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CHAIN COMPONENTS WITH THE STABLE SHADOWING PROPERTY FOR C^1 VECTOR FIELDS

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Abstract

Let M be a closed n-dimensional smooth Riemannian manifold, and let X be a C^1-vector field of M. Let γ be a hyperbolic closed orbit of X. In this paper, we show that X has the C^1-stably shadowing property on the chain component $C_X(\gamma)$ if and only if $C_X(\gamma)$ is the hyperbolic homoclinic class.

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1. Introduction

In differentiable dynamical systems, an important area of research in recent years has been the study of robust dynamical properties. These properties often have some close relation to hyperbolicity.

Many results obtained for diffeomorphisms can be extended to the case of vector fields, but not always. For instance, a diffeomorphism f is called a star diffeomorphism if there is a C^1-neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, every $p \in P(g)$ is hyperbolic. Denote by $\mathcal{F}(M)$ the set of all star diffeomorphisms. By Hayashi [8] and Aoki [3], if a diffeomorphism $f \in \mathcal{F}(M)$ then f satisfies both Axiom A and the no-cycle condition. However, there is a star flow with nonhyperbolic nonwandering set, for example the geometric Lorenz flow (see [7]).

Chain components are natural candidates to replace Smale's hyperbolic basic set in nonhyperbolic theory of dynamical systems. Many recent papers (see [1, 2, 5, 11–13, 16–27]), most of which are only for diffeomorphisms, explore their hyperbolic-like properties such as partial hyperbolicity and dominated splitting. For instance, in [13], Lee *et al.* showed that if f has the C^1-stably shadowing property on the chain

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components, then it is hyperbolic. However, it is still an open problem whether the above results can be extended to the case of vector fields.

In this paper, we will study the hyperbolic structure on the chain component of C^1-vector fields. More precisely, our main problem can be formally stated as follows.

PROBLEM. If a vector field has the C^1-stably shadowing property on the chain component, is it hyperbolic?

Let us recall two recent papers that motivate our result. In the first, Lee and Sakai [14] prove that if a nonsingular vector field X is in the C^1-interior of the set of vector fields whose flows have the shadowing property then the vector field satisfies both Axiom A and the strong transversality condition. In the second, Lee *et al.* [15] prove that, with an extra condition that the chain component does not contain nonhyperbolic singularities, if X has the C^1-robustly shadowing property on the chain component then the chain component is hyperbolic.

In this paper we will give a positive answer to our main problem without any extra condition. To do so, we adapt several techniques from [13] that, in turn, originate in work by Mañé [29].

2. Basic definition and statement of the results

Let M be a closed n-dimensional smooth Riemannian manifold, and let d be the distance on M induced from a Riemannian metric $\|\cdot\|$ on the tangent bundle TM. Denote by $\mathcal{X}^1(M)$ the set of C^1-vector fields on M endowed with the C^1-topology. Then every $X \in \mathcal{X}^1(M)$ generates a C^1-flow $X_t : M \times \mathbb{R} \to M$, that is, a C^1-map such that $X_t : M \to M$ is a diffeomorphism satisfying $X_0(x) = x$ and $X_{t+s}(x) = X_t(X_s(x))$ for all $s, t \in \mathbb{R}$ and $x \in M$.

For any $\delta > 0$, a sequence $\{(x_i, t_i) : x_i \in M, t_i \geq 1, \text{ and } -\infty \leq a < i < b \leq \infty\}$ is a δ-pseudo-orbit of X(or δ-chain of X) if

$$d(X_{t_i}(x_i), x_{i+1}) < \delta \quad \text{for any } a \leq i \leq b-1.$$

An increasing homeomorphism $h : \mathbb{R} \to \mathbb{R}$ with $h(0) = 0$ is called a *reparametrization* of $\mathbb{R}$. Denote by $\mathrm{Rep}(\mathbb{R})$ the set of reparametrizations of $\mathbb{R}$. Fix $\epsilon > 0$ and define $\mathrm{Rep}(\epsilon)$ as follows:

$$\mathrm{Rep}(\epsilon) = \left\{h \in \mathrm{Rep} : \left|\frac{h(t)}{t} - 1\right| < \epsilon\right\}.$$

DEFINITION 2.1. Let Λ be a closed X_t-invariant subset in M. We say that X has the *shadowing property* on Λ (or Λ is shadowable for X) if for any $\epsilon > 0$, there is $\delta > 0$ with the following property: given any δ-pseudo-orbit $\xi = \{(xi, ti) : x_i \in \Lambda, t_i \geq 1, i \in \mathbb{Z}\}$, there exist a point $y \in M$ and an increasing homeomorphism $h \in \mathrm{Rep}(\epsilon)$ such that

$$d(X_{h(t)}(y), X_{t-T_i}(x_i)) < \epsilon$$

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for any $T_i < t < T_{i+1}$, where T_i is defined as

$$T_i = \begin{cases} t_0 + t_1 + \cdots + t_{i-1} & \text{if } i > 0, \\ 0 & \text{if } i = 0 \\ -t_{-1} - t_{-2} - \cdots - t_i & \text{if } i < 0. \end{cases}$$

The point $y \in M$ is said to be a *shadowing point* of ξ.

We say that Λ is *isolated* if there is a compact neighborhood U of Λ such that

$$\bigcap_{t\in\mathbb{R}} X_t(U) = \Lambda.$$

We introduce the C^1-stably shadowing property for a closed X_t-invariant subset Λ of M.

Definition 2.2. Let $X \in \mathcal{X}^1(M)$, and let Λ be a closed subset of M. We say that X has the C^1*-stably shadowing property on* Λ if there are a C^1-neighborhood $\mathcal{U}(X)$ of X and a compact neighborhood U of Λ such that

(i) $\Lambda = \bigcap_{t\in\mathbb{R}} X_t(U)$,
(ii) for any $Y \in \mathcal{U}(X)$, Y has the shadowing property on $\Lambda_Y(U)$, where $\Lambda_Y(U) = \bigcap_{t\in\mathbb{R}} Y_t(U)$ is the continuation of Λ.

Let X_t be the flow of $X \in \mathcal{X}^1(M)$, and let Λ be a X_t-invariant compact set. The set Λ is called *hyperbolic* for X_t if there are constants $C > 0, \lambda > 0$ and a splitting $T_xM = E^s_x \oplus \langle X(x)\rangle \oplus E^u_x$ such that the tangent flow $DX_t : TM \to TM$ leaves invariant the continuous splitting and

$$\|DX_t|_{E^s_x}\| \le Ce^{-\lambda t} \quad \text{and} \quad \|DX_{-t}|_{E^u_x}\| \le Ce^{-\lambda t}$$

for $t > 0$ and $x \in \Lambda$.

We say that $X \in \mathcal{X}^1(M)$ is *Anosov* if M is hyperbolic for X. Hyperbolic singular points or hyperbolic periodic orbits are trivial examples of invariant compact subsets with the C^1-stably shadowing property.

For any $x, y \in M$, we say that $x \sim y$ if for any $\delta > 0$, there exist a δ-pseudo-orbit $\{(x_i, t_i) : 0 \le i < n\}$ with $n > 1$ such that $x_0 = x$ and $d(X_{t_n-1}(x_{n-1}), y) < \delta$ and a δ-pseudo-orbit $\{(z_i, s_i) : 0 \le i < m\}$ with $m > 1$ such that $z_0 = y$ and $d(X_{s_m-1}(z_{m-1}, x) < \delta$. It is easy to see that $\sim$ gives an equivalent relation on the chain recurrent set $\mathcal{R}(X)$. Each component of $\mathcal{R}(X)$ under the equivalence relation $\sim$ is called a chain component. The number of chain components may be finite or infinite. Each chain component may contain closed orbits or not. In this paper, we fix a hyperbolic closed orbit γ of the vector field X, and denote by $C_X(\gamma)$ the chain component that contains γ.

We define the *stable* and *unstable manifolds* of γ respectively by

$$W^s(\gamma) = \{y \in M : \omega(y) = \gamma\},$$
$$W^u(\gamma) = \{y \in M : \alpha(y) = \gamma\}.$$

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A point $x \in W^s(\gamma) \pitchfork W^u(\gamma)$ is called a *transversal homoclinic point* of X_t associated to γ. The closure of the transversal homoclinic points of X_t associated to γ is called the *homoclinic class* of X_t associated to γ, and it is denoted by

$$H_X(\gamma) = \overline{W^s(\gamma) \pitchfork W^u(\gamma)}.$$

It is clear that $H_X(\gamma)$ is a compact, transitive and X_t-invariant set. Note that $H_X(\gamma) \subset C_X(\gamma)$, but the converse is not true in general.

For two hyperbolic closed orbits γ_1 and γ_2 of X_t, we say γ_1 and γ_2 are *homoclinic related*, denoted by $\gamma_1 \sim \gamma_2$, if $W^s(\gamma_1) \pitchfork W^u(\gamma_2) \neq \emptyset$ and $W^u(\gamma_1) \pitchfork W^s(\gamma_2) \neq \emptyset$. By Smale's theorem, we know that

$$H_X(\gamma) = \overline{\{\gamma_1 : \gamma_1 \sim \gamma\}}.$$

It is clear that if $\gamma_1 \sim \gamma$ then $\mathrm{index}(\gamma_1) = \mathrm{index}(\gamma)$, where $\mathrm{index}(\gamma) = \dim W^s(\gamma)$. Since γ is a hyperbolic closed orbit of X_t then there exist a C^1-neighborhood $\mathcal{U}(X)$ of X and a neighborhood U of γ such that for any $Y \in \mathcal{U}(X)$, there exists a unique hyperbolic closed orbit γ_Y that equals $\bigcap_{t\in\mathbb{R}} Y_t(U)$. Moreover, we have $\mathrm{index}(\gamma) = \mathrm{index}(\gamma_Y)$. The hyperbolic closed orbit γ_Y is called the *continuation* of γ with respect to Y.

The main purpose of this paper is to characterize chain components $C_X(\gamma)$ containing a hyperbolic closed orbit γ by making use of the shadowing property under the C^1 open condition.

MAIN THEOREM. *Let $X \in \mathcal{X}^1(M)$, and let γ be a hyperbolic closed orbit of X. If X has the C^1-stably shadowing property on the chain component $C_X(\gamma)$, then $C_X(\gamma)$ is the hyperbolic homoclinic class $H_X(\gamma)$.*

3. Proof of main theorem

Let M be a closed smooth manifold, and let $X \in \mathcal{X}^1(M)$. Denote by $\mathrm{Sing}(X)$ the set of singularities of X and by $P(X)$ the set of periodic orbits of X. Let $\gamma \in P(X)$ be hyperbolic, and let $p \in \gamma$ be such that $X_{\pi(p)}(p) = p$, where $\pi(p)$ is the period of p. The strong stable manifold $W^{ss}(p)$ of p and the stable manifold $W^s(\gamma)$ of γ are defined as follows:

$$W^{ss}(p) = \{y \in M : d(X_t(y), X_t(p)) \to 0 \ \text{ as } t \to \infty\}.$$

If $\eta > 0$ then the local strong stable manifold $W^{ss}_{\eta(p)}(p)$ of p and the local stable manifolds $W^s_{\eta(\gamma)}(\gamma)$ of γ are defined by

$$W^{ss}_{\eta(p)}(p) = \{y \in M : d(X_t(y), X_t(p)) < \eta(p), \ \text{if } t \geq 0\},$$
$$W^{s}_{\eta(\gamma)}(\gamma) = \{y \in M : d(X_t(y), X_t(\gamma)) < \eta(\gamma), \ \text{if } t \geq 0\}.$$

By the stable manifold theorem, there is $\epsilon = \epsilon(p) > 0$ such that

$$W^{ss}(p) = \bigcup_{t\geq 0} X_{-t}(W^{ss}_{\epsilon}(X_t(p))).$$

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Let $\sigma \in \mathrm{Sing}(X)$ be hyperbolic. Then there is an $\epsilon(\sigma) > 0$ such that

$$W^s_{\epsilon(\sigma)}(\sigma) = \{y \in M : d(X_t(y), \sigma) < \epsilon(\sigma) \text{ if } t \geq 0\}$$

and

$$W^s(\sigma) = \bigcup_{t \geq 0} X_{-t}(W^s_\epsilon(\sigma)).$$

Similarly, we can define the strong unstable manifold, local strong unstable manifold and local unstable manifold.

A consequence of the shadowing property on the chain component $C_X(\gamma)$ is transversality between γ and critical points.

LEMMA 3.1. *If X has the shadowing property on $C_X(\gamma)$ then for any hyperbolic $\eta \in C_X(\gamma) \cap \mathrm{Crit}(X)$, we have*

$$W^s(\eta) \cap W^u(\gamma) \neq \emptyset \quad \text{and} \quad W^u(\eta) \cap W^s(\gamma) \neq \emptyset,$$

where $\mathrm{Crit}(X) = \mathrm{Sing}(X) \cup P(X)$.

PROOF. Let $\eta \in C_X(\gamma) \cap \mathrm{Crit}(X)$ be hyperbolic. We consider the case of hyperbolic singularity, that is, $\eta \in \mathrm{Sing}(X)$.

Choose $p \in \gamma$. Since η, p are hyperbolic, there are $\epsilon(\eta) > 0$ and $\epsilon(p) > 0$ such that if $x \in W^u_{\epsilon(\eta)}(\eta)$ then $d(X_t(x), X_t(\eta)) \leq \epsilon(\eta)$ for all $t \leq 0$ and if $x \in W^s_{\epsilon(p)}(p)$ then $d(X_t(x), X_t(p)) \leq \epsilon(p)$ for all $t \geq 0$.

Take $\epsilon = \min\{\epsilon(\eta), \epsilon(p)\}$, and let $0 < \delta = \delta(\epsilon) < \epsilon$ be as in the shadowing property. Since $C_X(\gamma)$ is the chain component, we can construct a finite δ-pseudo-orbit $\{(x_i, t_i) : t_i \geq 1, 0 \leq i \leq k\} \subset C_X(\gamma)$ as follows:

- $t_i = 1$ for $0 \leq i \leq k$;
- $x_0 = \eta$, $x_k = p (k \geq 1)$;
- $d(X_{t_i}(x_i), x_{i+1}) = d(X_i(x_i), x_{i+1}) < \delta$ for $0 \leq i \leq k-1$.

By gluing segments of real orbits,

- $x_{-i} = X_{-i}(x_0) = X_{-i}(\eta)$ for $i \geq 0$ and $t_i = 1$,
- $x_{k+i} = X_i(x_k) = X_i(p)$ for $i \geq 0$ and $t_i = 1$,

we get an infinite δ-pseudo-orbit $\{(x_i, t_i) : t_i = 1, i \in \mathbb{Z}\}$ in $C_X(\gamma)$.

Since X has the shadowing property on $C_X(\gamma)$, there are $z \in B_\epsilon(x_0)$ and an increasing homeomorphism $h : \mathbb{R} \to \mathbb{R}$ with $h(0) = 0$ such that

$$d(X_{h(t)}(z), X_{t-T_i}(x_i)) < \epsilon$$

for $T_i \leq t < T_{i+1}$, and $i \in \mathbb{Z}$. Since $d(z, \eta) < \epsilon$, by the shadowing property, $z \in W^u_{\epsilon(\eta)}(\eta)$. We have

$$d(X_{h(t)}(z), X_{i-T_k}(x_k)) = d(X_{h(t)}(z), X_{i-k}(p)) < \epsilon$$

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for all $i \geq k$, and

$$d(X_{h(t)}(z), X_{i-j}(p)) < \epsilon$$

for $j \leq i < j+1$. Then $X_{t'}(z) \in W^s_\epsilon(p)$, where $t' \in \mathrm{Rep}(\mathbb{R})$. Therefore, we have $\mathrm{Orb}(z) \cap W^u_\epsilon(\eta) \cap W^s_\epsilon(p) \neq \emptyset$, and so $W^u(\eta) \cap W^s(p) \neq \emptyset$.

The other case of a hyperbolic periodic orbit can be proved in a similar manner. □

We say that X is *Kupka–Smale* if every critical point of X is hyperbolic and their stable and unstable manifolds meet transversally. Denote by $\mathcal{KS}(M)$ the set of all Kupka–Smale vector fields on M. Let $C_X(\gamma)$ be the chain component with hyperbolic periodic orbit γ. Then there exist a C^1-neighborhood $\mathcal{U}(X)$ and a compact neighborhood U of $C_X(\gamma)$ such that for any $Y \in \mathcal{U}(X)$, $C_Y(\gamma_Y) \subset U$, where γ_Y is the continuation of γ. If a hyperbolic $\sigma \in C_X(\gamma) \cap \mathrm{Crit}(X)$ then $\sigma_Y \in C_Y(\gamma_Y) \cap \mathrm{Crit}(Y)$. Thus $\sigma_Y \in C_Y(\gamma_Y) \cap \mathrm{Crit}(Y) \subset U$, and so $\sigma_Y \in \Lambda_Y(U) = \bigcap_{t\in\mathbb{R}} Y_t(U)$.

The following result is useful because it rules any singular point out of the chain recurrent set under assumption of stable shadowability.

LEMMA 3.2. *If X has the C^1-stably shadowing property on $C_X(\gamma)$ then*

$$\mathrm{Sing}(X) \cap C_X(\gamma) = \emptyset.$$

PROOF. To derive a contradiction, we may assume that $\mathrm{Sing}(X) \cap C_X(\gamma) \neq \emptyset$. Let $\mathcal{U}(X)$ be a C^1-neighborhood of X. Since $\mathrm{Sing}(X) \cap C_X(\gamma) \neq \emptyset$, we can choose $\sigma \in \mathrm{Sing}(X) \cap C_X(\gamma)$. Note that for any $\sigma \in \mathrm{Sing}(X)$, we can see that there is Y C^1-close to X such that $\sigma_Y \in \mathrm{Sing}(Y)$ is hyperbolic (see [31]). Then we know $\dim W^s(\sigma) \neq 0, \dim M$, and $\dim W^s(\gamma) \neq 0, \dim M$. Since X has the C^1-stably shadowing property on $C_X(\gamma)$, there is $Y \in \mathcal{U}(X)$ such that Y has a hyperbolic singularity $\sigma_Y \in \mathrm{Sing}(Y) \cap C_Y(\gamma_Y)$ with index i and Y has the hyperbolic $\gamma_Y \in P(Y) \cap C_Y(\gamma_Y)$ with index j, where σ_Y, γ_Y are the continuations of σ, γ, respectively. Then

$$\dim W^{s,u}(\sigma) = \dim W^{s,u}(\sigma_Y) \quad \text{and} \quad \dim W^{s,u}(\gamma) = \dim W^{s,u}(\gamma_Y).$$

If $j < i$ then

$$\dim W^u(\sigma_Y) + \dim W^s(\gamma_Y) \leq \dim M.$$

Since X has the C^1-stably shadowing property on $C_X(\gamma)$, we can take $Z \in \mathcal{V}(Y) \cap \mathcal{KS}(M)$ such that Z has the shadowing property on $\Lambda_Z(U)$, where $\mathcal{V}(Y) \subset \mathcal{U}(X)$ is a C^1-neighborhood of Y. Since $C_Z(\gamma_Z) \subset \Lambda_Z(U)$, and Z has the shadowing property on $\Lambda_Z(U)$, we know that Z has the shadowing property on $C_Z(\gamma_Z)$. Since Z is a Kupka–Smale vector field, by [4, Lemma 3.4], if $\dim W^u(\sigma_Z) + \dim W^s(\gamma_Z) \leq \dim M$, where σ_Z, γ_Z are the continuations of σ_Y, γ_Y respectively, then

$$W^s(\sigma_Z) \cap W^u(\gamma_Z) = \emptyset.$$

Then we have

$$\dim W^u(\sigma) = \dim W^u(\sigma_Y) = \dim W^u(\sigma_Z)$$

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and

$$\dim W^s(\gamma) = \dim W^s(\gamma_Y) = \dim W^s(\gamma_Z).$$

By Lemma 3.1, we can take $x \in W^u(\sigma_Z) \cap W^s(\gamma_Z)$ such that

$$\mathrm{Orb}(x) \subset W^u(\sigma_Z) \cap W^s(\gamma_Z).$$

Then

$$T_x(W^u(\sigma_Z)) = T_x(\mathrm{Orb}(x)) \oplus \Delta^1$$

and

$$T_x(W^s(\gamma)) = T_x(\mathrm{Orb}(x)) \oplus \Delta^2.$$

Then we have

$$\dim\,(T_x(W^u(\sigma_Z)) + T_x(W^s(\gamma_Z))) < \dim W^u(\sigma_Z) + \dim W^s(\gamma_Z) = \dim M$$

since $\dim W^u(\sigma_Z) = \dim W^u(\sigma)$ and $\dim W^s(\gamma_Z) = \dim W^s(\gamma)$. Thus

$$\dim(T_x(W^u(\sigma)) + T_x(W^s(\gamma))) < \dim W^u(\sigma) + \dim W^s(\gamma) = \dim M.$$

This is a contradiction because Z is a Kupka–Smale vector field.

If $j \geq i$ then we have

$$\dim W^s(\sigma_Y) + \dim W^u(\gamma_Y) \leq \dim M.$$

By similar arguments to those above, we get a contradiction.

Therefore, if X has the C^1-stably shadowing property on $C_X(\gamma)$ then we have $\mathrm{Sing}(X) \cap C_X(\gamma) = \emptyset$. □

By Lemma 3.2, due to absence of singularity, we may combine with [15] to get hyperbolicity. However, we use another method with a shorter proof.

Hereafter we assume that an exponential map $\exp_p : T_pM(1) \to M$ is well defined for all $p \in M$, where $T_pM(\delta)$ denotes the ball $\{v \in T_pM : \|v\| \leq \delta\}$. For every regular point $x \in M(X(x) \neq 0)$, let $N_x = \langle X(x)\rangle^{\perp} \subset T_xM$, and let $N_x(\delta)$ be the δ-ball in N_x. Let $\mathcal{N}_{x,r} = \exp_x(N_x(r))$. Given any regular point $x \in M$ and $t \in \mathbb{R}$, there are $r > 0$ and a C^1-map $\tau : \mathcal{N}_{x,t} \to \mathbb{R}$ with $\tau(x) = t$ such that $X_{\tau(y)}(y) \in \mathcal{N}_{X_\tau(x),1}$ for any $y \in \mathcal{N}_{x,r}$. We define the *Poincaré map* $f_{x,t} : \mathcal{N}_{x,r} \to \mathcal{N}_{X_T(x),1}$ by $f_{x,t}(y) = X_{\tau(y)}(y)$. If $X_t(x) \neq x(0 < t \leq t_0)$ and r_0 is sufficiently small, then for $0 < r \leq r_0$, the map $(t, y) \mapsto X_t(y)$ is a C^1 embedding from

$$\{(t, y) \in \mathbb{R} \times \mathcal{N}_{x,r} : 0 \leq t \leq \tau(y)\}$$

to

$$F_x(X_t, r, t_0) := \{X_t(y) : y \in \mathcal{N}_{x,r} \ \text{ and } \ 0 \leq t \leq \tau(y)\}.$$

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For $\epsilon > 0$, let $B_\epsilon(\mathcal{N}_{x,r})$ be the set of all diffeomorphisms $\phi : \mathcal{N}_{x,r} \to \mathcal{N}_{x,r}$ such that $\mathrm{supp}(\phi) \subset \mathcal{N}_{x,r/2}$ and $d_{C^1}(\phi, \mathrm{id}) < \epsilon$. Here d_{C^1} is a standard C^1-metric, id is the identity map and the support of ϕ is the closure of the set where it differs from id.

LEMMA 3.3 [30]. *Let $X \in \mathcal{X}^1(M)$ have no singularities. Suppose $X_t(x) \neq x$ for $0 < t \leq t_0$, and let $f : \mathcal{N}_{x,r} \to \mathcal{N}_{x_1}$ be the Poincaré map ($r > 0$ is sufficiently small, $x_1 = X_{t_0}(x)$). Then, for every C^1-neighborhood $\mathcal{U}(X) \subset \mathcal{X}^1(M)$ of X and $0 < r_0 \leq r$, there is $\epsilon > 0$ with the property that for every $\phi \in B_\epsilon(\mathcal{N}_{x,r_0})$, there exists $Y \in \mathcal{U}(X)$ satisfying*

$$\begin{cases} Y(y) = X(y) & \text{if } y \notin F_x(X_t, r_0, t_0), \\ f_Y(y) = f \circ \phi(y) & \text{if } y \in \mathcal{N}_{x,r_0}. \end{cases}$$

Here $f_Y : \mathcal{N}_{x,r_0} \to \mathcal{N}_{x_1}$ is the Poincaré map defined by Y_t.

Let $X \in \mathcal{X}^1(M)$, and suppose $p \in \gamma \in P(X)$; $X_T(p) = p$, where $T > 0$ is the prime period. If $f : \mathcal{N}_{p,r_0} \to \mathcal{N}_p$ is the Poincaré map ($r_0 > 0$), then $f(p) = p$. Note that γ is hyperbolic if and only if p is a hyperbolic fixed point of f. The following lemma appears in [30].

LEMMA 3.4. *Let $X \in \mathcal{X}^1(M)$, $p \in \gamma \in P(X_t)$ ($X_T(p) = p, T > 0$), and let $f : \mathcal{N}_{p,r_0} \to \mathcal{N}_p$ be the Poincaré map for some $r_0 > 0$. Let $\mathcal{U}(X) \subset \mathcal{X}^1(M)$ be a C^1-neighborhood of X, and let $0 < r \leq r_0$ be given. Then there are $\delta_0 > 0$ and $0 < \epsilon_0 < r/2$ such that for a linear isomorphism $L : N_p \to N_p$ with $\|L - D_p f\| < \delta_0$, there is $Y \in \mathcal{U}(X)$ satisfying*

(a) $Y(x) = X(x)$ *if* $x \notin F_p(X_t, r, T/2)$,
(b) $p \in \gamma \in P(Y)$,
(c) $g(x) = \begin{cases} \exp_p \circ L \circ \exp_p^{-1}(x) & \text{if } x \in B_{\epsilon_0/4}(p) \cap \mathcal{N}_{p,r}, \\ f(x) & \text{if } x \notin B_{\epsilon_0}(p) \cap \mathcal{N}_{p,r}, \end{cases}$

where $B_\epsilon(x)$ is a closed ball in M centered at $x \in M$ with radius $\epsilon > 0$, and $g : \mathcal{N}_{p,r} \to \mathcal{N}_p$ is the Poincaré map defined by Y.

Let $X \in \mathcal{X}^1(M)$ have no singularities. For $x \in M$, let $\langle X(x) \rangle$ be the linear subspace spanned by $X(x)$ and $N_x = \langle X(x) \rangle^\perp$ be the orthogonal linear subspace of $\langle X(x) \rangle$ in $T_x M$. Denote $N = \langle N_x \rangle$ the subbundle of TM and $\pi : TN \to N$ the projection along X. We now define the *linear Poincaré flow* P_t^X as

$$P_t^X(v) = \pi(D_x X_t(v)),$$

for $v \in N_x$ and $x \in M$. It is well known that $P_t : N \to N$ is a one-parameter transformation group [28].

As before, we set $N_{x,r} = N_x \cap T_x M(r)$ ($r > 0$) for $x \in M$, and put $\mathcal{N}_{x,r} = \exp_x(N_{x,r})$. For any vector field Y that is C^1 nearby X, we get a Poincaré map with respect to $\mathcal{N}_{x,r}$ in the whole space M, for some $r > 0$. Let Λ be a closed X_t-invariant regular set. We say that Λ is *hyperbolic* if the bundle N_Λ has a P_t^X-invariant splitting $\Delta^s \oplus \Delta^u$ and there exists an $l > 0$ such that

$$\|P_l^X|_{\Delta_x^s}\| \leq \tfrac{1}{2} \quad \text{and} \quad \|P_{-l}^X|_{\Delta^u_{X^l(x)}}\| \leq \tfrac{1}{2},$$

for all $x \in \Lambda$. Then Doering [6] showed one way to obtain hyperbolicity of invariant subsets for flows.

PROPOSITION 3.5. *Let $\Lambda \subset M$ be a compact invariant set of X_t. Then Λ is a hyperbolic set of X_t if and only if the linear Poincaré flow restriction on Λ has a hyperbolic splitting $N_\Lambda = \Delta^s \oplus \Delta^u$, where $N = \bigcup_{x \in M_X} N_x$.*

Let Λ be a closed X_t-invariant regular set. A P_t^X-invariant splitting $N_\Lambda = \Delta^1 \oplus \Delta^2$ is a *dominated splitting* if there is an $l > 0$ such that

$$\|P_l^X|_{\Delta_x^1}\| \cdot \|P_{-l}^X|_{\Delta^2_{X_l(x)}}\| \le \tfrac{1}{2},$$

for all $x \in \Lambda$.

LEMMA 3.6. *Let $X \in \mathcal{X}^1(M)$, and let γ be a hyperbolic closed orbit of X_t. Suppose X has the C^1-stably shadowing property on Λ. Then there exists a C^1-neighborhood $\mathcal{V}(X) \subset \mathcal{U}(X)$ of X such that, for any $Y \in \mathcal{V}(X)$, every $\gamma_1 \in \Lambda_Y(U) \cap P(Y)$ is hyperbolic.*

PROOF. Suppose that X has the C^1-stably shadowing property on $C_X(\gamma)$. Then there exist a C^1-neighborhood $\mathcal{U}(X)$ of X and an isolated neighborhood U of $C_X(\gamma)$ such that for any $Y \in \mathcal{U}(X)$, $Y_t|_{\Lambda_Y(U)}$ has the shadowing property, where $\Lambda_Y(U) = \bigcap_{t \in \mathbb{R}} Y_t(U)$.

Suppose, for contradiction, that there is a nonhyperbolic periodic orbit $\gamma \in P(Y)$ for some $Y \in \mathcal{U}(X)$. Then one can see that every $p \in \gamma$ is nonhyperbolic. Let $p \in \gamma$ such that $Y_T(p) = p (T > 0)$. Then the flow Y_t defines the Poincaré map $f : \mathcal{N}_{p,r_0} \to \mathcal{N}_p$ (for some $r_0 > 0$).

Let $\delta_0 > 0$ and $0 < \epsilon_0 < r_0$ be given by Lemma 3.4 for $\mathcal{V}(X)$ and r_0 with $B_{\epsilon_0}(p) \subset U$. Take a linear isomorphism $L : N_p \to N_p$. Then there exists $Z \in \mathcal{V}(X)$ such that

- $Z(x) = Y(x)$ if $x \notin F_p(Y_t, r_0, T/2)$,
- $g(x) = \begin{cases} \exp_p \circ D_p f \circ \exp_p^{-1}(x) & \text{if } x \in B_{\epsilon_0/4}(p) \cap \mathcal{N}_{p,r_0}, \\ f(x) & \text{if } x \notin B_{\epsilon_0}(p) \cap \mathcal{N}_{p,r_0}. \end{cases}$

Then $f(p) = g(p) = p$. Since the periodic point $p \in \gamma$ is nonhyperbolic, there is an eigenvalue λ of $D_p g$ with $|\lambda| = 1$. We consider two cases: $\lambda \in \mathbb{R}$ and $\lambda \in \mathbb{C}$.

In the first case, we assume that $\lambda \in \mathbb{R}$ and let $v \in N_p$ be the associated nonzero eigenvector such that $\|v\| = \epsilon_0/8$. Take $\exp_p(v) \in B_{\epsilon_0/4}(p) \cap \mathcal{N}_{p,r_0} \setminus \{p\}$. Then

$$\begin{aligned} g(\exp_p(v)) &= \exp_p(D_p f(\exp_p^{-1}(\exp_p(v)))) \\ &= \exp_p(v). \end{aligned}$$

Put $\mathcal{J}_p = \{tv : -\epsilon_0/8 \le t \le \epsilon_0/8\}$. Then $g(\mathcal{J}_p) = \mathcal{J}_p$. Then we know that $\mathcal{J}_p \subset \Lambda_Y(U)$. Since Y has the shadowing property on $\Lambda_Y(U)$, Y has the shadowing property on $\mathcal{J}_p$. Take $\epsilon \in (0, \epsilon_0/16)$, and let $0 < \delta < \epsilon$ be as in the shadowing property. Let $x_0 = p$. Take $y \in B_{\epsilon_0/4}(p) \cap \mathcal{N}_{p,r_0}$ such that $d(p, y) > 2\epsilon$. Let $\pi(p)$ be the period of p and $t_i = \pi(p)$ for all $i \in \mathbb{Z}$.

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Then for every point $x_i \in \mathcal{J}_p$, we construct a δ-pseudo-orbit $\{(x_i, t_i) : t_i \geq 1, i \in \mathbb{Z}\}$ as follows:

(i) $x_i = Z_{\pi(p)}(p) = x_0$ for all $i \leq 0$,
(ii) $d(Z_{t_i}(x_i), x_{i+1}) = d(x_i, x_{i+1}) < \delta$ for all $0 < i \leq k$, and
(iii) $x_i = Z_{\pi(p)}(q)$ for all $i > k$.

It is clear that the δ-pseudo-orbit $\{(x_i, t_i) : t_i = 1, i \in \mathbb{Z}\} \subset \mathcal{J}_p$. If a shadowing point $z \in B_\epsilon(p) \setminus \mathcal{J}_p$, then by hyperbolicity there are $m > 0$ and an increasing homeomorphism $h : \mathbb{R} \to \mathbb{R}$ with $h(0) = 0$ such that for $T_m \leq t < T_{m+1}$,

$$d(Z_{h(t)}(z), Z_{t-T_m}(x_m)) = d(Z_{h(t)}(z), Z_{t-k}(x_m)) > \epsilon.$$

Since Z has the shadowing property on $\mathcal{J}_p$, the shadowing point z is in $\mathcal{J}_p$. Then for all $k \in \mathbb{Z}$, $X_{k\pi(p)}(z) = z$. By the shadowing property, we have

$$d(p, y) \leq d(p, z) + d(z, x_k) = d(p, z) + d(Z_{k\pi(p)}(z), x_k) < 2\epsilon,$$

which is a contradiction.

In the second case, we assume that $\lambda \in \mathbb{C}$. Then by Lemma 3.4 we can find $Z \in \mathcal{U}(X)$ such that P_t^Z is a rational rotation. Then there is $\tau \neq 0$ such that $P^Z_{\tau+\pi(p)} = \mathrm{id}$. As in the proof of the first case, we get a contradiction. □

We write $\mathcal{G}(\Lambda)$ if there exist a C^1-neighborhood $\mathcal{U}(X)$ and a neighborhood U of Λ such that for any $Y \in \mathcal{U}(X)$, every $p \in \Lambda_Y(U) \cap \mathrm{Crit}(Y)$ is hyperbolic, where $\mathrm{Crit}(X) = \mathrm{Sing}(X) \cup P(X)$. And we write $\mathcal{G}^*(\Lambda)$ if $\mathcal{G}(\Lambda) \setminus \mathrm{Sing}(X)$. Then by Lemma 3.6, we have the following argument. By Hayashi (see [9]), the family of periodic sequences of linear isomorphisms of $\mathbb{R}^{\dim M}$ generated by P_t^Y (Y close to X) along the hyperbolic periodic point $q \in \Lambda_Y(U) \cap P(Y)$ is uniformly hyperbolic. For two linear normed spaces E and F, and a linear operator $A : E \to F$, the min-norm $m(A)$ is defined by

$$m(A) = \inf_{v \in E, |v|=1} |Av|.$$

PROPOSITION 3.7 [28, Theorem 2.1][29, Lemma II.3]. *Let $X \in \mathcal{G}^*(C_X(\gamma))$, and let $\mathcal{V}(X)$ be a C^1-neighborhood of X as in Lemma 3.6. Then there are constants $0 < \lambda < 1, T > 0$ such that for any $Y \in \mathcal{V}(X)$ and any $p \in \gamma \in \Lambda_Y(U) \cap P(Y)$, the following properties hold.*

(a) *$\Lambda_Y(U)$ admits a dominated splitting $N_{\Lambda_Y(U)} = \Delta^1 \oplus \Delta^2$ such that for any $t \geq T$,*

$$\|P_t^Y|_{\Delta^1(p)}\| \cdot \|P_{-t}^Y|_{\Delta^2(Y_t(p))}\| \leq e^{-2\lambda t}.$$

(b) *If τ is the period of γ, m is any positive integer, and $0 = t_0 < t_1 < \cdots < t_k = m\tau$ is any partition of the time interval $[0, m\tau]$ with $t_{i+1} - t_i \geq T$, then*

$$\frac{1}{m\tau} \sum_{i=0}^{k-1} \log \|P^Y_{t_{i+1}-t_i}|_{\Delta^s(Y_{t_i}(p))}\| < -\lambda$$

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and

$$\frac{1}{m\tau}\sum_{i=0}^{k-1}\log\|P^Y_{-(t_{i+1}-t_i)}|_{\Delta^u(Y_{t_{i+1}}(p))}\| < -\lambda.$$

Then we rewrite Proposition 3.7 as follows.

PROPOSITION 3.8. *Let X be the C^1-stably shadowing property on $C_X(\gamma)$, and let $\mathcal{V}(X)$ be a C^1-neighborhood of X as in Lemma 3.6. Then there are constants $0 < \lambda < 1, T > 0$ such that, for any $Y \in \mathcal{V}(X)$ and any $p \in \gamma \in \Lambda_Y(U) \cap P(Y)$, the following properties hold.*

(a) *$\Lambda_Y(U)$ admits a dominated splitting $N_{\Lambda_Y(U)} = \Delta^1 \oplus \Delta^2$ such that for any $t \geq T$, $\|P^Y_t|_{\Delta^1(p)}\| \cdot \|P^Y_{-t}|_{\Delta^2(Y_t(p))}\| \leq e^{-2\lambda t}$.*

(b) *If τ is the period of γ, m is any positive integer, and $0 = t_0 < t_1 < \cdots < t_k = m\tau$ is any partition of the time interval $[0, m\tau]$ with $t_{i+1} - t_i \geq T$, then*

$$\frac{1}{m\tau}\sum_{i=0}^{k-1}\log\|P^Y_{t_{i+1}-t_i}|_{\Delta^s(Y_{t_i}(p))}\| < -\lambda$$

and

$$\frac{1}{m\tau}\sum_{i=0}^{k-1}\log\|P^Y_{-(t_{i+1}-t_i)}|_{\Delta^u(Y_{t_{i+1}}(p))}\| < -\lambda.$$

Let $X \in \mathcal{X}^1(M)$. For any $\epsilon > 0$, we denote by $C^u_\epsilon(y)$ the set of all points $x \in \mathcal{N}_{P,r_0}$ that have the following property: there is an increasing continuous map $h : (-\infty, 0] \to \mathbb{R}$ such that $d(X_{h(t)}(x), Y_t(y)) < \epsilon$ for all $t \in (-\infty, 0]$.

LEMMA 3.9. *Let $X \in \mathcal{X}^1(M)$, and let γ be a hyperbolic closed orbit of X. Suppose that X has the C^1-stably shadowing property on $C_X(\gamma)$. Then for every hyperbolic $\eta \in C_X(\gamma) \cap P(X)$, we have*

$$\text{index}(\eta) = \text{index}(\gamma).$$

PROOF. Let $\eta \in C_X(\gamma) \cap P(X)$ be hyperbolic. Suppose, by contradiction, that $\text{index}(\eta) \neq \text{index}(\gamma)$. Since X has the shadowing property on $C_X(\gamma)$, by Lemma 3.1 $W^s(\eta) \cap W^u(\gamma) \neq \emptyset$ and $W^u(\eta) \cap W^s(\gamma) \neq \emptyset$. Since $\text{index}(\eta) \neq \text{index}(\gamma)$, we have for $x \in W^s(\eta) \cap W^u(\gamma)$ and $y \in W^u(\eta) \cap W^s(\gamma)$,

$$T_xW^s(\eta) + T_xW^u(\gamma) \neq T_xM \quad \text{or} \quad T_yW^u(\eta) + T_yW^s(\gamma) \neq T_yM.$$

Without loss of generality, we may assume that $x, y \in \mathcal{N}_{p,r_0}$ for some $r_0 > 0$. Then by Lemma 3.3, there exists a vector field Y that is C^1-close to X such that

$$x_1 \in W^u(\eta_Y, Y) \cap W^s(\gamma_Y, Y),$$

$$y_1 \in W^s(\eta_Y, Y) \cap W^u(\gamma_Y, Y)$$

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and

$$T_{y_1}W^s(\eta_Y, Y) + T_{y_1}W^u(\gamma_Y, Y) \neq T_{y_1}M.$$

It can be seen that η_Y, x_1 and y_1 are elements of the chain component of Y that contains γ_Y. Let $g : \mathcal{N}_{p,r_0} \to \mathcal{N}_p$ be the Poincaré map associated to Y. Since $p \in \gamma_Y$ is hyperbolic, we have $N_p = \Delta_p^s \oplus \Delta_p^u$. Since g is locally linear at p, there is $\epsilon_1 > 0$ such that

$$W^s_{\epsilon_1}(\gamma_Y, Y) \cap \mathcal{N}_{p,r_0} \subset \exp_p(\Delta_p^s) \quad \text{and} \quad W^u_{\epsilon_1}(\gamma_Y, Y) \cap \mathcal{N}_{p,r_0} \subset \exp_p(\Delta_p^u).$$

Then we assume that $x_1 \in W^s_{\epsilon_1}(\gamma_Y, Y) \cap \mathcal{N}_{p,r_0}$. Let $C^u(x_1)$ be the connected component of $W^u(\eta_Y, Y) \cap \mathcal{N}_{p,r_0}$ containing x_1. Then

$$\exp_p(C^u(x_1)) \subset N_p \quad \text{and} \quad T_{x_1}C^u(x_1) = T_{x_1}(W^u(\eta_Y, Y) \cap N_p).$$

Let $L \subset N_p$ be an affine space tangent to $\exp_p^{-1}(C^u(x_1))$ at $\exp_p^{-1}(x_1)$. Denote by $\pi : N_p \to \Delta_p^u$ the natural projection parallel to Δ_p^s. Then we see that there is no transversality between $C^u(x_1)$ and $W^s(p, g)$ at x_1, and therefore, $\dim \pi(L) < \dim \Delta_p^u$.

For $\epsilon > 0$, let

$$L_\epsilon = \{v \in L : \|v - \exp_p^{-1}(x_1)\| < \epsilon\}.$$

Then we can find $\tau > 0$ such that $Y_{[-\tau,-\tau+1]}(x_1) \cap B_{r'}(\gamma_Y) = \emptyset$ for some $0 < r' < r$. By Lemma 3.3, there is Z C^1-close to Y such that:

- Z is different from Y in a small neighborhood of the arc $Y_{[-\tau,-\tau+1]}(x_1)$;
- there is $\epsilon > 0$ such that $\exp_p(L_\epsilon)$ is contained in the connected component of $W^u(\eta_Z, Z) \cap \mathcal{N}_{p,r_0}$ containing x_1.

By [15, Lemma 3.3], there is $\epsilon_2 > 0$ such that $C^u_\epsilon(x_1) \subset \exp_p(L_\epsilon)$ for any $0 < \epsilon \le \epsilon_2$. Since $C_Z(\gamma_Z)$ is upper semicontinuous, we have

$$C_Z(\gamma_Z) \subset \Lambda_Z(U) = \bigcap_{t \in \mathbb{R}} Z_t(U).$$

By the assumption, Z has the shadowing property on $\Lambda_Z(U)$. Thus Z has the shadowing property on $C_Z(\gamma_Z)$ and the shadowing point z must be contained in $C^u_\epsilon(x_1)$. Consequently, $X_t(z) \to \eta_Z$ as $t \to -\infty$. Finally, we can easily construct the pseudo-orbit containing x_1 which cannot be ϵ-shadowed by the point $z \in C^u_\epsilon(x_1)$. The proof is complete. □

LEMMA 3.10 [15, Lemma 3.5]. *Let $C_X(\gamma)$ be the chain component. If X has the shadowing property on $C_X(\gamma)$ then $C_X(\gamma) = H_X(\gamma)$.*

We now recall a flow version [9, 10, 32] of the Mañé ergodic closing lemma [29]. For $X \in \mathcal{X}^1(M)$, let $B_\epsilon(x, X) = \{y \in M : d(y, X_t(x)) \le \epsilon$ for some $t \in \mathbb{R}\}$. Define Σ_X as the set of points $x \in M$ such that for any C^1-neighborhood $\mathcal{U}(X)$ of X and every $\epsilon > 0$, there are $Y \in \mathcal{U}(X)$, $y \in P(Y)$, $T_0 > 0$ and $t_0, t_1 \in \mathbb{R}$ with $t_0 < t_1$ such that $Y_{T_0}(y) = y$, $X = Y$

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on $M \setminus B_\epsilon(x, X_t), d(Y_t(y), X_t(x)) \le \epsilon$ for all $0 \le t \le T_0$,

$$\{X_t(x) : t_0 \le t \le t_1\} \subset \{Y_t(y) : t \ge 0\}$$

and $(t_1 - t_0)/T_0 > 1 - \epsilon$. Note that Σ_X is X_t-invariant.

THEOREM 3.11 (Ergodic closing lemma, [32, Theorem 3.9]).

$$\mu(\Sigma_X \cup \mathrm{Sing}(X)) = 1,$$

for any X_t-invariant probability measure μ on the Borel sets of M, where $\mathrm{Sing}(X)$ *denotes the set of singularities of X.*

PROOF OF MAIN THEOREM. First, we prove the 'if' part. Suppose that $C_X(\gamma)$ is the homoclinic class $H_X(\gamma)$ and hyperbolic. Then there exist a C^1-neighborhood $\mathcal{U}(X)$ of X and a neighborhood U of $C_X(\gamma)$ such that:

- $C_X(\gamma) = \bigcap_{t\in\mathbb{R}} X_t(U)$, that is, locally maximal;
- for any $Y \in \mathcal{U}(X), \Lambda_Y(U) = \bigcap_{t\in\mathbb{R}} Y_t(U)$ and it is hyperbolic.

Consequently, Y has the shadowing property on $\Lambda_Y(U)$. This means that X has the C^1-stably shadowing property on $C_X(\gamma)$.

Next, we prove the 'only if' part. Suppose that X has the C^1-stably shadowing property on $C_X(\gamma)$, and let U be a compact neighborhood of $C_X(\gamma)$ as in the definition. Let $j = \mathrm{index}(\gamma)$ be an index of $C_X(\gamma)$. Then by Lemma 3.10,

$$H_X(\gamma) = C_X(\gamma) = \overline{P_j(X|_{H_X(\gamma)})} = \Lambda_j(X).$$

Let $\mathcal{V}(X)$ be the C^1-neighborhood of X given by Proposition 3.8. To get the result, it is sufficient to show that $\Lambda_j(X)$ is hyperbolic. Fix any neighborhood $U_j \subset U$ of $\Lambda_j(X)$. □

Claim. If $\mathcal{U}_0(X) \subset \mathcal{V}(X)$ is a small connected C^1-neighborhood of X containing Y such that $Y = X$ on $M \setminus U_j$, then $\mathrm{index}(\gamma) = \mathrm{index}(\gamma_1)$ for any $\gamma_1 \in \Lambda_Y(U) \cap P(Y)$.

PROOF. If not, there are $Y \in \mathcal{U}_0(X)$ and $\gamma_1 \in \Lambda_Y(U) \cap P(Y)$ such that $Y_t = X_t$ on $M \setminus U_j$ and $\mathrm{index}(\gamma_1) \neq \mathrm{index}(\gamma)$. Suppose that $Y_T(\gamma_1) = \gamma_1$, $i = \mathrm{index}(\gamma_1)$, and define $\Gamma : \mathcal{U}_0(X) \to \mathbb{Z}$ by

$$\Gamma(Y) = \sharp\{\eta \in \Lambda_Y(U) \cap P(Y) : Y_T(\eta) = \eta \text{ and } \mathrm{index}(\eta) = i\}.$$

By Lemma 3.6, the function Γ is continuous, and since $\mathcal{U}_0(X)$ is connected, it is constant. But the property of Y implies $\Gamma(Y) > \Gamma(X)$. This is a contradiction, so that the claim is proved.

We now show that $C_X(\gamma)$ is hyperbolic. By Proposition 3.8(a), $C_X(\gamma)$ admits a dominated splitting $N_{C_X(\gamma)} = \Delta^1 \oplus \Delta^2$ such that $\dim\Delta^1 = \mathrm{index}(\gamma)$. Let $\delta_0 > 0$ and $\mathcal{U}_1(X)$ be given by Lemma 3.4 with respect to $\mathcal{U}_0(X)$. Then, to get that $C_X(\gamma)$ is hyperbolic, it is enough to show that $N_{C_X(\gamma)}$ is hyperbolic. In turn, if

$$\liminf_{n\to\infty} \|P^X_t|_{\Delta^1_x}\| = 0 \quad \text{and} \quad \liminf_{n\to\infty} \|P^X_{-t}|_{\Delta^2_x}\| = 0$$

for all $x \in C_X(\gamma)$ then the splitting $N_{C_X(\gamma)} = \Delta^1 \oplus \Delta^2$ is hyperbolic; that is, $\Delta^1 = \Delta^s$ is contracting and $\Delta^2 = \Delta^u$ is expanding.

To prove, we suppose that there is a sequence $\{j_n\}$ with $j_n \to \infty$ as $n \to \infty$ such that

$$\lim_{j_n\to\infty} \log \|P^X_{j_n}|_{\Delta^1_x}\| \geq 0. \tag{3-1}$$

Let $C^0(C_X(\gamma))$ be the set of real continuous functions defined on $C_X(\gamma)$ with the C^0-topology, and define the sequence of continuous operators $\Gamma_n : C^0(C_X(\gamma)) \to \mathbb{R}$ by

$$\Gamma_n(\varphi) = \frac{1}{j_n}\int_0^{j_n} \varphi(P^X_s(x))\, ds.$$

Hence, there exists a convergent subsequence of Γ_n (again denoted by Γ_n) converging to a continuous map $\Gamma : C^0(C_X(\gamma)) \to \mathbb{R}$. Let $\mathcal{M}(C_X(\gamma))$ be the space of measures with support on $C_X(\gamma)$. By the Riesz theorem, there is $\mu \in \mathcal{M}(C_X(\gamma))$ such that

$$\int_{C_X(\gamma)} \varphi\, d\mu = \lim_{j_n\to\infty} \frac{1}{j_n}\int_0^{j_n} \varphi(P^X_s(x))\, ds = \Gamma(\varphi), \tag{3-2}$$

for every continuous map φ defined on $C_X(\gamma)$. Clearly, the μ is P^X_t-invariant. Define $\varphi_{P^X} : C^0(C_X(\gamma)) \to \mathbb{R}$ by

$$\varphi_{P^X}(z) = \partial_h(\log \|P^X_h|_{\Delta^1_p}\|)_{h=0} = \lim_{h\to 0}\frac{1}{h}\log \|P^X_h|_{\Delta^1_p}\|.$$

The map is continuous, and φ_{P^X} satisfies (2).

On the other hand, for any $T \in \mathbb{R}$,

$$\frac{1}{T}\int_0^T \varphi_{P^X}(P^X_s(p))\, ds = \frac{1}{T}\int_0^T \partial_h(\log \|P^X_h|_{\Delta^1_{P^X_s(p)}}\|)_{h=0}\, ds = \frac{1}{T}\log \|P^X_T|_{\Delta^1_p}\|. \tag{3-3}$$

By (3-1), (3-2) and (3-3), we have

$$\int_{C_X(\gamma)} \varphi_{P^X}\, d\mu \geq 0.$$

By the Birkhoff ergodic theorem,

$$\int_{C_X(\gamma)} \varphi_{P^X}\, d\mu = \int_{C_X(\gamma)} \lim_{T\to\infty}\frac{1}{T}\int_0^T \varphi_{P^X}(P^X_s(y))\, ds\, d\mu(y).$$

Since μ is invariant and $\mathrm{Supp}(\mu) \subset C_X(\gamma)$, by Theorem 3.11, we have

$$\mu(C_X(\gamma) \cap (\mathrm{Sing}(X) \cup \Sigma_X)) = 1.$$

Since X has the C^1-stably shadowing property on $C_X(\gamma)$, by Lemma 3.2 $\mathrm{Sing}(X) \cap C_X(\gamma) = \emptyset$. Therefore, we have $\mu(C_X(\gamma) \cap \Sigma_X) = 1$. By the ergodic decomposition

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theorem, we may assume that μ is ergodic. Then there is $z \in C_X(\gamma) \cap \Sigma_X$ such that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \varphi_{P^X}(P_s^X(z))\, ds \geq 0. \tag{3-4}$$

Since $z \in C_X(\gamma) \cap \Sigma_X$, there are, for all n, ϵ_n (with $\epsilon_n \to 0$ as $n \to \infty$), $Y^n \in \mathcal{U}_1(X)$, and $p_n \in \Lambda_{Y^n}(U) \cap P(Y^n)$ with period $\pi_n(p_n)$ such that

$$\|Y^n - X\| < \epsilon_n \quad \text{and} \quad d(Y_t^n(p_n), X_t(z)) < \epsilon_n$$

for $0 \leq t \leq \pi_n(p_n)$, where Y_t^n is the flow induced by Y^n. Obviously, $\pi_n(p_n) \to \infty$ as $n \to \infty$. Let $\lambda < 0$ be sufficiently small. By (3-4), there is $S(\lambda) > 0$ such that for $t \geq S(\lambda)$,

$$\frac{1}{t} \int_0^t \varphi_{P^X}(P_s^X(y))\, ds \geq \lambda.$$

Since $\pi_n(p_n) \to \infty$ as $n \to \infty$, we may assume that $\pi_n(p_n) > S(\lambda)$ for n sufficiently large. Since $\Delta^1 \oplus \Delta^2$ is continuous, we can choose Y^n and p_n such that

$$\frac{1}{\pi_n(p_n)} \int_0^{\pi_n(p_n)} \varphi^{P^{Y^n}}(P_s^X(p_n))\, ds \geq \lambda.$$

Thus we have

$$\|P^{Y^n}_{\pi_n(p_n)}|_{\Delta^1_{p_n}}\| > e^{\lambda \pi_n(p_n)}.$$

By Proposition 3.8(b), this is a contradiction. Thus $\Delta^1 = \Delta^s$. Analogously, we can show that $\Delta^2 = \Delta^u$. By Remark 3.5, $C_X(\gamma)$ is hyperbolic. □

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