


## CONSTRUCTION OF AGE-STRUCTURED BRANCHING PROCESSES BY STOCHASTIC EQUATIONS

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### Abstract

We provide constructions of age-structured branching processes without or with immigration as pathwise-unique solutions to stochastic integral equations. A necessary and sufficient condition for the ergodicity of the model with immigration is also given.

*Keywords:* Distribution-function-valued process; ergodicity; measure-valued; non-local; immigration

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### 1. Introduction

Branching processes were introduced to describe the evolution of populations undergoing random reproduction. For the classical continuous-time branching process, it is assumed that an individual has exponential life-length and gives birth to a random number of offspring at the end of its life. The age-dependent branching process introduced in [3] assumes that the individual may have a general life-length distribution. The model has been generalized further to allow the individual to give birth to offspring at any time during its life; see, e.g., [8, 9, 11, 15, 20]. Those age-dependent models are usually not Markovian if one only considers the evolution of the total number of individuals in the population. A measure-valued Markovian branching particle system was introduced in [4] to describe the evolution of a birth–death model with age structures; see also [5, 6, 10, 19]. In the models mentioned above, the death rate and the offspring distribution of an individual may depend on its age, but different individuals behave independently of each other. Several authors have also studied population models where the reproduction depends on the age structure of the whole population; see, e.g., [17, 18, 24, 25, 28]. We refer to [1, 12, 14, 16, 21] for systematic treatments of various classes of branching processes.

The approach of stochastic equations has played an important role in recent developments of the theory of branching processes. The reader may refer to [2, 22, 26] and the references therein for applications of this approach to continuous-state branching processes. Stochastic equations have also been introduced in the study of discrete-state branching models; see, e.g.,

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[7, 13, 28]. In particular, a stochastic equation for an age-structured birth–death process was proposed in [28] in the study of large population limits.

The purpose of this paper is to develop the approach of stochastic equations further for age-structured branching processes to allow general offspring distributions. For concreteness, we focus on the model where the death rate and the offspring distribution of an individual only depends on its own age. The model can be thought of as a typical special case of non-local branching particle systems; see, e.g., [5, 6, 10, 19, 21]. We give a construction of the age-structured branching process as the pathwise-unique solutions of a stochastic equation driven by a time–space Poisson random measure. The construction determines explicitly the behavior of the trajectory of the process. By a slight extension of the stochastic equation, an age-structured immigration model is constructed. We also prove a necessary and sufficient condition for the ergodicity of the process with immigration.

Let  $\mathcal{B}(\mathbb{R}_+)$  denote the Borel  $\sigma$ -algebra on  $\mathbb{R}_+ := [0, \infty)$ . Let  $\mathfrak{M}(\mathbb{R}_+)$  denote the set of finite Borel measures on  $\mathbb{R}_+$  with the weak convergence topology. Let  $\mathfrak{D}(\mathbb{R}_+)$  be set of bounded positive right-continuous increasing functions  $f$  on  $\mathbb{R}$  satisfying  $f(x) = 0$  for  $x < 0$ . We identify  $\nu \in \mathfrak{M}(\mathbb{R}_+)$  with its distribution function  $\nu \in \mathfrak{D}(\mathbb{R}_+)$  defined by  $\nu(x) = \nu[0, x]$  for  $x \geq 0$ . Let  $\mathfrak{N}(\mathbb{R}_+)$  be the subset of  $\mathfrak{M}(\mathbb{R}_+)$  consisting of integer-valued measures. Let  $B(\mathbb{R}_+)$  be the Banach space of bounded Borel functions on  $\mathbb{R}_+$  furnished with the supremum norm  $\|\cdot\|$ . Let  $C(\mathbb{R}_+)$  be the set of continuous functions in  $B(\mathbb{R}_+)$ , and let  $C^1(\mathbb{R}_+)$  be the set of functions in  $C(\mathbb{R}_+)$  with bounded continuous derivatives of the first order. We use the superscript ‘+’ to denote the subsets of positive elements and the subscript ‘0’ to denote the subsets of functions vanishing at infinity, e.g.  $B(\mathbb{R}_+)^+$ ,  $C_0(\mathbb{R}_+)^+$ , etc. For any  $f \in B(\mathbb{R}_+)$  and  $\nu \in \mathfrak{M}(\mathbb{R}_+)$  write  $\langle \nu, f \rangle = \int_{\mathbb{R}_+} f(x)\nu(dx)$ . In the integrals we use the convention that, for  $a \leq b \in \mathbb{R}$ ,

$$\int_a^b = \int_{(a,b]} \quad \text{and} \quad \int_a^\infty = \int_{(a,\infty)}.$$

The rest of this paper is organized as follows. In Section 2 we introduce the age-structured branching process and give some basic characterizations of its transition probabilities. In Section 3, the process is constructed as the pathwise-unique strong solution to a stochastic integral equation driven by a Poisson random measure. Similar results for the age-structured system with immigration are presented in Section 4, where the ergodicity of the model is also studied.

## 2. An age-structured branching process

In this section we introduce the age-structured branching process and give some basic characterizations of its transition probabilities. Most of the results presented here are essentially known, so we only sketch the proofs; see, e.g., [6, 19, 21, 23].

Let  $\alpha \in C^1(\mathbb{R}_+)^+$  be a function bounded away from zero. For each  $x \in \mathbb{R}_+$  let  $\{p(x, i) : i \in \mathbb{N}\}$  be a discrete probability distribution with generating function  $g(x, z) = \sum_{i=0}^\infty p(x, i)z^i$ ,  $z \in [0, 1]$ . We assume that  $p(\cdot, i) \in C^1(\mathbb{R}_+)^+$  for every  $i \in \mathbb{N}$ , and that  $\|\partial_z g(\cdot, 1-)\| = \sup_{x \geq 0} \sum_{i=1}^\infty p(x, i) < \infty$ , where  $\partial_z$  denotes the first derivative with respect to  $z$ . A branching particle system is characterized by the following properties:

- (i) The ages of the particles increase at unit speed, i.e. they move according to realizations of the deterministic process  $\xi = (\xi_t)_{t \geq 0}$  in  $\mathbb{R}_+$  defined by  $\xi_t = \xi_0 + t$ .

- (ii) For a particle which is alive at time  $r \geq 0$  with age  $x \geq 0$ , the conditional probability of survival in the time interval  $[r, t]$  is  $\exp\{-\int_0^{t-r} \alpha(x+s) ds\}$ .
- (iii) When a particle dies at age  $x \geq 0$ , it gives birth to a random number of offspring with age zero according to the probability law  $\{p(x, i) : i = 0, 1, \dots\}$  determined by the generating function  $g(x, \cdot)$ .

We assume that the lifetimes and the offspring productions of different particles are independent. Let  $X_t(B)$  denote the number of particles alive at time  $t \geq 0$  with ages belonging to the Borel set  $B \subset \mathbb{R}_+$ . If we assume  $X_0(\mathbb{R}_+) < \infty$ , then  $\{X_t : t \geq 0\}$  is a Markov process with state space  $\mathfrak{N}(\mathbb{R}_+)$ . We refer to [21, Section 4.3] for the formulation of general branching particle systems.

Let  $\sigma \in \mathfrak{N}(\mathbb{R}_+)$  and let  $\{X_t^\sigma : t \geq 0\}$  be the above system with initial value  $X_0^\sigma = \sigma$ . Let  $\delta_x$  denote the unit measure concentrated at  $x \in \mathbb{R}_+$ . Suppose that the process is defined on a probability space  $(\Omega, \mathcal{G}, \mathbb{P})$ . Properties (i)–(iii) imply that, for  $f \in B(\mathbb{R}_+)^+$ ,

$$\mathbb{E}[\exp\{-\langle X_t^\sigma, f \rangle\}] = \exp\{-\langle \sigma, u_t f \rangle\}, \quad u_t f(x) = -\log \mathbb{E}[\exp\{-\langle X_t^{\delta_x}, f \rangle\}].$$

From properties (i)–(iii) we derive, as in [19, Section 3] and [21, Section 4.3], the following renewal equation:

$$\begin{aligned} e^{-u_t f(x)} &= \exp\left\{-f(x+t) - \int_0^t \alpha(x+s) ds\right\} \\ &+ \int_0^t \exp\left\{-\int_0^s \alpha(x+r) dr\right\} \alpha(x+s) g(x+s, e^{-u_{t-s} f(0)}) ds. \end{aligned} \quad (2.1)$$

From [21, Proposition 2.9], the above equation implies

$$e^{-u_t f(x)} = e^{-f(x+t)} + \int_0^t \alpha(x+t-s) [g(x+t-s, e^{-u_s f(0)}) - e^{-u_s f(x+t-s)}] ds. \quad (2.2)$$

The uniqueness of the solution to (2.1) and (2.2) follows by Gronwall's inequality. The two equations are therefore equivalent.

We call any Markov process  $(X_t : t \geq 0)$  with state space  $\mathfrak{N}(\mathbb{R}_+)$  an  $(\alpha, g)$ -age-structured branching process if it has a transition semigroup  $(Q_t)_{t \geq 0}$  defined by

$$\int_{\mathfrak{N}(\mathbb{R}_+)} e^{-\langle \nu, f \rangle} Q_t(\sigma, d\nu) = \exp\{-\langle \sigma, u_t f \rangle\}, \quad f \in B(\mathbb{R}_+)^+, \quad (2.3)$$

where  $u_t f(x)$  is the unique solution to (2.2).

**Proposition 2.1.** For any  $x \geq 0$ ,  $t \geq 0$ , and  $f \in C_0^1(\mathbb{R}_+)^+$ , we have

$$\partial_t u_t f(x) = \partial_x u_t f(x) + \alpha(x) [1 - e^{u_t f(x)} g(x, e^{-u_t f(0)})]. \quad (2.4)$$

*Proof.* For any  $t \geq 0$  let  $T_t$  be the operator on the Banach space  $C_0(\mathbb{R}_+)$  defined by  $T_t f(x) = f(x+t)$ . By (2.2) it is easy to see that  $U_t f(x) = 1 - e^{-u_t f(x)}$  solves the evolution integral equation

$$U_t f(x) = T_t U_0 f(x) - \int_0^t T_{t-s} \phi(\cdot, U_s f)(x) ds, \quad (2.5)$$

where  $\phi(x, f) = \alpha(x)[g(x, 1 - f(0)) - 1 + f(x)]$ . By a general result on semi-linear evolution equations, we know that  $t \mapsto U_t f(x)$  is continuously differentiable and solves the differential evolution equation  $\partial_t U_t f(x) = \partial_x U_t f(x) - \phi(x, U_t f)$ ,  $U_0 f(x) = 1 - e^{-f(x)}$ ; see, e.g., [27, Theorem 6.1.5, p. 187]. Then  $t \mapsto \partial_t u_t f(x)$  is also continuously differentiable. By differentiating both sides of (2.2) we have

$$\begin{aligned} e^{-u_t f(x)} \partial_t u_t f(x) &= \partial_x e^{-f(x+t)} \\ &+ \int_0^t \partial_x [\alpha(x+t-s)(g(x+t-s, e^{-u_s f(0)}) - e^{-u_s f(x+t-s)})] ds \\ &+ \alpha(x)[e^{-u_t f(x)} - g(x, e^{-u_t f(0)})] \\ &= e^{-u_t f(x)} \partial_x u_t f(x) + \alpha(x)[e^{-u_t f(x)} - g(x, e^{-u_t f(0)})], \end{aligned}$$

which proves (2.4). □

**Proposition 2.2.** For any  $t \geq 0$  and  $\sigma \in \mathfrak{N}(\mathbb{R}_+)$  we have

$$\int_{\mathfrak{N}(\mathbb{R}_+)} \langle v, f \rangle Q_t(\sigma, dv) = \langle \sigma, \pi_t f \rangle, \quad f \in B(\mathbb{R}_+), \tag{2.6}$$

where  $(\pi_t)_{t \geq 0}$  is the semigroup of bounded kernels on  $\mathbb{R}_+$  defined by

$$\pi_t f(x) = f(x+t) + \int_0^t \alpha(x+s)[\partial_z g(x+s, 1-) \pi_{t-s} f(0) - \pi_{t-s} f(x+s)] ds. \tag{2.7}$$

*Proof.* The existence and uniqueness of the locally bounded solution to (2.7) follows by a general result; see, e.g., [21, Lemma 2.17]. For  $f \in B(\mathbb{R}_+)^+$  we can use (2.2) to see that the unique solution of (2.7) is given by  $\pi_t f(x) = \partial_\theta u_t(\theta f)(x)|_{\theta=0}$ . By differentiating both sides of (2.3) we get (2.6). The extension to  $f \in B(\mathbb{R}_+)$  is immediate by linearity. □

**Proposition 2.3.** Let  $c_* = \inf_{y \geq 0} \alpha(y)[1 - \partial_z g(y, 1-)]$ . Then  $\|\pi_t f\| \leq e^{-c_* t} \|f\|$  for  $t \geq 0$ ,  $f \in B(\mathbb{R}_+)$ .

*Proof.* This follows from (2.7) and [23, Theorem 3.1]. □

**Proposition 2.4.** Let  $c_1 = \sup_{y \geq 0} \alpha(y)$ . Then, for any  $t \geq 0$  and  $x \geq 0$ ,

$$\pi_t f(x) \geq u_t f(x) \geq (1 - e^{-f(x+t)}) e^{-c_1 t}, \quad f \in B(\mathbb{R}_+)^+. \tag{2.8}$$

*Proof.* By taking  $\sigma = \delta_x$  in (2.3) and (2.6) and using Jensen’s inequality we get the first inequality in (2.8). Let  $U_t f(x)$  be as in the proof of Proposition 2.1. By (2.5) and a comparison theorem we have  $U_t f(x) \geq v_t f(x)$ , where  $(t, x) \mapsto v_t f(x)$  solves

$$v_t f(x) = 1 - e^{-f(x+t)} - \int_0^t \alpha(x+s) v_{t-s} f(x+s) ds.$$

The unique locally bounded solution to the above equation is given by

$$v_t f(x) = (1 - e^{-f(x+t)}) \exp \left\{ - \int_0^t \alpha(x+s) ds \right\}.$$

Then we have the estimate (2.8). □

### 3. Construction by stochastic equations

In this section we give a construction of the age-structured branching process by solving a stochastic equation driven by a Poisson random measure. Recall that  $\mathcal{D}(\mathbb{R}_+)$  denotes the set of bounded positive right-continuous increasing functions  $f$  on  $\mathbb{R}$  satisfying  $f(x) = 0$  for  $x < 0$ . For  $\mu \in \mathcal{D}(\mathbb{R}_+)$  and  $\alpha \in B(\mathbb{R}_+)^+$  define  $A_\alpha(\mu, y) = \inf\{z \geq 0 : \langle \mu, \alpha \mathbf{1}_{[0,z]} \rangle > \langle \mu, \alpha \rangle y\}$ ,  $0 \leq y \leq 1$ , with  $\inf \emptyset = \infty$  by convention. Then  $\langle \mu, \alpha \rangle = 0$  implies  $A_\alpha(\mu, y) = \infty$  for all  $0 \leq y \leq 1$ . By an elementary result in probability theory, we have the following lemma.

**Lemma 3.1.** *If  $\langle \mu, \alpha \rangle > 0$  and if  $\xi$  is a random variable with uniform distribution on  $(0, 1]$ , then  $\mathbb{P}\{A_\alpha(\mu, \xi) \in dx\} = \langle \mu, \alpha \rangle^{-1} \alpha(x) \mu(dx)$ ,  $x \geq 0$ .*

Suppose that  $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$  is a filtered probability space satisfying the usual hypotheses. Let  $M(dt, du, dy, dz, dv)$  be an  $(\mathcal{F}_t)$ -Poisson random measure on  $(0, \infty)^2 \times (0, 1] \times \mathbb{N} \times (0, 1]$  with intensity  $dt du dy \pi(dz) dv$ , where  $\pi(dz)$  denotes the counting measure on  $\mathbb{N}$ . Given an  $\mathcal{F}_0$ -measurable random function  $X_0 \in \mathcal{D}(\mathbb{R}_+)$ , we consider the stochastic integral equation, for  $t \geq 0$  and  $x \geq 0$ ,

$$\begin{aligned}
 X_t(x) = & X_0(x-t) + \int_0^t \int_0^{\langle X_{s-}, \alpha \rangle} \int_0^1 \int_{\mathbb{N}} \int_0^{p(A_\alpha(X_{s-}, y), z)} z \mathbf{1}_{\{t-s \leq x\}} M(ds, du, dy, dz, dv) \\
 & - \int_0^t \int_0^{\langle X_{s-}, \alpha \rangle} \int_0^1 \int_{\mathbb{N}} \int_0^{p(A_\alpha(X_{s-}, y), z)} \mathbf{1}_{\{A_\alpha(X_{s-}, y) + t - s \leq x\}} M(ds, du, dy, dz, dv). \quad (3.1)
 \end{aligned}$$

Heuristically, the left-hand side  $X_t(x)$  is the number of individuals at time  $t$  with ages less than  $x$ . On the right-hand side, the first term  $X_0(x-t)$  counts the number of individuals having ages less than  $x-t$  at time 0, and thus having ages less than  $x$  at time  $t$ . A death in the population occurs at time  $s \in [0, t]$  at rate  $\langle X_{s-}, \alpha \rangle ds$ . In that case, the age of the dying individual is distributed according to the probability measure  $\langle X_{s-}, \alpha \rangle^{-1} \alpha(x) X_{s-}(dx)$  and is realized as  $A_\alpha(X_{s-}, y)$ , where  $y \in (0, 1]$  is taken according to the uniform distribution by the Poisson random measure. The number of offspring of the individual takes the value  $z \in \mathbb{N}$  with probability  $p(A_\alpha(X_{s-}, y), z)$  and contributes to the number  $X_t(x)$  if and only if  $t-s \leq x$ , which is recorded by the second term. The death of the individual affects  $X_t(x)$  if and only if  $A_\alpha(X_{s-}, y) + t-s \leq x$ , which is recorded by the third term.

Let  $\zeta_a(x) = \mathbf{1}_{\{a \leq x\}}$  for  $a, x \in \mathbb{R}$ . Given a function  $f$  on  $\mathbb{R}$  define  $f \circ \theta_t(x) = f(x+t)$  for  $x, t \in \mathbb{R}$ . Then we may rewrite (3.1) equivalently, for  $t \geq 0$  and  $x \geq 0$ , as

$$\begin{aligned}
 X_t(x) = & X_0 \circ \theta_{-t}(x) + \int_0^t \int_0^{\langle X_{s-}, \alpha \rangle} \int_0^1 \int_{\mathbb{N}} \int_0^{p(A_\alpha(X_{s-}, y), z)} [z \zeta_0 \circ \theta_{s-t}(x) \\
 & - \zeta_{A_\alpha(X_{s-}, y)} \circ \theta_{s-t}(x)] M(ds, du, dy, dz, dv). \quad (3.2)
 \end{aligned}$$

A pathwise solution to (3.2) is constructed by unscrambling the equation as follows. Let  $\tau_0 = 0$ . Given  $\tau_{k-1} \geq 0$  and  $X_{\tau_{k-1}} \in \mathcal{D}(\mathbb{R}_+)$ , we first define  $\tau_k = \tau_{k-1} + \inf\{t > 0 : M((\tau_{k-1}, \tau_{k-1} + t] \times (0, \langle X_{\tau_{k-1}}, \alpha \rangle] \times H_k) > 0\}$ , where  $H_k = \{(y, z, v) : y \in (0, 1], z \in \mathbb{N}, 0 < v \leq p(A_\alpha(X_{\tau_{k-1}}, y), z)\}$ , and

$$X_t(x) = X_{\tau_{k-1}} \circ \theta_{\tau_{k-1}-t}(x), \quad \tau_{k-1} \leq t < \tau_k, \quad x \geq 0. \quad (3.3)$$

Then we define

$$X_{\tau_k}(x) = X_{\tau_k-}(x) + z_k \zeta_0(x) - \zeta_{A_\alpha(X_{\tau_k-}, y_k)}(x), \quad x \geq 0, \quad (3.4)$$

where  $X_{\tau_k-}(x) = X_{\tau_{k-1}} \circ \theta_{\tau_{k-1}-\tau_k}(x)$ , and  $(u_k, y_k, z_k, v_k) \in (0, \infty) \times (0, 1] \times \mathbb{N} \times (0, 1]$  is the point such that  $(\tau_k, u_k, y_k, z_k, v_k) \in \text{supp}(M)$ . Since  $A_\alpha(X_{\tau_k-}, y_k) \in \text{supp}(X_{\tau_k-})$ , we have  $X_{\tau_k} \in \mathfrak{D}(\mathbb{R}_+)$ . Equation (3.4) means that at time  $\tau_k$  an individual at age  $A_\alpha(X_{\tau_k-}, y_k)$  dies and gives birth to  $z_k$  offspring with starting age  $0 \in \mathbb{R}_+$ .

It is clear that (3.3) and (3.4) uniquely determine the behavior of the trajectory  $t \mapsto X_t$  on the time intervals  $[\tau_{k-1}, \tau_k]$ ,  $k = 1, 2, \dots$ . Let  $\tau = \lim_{k \rightarrow \infty} \tau_k$  and let  $X_t = \infty$  for  $t \geq \tau$ . Then  $\{X_t : t \geq 0\}$  is the pathwise-unique solution to (3.2) up to the lifetime  $\tau$ . More precisely, the equations hold almost surely with  $t$  replaced by  $t \wedge \tau_k$  for every  $k = 1, 2, \dots$ . Let  $n(t) = \sup\{k \geq 0 : \tau_k \leq t\}$  for  $t \geq 0$ , and  $\beta = \|\alpha \partial_z g(\cdot, 1-)\| < \infty$ .

**Lemma 3.2.** *Suppose that  $\mathbb{E}[X_0(\infty)] < \infty$ . Then, for any  $k \geq 1$ ,*

$$\mathbb{E}[\sup_{0 \leq s \leq t \wedge \tau_k} X_s(\infty)] \leq \mathbb{E}[X_0(\infty)]e^{\beta t}, \quad t \geq 0. \tag{3.5}$$

*Proof.* Recall that  $X_t(\infty) = \lim_{x \rightarrow \infty} X_t(x) = X_t(\mathbb{R}_+)$ . Let  $\eta_i = \inf\{s \geq 0 : X_s(\infty) \geq i\}$  for  $i \geq 1$ . It is clear that  $\lim_{i \rightarrow \infty} \eta_i = \tau$ . Let  $\zeta_{i,k} = \eta_i \wedge \tau_k$ . In view of (3.2), we have

$$X_t(\infty) = X_0(\infty) + \int_0^t \int_0^{(X_{s-}, \alpha)} \int_0^1 \int_{\mathbb{N}} \int_0^1 p(A_\alpha(X_{s-}, y), z) (z - 1) M(ds, du, dy, dz, dv).$$

It follows that

$$\begin{aligned} \mathbb{E}[\sup_{0 \leq s \leq t \wedge \zeta_{i,k}} X_s(\infty)] &\leq \mathbb{E}[X_0(\infty)] + \sum_{z \in \mathbb{N}} \mathbb{E}\left[\int_0^{t \wedge \zeta_{i,k}} (X_{s-}, \alpha) ds \int_0^1 p(A_\alpha(X_{s-}, y), z) z dy\right] \\ &= \mathbb{E}[X_0(\infty)] + \sum_{z \in \mathbb{N}} \mathbb{E}\left[\int_0^{t \wedge \zeta_{i,k}} ds \int_{\mathbb{R}_+} p(y, z) z \alpha(y) dX_{s-}(y)\right] \\ &= \mathbb{E}[X_0(\infty)] + \mathbb{E}\left[\int_0^{t \wedge \zeta_{i,k}} ds \int_{\mathbb{R}_+} \alpha(y) \partial_z g(y, 1-) dX_{s-}(y)\right] \\ &\leq \mathbb{E}[X_0(\infty)] + \beta \mathbb{E}\left[\int_0^{t \wedge \zeta_{i,k}} X_{s-}(\infty) ds\right]. \end{aligned}$$

Clearly, we have  $X_{s-}(\infty) \leq i$  for  $0 < s \leq t \wedge \zeta_{i,k}$ . Then  $\mathbb{E}[\sup_{0 \leq s \leq t \wedge \zeta_{i,k}} X_s(\infty)]$  is locally bounded in  $t \geq 0$ . Since the trajectory  $s \mapsto X_s(\infty)$  has at most countably many jumps, it follows that

$$\begin{aligned} \mathbb{E}[\sup_{0 \leq s \leq t \wedge \zeta_{i,k}} X_s(\infty)] &\leq \mathbb{E}[X_0(\infty)] + \beta \mathbb{E}\left[\int_0^{t \wedge \zeta_{i,k}} X_s(\infty) ds\right] \\ &\leq \mathbb{E}[X_0(\infty)] + \beta \int_0^t \mathbb{E}[X_{s \wedge \zeta_{i,k}}(\infty)] ds \\ &\leq \mathbb{E}[X_0(\infty)] + \beta \int_0^t \mathbb{E}[\sup_{0 \leq r \leq s \wedge \zeta_{i,k}} X_r(\infty)] ds. \end{aligned}$$

By Gronwall's inequality we have  $\mathbb{E}[\sup_{0 \leq s \leq t \wedge \zeta_{i,k}} X_s(\infty)] \leq \mathbb{E}[X_0(\infty)]e^{\beta t}$ . Then, letting  $i \rightarrow \infty$  and using Fatou's lemma, we obtain (3.5). □

**Proposition 3.1.** *Suppose that  $\mathbb{E}[X_0(\infty)] < \infty$ . Then  $\mathbb{P}\{\tau = \infty\} = 1$  and*

$$\mathbb{E}[n(t)] \leq \|\alpha\| \mathbb{E}[X_0(\infty)] \int_0^t e^{\beta s} ds, \quad t \geq 0. \quad (3.6)$$

*Proof.* By (3.2) and monotone convergence we have

$$\begin{aligned} \mathbb{E}[n(t)] &= \lim_{k \rightarrow \infty} \mathbb{E} \left[ \int_0^{t \wedge \tau_k} \int_0^{\langle X_{s-}, \alpha \rangle} \int_0^1 \int_{\mathbb{N}} \int_0^{p(A_\alpha(X_{s-}, y), z)} M(ds, du, dy, dz, dv) \right] \\ &= \lim_{k \rightarrow \infty} \mathbb{E} \left[ \sum_{z \in \mathbb{N}} \int_0^{t \wedge \tau_k} \langle X_{s-}, \alpha \rangle ds \int_0^1 p(A_\alpha(X_{s-}, y), z) dy \right] \\ &= \lim_{k \rightarrow \infty} \mathbb{E} \left[ \sum_{z \in \mathbb{N}} \int_0^{t \wedge \tau_k} ds \int_{\mathbb{R}_+} p(y, z) \alpha(y) dX_{s-}(y) \right] \\ &\leq \lim_{k \rightarrow \infty} \|\alpha\| \mathbb{E} \left[ \int_0^{t \wedge \tau_k} X_s(\infty) ds \right] = \lim_{k \rightarrow \infty} \|\alpha\| \int_0^t \mathbb{E}[X_{s \wedge \tau_k}(\infty)] ds. \end{aligned}$$

Then (3.6) follows by (3.5). In particular, we have  $\mathbb{P}\{\tau > t\} = \mathbb{P}\{n(t) < \infty\} = 1$  for every  $t \geq 0$ , which implies  $\mathbb{P}\{\tau = \infty\} = 1$ .  $\square$

**Proposition 3.2.** *Suppose  $\mathbb{E}[X_0(\infty)] < \infty$ . Then  $\mathbb{E}[\sup_{0 \leq s \leq t} X_s(\infty)] \leq \mathbb{E}[X_0(\infty)]e^{\beta t}$ ,  $t \geq 0$ .*

*Proof.* Since  $\mathbb{P}\{\tau = \infty\} = 1$  by Proposition 3.1, we obtain the result from (3.5) by using monotone convergence.  $\square$

By Proposition 3.1 the solution of (3.1) or (3.2) has infinite lifetime and determines a measure-valued strong Markov process  $\{X_t : t \geq 0\}$ . The following propositions give some useful characterization of the process.

**Proposition 3.3.** *For any  $t \geq 0$  and  $f \in B(\mathbb{R}_+)$ ,*

$$\begin{aligned} \langle X_t, f \rangle &= \langle X_0, f \circ \theta_t \rangle + \int_0^t \int_0^{\langle X_{s-}, \alpha \rangle} \int_0^1 \int_{\mathbb{N}} \int_0^{p(A_\alpha(X_{s-}, y), z)} [zf \circ \theta_{t-s}(0) \\ &\quad - f \circ \theta_{t-s}(A_\alpha(X_{s-}, y))] M(ds, du, dy, dz, dv). \end{aligned} \quad (3.7)$$

*Proof.* Let  $C_0^1(\mathbb{R}_+)$  denote the subspace of  $C^1(\mathbb{R}_+)$  consisting of functions vanishing at infinity. For any fixed integer  $n \geq 1$ , let  $x_i = in/2^n$  with  $i = 0, 1, \dots, 2^n$ . By (3.2), almost surely for any  $f \in C_0^1(\mathbb{R}_+)$ ,

$$\begin{aligned} \sum_{i=1}^{2^n} f'(x_i) X_t(x_i) &- \sum_{i=1}^{2^n} f'(x_i) X_0 \circ \theta_{-t}(x_i) \\ &= \int_0^t \int_0^{\langle X_{s-}, \alpha \rangle} \int_0^1 \int_{\mathbb{N}} \int_0^{p(A_\alpha(X_{s-}, y), z)} \sum_{i=1}^{2^n} f'(x_i) [z \zeta_0 \circ \theta_{s-t}(x_i) \\ &\quad - \zeta_{A_\alpha(X_{s-}, y)} \circ \theta_{s-t}(x_i)] M(ds, du, dy, dz, dv). \end{aligned}$$

Then we multiply the above equation by  $2^{-n}$  and let  $n \rightarrow \infty$  to get, almost surely,

$$\begin{aligned} \int_0^\infty f'(x)X_t(x) \, dx - \int_0^\infty f'(x)X_0 \circ \theta_{-t}(x) \, dx &= \int_0^t \int_0^{(X_{s-}, \alpha)} \int_0^1 \int_{\mathbb{N}} \int_0^{p(A_\alpha(X_{s-}, y), z)} \left\{ \int_0^\infty f'(x) [z\zeta_0 \circ \theta_{s-t}(x) \right. \\ &\quad \left. - \zeta_{A_\alpha(X_{s-}, y)} \circ \theta_{s-t}(x)] \, dx \right\} M(ds, du, dy, dz, dv) \\ &= - \int_0^t \int_0^{(X_{s-}, \alpha)} \int_0^1 \int_{\mathbb{N}} \int_0^{p(A_\alpha(X_{s-}, y), z)} [zf \circ \theta_{t-s}(0) \\ &\quad - f \circ \theta_{t-s}(A_\alpha(X_{s-}, y))] M(ds, du, dy, dz, dv). \end{aligned} \tag{3.8}$$

By integration by parts we have  $\langle X_t, f \rangle = - \int_0^\infty f'(x)X_t(x) \, dx$ . From this and (3.8) we see that (3.7) holds for any  $f \in C_0^1(\mathbb{R}_+)$ . Then the relation also holds for any  $f \in B(\mathbb{R}_+)$  by a monotone class argument.  $\square$

**Proposition 3.4.** For any  $t \geq 0$  and  $f \in C^1(\mathbb{R}_+)$ ,

$$\begin{aligned} \langle X_t, f \rangle &= \langle X_0, f \rangle + \int_0^t \langle X_{s-}, f' \rangle \, ds + \int_0^t \int_0^{(X_{s-}, \alpha)} \int_0^1 \int_{\mathbb{N}} \int_0^{p(A_\alpha(X_{s-}, y), z)} [zf(0) \\ &\quad - f(A_\alpha(X_{s-}, y))] M(ds, du, dy, dz, dv). \end{aligned} \tag{3.9}$$

*Proof.* For  $n \geq 1$  we consider a partition  $\Delta_n = \{0 = t_0 < t_1 < \dots < t_n = t\}$  of  $[0, t]$ . Notice that  $\partial_t(f \circ \theta_t)(x) = f'(x + t)$ . By (3.7) we have

$$\begin{aligned} \langle X_t, f \rangle &= \langle X_0, f \rangle + \sum_{i=1}^n [\langle X_{t_i}, f \rangle - \langle X_{t_{i-1}}, f \circ \theta_{t_i-t_{i-1}} \rangle] \\ &\quad + \sum_{i=1}^n [\langle X_{t_{i-1}}, f \circ \theta_{t_i-t_{i-1}} \rangle - \langle X_{t_{i-1}}, f \rangle] \\ &= \langle X_0, f \rangle + \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \int_0^{(X_{s-}, \alpha)} \int_0^1 \int_{\mathbb{N}} \int_0^{p(A_\alpha(X_{s-}, y), z)} [zf \circ \theta_{t_i-s}(0) \\ &\quad - f \circ \theta_{t_i-s}(A_\alpha(X_{s-}, y))] M(ds, du, dy, dz, dv) \\ &\quad + \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \langle X_{t_{i-1}}, f' \circ \theta_{s-t_{i-1}} \rangle \, ds. \end{aligned}$$

By letting  $|\Delta_n| := \max_{1 \leq i \leq n} (t_i - t_{i-1}) \rightarrow 0$  and using the right continuity of  $s \rightarrow X_s$  and the continuity of  $s \rightarrow f \circ \theta_s$ , we obtain (3.9).  $\square$

**Proposition 3.5.** For  $f, G \in C^1(\mathbb{R}_+)$  let  $G_f(\mu) = G(\langle \mu, f \rangle)$ , and let

$$\begin{aligned} \mathcal{L}_0 G_f(\mu) &= \langle \mu, f' \rangle G'(\langle \mu, f \rangle) \\ &\quad - \sum_{z \in \mathbb{N}} \int_{\mathbb{R}_+} \alpha(y)p(y, z) [G(\langle \mu, f \rangle) - G(\langle \mu, f \rangle + zf(0) - f(y))] \mu(dy). \end{aligned}$$



Then we have

$$G_f(X_t) = G_f(X_0) + \int_0^t \mathcal{L}_0 G_f(X_s) \, ds + mart. \tag{3.10}$$

*Proof.* Let  $\tilde{M}$  denote the compensated measure of  $M$ . Since the process  $s \mapsto X_s$  has at most countably many jumps, by Proposition 3.4 and Itô's formula we have

$$\begin{aligned} G(\langle X_t, f \rangle) &= G(\langle X_0, f \rangle) + \int_0^t G'(\langle X_s, f \rangle) \langle X_s, f' \rangle \, ds \\ &\quad - \int_0^t \int_0^{\langle X_{s-}, \alpha \rangle} \int_0^1 \int_{\mathbb{N}} \int_0^{p(A_\alpha(X_{s-}, y), z)} [G(\langle X_{s-}, f \rangle) \\ &\quad - G(\langle X_{s-}, f \rangle + zf(0) - f(A_\alpha(X_{s-}, y)))] M(ds, du, dy, dz, dv) \\ &= G(\langle X_0, f \rangle) + \int_0^t G'(\langle X_s, f \rangle) \langle X_s, f' \rangle \, ds - M_t^G(f) \\ &\quad - \int_0^t ds \int_{\mathbb{R}_+} \alpha(y) \sum_{z \in \mathbb{N}} p(y, z) [G(\langle X_{s-}, f \rangle) \\ &\quad - G(\langle X_{s-}, f \rangle + zf(0) - f(y))] X_{s-}(dy) \\ &= G(\langle X_0, f \rangle) + \int_0^t \mathcal{L}_0 G_f(X_s) \, ds - M_t^G(f), \end{aligned}$$

where

$$\begin{aligned} M_t^G(f) &= \int_0^t \int_0^{\langle X_{s-}, \alpha \rangle} \int_0^1 \int_{\mathbb{N}} \int_0^{p(A_\alpha(X_{s-}, y), z)} [G(\langle X_{s-}, f \rangle) \\ &\quad - G(\langle X_{s-}, f \rangle + zf(0) - f(A_\alpha(X_{s-}, y)))] \tilde{M}(ds, du, dy, dz, dv). \end{aligned}$$

By Proposition 3.2 we can check that  $\{M_t^G(f) : t \geq 0\}$  is a martingale. □

**Theorem 3.1.** *The measure-valued process  $\{X_t : t \geq 0\}$  defined by the stochastic equation (3.1) or (3.2) is an  $(\alpha, g)$ -age-structured branching process.*

*Proof.* Recall that  $\{X_t : t \geq 0\}$  is a càdlàg process. Let  $e_f(\mu) = e^{-\langle \mu, f \rangle}$  and let  $\mathcal{L}_0$  be defined as in Proposition 3.5. It is elementary to see that

$$\begin{aligned} \mathcal{L}_0 e_f(\mu) &= -\langle \mu, f' \rangle e^{-\langle \mu, f \rangle} - e^{-\langle \mu, f \rangle} \int_{\mathbb{R}_+} \alpha(y) \left[ 1 - e^{f(y)} \sum_{z \in \mathbb{N}} p(y, z) e^{-zf(0)} \right] \mu(dy) \\ &= -e^{-\langle \mu, f \rangle} \langle \mu, f' \rangle - e^{-\langle \mu, f \rangle} \int_{\mathbb{R}_+} \alpha(y) (1 - e^{f(y)} g(y, e^{-f(0)})) \mu(dy). \end{aligned}$$

Let  $\mathcal{F}_t = \sigma\{X_s : 0 \leq s \leq t\}$ . By (2.4), (3.10), and the mean-value theorem, we have

$$\begin{aligned} e^{-\langle X_t, u_T - if \rangle} &= e^{-\langle X_0, u_T f \rangle} + \sum_{i=0}^{\infty} \left[ e^{-\langle X_{t \wedge (i+1)/k}, u_{T-t \wedge i/k} f \rangle} - e^{-\langle X_{t \wedge i/k}, u_{T-t \wedge i/k} f \rangle} \right] \\ &\quad + \sum_{i=0}^{\infty} \left[ e^{-\langle X_{t \wedge (i+1)/k}, u_{T-t \wedge (i+1)/k} f \rangle} - e^{-\langle X_{t \wedge (i+1)/k}, u_{T-t \wedge i/k} f \rangle} \right] \end{aligned}$$

$$\begin{aligned}
 &= e^{-\langle X_0, u_T f \rangle} - \sum_{i=0}^{\infty} \int_{t \wedge i/k}^{t \wedge (i+1)/k} e^{-\langle X_s, u_{T-t \wedge i/k} f \rangle} \left[ \langle X_s, \partial_x u_{T-t \wedge i/k} f \rangle \right. \\
 &\quad \left. + \int_{\mathbb{R}_+} \alpha(y) [1 - e^{u_{T-t \wedge i/k} f(y)} g(y, e^{-u_{T-t \wedge i/k} f(0)})] X_s(dy) \right] ds \\
 &\quad + M_k(t) + \sum_{i=0}^{\infty} e^{-\xi_k(t)} \langle X_{t \wedge (i+1)/k}, u_{T-t \wedge i/k} f - u_{T-t \wedge (i+1)/k} f \rangle \\
 &= e^{-\langle X_0, u_T f \rangle} - \sum_{i=0}^{\infty} \int_{t \wedge i/k}^{t \wedge (i+1)/k} e^{-\langle X_s, u_{T-t \wedge i/k} f \rangle} \left[ \langle X_s, \partial_x u_{T-t \wedge i/k} f \rangle \right. \\
 &\quad \left. + \int_{\mathbb{R}_+} \alpha(y) [1 - e^{u_{T-t \wedge i/k} f(y)} g(y, e^{-u_{T-t \wedge i/k} f(0)})] X_s(dy) \right] ds \\
 &\quad + M_k(t) + \sum_{i=0}^{\infty} \int_{t \wedge i/k}^{t \wedge (i+1)/k} e^{-\xi_k(t)} \left[ \langle X_{t \wedge (i+1)/k}, \partial_x u_{T-s} f \rangle \right. \\
 &\quad \left. + \int_{\mathbb{R}_+} \alpha(y) [1 - e^{u_{T-s} f(y)} g(y, e^{-u_{T-s} f(0)})] X_{t \wedge (i+1)/k}(dy) \right] ds,
 \end{aligned}$$

where  $t \mapsto M_k(t)$  is an  $(\mathcal{F}_t)$ -martingale and

$$\langle X_{t \wedge (i+1)/k}, u_{T-t \wedge i/k} f \rangle \leq \xi_k(t) \leq \langle X_{t \wedge (i+1)/k}, u_{T-t \wedge (i+1)/k} f \rangle$$

or

$$\langle X_{t \wedge (i+1)/k}, u_{T-t \wedge (i+1)/k} f \rangle \leq \xi_k(t) \leq \langle X_{t \wedge (i+1)/k}, u_{T-t \wedge i/k} f \rangle.$$

By letting  $k \rightarrow \infty$  we see that  $t \mapsto e^{-\langle X_t, u_{T-t} f \rangle}$  is an  $(\mathcal{F}_t)$ -martingale. In particular, we have  $\mathbb{E}[e^{-\langle X_T, f \rangle} | \mathcal{F}_t] = e^{-\langle X_t, u_{T-t} f \rangle}$ ,  $T \geq t \geq 0$ . Then  $\{X_t : t \geq 0\}$  is a Markov process with transition semigroup  $(Q_t)_{t \geq 0}$  given by (2.3). □

A calculation of the generator for an age-structured birth–death process was given in [4, (3.1)]; see also [5]. A stochastic equation for a similar model was proposed in [28, (2.5)], which assumed that the death rate of an individual may depend on the whole population and also calculated the generator of the model.

#### 4. The branching process with immigration

In this section we introduce an age-structured branching process with immigration and discuss its ergodicity. Let  $\psi$  be a functional on  $B(\mathbb{R}_+)^+$  given by  $\psi(f) = \int_{\mathfrak{N}(\mathbb{R}_+)} (1 - e^{-(v, f)}) L(dv)$ ,  $f \in B(\mathbb{R}_+)^+$ , where  $L(dv)$  is a finite measure on  $\mathfrak{N}(\mathbb{R}_+)^\circ := \mathfrak{N}(\mathbb{R}_+) \setminus 0$ , and 0 denotes the null measure.

A Markov process  $Y = (Y_t : t \geq 0)$  with state space  $\mathfrak{N}(\mathbb{R}_+)$  is called an  $(\alpha, g, \psi)$ -age-structured branching process with immigration if it has the transition semigroup  $(P_t)_{t \geq 0}$  defined by

$$\int_{\mathfrak{N}(\mathbb{R}_+)} e^{-(v, f)} P_t(\sigma, dv) = \exp \left\{ -\langle \sigma, u_t f \rangle - \int_0^t \psi(u_s f) ds \right\}, \tag{4.1}$$

where  $u_t f(x)$  is the unique solution to (2.2). Such a process is characterized by the properties (i)–(iii) given in Section 2, along with the following:

- (iv) The immigrants come according to a Poisson random measure on  $(0, \infty) \times \mathfrak{N}(\mathbb{R}_+)^{\circ}$  with intensity  $dsL(dv)$ .

Suppose that  $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$  is a filtered probability space satisfying the usual hypotheses. Let  $M(dt, du, dy, dz, dv)$  be a Poisson random measure as in Section 3, and let  $N(dt, dv)$  be an  $(\mathcal{F}_t)$ -Poisson random measure on  $(0, \infty) \times \mathfrak{N}(\mathbb{R}_+)^{\circ}$  with intensity  $dtL(dv)$ . We assume that the two random measures are independent of each other. Consider the stochastic integral equation

$$\begin{aligned}
 Y_t(x) = & Y_0 \circ \theta_{-t}(x) + \int_0^t \int_0^{\langle Y_{s-}, \alpha \rangle} \int_0^1 \int_{\mathbb{N}} \int_0^{p(A_\alpha(Y_{s-}, y), z)} [z \zeta_0 \\
 & - \zeta_{A_\alpha(Y_{s-}, y)}] \circ \theta_{s-t}(x) M(ds, du, dy, dz, dv) \\
 & + \int_0^t \int_{\mathfrak{N}(\mathbb{R}_+)^{\circ}} \nu \circ \theta_{s-t}(x) N(ds, dv). \tag{4.2}
 \end{aligned}$$

Here, the first two terms on the right-hand side are as explained for (3.1) and the last term represents the immigration. A pathwise-unique solution to (4.2) is constructed as follows. Let  $\sigma_0 = 0$ . Given  $\sigma_{k-1}$ , we define  $\sigma_k = \sigma_{k-1} + \inf \{t > 0 : N((\sigma_{k-1}, \sigma_{k-1} + t] \times N(\mathbb{R}_+^{\circ})) > 0\}$  and  $Y_t(x) = X_{k,t-\sigma_{k-1}}(x)$ ,  $\sigma_{k-1} \leq t < \sigma_k$ , where  $\{X_{k,t}(x) : t \geq 0\}$  is the pathwise-unique solution to the following equation:

$$\begin{aligned}
 X_t(x) = & Y_{\sigma_{k-1}} \circ \theta_{-t}(x) + \int_0^t \int_0^{\langle X_{s-}, \alpha \rangle} \int_0^1 \int_{\mathbb{N}} \int_0^{p(A_\alpha(X_{s-}, y), z)} [z \zeta_0 \circ \theta_{s-t}(x) \\
 & - \zeta_{A_\alpha(X_{s-}, y)}] \circ \theta_{s-t}(x) M(\sigma_{k-1} + ds, du, dy, dz, dv).
 \end{aligned}$$

Then we define  $Y_{\sigma_k}(x) = Y_{\sigma_{k-1}}(x) + \int_{\{\sigma_k\}} \int_{\mathfrak{N}(\mathbb{R}_+)^{\circ}} \nu(x) N(ds, dv)$ . It is easy to see that  $\lim_{k \rightarrow \infty} \sigma_k = \infty$  and  $\{Y_t : t \geq 0\}$  is the pathwise-unique solution to (4.2).

The solution to (4.2) determines a measure-valued strong Markov process  $\{Y_t : t \geq 0\}$  with state space  $\mathfrak{N}(\mathbb{R}_+)$ . We omit the proofs of some of the following results since the arguments are similar to those for the corresponding results in Section 3.

**Proposition 4.1.** For any  $f \in B(\mathbb{R}_+)^+$ ,

$$\begin{aligned}
 \langle Y_t, f \rangle = & \langle Y_0, f \circ \theta_t \rangle + \int_0^t \int_0^{\langle Y_{s-}, \alpha \rangle} \int_{[0,1]} \int_{\mathbb{N}} \int_0^{p(A_\alpha(Y_{s-}, y), z)} [zf \circ \theta_{t-s}(0) \\
 & - f \circ \theta_{t-s}(A_\alpha(Y_{s-}, y))] M(ds, du, dy, dz, dv) \\
 & + \int_0^t \int_{\mathfrak{N}(\mathbb{R}_+)^{\circ}} \langle \nu, f \circ \theta_{t-s} \rangle N(ds, dv).
 \end{aligned}$$

**Proposition 4.2.** For any  $t \geq 0$  and  $f \in C^1(\mathbb{R}_+)^+$ ,

$$\begin{aligned}
 \langle Y_t, f \rangle = & \langle Y_0, f \rangle + \int_0^t \langle Y_{s-}, f' \rangle ds + \int_0^t \int_0^{\langle Y_{s-}, \alpha \rangle} \int_{[0,1]} \int_{\mathbb{N}} \int_0^{p(A_\alpha(Y_{s-}, y), z)} [zf(0) \\
 & - f(A_\alpha(Y_{s-}, y))] M(ds, du, dy, dz, dv) \\
 & + \int_0^t \int_{\mathfrak{N}(\mathbb{R}_+)^{\circ}} \langle \nu, f \rangle N(ds, dv). \tag{4.3}
 \end{aligned}$$

**Proposition 4.3.** For any  $f, G \in C^1(\mathbb{R}_+)$ , let  $G_f(\mu) = G(\langle \mu, f \rangle)$  and

$$\begin{aligned} \mathcal{L}G_f(\mu) &= \langle \mu, f' \rangle G'(\langle \mu, f \rangle) - \int_{\mathbb{R}_+} \alpha(y) \sum_{z \in \mathbb{N}} p(y, z) [G(\langle \mu, f \rangle) \\ &\quad - G(\langle \mu, f \rangle + zf(0) - f(y))] \mu(dy) \\ &\quad + \int_{\mathfrak{N}(\mathbb{R}_+)} \circ [G(\langle \mu, f \rangle + \langle \nu, f \rangle) - G(\langle \mu, f \rangle)] L(d\nu). \end{aligned}$$

Then  $G(\langle Y_t, f \rangle) = G(\langle Y_0, f \rangle) + \int_0^t \mathcal{L}G_f(Y_s) ds + mart.$

*Proof.* Let  $\tilde{M}$  and  $\tilde{N}$  denote the compensated measure of  $M$  and  $N$ , respectively. By (4.3) and Itô's formula we have

$$\begin{aligned} G(\langle Y_t, f \rangle) &= G(\langle Y_0, f \rangle) + \int_0^t G'(\langle Y_s, f \rangle) \langle Y_s, f' \rangle ds \\ &\quad + \int_0^t \int_0^{\langle Y_{s-}, \alpha \rangle} \int_0^1 \int_{\mathbb{N}} \int_0^{p(A_\alpha(Y_{s-}, y), z)} [G(\langle Y_{s-}, f \rangle + zf(0) \\ &\quad - f(A_\alpha(Y_{s-}, y))) - G(\langle Y_{s-}, f \rangle)] M(ds, du, dy, dz, d\nu) \\ &\quad + \int_0^t \int_{\mathfrak{N}(\mathbb{R}_+)} \circ [G(\langle Y_{s-}, f \rangle + \langle \nu, f \rangle) - G(\langle Y_{s-}, f \rangle)] N(ds, d\nu) \\ &= G(\langle Y_0, f \rangle) + \int_0^t G'(\langle Y_s, f \rangle) \langle Y_s, f' \rangle ds + N_t^G(f) \\ &\quad + \int_0^t ds \int_{\mathbb{R}_+} \alpha(y) \sum_{z \in \mathbb{N}} p(y, z) [G(\langle Y_{s-}, f \rangle + zf(0) - f(y)) \\ &\quad - G(\langle Y_{s-}, f \rangle)] Y_{s-}(dy) \\ &\quad + \int_0^t ds \int_{\mathfrak{N}(\mathbb{R}_+)} \circ [G(\langle Y_{s-}, f \rangle + \langle \nu, f \rangle) - G(\langle Y_{s-}, f \rangle)] L(d\nu), \end{aligned}$$

where

$$\begin{aligned} N_t^G(f) &= \int_0^t \int_0^{\langle Y_{s-}, \alpha \rangle} \int_0^1 \int_{\mathbb{N}} \int_0^{p(A_\alpha(Y_{s-}, y), z)} [G(\langle Y_{s-}, f \rangle + zf(0) - f(A_\alpha(Y_{s-}, y))) \\ &\quad - G(\langle Y_{s-}, f \rangle)] \tilde{M}(ds, du, dy, dz, d\nu) \\ &\quad + \int_0^t \int_{\mathfrak{N}(\mathbb{R}_+)} \circ [G(\langle Y_{s-}, f \rangle + \langle \nu, f \rangle) - G(\langle Y_{s-}, f \rangle)] \tilde{N}(ds, d\nu). \end{aligned}$$

A first moment estimate can check that  $\{N_t^G(f) : t \geq 0\}$  is a martingale. □

**Theorem 4.1.** The measure-valued process  $\{Y_t : t \geq 0\}$  defined by (4.2) is an  $(\alpha, g, \psi)$ -age-structured branching process with immigration.

*Proof.* Let  $\mathcal{F}_t = \sigma\{Y_s : 0 \leq s \leq t\}$ . By a modification of the proof of Theorem 3.1, we can see that  $t \mapsto \exp\left\{-\langle Y_t, u_{T-t} f \rangle - \int_0^{T-t} \psi(u_s f) ds\right\}$  is an  $(\mathcal{F}_t)$ -martingale. Then  $\{Y_t : t \geq 0\}$  is a Markov process with transition semigroup  $(P_t)_{t \geq 0}$  defined by (4.1), which completes the proof. □

A necessary and sufficient condition for the ergodicity of the  $(\alpha, g, \psi)$ -age-structured branching process with immigration is given in the next theorem.

**Theorem 4.2.** *Suppose that  $c_* = \inf_{y \geq 0} \alpha(y)[1 - \partial_z g(y, 1-)] > 0$ . Then  $P_t(\sigma, \cdot)$  converges as  $t \rightarrow \infty$  to a probability measure  $\eta$  on  $\mathfrak{N}(\mathbb{R}_+)$  for every  $\sigma \in \mathfrak{N}(\mathbb{R}_+)$  if and only if*

$$\int_{\mathfrak{N}(\mathbb{R}_+)} \mathbf{1}_{\{(v,1) \geq 1\}} \log \langle v, 1 \rangle L(dv) < \infty. \tag{4.4}$$

In this case, the Laplace transform of  $\eta$  is given by

$$\int_{\mathfrak{N}(\mathbb{R}_+)} e^{-(v,f)} \eta(dv) = \exp \left\{ - \int_0^\infty \psi(u_s f) ds \right\}. \tag{4.5}$$

*Proof.* First, for  $f \in B(\mathbb{R}_+)^+$  let  $f_* = \inf_{x \geq 0} f(x)$ . From Propositions 2.3 and 2.4 we have

$$(1 - e^{-f_*})e^{-c_1 t} \leq u_t f(x) \leq \|u_t f\| \leq \|\pi_t f\| \leq e^{-c_* t} \|f\|, \quad x \geq 0. \tag{4.6}$$

Moreover, for  $a, c > 0$  it is elementary to see that

$$\begin{aligned} \int_0^\infty ds \int_{\mathfrak{N}(\mathbb{R}_+)} (1 - e^{-ae^{-cs}(v,1)}) L(dv) &= \int_{\mathfrak{N}(\mathbb{R}_+)} L(dv) \int_0^\infty (1 - e^{-ae^{-cs}(v,1)}) ds \\ &= c^{-1} \int_{\mathfrak{N}(\mathbb{R}_+)} L(dv) \int_0^{a(v,1)} (1 - e^{-z}) z^{-1} dz. \end{aligned} \tag{4.7}$$

Second, suppose that (4.4) holds. For any  $f \in B(\mathbb{R}_+)^+$  we see from (4.6) that  $\sigma(u_t f) \rightarrow 0$  as  $t \rightarrow \infty$ . Take any  $a > \|f\|$ . By (4.6) and (4.7) we have

$$\begin{aligned} \int_0^\infty \psi(u_s f) ds &= \int_0^\infty ds \int_{\mathfrak{N}(\mathbb{R}_+)} (1 - e^{-(v, u_s f)}) L(dv) \\ &\leq \int_0^\infty ds \int_{\mathfrak{N}(\mathbb{R}_+)} (1 - e^{-ae^{-c_* s}(v,1)}) L(dv) \\ &\leq c_*^{-1} \int_{\mathfrak{N}(\mathbb{R}_+)} \left[ 1 + \int_1^{1 \vee (a(v,1))} z^{-1} dz \right] L(dv) \\ &\leq c_*^{-1} \int_{\mathfrak{N}(\mathbb{R}_+)} [1 + 0 \vee \log(a(v,1))] L(dv) < \infty. \end{aligned}$$

Then, as  $t \rightarrow \infty$ ,

$$\begin{aligned} \int_0^t \psi(u_s f) ds &= \int_0^t ds \int_{\mathfrak{N}(\mathbb{R}_+)} (1 - e^{-(v, u_s f)}) L(dv) \\ &\rightarrow \int_0^\infty ds \int_{\mathfrak{N}(\mathbb{R}_+)} (1 - e^{-(v, u_s f)}) L(dv) = \int_0^\infty \psi(u_s f) ds < \infty. \end{aligned}$$

From (4.1) we infer that  $P_t(\sigma, \cdot)$  converges as  $t \rightarrow \infty$  to a probability measure  $\eta$  defined by (4.5); see, e.g., [21, Theorem 1.20].

Third, suppose that  $P_t(\sigma, \cdot)$  converges as  $t \rightarrow \infty$  to a probability measure  $\eta$  on  $\mathfrak{N}(\mathbb{R}_+)$  for every  $\sigma \in \mathfrak{N}(\mathbb{R}_+)$ . By (4.1) we see that  $\eta$  has a Laplace functional given by (4.5); see, e.g., [21, Theorems 1.17 and 1.18]. In particular, we have  $\int_0^\infty \psi(u_s f) ds < \infty$  for every  $f \in B(\mathbb{R}_+)^+$ . Let  $a = (1 - e^{-1})$ . By (4.6) and (4.7) we have

$$\begin{aligned} \int_0^\infty \psi(u_s 1) ds &= \int_0^\infty ds \int_{\mathfrak{N}(\mathbb{R}_+)} (1 - e^{-(v, u_s 1)}) L(dv) \\ &\geq \int_0^\infty ds \int_{\mathfrak{N}(\mathbb{R}_+)} (1 - e^{-ae^{-c_1 s}(v, 1)}) L(dv) \\ &\geq ac_1^{-1} \int_{\{(v, 1) \geq 1\}} L(dv) \int_1^{a(v, 1)} z^{-1} dz \\ &\geq ac_1^{-1} \int_{\{(v, 1) \geq 1\}} [\log a + \log(v, 1)] L(dv). \end{aligned}$$

This gives us (4.4). □

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There were no competing interests to declare which arose during the preparation or publication process of this article.

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