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The Local Möbius Equation and Decomposition Theorems in Riemannian Geometry

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Abstract. A partial differential equation, the local Möbius equation, is introduced in Riemannian geometry which completely characterizes the local twisted product structure of a Riemannian manifold. Also the characterizations of warped product and product structures of Riemannian manifolds are made by the local Möbius equation and an additional partial differential equation.

1 Introduction

It is clear that local structure of a Riemannian manifold cannot determine its global structure. Yet it may be expected that global properties of a Riemannian manifold may determine its local structure. One important local structure of a Riemannian manifold is its local decomposition to certain product structures such as, most importantly, twisted product, warped product or product structures. In this paper, we give some results showing global analytic structure of a Riemannian manifold can determine its local structure as a decomposition to certain product structures as in the above. Here we introduce a (global) partial differential equation on a Riemannian manifold, called the local Möbius equation, and in the case of a solution to this equation, manifold locally decomposes to certain products of Riemannian manifolds. In fact, local Möbius equation completely characterizes the twisted product and product Riemannian manifolds, and in particular, with an additional partial differential equation, it completely characterizes the locally warped product Riemannian manifolds. In this paper we give these results in the setting of Riemannian geometry. Yet it may be easily seen that the results remain valid in the setting of semi-Riemannian geometry as well, under certain regularity assumptions reducing the cases similar to Riemannian case by ruling out degenerate spaces occurring because of an indefinite metric tensor.

In Section 2, we give preliminaries and motivation to local Möbius equation. In Section 3, we state main decomposition results of this paper.

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2 Preliminaries and Motivation

Let (M, g) be an n-dimensional Riemannian manifold with Levi-Civita connection ∇. We also denote the gradient of a function on (M, g) by ∇. As a notation, we define the Hessian tensor, Hessian form and Laplacian of a function $f\colon (M, g) \to \mathbb{R}$ by $h_f(X) = \nabla_X \nabla f$, $H_f(X, Y) = g\big(h_f(X), Y\big)$ and $\Delta f = \operatorname{div} \nabla f$, respectively, where $X, Y \in \Gamma TM$. In [2], a function $f\colon (M, g) \to \mathbb{R}$ is said to satisfy the *Möbius equation* if

$$H_f - df \otimes df - \frac{1}{n}[\Delta f - g(\nabla f, \nabla f)]g = 0$$

on (M, g). It is also shown in [2] that, if $f\colon (M, g) \to \mathbb{R}$ satisfies the Möbius equation then the function $t = e^{-f}$ satisfies the equation, which we call the *localized Möbius equation*,

$$H_t = \frac{\Delta t}{n} g$$

on (M, g). We call the latter equation localized Möbius equation since if the localized Möbius equation has a solution t then the Möbius equation has a local solution given by $f = -\log t$, provided that $t > 0$. The localized Möbius equation is studied in Riemannian geometry in connection to its relation to warped product decompositions in [2], and as well, in a similar way in semi-Riemannian geometry in [1].

Now we modify and extend local Möbius equation in a way that to characterize twisted products in Riemannian geometry. Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds of dimensions n_1 and n_2 with Levi-Civita connections $\overset{1}{\nabla}$ and $\overset{2}{\nabla}$, respectively. *The second fundamental form* of a map $f\colon (M_1, g_1) \to (M_2, g_2)$ is defined by

$$(\nabla f_*)(X, Y) = \overset{2}{\nabla}_X f_* Y - f_*(\overset{1}{\nabla}_X Y),$$

where $X, Y \in \Gamma TM_1$ and $\overset{2}{\nabla}$ also denotes the pullback of $\overset{2}{\nabla}$ along f. Also the *tension field* of f is defined by

$$\tau(f) = \operatorname{trace} \nabla f_*$$

with respect to g_1.

Here we could call a map $f\colon (M_1, g_1) \to (M_2, g_2)$ satisfy *local Möbius equation* if

$$\nabla f_* = \frac{\tau(f)}{n_1} g_1.$$

However in order to make it characterize twisted products we adopt the following definition.

Definition 1 Let $f\colon (M_1, g_1) \to (M_2, g_2)$ be a submersion between Riemannian manifolds (M_1, g_1) and (M_2, g_2) with dimensions $n_1 > n_2 \geq 1$. Then f is said to satisfy the *local Möbius equation* if

$$(\nabla f_*)(X, Y) = \frac{\tau(f)}{n_1 - n_2} g_1(X, Y)$$

and

$$(\nabla f_*)(X,U) = 0$$

where $X, Y \in \Gamma \ker f_*$ and $U \in \Gamma(\ker f_*)^\perp$.

Remark 1 Note that if $(M_1, g_1) = (M, g)$ and $(M_2, g_2) = (\mathbb{R}, dx \otimes dx)$ then the local Möbius equation defined above becomes

$$H_f(X,Y) = \frac{\Delta f}{n-1} g(X,Y)$$

and

$$H_f(X,U) = 0,$$

or equivalently,

$$h_f(X) = \frac{\Delta f}{n-1} X,$$

where $X, Y \in \Gamma \ker f_*$ and $U \in \Gamma(\ker f_*)^\perp$. Note here that, in this case, local Möbius equation is essentially different than what we called localized Möbius equation.

3 Decomposition Theorems

In this section, first we show that the local Möbius equation completely characterizes the locally twisted product Riemannian manifolds. Recall that, if (M^1, g^1) and (M^2, g^2) are Riemannian manifolds and $\varphi\colon M^1 \times M^2 \to (0,\infty)$ is a function then the product manifold $(M^1 \times M^2, g^1 \oplus \varphi g^2)$ is called the *twisted product* of (M^1, g^1) and (M^2, g^2) with twisting function φ.

Theorem 1 *Let a submersion $f\colon (M_1, g_1) \to (M_2, g_2)$ between Riemannian manifolds (M_1, g_1) and (M_2, g_2) satisfy the local Möbius equation. Then (M_1, g_1) is locally a twisted product $(M_1^1 \times M_1^2, g_1^1 \oplus \varphi g_1^2)$, where (M_1^1, g_1^1) and (M_1^2, g_1^2) are Riemannian manifolds, and $f\colon M_1^1 \times M_1^2 \to M_1^1$ is the projection map.*

Proof Note that $\ker f_*$ and $(\ker f_*)^\perp$ are orthogonal distributions on (M_1, g_1) with $\ker f_*$ is integrable. First we show that $(\ker f_*)^\perp$ is integrable with totally geodesic integral manifolds. For this, it suffices to show that $\overset{1}{\nabla}_U V \in \Gamma(\ker f_*)^\perp$ for every $U, V \in \Gamma(\ker f_*)^\perp$. Now let $U, V \in \Gamma(\ker f_*)^\perp$ and $X \in \Gamma \ker f_*$. Then since

$$g_1(\overset{1}{\nabla}_U V, X) = -g_1(V, \overset{1}{\nabla}_U X) = -g_1\big(V, (\overset{1}{\nabla}_U X)^\perp\big),$$

where $(\overset{1}{\nabla}_U X)^\perp$ is the component of $\overset{1}{\nabla}_U X$ in $(\ker f_*)^\perp$, it follows that

$$g_1(\overset{1}{\nabla}_U V, X) = 0 \quad \text{iff} \quad (\overset{1}{\nabla}_U X)^\perp = 0 \quad \text{iff} \quad f_*(\overset{1}{\nabla}_U X) = 0$$

for every $X \in \Gamma \ker f_*$. On the other hand, since

$$0 = (\nabla f_*)(U, X) = \overset{2}{\nabla}_U f_* X - f_*(\overset{1}{\nabla}_U X) = -f_*(\overset{1}{\nabla}_U X)$$

for every $X \in \Gamma \ker f_*$, it follows that $\overset{1}{\nabla}_U V \in \Gamma(\ker f_*)^\perp$ and hence $(\ker f_*)^\perp$ is integrable with totally geodesic integral manifolds.

Next we show that integral manifolds of $\ker f_*$ are totally umbilic. Let $\mathbb{I}_f$ be the second fundamental form tensor of $\ker f_*$. Then, for $X, Y \in \Gamma \ker f_*$,

$$\begin{aligned} f_* \mathbb{I}_f(X,Y) &= f_*(\overset{1}{\nabla}_X Y)^\perp = f_*(\overset{1}{\nabla}_X Y) \\ &= -(\nabla f_*)(X,Y) + \overset{2}{\nabla}_X f_* Y \\ &= -(\nabla f_*)(X,Y) \\ &= -\frac{\tau(f)}{n_1 - n_2} g_1(X,Y). \end{aligned}$$

Thus, if $\tilde{\tau}(f)$ is the lift of $\tau(f)$ to $(\ker f_*)^\perp$ defined by $f_*\big(\tilde{\tau}(f)\big) = \tau(f)_{f(p_1)}$,

$$\mathbb{I}_f(X,Y) = -\frac{\tilde{\tau}(f)}{n_1 - n_2} g_1(X,Y).$$

That is, integral manifolds of $\ker f_*$ are totally umbilic with the mean curvature vector field $N = -\frac{\tilde{\tau}(f)}{n_1 - n_2}$. Hence by [3, Prop. 3-b], (M_1, g_1) is locally a twisted product. ∎

Now we complete showing that the local Möbius equation completely characterizes the local twisted product decomposition. First we need the following lemma.

Let $(M_1, g_1) = (M_1^1 \times M_1^2, g_1^1 \oplus \varphi g_1^2)$ be the twisted product of Riemannian manifolds (M_1^1, g_1^1) and (M_1^2, g_1^2) with twisting function φ, and let $\overset{1}{\nabla}$, $\overset{1}{\nabla}{}^1$ and $\overset{1}{\nabla}{}^2$ be the Levi-Civita connections of (M_1, g_1), (M_1^1, g_1^1) and (M_1^2, g_1^2), respectively. Given vector fields X_1, Y_1, Z_1 on M_1^1 and X_2, Y_2, Z_2 on M_1^2, we can lift them to $M_1^1 \times M_1^2$ and obtain vector fields $X = (X_1, 0) + (0, X_2) = (X_1, X_2)$, $Y = (Y_1, 0) + (0, Y_2) = (Y_1, Y_2)$ and $Z = (Z_1, 0) + (0, Z_2) = (Z_1, Z_2)$ on $(M_1^1 \times M_1^2)$.

Lemma 1 *Let $(M_1, g_1) = (M_1^1 \times M_1^2, g_1^1 \oplus \varphi g_1^2)$ be the twisted product of Riemannian manifolds (M_1^1, g_1^1) and (M_1^2, g_1^2) with twisting function φ. Then for vector fields X, Y, Z on $M_1^1 \times M_1^2$ as in the above, we have*

$$\begin{aligned} g_1(\overset{1}{\nabla}_X Y, Z) = g_1\big((\overset{1}{\nabla}{}^1{}_{X_1} Y_1, \overset{1}{\nabla}{}^2{}_{X_2} Y_2), (Z_1, Z_2) \big) \\ + \frac{1}{2}[X(\lambda) g_1^2(Y_2, Z_2) + Y(\lambda) g_1^2(X_2, Z_2) - Z(\lambda) g_1^2(X_2, Y_2)], \end{aligned}$$

where $\lambda = \log \varphi$.

Proof The proof can be obtained from Koszul formula by a straightforward computation. ∎

Theorem 2 *Let $(M_1, g_1) = (M_1^1 \times M_1^2, g_1^1 \oplus \varphi g_1^2)$ be the twisted product of Riemannian manifolds (M_1^1, g_1^1) and (M_1^2, g_1^2) with twisting function φ. Then the projection map $\pi_1 \colon M_1^1 \times M_1^2 \to M_1^1$ satisfies the local Möbius equation.*

Proof Since $\nabla \pi_{1*}$ is tensorial, it suffices to show that π_1 satisfies the local Möbius equation for the vector fields as in the above of Lemma 1. First note that, for $X = (0, X_2)$, $Y = (0, Y_2)$ and $U = (U_1, 0)$ on $M_1^1 \times M_1^2$, by Lemma 1,

$$g_1(\overset{1}{\nabla}_X Y, U) = -\frac{1}{2} U(\lambda) g_1^2(X_2, Y_2) = -\frac{1}{2} g_1(\overset{1}{\nabla}\lambda, U) g_1^2(X_2, Y_2).$$

Thus

$$(\overset{1}{\nabla}_X Y)^{\perp} = -\frac{1}{2} g_1^2(X_2, Y_2)(\overset{1}{\nabla}\lambda)^{\perp},$$

where $(\overset{1}{\nabla}_X Y)^{\perp}$ and $(\overset{1}{\nabla}\lambda)^{\perp}$ are the components of $\overset{1}{\nabla}_X Y$ and $\overset{1}{\nabla}\lambda$ tangent to the copies of M_1^1 in $M_1^1 \times M_1^2$, respectively. Next note that, for $U = (U_1, 0)$ and $V = (V_1, 0)$ on $M_1^1 \times M_1^2$, by Lemma 1, $\overset{1}{\nabla}_U V = (\overset{1}{\nabla}{}^1{}_{U_1} V_1, 0)$. Hence, if $\{V_1^1, \dots, V_1^{n_1^1}\}$ and $\{X_2^1, \dots, X_2^{n_1^2}\}$ are orthonormal basis frames on (M_1^1, g_1^1) and (M_1^2, g_1^2), respectively, then $\{V^1 = (V_1^1, 0), \dots, V^{n_1^1} = (V_1^{n_1^1}, 0), \frac{X^1}{\varphi^{1/2}} = (0, \frac{X_2^1}{\varphi^{1/2}}), \dots, \frac{X^{n_1^2}}{\varphi^{1/2}} = (0, \frac{X_2^{n_1^2}}{\varphi^{1/2}})\}$ is an orthonormal basis frame on (M_1, g_1) and, since

$$\begin{aligned}
(\nabla \pi_{1*})\left(\frac{X^i}{\varphi^{1/2}}, \frac{X^i}{\varphi^{1/2}}\right) &= \overset{1}{\nabla}{}^1{}_{\frac{X^i}{\varphi^{1/2}}} \pi_{1*} \frac{X^i}{\varphi^{1/2}} - \pi_{1*}\left(\overset{1}{\nabla}_{\frac{X^i}{\varphi^{1/2}}} \frac{X^i}{\varphi^{1/2}}\right) \\
&= -\frac{1}{\varphi^{1/2}} \pi_{1*}\left(\overset{1}{\nabla}_{X^i} \frac{X^i}{\varphi^{1/2}}\right)^{\perp} \\
&= -\frac{1}{\varphi} \pi_{1*}(\overset{1}{\nabla}_{X^i} X^i)^{\perp} \\
&= \frac{1}{2\varphi} g_1^2(X_2^i, X_2^i) \pi_{1*}(\overset{1}{\nabla}\lambda)^{\perp} \\
&= \frac{1}{2\varphi^2} \pi_{1*}(\overset{1}{\nabla}\varphi)^{\perp}
\end{aligned}$$

and

$$\begin{aligned}
(\nabla \pi_{1*})(V^i, V^i) &= \overset{1}{\nabla}{}^1{}_{V^i} \pi_{1*} V^i - \pi_{1*}(\overset{1}{\nabla}_{V^i} V^i) \\
&= \overset{1}{\nabla}{}^1{}_{V^i}(V_1^i \circ \pi_1) - (\overset{1}{\nabla}{}^1{}_{V_1^i} V_1^i) \circ \pi_1 \\
&= (\overset{1}{\nabla}{}^1{}_{V_1^i} V_1^i) \circ \pi_1 - (\overset{1}{\nabla}{}^1{}_{V_1^i} V_1^i) \circ \pi_1 = 0,
\end{aligned}$$

it follows that

$$\tau(\pi_1) = \operatorname{trace} \nabla \pi_{1*} = \frac{n_1^2}{2\varphi^2} \pi_{1*}(\overset{1}{\nabla}\varphi)^{\perp}.$$

On the other hand, for $X = (0, X_2)$ and $Y = (0, Y_2)$, since

$$\begin{aligned}(\nabla\pi_{1*})(X,Y) &= \overset{1}{\nabla}{}^1{}_X\pi_{1*}Y - \pi_{1*}(\overset{1}{\nabla}_X Y)\\ &= -\pi_{1*}(\overset{1}{\nabla}_X Y)^{\perp}\\ &= \frac{1}{2}g_1^2(X_2,Y_2)\pi_{1*}(\overset{1}{\nabla}\lambda)^{\perp}\\ &= \frac{1}{2\varphi^2}g_1(X,Y)\pi_{1*}(\overset{1}{\nabla}\varphi)^{\perp}\end{aligned}$$

it follows that

$$(\nabla\pi_{1*})(X,Y) = \frac{\tau(\pi_1)}{n_1^2}g_1(X,Y)$$

for vector fields X, Y tangent to the copies of M_1^2 in $M_1^1 \times M_1^2$.

To show the second equation in the local Möbius equation is also satisfied by π_1, let $U = (U_1, 0)$ and $X = (0, X_2)$ on $M_1^1 \times M_1^2$. Then for $Z = (Z_1, 0)$ on $M_1^1 \times M_1^2$, by Lemma 1, $g_1(\overset{1}{\nabla}_U X, Z) = 0$. Hence $(\overset{1}{\nabla}_U X)^{\perp} = 0$, where $(\overset{1}{\nabla}_U X)^{\perp}$ is the component of $\overset{1}{\nabla}_U X$ tangent to the copies of M_1^1 in $M_1^1 \times M_1^2$. Thus

$$\begin{aligned}(\nabla\pi_{1*})(U,X) &= \overset{1}{\nabla}{}^1{}_U\pi_{1*}X - \pi_{1*}(\overset{1}{\nabla}_U X)\\ &= -\pi_{1*}(\overset{1}{\nabla}_U X)^{\perp} = 0.\end{aligned}$$

That is, π_1 satisfies the local Möbius equation. ■

Remark 2 Note in the proof of the above theorem that if a Riemannian submersion $f\colon (M_1, g_1) \to (M_2, g_2)$ between Riemannian manifolds (M_1, g_1) and (M_2, g_2) satisfies the local Möbius equation then $(\nabla f_*)(U, V) = 0$ for every $U, V \in \Gamma(\ker f_*)^{\perp}$ since f is the projection map onto the first factor of local twisted product decomposition of (M_1, g_1).

Recall that a map $f\colon (M_1, g_1) \to (M_2, g_2)$ between Riemannian manifolds (M_1, g_1) and (M_2, g_2) is called *harmonic* if $\tau(f) = 0$. Here also observe that, by Remark 2, a Riemannian submersion between Riemannian manifolds is a totally geodesic map if and only if it is harmonic and satisfies the local Möbius equation.

Corollary 1 *If a harmonic submersion $f\colon (M_1, g_1) \to (M_2, g_2)$ between Riemannian manifolds (M_1, g_1) and (M_2, g_2) satisfies the local Möbius equation then (M_1, g_1) is locally a product $(M_1^1 \times M_1^2, g_1^1 \oplus g_1^2)$, where (M_1^1, g_1^1) and (M_1^2, g_1^2) are Riemannian manifolds and $f\colon M_1^1 \times M_1^2 \to M_1^1$ is the projection map. Conversely, if $(M_1, g_1) = (M_1^1 \times M_1^2, g_1^1 \oplus g_1^2)$ is a product of Riemannian manifolds (M_1^1, g_1^1) and (M_1^2, g_1^2) then the projection map $\pi_1\colon M_1^1 \times M_1^2 \to M_1^1$ is harmonic and satisfies the local Möbius equation.*

Proof By Theorem 1, (M_1, g_1) is locally a twisted product with totally umbilic integral manifolds of $\ker f_*$ whose mean curvature vector field is given by $N = -\frac{\tilde{\tau}(f)}{n_1 - n_2}$, where $\tilde{\tau}(f)$ is the lift of $\tau(f)$ to $(\ker f_*)^\perp$. Hence, since $\tau(f) = 0$, $\tilde{\tau}(f) = 0$ and it follows that $N = 0$, that is, integral manifolds of $\ker f_*$ are totally geodesic. Then from [3, Prop. 3-d], (M_1, g_1) is locally a product $(M_1^1 \times M_1^2, g_1^1 \oplus g_1^2)$. Conversely, by Theorem 2, π_1 satisfies the local Möbius equation. Also by the proof of Theorem 2, since $(M_1, g_1) = (M_1^1 \times M_1^2, g_1^1 \oplus g_1^2)$ is a twisted product with twisting function $\varphi = 1$, it follows that $\tau(\pi_1) = \operatorname{trace} \nabla \pi_{1*} = \frac{n_1^2}{2\varphi^2} \pi_{1*}(\nabla\varphi)^\perp = 0$. Thus π_1 is harmonic. (In fact, π_1 is totally geodesic by the local Möbius equation). ■

Remark 3 In Vilms [4] the following theorem is proven:

***Theorem 3* ([4])** *Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds with dimensions $n_1 > n_2 \geq 1$, where (M_1, g_1) is complete, connected and simply connected. If there is a surjective, totally geodesic Riemannian submersion $f\colon (M_1, g_1) \to (M_2, g_2)$ then (M_1, g_1) is a product $(M_2 \times M_3, g_2 \oplus g_3)$, where (M_3, g_3) is a Riemannian manifold.*

Here recall that a totally geodesic submersion satisfies the local Möbius equation and is harmonic. Hence, in the view of Corollary 1, it can be easily seen that the assumption of totally geodesic Riemannian submersion in the above theorem can be weakened to totally geodesic submersion and still obtain a local product decomposition on M_1.

Now we show that the local Möbius equation with an additional assumption completely characterizes the locally warped product Riemannian manifolds. Recall that a twisted product $(M_1^1 \times M_1^2, g_1^1 \oplus \varphi g_1^2)$ is called a *warped product* if φ is a function on M_1^1 only.

Theorem 4 *Let a submersion $f\colon (M_1, g_1) \to (M_2, g_2)$ between Riemannian manifolds (M_1, g_1) and (M_2, g_2) satisfy the local Möbius equation. If, in addition, $\nabla\nabla f_*(X, Y, Z) = 0$ for every $X, Y, Z \in \Gamma \ker f_*$, then (M_1, g_1) is locally a warped product $(M_1^1 \times M_1^2, g_1^1 \oplus \varphi g_1^2)$, where (M_1^1, g_1^1) and (M_1^2, g_1^2) are Riemannian manifolds and $f\colon M_1^1 \times M_1^2 \to M_1^1$ is the projection map.*

Proof First note that, by Theorem 1, (M_1, g_1) is locally a twisted product with totally umbilic integral manifolds of $\ker f_*$ whose mean curvature vector field is given by $N = -\frac{\tilde{\tau}(f)}{n_1 - n_2}$, where $\tilde{\tau}(f)$ is the lift of $\tau(f)$ to $(\ker f_*)^\perp$. Hence to show that (M_1, g_1) is locally a warped product, it suffices to show that N is normal parallel along the integral manifolds of $\ker f_*$. For, first note that if $X \in \Gamma \ker f_*$,

$$\begin{aligned} 0 = (\nabla f_*)\big(X, \tilde{\tau}(f)\big) &= \overset{2}{\nabla}_X f_* \tilde{\tau}(f) - f_*\big(\overset{1}{\nabla}_X \tilde{\tau}(f)\big) \\ &= \overset{2}{\nabla}_X \tau(f) - f_*\big(\overset{1}{\nabla}_X \tilde{\tau}(f)\big). \end{aligned}$$

Thus

$$\overset{2}{\nabla}_X \tau(f) = f_*\big(\overset{1}{\nabla}_X \tilde{\tau}(f)\big) = f_*\Big(\big(\overset{1}{\nabla}_X \tilde{\tau}(f)\big)^\perp\Big)$$

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and hence, to show that N is normal parallel along the integral manifolds of $\ker f_*$, it suffices to show that $\overset{2}{\nabla}_X \tau(f) = 0$ for every $X \in \Gamma \ker f_*$. For this, note that, for $X, Y, Z \in \Gamma \ker f_*$,

$$
\begin{aligned}
0 = (\nabla\nabla f_*)(X,Y,Z) &= \overset{2}{\nabla}_X\big((\nabla f_*)(Y,Z)\big) - (\nabla f_*)(\overset{1}{\nabla}_X Y, Z) \\
&\quad - (\nabla f_*)(Y, \overset{1}{\nabla}_X Z) \\
&= \overset{2}{\nabla}_X\big((\nabla f_*)(Y,Z)\big) - (\nabla f_*)\big((\overset{1}{\nabla}_X Y)^T, Z\big) \\
&\quad - (\nabla f_*)\big(Y, (\overset{1}{\nabla}_X Z)^T\big) \\
&= \overset{2}{\nabla}_X\left(\frac{\tau(f)}{n_1 - n_2} g_1(Y,Z)\right) - \frac{\tau(f)}{n_1 - n_2} g_1(\overset{1}{\nabla}_X Y, Z) \\
&\quad - \frac{\tau(f)}{n_1 - n_2} g_1(Y, \overset{1}{\nabla}_X Z) \\
&= \frac{1}{n_1 - n_2}\big(\overset{2}{\nabla}_X \tau(f)\big) g_1(Y,Z),
\end{aligned}
$$

where $(\overset{1}{\nabla}_X Y)^T$ and $(\overset{1}{\nabla}_X Z)^T$ are the components of $\overset{1}{\nabla}_X Y$ and $\overset{1}{\nabla}_X Z$ in $\ker f_*$, respectively. Thus, $\overset{2}{\nabla}_X \tau(f) = 0$ for every $X \in \Gamma \ker f_*$ and it follows from [3, Prop. 3-c] that (M_1, g_1) is locally a warped product. ■

Conversely, we have the following.

Theorem 5 *Let $(M_1, g_1) = (M_1^1 \times M_1^2, g_1^1 \oplus \varphi g_1^2)$ be a warped product of Riemannian manifolds (M_1^1, g_1^1) and (M_1^2, g_1^2) with warping function φ. Then the projection map $\pi_1 \colon M_1^1 \times M_1^2 \to M_1^1$ satisfies the local Möbius equation with $(\nabla\nabla\pi_{1*})(X,Y,Z) = 0$ for every X, Y, Z tangent to the copies of M_1^2 in $M_1^1 \times M_1^2$.*

Proof Note that, by Theorem 2, π_1 satisfies the local Möbius equation and

$$(\nabla\pi_{1*})(Y,Z) = \frac{1}{2\varphi^2} g_1(Y,Z)\pi_{1*}(\overset{1}{\nabla}\varphi)^\perp,$$

where $Y = (0, Y_2)$, $Z = (0, Z_2)$ and $(\overset{1}{\nabla}\varphi)^\perp$ is the component of $\overset{1}{\nabla}\varphi$ tangent to the copies of M_1^1 in $M_1^1 \times M_1^2$. Thus, since φ is a function on M_1^1 only, $(\overset{1}{\nabla}\varphi)^\perp = (\overset{1}{\nabla}{}^1\varphi, 0)$, and it follows from

$$\tau(\pi_1) = \frac{n_1^2}{2\varphi^2}\pi_{1*}(\overset{1}{\nabla}\varphi)^\perp = \frac{n_1^2}{2\varphi^2}(\overset{1}{\nabla}{}^1\varphi) \circ \pi_1$$

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(see the proof of Theorem 2) that $\overset{1}{\nabla}{}^1{}_X\tau(\pi_1) = 0$ for every $X = (0, X_2)$. Then as in the proof of Theorem 4,

$$(\nabla\nabla\pi_{1*})(X, Y, Z) = \frac{1}{n_1^2}\big(\overset{1}{\nabla}_X\tau(\pi_1)\big)\, g_1(Y, Z) = 0$$

for every $X = (0, X_2)$, $Y = (0, Y_2)$ and $Z = (0, Z_2)$. Hence since $\nabla\nabla\pi_{1*}$ is tensorial, $(\nabla\nabla\pi_{1*})(X, Y, Z) = 0$ for every X, Y, Z tangent to the copies of M_1^2 in $M_1^1 \times M_1^2$. ■

In the statement of Theorem 4, the assumption on $\nabla\nabla f_*$ can be replaced by an assumption on the Ricci tensor for the following special case.

Let $(M_1, g_1) = (M, g)$ and $(M_2, g_2) = (\mathbb{R}^m, \langle\ ,\ \rangle)$, where $\langle\ ,\ \rangle = \sum_{i=1}^m dx^i \otimes dx^i$ is the Euclidean metric tensor on $\mathbb{R}^m$. Then note that, if $f = (f_1, \dots, f_m) \colon (M, g) \to (\mathbb{R}^m, \langle\ ,\ \rangle)$ is a map then

$$\nabla f_* = \sum_{i=1}^m H_f \frac{\partial}{\partial x^i} \circ f \quad \text{and} \quad \tau(f) = \sum_{i=1}^m (\Delta f_i) \frac{\partial}{\partial x^i} \circ f.$$

Corollary 2 *Let a submersion $f = (f_1, \dots, f_m) \colon (M, g) \to (\mathbb{R}^m, \langle\ ,\ \rangle)$ satisfy the local Möbius equation with* $\mathrm{Ric}(U, X) = 0$ *for every $U \in \Gamma(\ker f_*)^{\perp}$ and $X \in \Gamma \ker f_*$, where* Ric *is the Ricci tensor of (M, g). Then (M, g) is locally a warped product $(M^1 \times M^2, g^1 \oplus \varphi g^2)$, where (M^1, g^1) and (M^2, g^2) are Riemannian manifolds and $f \colon M^1 \times M^2 \to M^1$ is the projection map.*

Proof By Theorem 4, it suffices to show that $(\nabla\nabla f_*)(X, Y, Z) = 0$ for every $X, Y, Z \in \Gamma \ker f_*$. Also note that, by the proof of Theorem 4, since

$$(\nabla\nabla f_*)(X, Y, Z) = \frac{1}{n - m}\big(D_X\tau(f)\big)\, g(Y, Z),$$

where D is the Levi-Civita connection of $(\mathbb{R}^m, \langle\ ,\ \rangle)$, for this, it suffices to show that $D_X\tau(f) = 0$ for every $X \in \Gamma \ker f_*$. Thus, for $X \in \Gamma \ker f_*$, since

$$D_X\tau(f) = \sum_{i=1}^m X(\Delta f_i)\frac{\partial}{\partial x^i} \circ f,$$

we need to show that $X(\Delta f_i) = 0$ for every $X \in \Gamma \ker f_*$. Now first note that, since $(\nabla f_*) = H_{f_i}\frac{\partial}{\partial x^i} \circ f$, as in Remark 1, $h_{f_i}(X) = \frac{\Delta f_i}{n-m} X$ for every $X \in \Gamma \ker f_*$. Hence for $X, Y \in \Gamma \ker f_*$,

$$\begin{aligned} R(X, Y)\nabla f_i &= \nabla_X\nabla_Y\nabla f_i - \nabla_Y\nabla_X\nabla f_i - \nabla_{[X,Y]}\nabla f_i \\ &= \nabla_X\left(\frac{\Delta f_i}{n - m}Y\right) - \nabla_Y\left(\frac{\Delta f_i}{n - m}X\right) - \frac{\Delta f_i}{n - m}[X, Y] \\ &= \frac{1}{n - m}\big(X(\Delta f_i)Y - Y(\Delta f_i)X\big), \end{aligned}$$

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where R is the curvature tensor of (M, g).

Also for $Y \in \Gamma \ker f_*$ and $U \in \Gamma(\ker f_*)^\perp$, since $(\ker f_*)^\perp$ is totally geodesic,

$$g\big(R(U,Y)\nabla f_i, U\big) = g\big(R(\nabla f_i, U)U, Y\big) = 0.$$

Thus, if $\{X_1, \dots, X_{n-m-1}, X_{n-m} = Y\}$ is a local orthonormal frame in $\ker f_*$, we have

$$\begin{aligned} 0 = \operatorname{Ric}(Y, \nabla f_i) &= \frac{1}{n-m} \sum_{i=1}^{n-m} g\big(X_i(\Delta f_i)Y - Y(\Delta f_i)X_i, X_i\big) \\ &= -Y(\Delta f_i). \end{aligned}$$

That is, $X(\Delta f_i) = 0$ for every $X \in \Gamma \ker f_*$ and hence $(\nabla\nabla f_*)(X, Y, Z) = 0$ for every $X, Y, Z \in \Gamma \ker f_*$. ■

Remark 4 We finalize this paper with a note on the question of "when a local diffeomorphism between two Riemannian manifolds is a local isometry?". That is, "is there a global analytic condition on a local diffeomorphism between two Riemannian manifolds, such as a partial differential equation satisfied by it, so that it becomes a local isometry?". In this paper we investigated a partial answer to this question in terms of a submersion between two Riemannian manifolds satisfying a local Möbius equation.

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