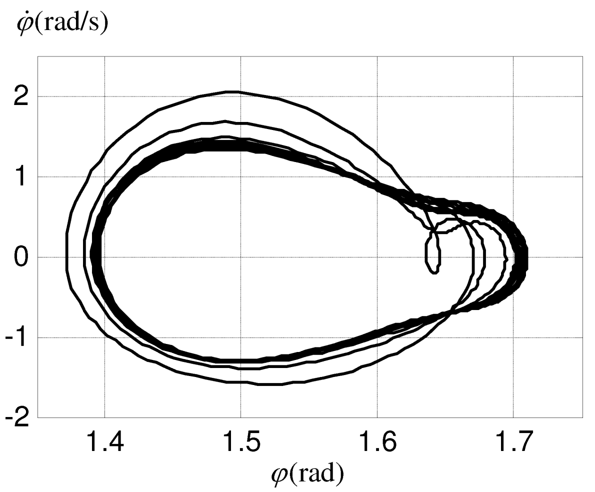

Fig. 7. The phase trajectory of variable φ with the last 10 s data of Fig. 5 (b).

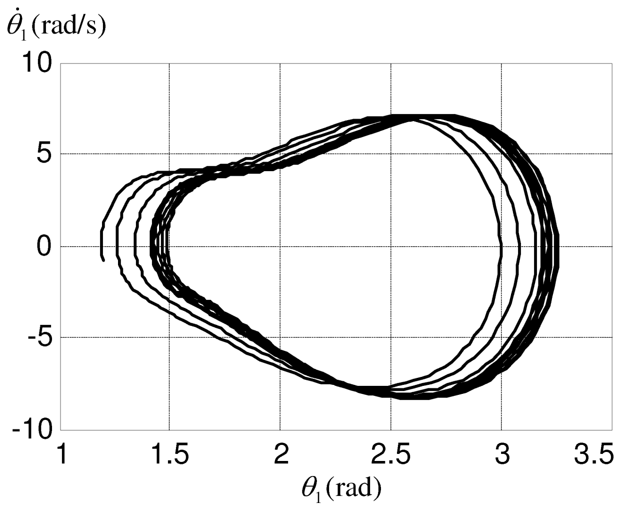

Fig. 8. The phase trajectory of variable θ_1 with the last 10 s data of Fig. 5(c).

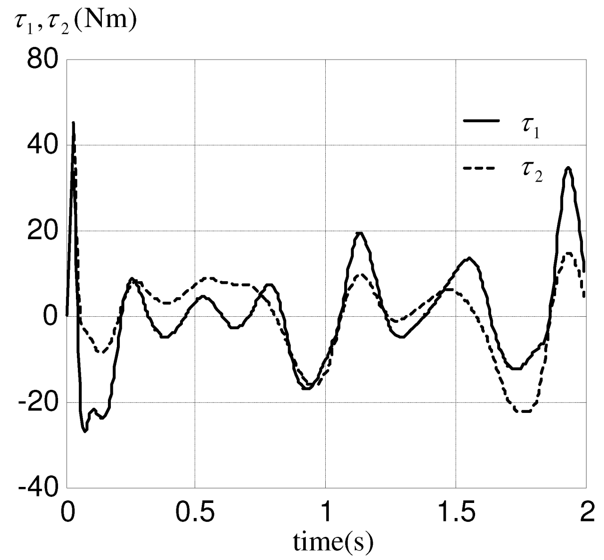

Fig. 9. The torque of the actuators with the first 2 s data of Fig. 5(d).

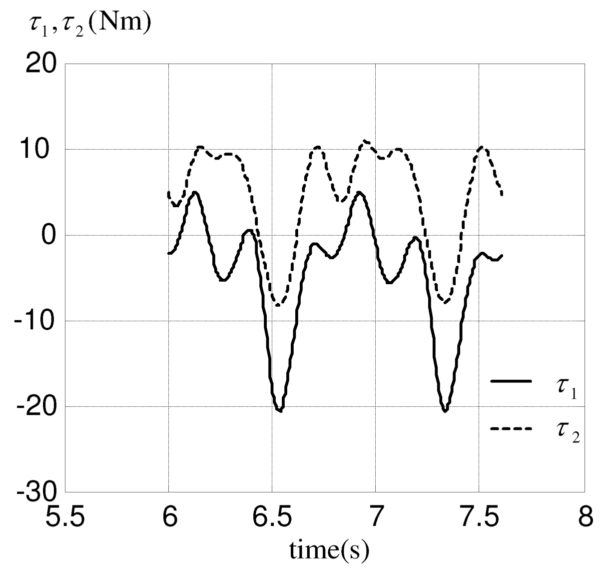

Fig. 10. The torque of the actuators during the steady motion of Fig. 5.

exceed the telescopic range of the passive leg, and the phase angle of the vibration of the elastic leg should be selected to be on the beat in the sense of the resonance of the system. As shown by Ahmadi and Buehler,[31] a rationally planned motion, such as the passive dynamic motion can considerably reduce the torque of inputs of the hopping systems. Whereas this is another complex problem especially for a non-SLIP model-based underactuated legged robot system.

6. Conclusions

Comparing to the SLIP model hopping robot systems, the non-SLIP model-based hopping robots release the restriction that the CM of the system has to be coincident with the hip joint. This permits one to understand the behaviors of the nonlinear dynamic coupling and search a reasonable way to utilizing the inherent properties to improve the energy-efficiency as well as the mobility of the animalized robot

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systems. Trough a specific hopping system just actuated by two arms, while it holds a rather general dynamics of the non-SLIP model systems, it is shown that the underactuated hopping system can be stabilized in stance phase. We show that the dynamics of the underactuated hopping robot can be transformed into a strict feedback form with bounded perturbation terms based on the Olfati-Saber transformation if there exists nonconstant potential field, such that the backstepping control could be used to stabilize the robot system in stance phase. Since the normalized form is approximate, we proposed a robust backstepping controller for stabilizing the robot system. Both theoretical proof and illustrational numerical simulations verify that the control scheme presented in this paper is effective for the specific systems. These results encourage us to invent a new non-SLIP model-based legged running robot with better energy-efficiency by some optimization methods in the future, as well as to develop the applications of robust nonlinear control techniques in biological mechanical systems with highly nonlinear dynamics.

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Appendix A. The Inertia Matrix of the Robot Prototype

The positions of CM of the links for the underactuated one-legged hopping robot are given as follows:

The CM of the leg

$$\begin{aligned} x_{c1} &= (0.5l_1 + l_2)\cos\varphi \\ z_{c1} &= (0.5l_1 + l_2)\sin\varphi \end{aligned}. \tag{A1}$$

The first arm

$$\begin{aligned} x_{c2} &= (l_1 + l_2)\cos\varphi + r\cos(\varphi + \theta_1) \\ z_{c2} &= (l_1 + l_2)\sin\varphi + r\sin(\varphi + \theta_1) \end{aligned}. \tag{A2}$$

The second arm

$$\begin{aligned} x_{c3} &= (l_1 + l_2)\cos\varphi + r\cos(\varphi + \theta_2) \\ z_{c3} &= (l_1 + l_2)\sin\varphi + r\sin(\varphi + \theta_2) \end{aligned}. \tag{A3}$$

The velocities of the CMs of the links are

$$\begin{aligned} \dot{x}_{c1} &= -(0.5l_1 + l_2)\dot{\varphi}\sin\varphi + \dot{l}_2\cos\varphi \\ \dot{z}_{c1} &= (0.5l_1 + l_2)\dot{\varphi}\cos\varphi + \dot{l}_2\sin\varphi \end{aligned}, \tag{A4}$$

$$\begin{aligned} \dot{x}_{c2} &= -(l_1 + l_2)\dot{\varphi}\sin\varphi + \dot{l}_2\cos\varphi - r(\dot{\varphi} + \dot{\theta}_1)\sin(\varphi + \theta_1) \\ \dot{z}_{c2} &= (l_1 + l_2)\dot{\varphi}\cos\varphi + \dot{l}_2\sin\varphi + r(\dot{\varphi} + \dot{\theta}_1)\cos(\varphi + \theta_1) \end{aligned}, \tag{A5}$$

$$\begin{aligned} \dot{x}_{c3} &= -(l_1 + l_2)\dot{\varphi}\sin\varphi + \dot{l}_2\cos\varphi - r(\dot{\varphi} + \dot{\theta}_2)\sin(\varphi + \theta_2) \\ \dot{z}_{c3} &= (l_1 + l_2)\dot{\varphi}\cos\varphi + \dot{l}_2\sin\varphi + r(\dot{\varphi} + \dot{\theta}_2)\cos(\varphi + \theta_2) \end{aligned}. \tag{A6}$$

The kinetic energy of the robot system is

$$K = K_t + K_r, \tag{A7}$$

where

$$\begin{aligned} K_r &= \frac{1}{2}\left[I_1\dot{\varphi}^2 + I_2(\dot{\varphi} + \dot{\theta}_1)^2 + I_2(\dot{\varphi} + \dot{\theta}_1)^2\right], \\ K_t &= \frac{1}{2}\left[m_1\left(\dot{x}_{c1}^2 + \dot{z}_{c1}^2\right) + m_2\left(\dot{x}_{c2}^2 + \dot{z}_{c2}^2 + \dot{x}_{c3}^2 + \dot{z}_{c3}^2\right)\right]. \end{aligned}$$

The potential energy of the robot system is

$$V = m_1 g z_{c1} + m_2 g(z_{c2} + z_{c3}) + \frac{1}{2}k(l_2 - l_0)^2, \tag{A8}$$

where k is the stiffness of the spring and g is the gravitational acceleration. Let $\boldsymbol{q} = [l_2 \quad \varphi \quad \theta_1 \quad \theta_2]^{\mathrm{T}}$, then the kinetic energy has form

$$K = \frac{1}{2}\dot{\boldsymbol{q}}^{\mathrm{T}}\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}, \tag{A9}$$

where

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix},$$

$$\begin{aligned} m_{11} &= m_1 + 2m_2 \\ m_{12} &= -m_2 r\,(\sin\theta_1 + \sin\theta_2) \\ m_{13} &= -m_2 r\sin\theta_1 \\ m_{14} &= -m_2 r\sin\theta_2 \\ m_{21} &= m_{12} \\ m_{22} &= m_1\left(\frac{1}{2}l_1 + l_2\right)^2 + I_1 + 2m_2 r^2 \\ &\quad + 2m_2\,(l_1 + l_2)^2 + 2m_2(l_1 + l_2)r\,(\cos\theta_1 + \cos\theta_2) \\ m_{23} &= m_2 r^2 + m_2(l_1 + l_2)r\cos\theta_1 \\ m_{24} &= m_2 r^2 + m_2(l_1 + l_2)r\cos\theta_2 \\ m_{31} &= m_{13} \\ m_{32} &= m_{23} \\ m_{33} &= m_2 r^2 \\ m_{34} &= 0 \\ m_{41} &= m_{14} \\ m_{42} &= m_{24} \\ m_{43} &= 0 \\ m_{44} &= m_{33}. \end{aligned}$$

In expression (3), the submatrices are defined by

$$\boldsymbol{m}_{\mathrm{pp}} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}, \quad \boldsymbol{m}_{\mathrm{pa}} = \begin{bmatrix} m_{13} & m_{14} \\ m_{23} & m_{24} \end{bmatrix},$$

$$\boldsymbol{m}_{\mathrm{ap}} = \boldsymbol{m}_{\mathrm{ap}}^{\mathrm{T}}, \quad \boldsymbol{m}_{\mathrm{aa}} = \begin{bmatrix} m_{33} & m_{34} \\ m_{43} & m_{44} \end{bmatrix}.$$

Appendix B. The Physical Parameters of the Robot Prototype

The nature frequency: $f_n = 2.5$ (Hz);
The mass of the model: $m_1 = 1.2$ (kg), $m_2 = 1.0$ (kg);
The stiffness of spring: $k = 4\pi^2 m f_n^2 \approx 790$ (N/m);
The length of massless leg segment: $l_0 = 0.4$ (m);
The length of nonzero mass leg segment: $l_1 = 0.3$ (m);
The length of arms: $r = 0.4$ (m);
The initial length of spring leg: $l_{20} \approx l_0 - (m_1 + 2m_2)g/k \approx 0.360$ (m).

References

1. M. H. Raibert, *Legged Robots That Balance* (MIT Press, Boston, 1986).
2. J. E. Seipel and P. Holmes, "Running in three dimensions: Analysis of a point-mass sprung-leg model," *Int. J. Robot. Res.* **24**(8), 657–674 (2005).
3. I. Poulakakis, E. Papadopoulos and M. Buhler, "On the stability of the passive dynamics of quadrupedal running with a bounding gait," *Int. J. Robot. Res.* **25**(7), 669–687 (2006).
4. M. H. Raibert, "Hopping in legged systems-modeling and simulation for the two-dimensional one-legged case," *IEEE Trans. Syst. Man Cybern.* **SMC-14**(3), 451–463 (1984).
5. M. H. Raibert, H. Benjamin Brown, Jr. and Michael Cheppoins, "Experiments in balance with 2 3D one-legged hopping machine," *Int. J. Robot. Res.* **3**(2), 75–92 (1984).
6. M. H. Raibert, M. Chepponis and H. B. Brown, Jr., "Running on four legs as though they were one," *IEEE J. Robot. Autom.* **RA-2**(2), 70–82 (1986).
7. J. K. Hodgins and M. H. Raibert, "Biped gymnastics," *Int. J. Robot. Res.* **9**(2), 115–132 (1990).
8. J. K. Hodgins and M. H. Raibert, "Adjusting step length for rough terrain locomotion," *IEEE Trans. Robot. Autom.* **7**(3), 289–298 (1991).
9. S. H. Hyon and T. Mita, "Development of a Biologically Inspired Hopping Robots-Kenken," *IEEE International Conference on Robotics and Automation,* Washington, DC (2002) pp. 3984–3991.
10. G. J. Zeglin, Uniroo: A One-Legged Dynamic Hopping Robot *BS thesis* (Boston, MA: Department of Mechanical Engineering, Massachusetts Institute of Technology, 1991).
11. A. Isidori, *Nonlinear Control Systems* (Springer-Verlag, Berlin, 1995).
12. S. Sastry, *Nonlinear Systems Analysis, Stability and Control* (Springer-Verlag, New York, 1999).
13. R. Olfati-Saber, Nonlinear Control of Underactuated Mechanical Systems with Application to Robotics and Aerospace Vehicles *Doctoral Dissertation* (Massachusetts Institute of Technology, Cambridge, 2001).
14. C. Samson, "Control of chained systems application to path following and time-vary point-stabilization of mobile robots," *IEEE Trans. Autom. Control* **40**(1), 64–77 (1995).
15. P. Lucibello and G. Oriolo, "Robust stabilization via iterative state steering with an application to chained-form systems," *Automatica* **37**, 71–79 (2001).
16. N. Marchand and M. Alamir, "Discontinuous exponential stabilization of chained form systems," *Automatica* **39**, 343–348 (2003).
17. G. Oriolo and M. Vendittelli, "A framework for the stabilization of general nonholonomic systems with an application to the plate-ball mechanism," *IEEE Trans. Robot.* **21**(2), 162–175 (2005).
18. M. W. Spong, "The swing up control problem for the acrobot," *IEEE Control Syst. Mag.* **15**, 49–55 (1995).
19. M. Kristić, I. Kanellakopoulos and P. Kokotović, *Nonlinear and Adaptive Control Design* (John Wiley & Sons, San Francisco, 1995).
20. R. A. Freeman and P. V. Kokotović, *Robust Nonlinear Control Design: State-Space and Lyapunov Techniques* (Birkhäuser, Boston, 1996).
21. A. M. Bloch, J. Baillieul, P. Crouch and J. Marsden, *Nonholonomic Mechanics and Control* (Springer-Verlag, New York, 2003).
22. K. D. Do, Z. P. Jiang and J. Pan. "Universal controllers for stabilization and tracking of underactuated ships," *Syst. Control Lett.* **47**, 299–317 (2002).
23. A. D. Luca, R. Mattone and G. Oriolo, "Stabilization of an underactuated planar 2R manipulator," *Int. J. Robust Nonlinear Control* **10**, 181–198 (2000).
24. R. M. Murray, Zexiang Li and S. Shankar Sastry, *A Mathematical Introduction to Robotics Manipulation* (CRC Press, Boca Raton, 1994).
25. R. W. Brockett, "Asymptotic Stability and Feedback Stabilization," **In**: *Differential Geometric Control Theory* (R. W. Brockett, R. S. Millman and H. J. Sussmann, eds.) (Birkhauser, Boston, 1983) pp. 181–191.
26. A. D. Luca and G. Oriolo, "Trajectory planning and control for planar robots with passive last joint," *Int. J. Robot. Res.* **21**(5–6), 575–590 (2002).
27. M. Fliess, J. Lévine, P. Martin and P. Rouchon, "Flatness and defect of nonlinear systems: Introductory theory and examples," *Int. J. Control* **61**(6), 1327–1361 (1995).
28. F. Mazenc and L. Praly, "Adding integrations, saturated controls, and stabilization for feedforward systems," *IEEE Trans. Autom. Control* **40**, 1559–1578 (1996).
29. E. Papadopoulos and N. Cherouvim, "On Increasing Energy Autonomy for a One-Legged Hopping Robot," *IEEE International Conference on Robotics & Automation,* New Orleans (2004) pp. 4645–4650.
30. G.-P. He, X.-L. Tan, X.-H. Zhang and Z. Lu, "Modeling, motion planning, and control of one-legged hopping robot actuated by two arms," *Mech. Mach. Theory* **43**(1),33–49 (2008).
31. M. Ahmadi and M. Buhler, "Controlled passive dynamic running experiments with the ARL-Monopod II," *IEEE Trans. Robot.* **22**(2), 974–986 (2006).