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COMPARISON OF QUEUES WITH DIFFERENT DISCRETE-TIME ARRIVAL PROCESSES

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Traveling times in a FIFO-stochastic event graph are compared in increasing convex ordering for different arrival processes. As a special case, a stochastic lower bound is obtained for the sojourn time in a tandem network of FIFO queues with a Markov arrival process. A counterexample shows that the extended Ross conjecture is not true for discrete-time arrival processes.

1. INTRODUCTION

In this article, we consider an open stochastic queueing network with one input node. The network dynamics are supposed to satisfy a linear recursion in the so-called (max, +)-algebra on $\mathbb{R}^L$ (see [9]). It is well known that the epoch of the beginning of the nth firing time of a FIFO-stochastic event graph (FSEG) satisfies such a linear recursion for each transition (see [8]).

A special case is a stochastic network of L single-server FIFO queues in tandem, with infinite buffer capacity in the first queue and finite buffers with manufactoring blocking or infinite buffers in the other queues. Readers not familiar with (max, +)-linear systems might read the article with this specific model in mind; the firing time then is the service time and transitions $1, \dots, L$ of the FSEG become the servers $1, \dots, L$. Note that the sojourn time in this tandem network of L single-server FIFO queues is the traveling time to server L plus its service time at L. Hence, the comparison results below hold also for the sojourn time.

Let T_n, $n = 1, 2, \dots$, be a stationary sequence of potential arrival epochs. The number of arrivals at T_n will be denoted by A_n. In general A_n, $n = 1, 2, \dots$, may be a

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stochastic sequence, and $A_n = l$ means that l customers arrive at T_n. Note that $A_n = 0$ implies that T_n is not an actual arrival epoch.

Two arrival processes are compared with respect to their implied performance of the stochastic network. We assume that both have the same potential arrival epochs but different A sequences, say A_n^1 and A_n^2. Let us call these the admission sequences. Let S_n^j be the firing time (service time) of the nth token (customer) in transition (server) j. We assume that

$$S_n = (S_n^1, \ldots, S_n^L)$$

is a stationary sequence of stochastic vectors. Note that no independence assumption is made on the firing times; stationarity is sufficient. However, we assume that for $i = 1$ and for $i = 2$, every couple of two sequences from $\{A_n^i, T_n, S_n\}$ are stochastically independent.

Let ${}_bW_n^q$ denote the traveling time of the nth arrival to transition q (i.e., the time between its entrance in the stochastic network and the beginning of its firing time at transition q). Let ${}_aW_n^q$ be the same time of a potential arrival at T_n. Recall that at T_n, there may be no arrival, and the arrival time of the nth customer is, in general, not T_n. With Z_n^i, we denote the nth arrival epoch for arrival process i, $i = 1,2$, that is,

$$Z_n^i = \min\left\{k : \sum_{l=1}^{k} A_l^i \geq n\right\}.$$

Then, the arrival time of the nth customer is $T_{Z_n^i}$.

We also need the following notation:

$$B_n^i \triangleq Z_n^i - Z_{n-1}^i, \quad i = 1,2,\ n = 1,2,\ldots,$$

where we take $Z_0^i = 0$. Note that B_n^i is the nth interarrival length: in general, this is not equal to the interarrival time. Of course, B_n^i is a function of $A_1^i, A_2^i, \ldots, A_{Z_n^i}^i$; we suppress this in our notation. It is shown in [1] for a (max,+)-linear system that for any transition q, and $n = 1,2,\ldots,$

$$\mathbb{E}\ {}_bW_n^q \text{ is a multimodular function of } (B_1, \ldots, B_n)$$

$$\mathbb{E}\ {}_aW_n^q \text{ is a multimodular function of } (A_1, \ldots, A_n),$$

where the expectation is with respect to T_n and S_n, $n \in \mathbb{N}$. These multimodularity properties induce the convexity results which we use to prove our comparison results. The arrival processes in this article will be generated by a Markov arrival process (MAP), for which we assume a Markov process on E, a finite state space with intensities λ_{xy}, $x, y \in E$, and an arrival occurs with probability r_{xy} when a transition from state x to state y happens.

In [14], it is explained that a MAP is more general than the Markov-modulated Poisson process or the phase-type renewal process. In [5], it is shown that any arrival process can be approximated arbitrarily close by a MAP.

Let us mention the stochastic orders we use in this article. Random vectors $X^1 = (X_1^1, \ldots, X_n^1)$ and $X^2 = (X_1^2, \ldots, X_n^2)$ are ordered with respect to the convex ordering ($X^1 \leq_{\text{cx}} X^2$) [resp. increasing convex ordering ($X^1 \leq_{\text{icx}} X^2$)] if

$$\mathbb{E}h(X^1) \leq \mathbb{E}h(X^2)$$

for all convex [resp. increasing convex] functions

$$h: \mathbb{R}^n \to \mathbb{R}.$$

In this article, increasing and decreasing are always in the nonstrict sense.

In Section 2, we give a counterexample which shows that the extension of the Ross conjecture is not true in our comparison of queues with different admission sequences. In Section 3, a first comparison lemma is derived for admission (interarrival) sequences which are comparable in the convex ordering. It is shown that the potential (actual) traveling times are ordered in the increasing convex ordering. Similar comparison results hold for the stationary traveling times. As applications of the lemma, we derive the following results:

1. Independent sources have a better performance (in increasing convex ordering sense) than coupled sources.
2. Fixed batch sizes are better than random batch sizes.
3. Fluid scaling improves the performance.

In Comparison Lemma 2, derived in Section 4, actual (potential) traveling times are ordered in the increasing convex (icx) ordering for noninteger admission (interarrival) sequences. Here, a regularization procedure is given, which has been used in the theory on balanced sequences and optimal routing (cf. [3]).

In Section 5, we construct the most regular arrival process for a fixed arrival intensity, and we call it the regular arrival process (RAP). We show that the RAP provides a stochastic lower bound for any MAP source with the same arrival intensity. This result (Theorem 1) can be seen as the Ross conjecture theorem in the comparison of discrete-time arrival processes.

In the literature on optimal routing to parallel queues, it was claimed (cf. [12]) that good approximations could be obtained through replacing the MAP by a renewal process with approximately the same arrival intensity. Theorem 2 and Corollary 3 provide the proof of these claims. Indeed, the performance of a RAP has stochastic lower bounds for arrival processes which are (approximately) renewal processes. For a rational arrival intensity, there is a RAP which is renewal with Erlang-distributed interarrival times. By a continuity argument, we obtain that the renewal-arrival process with constant interarrival times gives, for any real stationary arrival intensity, a stochastic lower bound on the performance.

2. ON THE ROSS CONJECTURE IN DISCRETE TIME

In his inspiring article, Ross [17] conjectured that the mean waiting time in a $\cdot/G/1/\infty$ queue with a nonstationary Poisson arrival process is larger than or equal

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to the mean waiting time of the $M/G/1/\infty$ queue with the same arrival intensity. This article initiated a long sequence of research papers on this and related problems. Rolski proved the Ross conjecture in [16]. Recent publications on this and related topics are [4], [6], and [10].

Suppose a $\cdot/G/1/\infty$ queue with potential arrival epochs given by sequence T_n but with different admission sequences A^1 and A^2. Because the potential arrival times are fixed and form a sequence of discrete epochs, we prefer to call this a discrete-time model, although the T_n may have continuous distributions. Suppose A^1 is time stationary and A^2 not, but they have the same intensity. Is the mean waiting time for the A^1 sequence smaller than or equal to that of A^2? The A sequence can be seen as a random environment and one may expect that an extended Ross conjecture holds (cf. [7], where the service process has a random environment) which states: "The $G/G/1/\infty$ queue in a random environment should be bounded below by the corresponding queue where the environment process is 'frozen' to its mean values."

For the setting of this article, it is not true, as the following counterexample shows. Let $T = \{T_n\}$ be a Poisson process with rate 1. We consider the $\cdot/M/1/\infty$ queue, and we assume that the S_n are i.i.d. with exponential distribution with mean 1. Let A_n^1 be distributed as i.i.d. Bernoulli random variables with mean $\frac{1}{2}$. Let A_n^2 be distributed as independent Bernoulli random variables and let the mean of A_n be p_n. We assume that p_n is random with mean $\frac{1}{2}$. Then, A_n^1 is the arrival process for which the random environment of A_n^2 is frozen to its mean. So the extended Ross conjecture would claim that for W^i, the stationary waiting time for sequence A^i, $i = 1,2$, is

$$\mathbb{E}W^1 \leq \mathbb{E}W^2. \tag{1}$$

Suppose that the sequence $(p_1, p_2, p_3, \dots)$ is with probability $\frac{1}{2}$ equal to $\frac{1}{2}+\epsilon$, $\frac{1}{2}-\epsilon$, $\frac{1}{2}+\epsilon, \dots$ and with probability $\frac{1}{2}$ equal to $\frac{1}{2}-\epsilon$, $\frac{1}{2}+\epsilon$, $\frac{1}{2}-\epsilon, \dots$, where $0 \leq \epsilon \leq \frac{1}{2}$. Then, $\mathbb{E}p_n = \frac{1}{2}$, which is the probability in the A_n^1 sequence.

It is easily seen that for $\epsilon = \frac{1}{2}$, we have that W^1 is the stationary waiting time of the $M/M/1/\infty$ queue with traffic intensity $\rho = \frac{1}{2}$ and that W^2 is the stationary waiting time of the $GI/M/1/\infty$ queue with interarrival times which have an Erlang distribution with two phases of exponential length with mean 1. It is well known that

$$\mathbb{E}W^1 > \mathbb{E}W^2,$$

which contradicts relation (1).

In Section 5, we will derive a discrete-time analogon of the Ross conjecture theorem. The above example explains why we use regular sequences there.

3. COMPARISON LEMMA 1 AND APPLICATIONS

In this section, we derive a first comparison lemma which is a rather direct consequence of the multimodularity of the traveling times as a function of the admission sequence. As we will see, the comparison lemma has some nice implications.

We recall that ${}_bW_n^i$ $({}_aW_n^i)$, $i = 1,2$, is the traveling time of the nth arrival (potential arrival at T_n) to a fixed but arbitrarily chosen transition q in the FSEG.

COMPARISON LEMMA 1: *The following implications hold for any* $n = 1,2,\ldots$:

(a) $(A_1^1,\ldots,A_n^1) \le_{\mathrm{cx}} (A_1^2,\ldots,A_n^2)$ *implies that*

$$({}_aW_1^1,\ldots,{}_aW_n^1) \le_{\mathrm{icx}} ({}_aW_1^2,\ldots,{}_aW_n^2).$$

(b) $(B_1^1,\ldots,B_n^1) \le_{\mathrm{cx}} (B_1^2,\ldots,B_n^2)$ *implies that*

$$({}_bW_1^1,\ldots,{}_bW_n^1) \le_{\mathrm{icx}} ({}_bW_1^2,\ldots,{}_bW_n^2).$$

PROOF: The proof is given for (a); the proof of (b) is similar. Note that $(A_1^i,\ldots,A_n^i) \in \mathbb{N}^n$, $i = 1,2$; let $h:\mathbb{R} \to \mathbb{R}$ be an increasing convex function. Then, for $i = 1,2$,

$$W_k(A_1^i,\ldots,A_n^i) \triangleq \mathbb{E}h({}_aW_k^i(A_1^i,\ldots,A_n^i))$$

is a function from $\mathbb{N}^n$ to $\mathbb{R}$.

Theorem 5.4 of [1] (for the proof of (b), use Theorem 6.4 of [1]) shows that $W_k(A_1^i,\ldots,A_k^i)$ is multimodular in $(A_1^i,\ldots,A_k^i)$ as a function on $\mathbb{N}^k$ for $k = 1,2,\ldots$.

Since $W_k(A_1^i,\ldots,A_k^i)$ is independent of A_n^i for $n > k$, it trivially follows that $W_k(A_1^i,\ldots,A_n^i)$ is also multimodular in $(A_1^i,\ldots,A_n^i)$ if $n \ge k$. It then follows from Theorem 2.2 of [2] that W_k^i is integer convex on $\mathbb{N}^n$, $i = 1,2$, $k = 1,\ldots,n$.

The rest of the proof is more or less standard. Since $(A_1^1,\ldots,A_n^1) \le_{\mathrm{cx}} (A_1^2,\ldots,A_n^2)$, we may, by Strassen's representation theorem, assume without loss of generality that

$$\mathbb{E}((A_1^2,\ldots,A_n^2)|(A_1^1,\ldots,A_n^1)) = (A_1^1,\ldots,A_n^1).$$

From Jensen's inequality, we then have,

$$\begin{aligned} W_k(A_1^1,\ldots,A_n^1) &= W_k(\mathbb{E}((A_1^2,\ldots,A_n^2)|(A_1^1,\ldots,A_n^1))) \\ &\le \mathbb{E}(W_k(A_1^2,\ldots,A_n^2)|(A_1^1,\ldots,A_n^1)). \end{aligned} \quad \textbf{(2)}$$

Hence,

$$\mathbb{E}W_k(A_1^1,\ldots,A_n^1) \le \mathbb{E}W_k(A_1^2,\ldots,A_n^2).$$

Let h: $\mathbb{R}^n \to \mathbb{R}$ be an increasing convex function. Then,

$$h(W_1(A_1^i,\ldots,A_n^i),\ldots,W_n(A_1^i,\ldots,A_n^i))$$

is an increasing convex function of $(A_1^i,\ldots,A_n^i)$, $i = 1,2$.

The first inequality below is now a consequence of (2) and the increasingness of h; the second inequality follows from Jensen's inequality:

$$\begin{aligned} &h(W_1(A_1^1,\ldots,A_n^1),\ldots,W_n(A_1^1,\ldots,A_n^1)) \\ &\quad \le h(\mathbb{E}(W_1(A_1^2,\ldots,A_n^2)|(A_1^1,\ldots,A_n^1),\ldots,\mathbb{E}(W_n(A_1^2,\ldots,A_n^2)|(A_1^1,\ldots,A_n^1)))) \\ &\quad \le \mathbb{E}(h(W_1(A_1^2,\ldots,A_n^2),\ldots,W_n(A_1^2,\ldots,A_n^2))|(A_1^1,\ldots,A_{n1})). \end{aligned}$$

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Hence,

$$\mathbb{E}h(W_1(A_1^1,\dots,A_n^1),\dots,W_n(A_1^1,\dots,A_n^1)) \\ \leq \mathbb{E}h(W_1(A_1^2,\dots,A_n^2),\dots,W_n(A_1^2,\dots,A_n^2)). \qquad \blacksquare$$

Let us assume the following:

ASSUMPTION 1: *A_n^i is a stationary sequence in $n \in \mathbb{Z}$ for $i = 1,2$.*

The following sequences with $i = a$ or b and $j = 1$ or 2 are well known as the Loynes sequences (cf. [7]):

$$_i\overline{W}_n^j \triangleq {}_iW_n(A_{-n}^j, A_{-n+1}^j, \dots, A_{-1}^j).$$

They are monotone increasing in n. Consequently, they have a limit as n tends to infinity, which is possibly ∞. These limits are called Loynes variables; we denote them as

$$_iW_\infty^j \triangleq \lim_{n\to\infty} {}_i\overline{W}_n^j.$$

It is well known (cf. [7, 11]) that under strong coupling or renovation, it holds that

$$_iW_\infty^j = \lim_{n\to\infty} {}_iW_n^j;$$

that is, it is the time-forward limit.

As an immediate consequence of Comparison Lemma 1, we have the following corollary:

COROLLARY 1: *Suppose that Assumption* 1 *holds:*

(a) $(A_1^1,\dots,A_n^1) \leq_{\text{cx}} (A_1^2,\dots,A_n^2)$ *for all* $N \in \mathbb{N}$ *implies that* $_aW_\infty^1 \leq {}_aW_\infty^2$
(b) $(B_1^1,\dots,B_n^1) \leq_{\text{cx}} (B_1^2,\dots,B_n^2)$ *for all* $n \in \mathbb{N}$ *implies that* $_bW_\infty^1 \leq {}_bW_\infty^2$.

PROOF: We prove part (a); the proof of part (b) is similar. From Assumption 1, we have for all n,

$$(A_{-n}^1, A_{-n+1}^1, \dots, A_{-1}^1) \overset{d}{=} (A_1^1, A_2^1, \dots, A_n^1) \\ \leq_{\text{cx}} (A_1^2, A_2^2, \dots, A_n^2) \overset{d}{=} (A_{-n}^2, A_{-n+1}^2, \dots, A_{-1}^2).$$

From Comparison Lemma 1,

$$_a\overline{W}_n^1 \leq_{\text{icx}} {}_a\overline{W}_n^2.$$

The monotone convergence theorem then gives

$$_aW_\infty^1 = \lim_{n\to\infty} {}_a\overline{W}_n^1 \leq_{\text{icx}} \lim_{n\to\infty} {}_a\overline{W}_n^2 = {}_aW_\infty^2. \qquad \blacksquare$$

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It is well known that in case of stability of the stochastic networks, the Loynes variables are a.s. finite and represent the stationary versions of the traveling times. Thus, in the case of stability also, the stationary versions are icx ordered.

Also, a multidimensional marginal distribution of the stationary processes can be shown (also as a consequence of Comparison Lemma 1) to be ordered in the icx ordering.

3.1. Application 1: Two i.i.d. MAP Sources Perform Better Than Two Completely Coupled MAP Sources

Consider two MAPs, say MAP^i, $i = 1,2$, which are independent and have the same distribution. Denote by T^i the transition epochs of MAP^i, $i = 1,2$, and let $T = T^1 \cup T^2$ be the superposition of T^1 and T^2. Define $A_n^1 = 1$ if $T_n \in T$ is an arrival epoch of MAP^1 or MAP^2. Then, $T = \{T_n\}$ are the potential arrival epochs, and the admission sequence A_n^1 generates all arrivals of the two independent MAPs.

Consider now two completely coupled MAP sources, which are equivalent to one MAP source which generates two arrivals at any of its arrival epochs. Define the MAP source with probability $\frac{1}{2}$ as the MAP^1 source and that with probability $\frac{1}{2}$ as the MAP^2 source. Define, for $i = 1,2$,

$$E_n^i = \begin{cases} 1 & \text{if } T_n \text{ is the arrival epoch of } \mathrm{MAP}^i \\ 0 & \text{otherwise.} \end{cases}$$

Then,

$$A_n^1 = E_n^1 + E_n^2,$$

and for A_n^2, the admission sequence of the coupled MAP sources, we have for $1 \le m \le n$ that A_m^2 given $A_1^1, \ldots, A_n^1$ with probability $\frac{1}{2}$ equal to $2E_m^1$ and with probability $\frac{1}{2}$ equal to $2E_m^2$. Hence,

$$\mathbb{E}[A_m^2 | A_1^1, \ldots, A_n^1] = A_m^1,$$

and, therefore,

$$(A_1^1, \ldots, A_n^1) \le_{\mathrm{cx}} (A_1^2, \ldots, A_n^2) \quad \text{for all } n.$$

Clearly, the events E_n^i in MAP^i are independent of the transition epochs $T_n^i, i = 1,2$. This implies that $\{A_n^i\}$ and $\{T_n\}$ are independent for $i = 1,2$. Hence, we can apply Comparison Lemma 1 and Corollary 1 and find that the potential (stationary) traveling times for the i.i.d. MAP sources are in icx ordering smaller than those of two completely coupled MAP sources.

It is possible to extend this result to k i.i.d. MAP sources which generate l customers at each of their arrival epochs that perform better in icx order than $l \le k$ i.i.d. MAP sources which generate k customers at each of their arrival epochs.

3.2. Application 2: A Fixed Batch Size Is Better Than Random Batch Sizes

Consider a MAP source and assume that at each of its arrival epochs, a random batch number of customers arrive, say N_n at arrival epoch T_n. We assume that $\{N_n\}$ is independent of $\{T_n\}$ and that $\{N_n\}$ is stationary and $\mathbb{E}N_1 = l$.

Take $A_n^1 = l$ and $A_n^2 = N_n$, $n \in \mathbb{N}$; then, A_n^1 is the admission sequence with fixed (or frozen) batch size. Clearly, $(A_1^1, \ldots, A_n^1) \leq_{\text{cx}} (A_1^2, \ldots, A_n^2)$, $n \in \mathbb{N}$, and part (a) of Comparison Lemma 1 and Corollary 1 applies.

If $l = 1$, then $(B_1^1, \ldots, B_n^1) = (1, \ldots, 1)$, and in order to show that

$$(B_1^1, \ldots, B_n^1) \leq_{\text{cx}} (B_1^2, \ldots, B_n^2),$$

it suffices to verify that for $k = 1, \ldots, n$,

$$\mathbb{E}B_k^2 = 1.$$

Since A_n^2 is stationary, it follows that

$$\mathbb{E}A_1^2 \cdot \mathbb{E}B_1^2 = 1,$$

and $\mathbb{E}B_k = \mathbb{E}B_1 = 1$. Hence, in this case also, the actual traveling times are smaller in icx order for the fixed batch sizes.

3.3. Application 3: Fluid Scaling Improves the Performance

Consider the following transformations of the time variable t and the state variable x:

$$t \to Nt,$$

$$x \to \frac{x}{N}.$$

Fluid limits are obtained by taking limits for $N \to \infty$. Here, we take a fixed $N \in \mathbb{N}$. If we have a MAP^1 with finite state space E and transition rates λ_{xy}, $x, y \in E$, and if we divide the time variable by N, then we get a MAP^2 with transition rates $(1/N)\lambda_{xy}$. After uniformizing both processes such that the transition times in both processes are a Poisson process with the same parameter λ, say $\text{MAP}^1(\lambda)$ and $\text{MAP}^2(\lambda)$, we have that a real transition in $\text{MAP}^1(\lambda)$ (i.e., a transition in MAP^1) is, with probability $(1/N)$, a real transition in $\text{MAP}^2(\lambda)$ (i.e., a transition of MAP^2). Clearly, we can couple the $\text{MAP}^1(\lambda)$ and $\text{MAP}^2(\lambda)$ such that if T_n are the arrival epochs of $\text{MAP}^1(\lambda)$, then the potential arrival epochs of $\text{MAP}^2(\lambda)$ are $\{T_n\}$ and the admission sequence is

$$A_n^2 = \begin{cases} 1 & \text{with probability } \dfrac{1}{N} \\ 0 & \text{otherwise.} \end{cases}$$

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where the A_n^2 are i.i.d. and independent of T_n. If we take $A_n^1 = 1$, then A_n^1 is the admission sequence for $\text{MAP}^1(\lambda)$.

As in Application 2, we have

$$(A_1^1,\dots,A_n^1) \leq_{\text{cx}} (\bar{A}_1^2,\dots,\bar{A}_n^2),$$

where $\bar{A}_k^2 = NA_k^2$, $k \in \mathbb{N}$.

The scaling of the state can be done by considering the original service requirements as a number of packets (possibly of random size), and taking N arrivals instead of one arrival. This gives the $\bar{A}_n^2$ as admission sequence. So, the process corresponding to the A_n^1 can be seen as a fluid scaling of the $\bar{A}_n^2$-induced process. Mathematically, it is the same comparison as in Application 2. The Comparison Lemma 1 and Corollary 1 imply that the performance of the fluid scaled process is better than that of the original process in icx ordering.

4. COMPARISON LEMMA 2

In Comparison Lemma 1, we had admission sequences $\{A_n^i\}, i = 1,2$, where A_n^i defined the number of arrivals at T_n. This means that A_n^i is an integer. In Comparison Lemma 2, the admission sequences are $\{p_n^i\}$, where p_n^i may be any nonnegative real number. For $p^i = (p_1^i, p_2^i,\dots)$ with $p_n^i \geq 0$, $n = 1,2,\dots$, we define an integer admission sequence $\{A_n^i(p)\}$ by

$$A_n(p^i) \triangleq \lfloor \sum_{j=1}^{n} p_j^i + \theta \rfloor - \lfloor \sum_{j=1}^{n-1} p_j^i + \theta \rfloor,$$

where θ is a random variable, uniformly distributed on $[0,1)$, and where $\lfloor x \rfloor$ denotes the largest integer smaller than or equal to $x \in \mathbb{R}_+$. Note that $A_n^i(p)$ is random and integer-valued; it gives the number of arrivals at T_n.

For the interarrival lengths, we proceed similarly given $q^i = (q_1^i, q_2^i,\dots)$ with $q_n^i \geq 0$, $n = 1,2,\dots$; we define

$$B_n(q^i) \triangleq \lfloor \sum_{j=1}^{n} q_j^i + \theta \rfloor - \lfloor \sum_{j=1}^{n-1} q_j^i + \theta \rfloor,$$

where, again, θ is uniformly distributed on $[0,1)$. Note that if $p_1^i, p_2^i,\dots, p_n^i$ are all integer-valued, then $A_k(p^i) = p_k^i$ for $1 \leq k \leq n$ and any $\theta \in [0,1)$.

Similarly, as in Comparison Lemma 1, we consider the potential traveling times

$${}_aW_n^i \triangleq {}_aW_n(A_1(p^i),\dots,A_n(p^i)), \quad i = 1,2,$$

and the actual traveling times

$${}_bW_n^i \triangleq {}_bW_n(B_1(q^i),\dots,B_n(q^i)), \quad i = 1,2.$$

Comparison Lemma 2: *The following implications hold for any* $n = 1,2,\dots$:

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(a) $(p_1^1,\dots,p_n^1) \leq_{\mathrm{cx}} (p_1^2,\dots,p_n^2)$ *implies that*

$$({}_aW_1^1,\dots,{}_aW_n^1) \leq_{\mathrm{icx}} ({}_aW_1^2,\dots,{}_aW_n^2).$$

(b) $(q_1^1,\dots,q_n^1) \leq_{\mathrm{cx}} (q_1^2,\dots,q_n^2)$ *implies that*

$$({}_bW_1^1,\dots,{}_bW_n^1) \leq_{\mathrm{icx}} ({}_bW_1^2,\dots,{}_bW_n^2).$$

PROOF: We prove part (a); the proof of part (b) is similar. Since

$$(p_1^1,\dots,p_n^1) \leq_{\mathrm{cx}} (p_1^2,\dots,p_n^2),$$

we may, by Strassen's representation theorem, assume without loss of generality that

$$\mathbb{E}((p_1^2,\dots,p_n^2)|(p_1^1,\dots,p_n^1)) = (p_1^1,\dots,p_n^1).$$

Theorem 5.4 of [1] together with Theorem 2.1 of [2] (which is an extension of Hajek's theorem on multimodularity in [13]) imply that for h an increasing convex function,

$$ {}_aW_k(p_1^i,\dots,p_n^i) \triangleq \mathbb{E}h({}_aW_k^i(A_1(p^i),\dots,A_n(p^i)))$$

is a convex function of $(p_1^i,\dots,p_n^i)$.

The rest of the proof is similar to the proof of Comparison Lemma 1 with $(p_1^i,\dots,p_n^i)$ substituted for $(A_1^i,\dots,A_n^i)$. ■

Also in this setting, we can consider the Loynes stochastic variables, assuming that p_n^i is defined for all $n \in \mathbb{Z}$,

$$ {}_i\overline{W}_n^j \triangleq {}_iW_n(p_{-n}^j,\dots,p_{-1}^j)$$

and

$$ {}_iW_\infty^j \triangleq \lim_{n\to\infty} {}_i\overline{W}_n^j.$$

ASSUMPTION 2: *$A_n^i(p)$ is a stationary sequence in $n \in \mathbb{Z}$ for $i = 1,2$.*

With the same proof as for Corollary 1, we then have the following corollary.

COROLLARY 2: *Under Assumption 2, the following hold*:

(a) $(p_1^1,\dots,p_n^1) \leq_{\mathrm{cx}} (p_1^2,\dots,p_n^2)$ *for all $n \in \mathbb{N}$ implies that*

$$ {}_aW_\infty^1 \leq_{\mathrm{icx}} {}_aW_\infty^2.$$

(b) $(q_1^1,\dots,q_n^1) \leq_{\mathrm{cx}} (q_1^2,\dots,q_n^2)$ *for all $n \in \mathbb{N}$ implies that*

$$ {}_bW_\infty^1 \leq_{\mathrm{icx}} {}_bW_\infty^2.$$

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5. A STOCHASTIC LOWER BOUND ON THE TRAVELING TIMES

In Section 2, we found that the intuitive argument that queues in a random environment should be bounded below by the corresponding queues where the environment is "frozen" to its mean values is not generally true.

As an application of Comparison Lemma 2, we will derive in this section a lower bound in the icx ordering. The queuing model is a FSEG with a MAP source. We will construct a more regular (in fact, the most regular) arrival process with the same arrival intensity as the MAP source. This will provide the lower bound. As we will see, this regular arrival process can be approximated by a renewal arrival process with Erlang-distributed interarrival times.

Without loss of generality, we may assume that the MAP has transition times $\{T_n\}$ that form a Poisson(λ) process. Let $\{X_n\}$ be the Markov process with transition probabilities λ_{xy} which governs the transitions of the MAP (i.e., X_n is the state at T_n). We assume that the Markov process is stationary and we denote by π_x the stationary probability on state $x \in E$. The probability on an arrival at T_n is

$$p \triangleq \sum_x \sum_y \pi_x \lambda_{xy} r_{xy}. \tag{3}$$

The arrival intensity of the stationary MAP is then $\bar{\lambda} \triangleq p\lambda$. The MAP corresponds to the following admission sequence:

$$A_n^2 = \begin{cases} 1 & \text{with probability } r_{X_{n-1}X_n} \\ 0 & \text{otherwise.} \end{cases}$$

Because A_n^2 is 0 or 1, hence integer-valued for all n, we can also use the p representation (i.e., take $p_n^2 = A_n^2$, $n \in \mathbb{N}$); then,

$$A_n(p^2) = A_n^2 \quad (\text{for all } \theta).$$

Take $p^1 = (p, p, \dots)$ and $A_n^1 \triangleq A_n(p^1)$. Then, A_n^1 (for fixed θ) is called the (most) regular sequence with rate p. Regular sequences are a subclass of the balanced sequences, on which there is a extensive literature in combinatorics (see [3] for a recent article in which balanced sequences are applied to optimal routing to queues).

The lower bound in icx ordering is obtained if we use the arrival process on T_n with A_n^1 as the admission sequence. Let us call this the regular arrival process with parameters (p, λ) (RAP(p, λ)). The MAP with stationary distribution π_x, $x \in E$, and arrival probabilities r_{xy}, $x, y \in E$, we denote by MAP(π, r). Analogous to the $\cdot/G/1$ notation, let us denote the FSEG (with stationary sequences T_n and S_n) by $\cdot/G/$SEG and ${}_iW_\infty(\cdot/G/\text{SEG})$, $i = a, b$ for the potential $(i = a)$ or actual traveling time $(i = b)$ (to a fixed transition). Then, the main result of this article, which is an application of Comparison Lemma 2, is Theorem 1.

THEOREM 1: *For $i = a, b$,*

$${}_iW_\infty(\text{RAP}(p, \lambda)/G/\text{SEG}) \leq_{\text{icx}} {}_iW_\infty(\text{MAP}(\pi, r)/G/\text{SEG}).$$

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PROOF: For $i = a$, we apply part (a) of Comparison Lemma 2. Therefore, we have to show that

$$\mathbb{E}((p_1^2,\ldots,p_n^2)|(p_1^1,\ldots,p_n^1)) = (p_1^1,\ldots,p_n^1).$$

However, $p_k^1 = p$ for all k; therefore, it suffices to show that $\mathbb{E}p_k^2 = p$ for all k. Indeed, this holds since

$$\mathbb{E}p_k^2 = \sum_x \sum_y \pi_x \lambda_{xy} r_{xy} = p.$$

In order to apply Corollary 2, we have to verify Assumption 2. Indeed, the admission sequence A_n^2 is stationary since X_n is assumed to be stationary. Because $\int_0^1 \lfloor x + \theta \rfloor \, d\theta = x$ implies

$$\mathbb{E}A_n^1(p) = \int_0^1 (\lfloor np + \theta \rfloor - \lfloor (n-1)p + \theta \rfloor) \, d\theta = p,$$

it follows that $A_n^1(p)$ is a stationary sequence and Assumption 2 applies for both sequences. For $i = b$, we consider the B_n sequence corresponding to the $A_n^1(p)$ sequence. Lemma 7.3 in [3] guarantees that it is balanced with rate $1/p$. Since $A_n^1(p)$ is stationary, B_n^1 is also stationary. Hence, by a result of Morse and Hedlund (see Thm. 7.2 of [3]),

$$B_n^1 = B_n(q) \quad \text{with } q = (1/q, 1/q, \ldots).$$

Since the A_n^2 sequence is integer-valued, the corresponding B_n^2 sequence is also integer-valued. Hence, it remains to verify that $\mathbb{E}B_n^2 = 1/p$, but this is a standard result for stationary MAP processes. ■

It is well known that a MAP process with transition times $\{T_n\} \stackrel{d}{=} \text{Poisson}(\lambda)$ can be represented also as one with transition times $\{T_n^1\} \stackrel{d}{=} \text{Poisson}(N\lambda)$ with the same π as the stationary distribution. If we want to keep the arrival intensity equal to $\bar{\lambda}$, then we have to divide the p and the r_{xy} by N; hence, $1/p$ is multiplied by N.

Now, suppose $1/p$ is rational, say N_1/N_2, then $N_2/p = N_1$ is an integer. The corresponding regular arrival process has interarrival lengths of N_1 steps; hence, its interarrival times are Erlang distributed with N_1 phases of exponential-distributed length with parameter $N_2\lambda$.

Using Theorem 1, we will show Theorem 2.

THEOREM 2: *For $i = a, b$ and any real number $0 < c \leq 1$, it holds that*

$${}_iW_\infty(\text{RAP}(p,\lambda)/G/\text{SEG}) \leq_{\text{icx}} {}_iW_\infty(\text{RAP}(p/c,\lambda c)/G/\text{SEG}).$$

PROOF: $\text{RAP}(p/c,\lambda c)$ can be seen as a MAP with transition times $\{T_n\}$ which are a $\text{Poisson}(\lambda)$ process. With probability c, a transition is a real transition (i.e., a transition of the $\text{RAP}(p/c,\lambda c)$ process). The stationary admission sequence in $\text{RAP}(p/c,\lambda c)$ has rate p/c on an arrival at a (real) transition. Hence, the stationary probability on an arrival at T_n is $c \cdot p/c = p$, and Theorem 1 applies. ■

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As a consequence of Theorem 2, we have that for a $\text{MAP}(\pi, r)$ with p (as in (3)) rational, say $p = N_2/N_1$, the FSEG with renewal input with N_1 phases of exponential-distributed length with parameter $N_2\lambda$ provides a icx stochastic lower bound on the actual and potential stationary traveling times.

By Theorem 2, $\text{RAP}(pc, \lambda/c)$ for any $0 < c \leq 1$ provides also a icx lower bound. Hence, since $\text{RAP}(pc, \lambda/c)$ is arbitrarily close to a renewal input for c sufficiently small, we get, for p is irrational, an approximation. These facts have been used, without proof, in articles on optimal routing to parallel queues (cf. [12]).

Clearly, the limit process of $\text{RAP}(p/c, \lambda c)$ for $c \to \infty$ is the renewal process with a constant interarrival time equal to $1/p\lambda$ (notation $\mathcal{D}(1/p\lambda)$). Hence, by a continuity argument (cf. [15]) we have Corollary 3 as a consequence of Theorems 1 and 2.

Corollary 3: *For $i = a, b$, the following hold*:

(a) ${}_iW_\infty(\text{RAP}(p/c, \lambda c)/G/\text{SEG})$ *is monotone decreasing in c.*
(b) *For any $c \geq 1$,*

$$\begin{aligned} {}_iW_\infty(\mathcal{D}(1/p\lambda)/G/\text{SEG}) &\leq_{\text{icx}} {}_iW_\infty(\text{RAP}(p/c, \lambda c)/G/\text{SEG}) \\ &\leq_{\text{icx}} {}_iW_\infty(\text{MAP}(\pi, r)/G/\text{SEG}). \end{aligned}$$

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