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INTEGRAL FUNCTIONALS UNDER THE EXCURSION MEASURE

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Abstract

A new approach to the problem of finding the distribution of integral functionals under the excursion measure is presented. It is based on the technique of excursion straddling a time, stochastic analysis, and calculus on local time, and it is done for Brownian motion with drift reflecting at 0, and under some additional assumptions for some class of Itó diffusions. The new method is an alternative to the classical potential-theoretic approach and gives new specific formulas for distributions under the excursion measure.

Keywords: Brownian motion with drift; excursions of Markov process; excursion measure; first hitting time of 0; Itó diffusion

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1. Introduction

Excursion theory of Markov processes, started originally from ideas of Itô [10], has provided a clever distributional description of paths for some regular Markov processes through dividing the trajectories into independent random fragments (excursions), considered then to be the values of the associated (possibly Poisson) point process. So the process is fragmented, moved to the path space, and considered under some specific measure called the excursion measure. Such a presentation can be realized by parametrization in terms of the local time, and this concept is well known and developed in the books of Itô and McKean [11], Rogers and Williams [21], or Revuz and Yor [20]. For a comprehensive lecture on this topic we recommend Blumenthal's book [2].

A study on excursion theory for linear diffusions and a case study of the Ornstein–Uhlenbeck process were presented in [22]. Excursion theory can be successfully applied to studies on functionals of Brownian motion and Bessel processes, and it was the subject of extensive work by Pitman and Yor [16, 17, 18, 19]. In particular, excursion theory for a Brownian motion is used in [5] for pricing Parisian barrier options.

The goal of this work is to present a new approach to the problem of finding the distribution of integral functionals under the excursion measure. Originally, [16, Section 3] considered the excursion measure for 0-diffusions (regular diffusions with state space $[0, \infty)$ and 0 being an absorbing boundary). They assumed that 0 is attained from any point of the state space with positive probability and that it is an exit point. They showed that the coordinate process under the excursion measure $\widehat{\mathrm{P}}$ is Markov with transition probabilities p_t of the original process and entrance law given by

$$\widehat{\mathrm{P}}(X_t \in \mathrm{d}z) = \lim_{x \downarrow 0} \frac{p_t(x, z)m(\mathrm{d}z)}{s(x)},$$

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where m is a speed measure and s is a scale function. Following [16, (3a)], for A measurable with respect to the element of filtration generated by a coordinate process and away from neighborhoods of the trajectory identically equal zero, we may write

$$\widehat{\mathrm{P}}(A) = \lim_{x \downarrow 0} \frac{\mathrm{P}_x(A)}{s(x)}, \tag{1}$$

where the limit is in the sense of weak convergence, and on the right-hand side P_x denotes the distribution on the canonical space of 0-diffusion started at x and killed at the first hitting time of 0. To describe the measure $\widehat{\mathrm{P}}$, Pitman and Yor used the auxiliary process arising as a 0-diffusion conditioned never to hit 0. Their approach is based on the theory of semi-groups and inspired by ideas of Doob [6] for Brownian motion and the results of Itó–McKean [11, Section 6.2].

Since [16] did not discuss the choice of scale function s, we think that some comment on this is needed. Following the description in [22], for each $x \in E$ the measure P_x is given and associated with X starting from x. By (1), the measure $\widehat{\mathrm{P}}$ depends on the choice of s. The last in turn is usually defined up to a multiplicative constant (see [20, Chapter VII, Proposition 3.2]). In our approach we use local time normalized by the occupation time formula and local time normalized by the Tanaka formula. It is well known that both local times differ by a multiplicative constant c^* (see [2, Chapter III, 3(c)]). Both local times meet in the master formula, so the multiplicative constant c^* determines $\widehat{\mathrm{P}}$ and thus, by (1), it determines s. The circle closes. For this reason $\widehat{\mathrm{P}}$ is determined either by the choice of s or by the choice of normalizing constant of local time (see also the discussion at the end of Section 2).

We divide our presentation into two parts. The first concerns a Brownian motion with drift, reflecting at 0; the second, under some additional assumptions, concerns two classes of Itó diffusions. The first class corresponds (at least to some extent) to the 0-diffusion studied by Pitman and Yor, while considering the second class we relax the assumption on X to be non-negative, and thus go beyond the setup of Pitman and Yor. However, for the second class we need some additional assumptions on the diffusion's coefficients.

Our new approach is based on the technique of excursion straddling a time, the distributional Lévy's theorem, and calculus on local time. The technique of local time is inspired by Peskir's brilliant method of solution of the Stroock–Williams equation [14]. We obtain new formulas describing the distribution of specific integral functionals under the excursion measure. We present the absolute-continuity relationship between excursion measures for processes which are local martingales. It turns out that for two different excursion measures $\widehat{\mathrm{P}}$ and $\widehat{\mathrm{Q}}$ associated with two reflected at 0 Markov processes on $[0, \infty)$ which are local martingales, if m_1', m_2' denote the densities with respect to the Lebesgue measure of their speed measures, respectively, then for any $b > 0$ and $\widehat{X}$ denoting the coordinate process killed at R we have

$$c_2^* \, \widehat{\mathrm{P}}\biggl(\int_0^{\sigma_b \wedge R} f(\widehat{X}_s) m_2'(\widehat{X}_s) \, \mathrm{d}s \biggr) = c_1^* \, \widehat{\mathrm{Q}}\biggl(\int_0^{\sigma_b \wedge R} f(\widehat{X}_s) m_1'(\widehat{X}_s) \, \mathrm{d}s \biggr),$$

where c_1^*, c_2^* are constants, σ_b denotes the first hitting time of b, and f is a given non-negative, measurable function.

2. Preliminaries

Let X be a diffusion on $\mathcal{C}(E, \mathcal{E})$, the canonical space of all continuous functions on E with an associated σ-field of Borel subsets of E. $X_0 = x$ almost surely (a.s.) under P_x, and we

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denote by $\mathrm{P} := \mathrm{P}_0$ the measure under which $X_0 = 0$ a.s. The filtration $(\mathcal{F}_t^X)_{t\geq 0}$ generated by X is augmented to satisfy the usual conditions. By θ we denote the shift operator. We assume that the state space E of the process X is a subset of the real line, and that $0 \in E$ in the sense that X attains 0 during the lifetime with a positive probability. If the diffusion is recurrent we may follow the description of Revuz and Yor [20, Chapter XII] for excursions of recurrent processes. Otherwise we may follow the more general approach of [2].

Let U be the subspace of the canonical space consisting of functions $u : E \to \mathbb{R}$ such that $u(0) = 0$, $R(u) = \inf\{t > 0 : u(t) = 0\}$, and $u(t) = 0$ for all $t \geq R(u)$. If the process is transient, R may be infinite. Set $\mathcal{U}$ to be the σ-algebra generated by the coordinate mappings. Let $\delta \equiv 0$. By U^δ we denote the space $U \cup \{\delta\}$, and $\mathcal{U}^\delta = \sigma(\mathcal{U}, \{\delta\})$. If the process is recurrent, an associated point process is a Poisson point process (PPP) $(e_s)_{s>0}$. The excursion measure on the path space (a subspace of the canonical space) will be denoted $\widehat{\mathrm{P}}$ after [2, Chapter III, Theorem 3.24]. It is well known [20, Chapter XII, Theorem 4.1] that the coordinate process restricted to $(0, R)$ is, under $\widehat{\mathrm{P}}$, a homogeneous strong Markov process with a transition semi-group of a process killed at 0 and an entrance law given by

$$\eta_t(\mathrm{d}y) = g_{y0}(t)m(\mathrm{d}y), \qquad t > 0,$$

where g_{y0} denotes the density of the first hitting time of 0 under P_y, and m is the speed measure of the original Markov process (see [22, Theorem 2]). For example, the entrance law of Brownian motion with non-negative drift μ (with speed measure given by $m(\mathrm{d}y) = 2\mathrm{e}^{2\mu y}\mathrm{d}y$) is given on $(0, \infty)$ by

$$\eta_t(\mathrm{d}y) = \frac{y}{\sqrt{2\pi t^3}}\mathrm{e}^{-\frac{(y+\mu t)^2}{2t}}2\mathrm{e}^{2\mu y}\mathrm{d}y = y\sqrt{\frac{2}{\pi t^3}}\mathrm{e}^{-\frac{(y-\mu t)^2}{2t}}\mathrm{d}y,$$

(see [3, pp. 127, 295]). It is also well known that the distribution of R under $\widehat{\mathrm{P}}$ is given on $\mathbb{R}_+$ by $\widehat{\mathrm{P}}(R > t) = \int_0^\infty g_{y0}(t)m(\mathrm{d}y)$. Define $\sigma_0 = \inf\{t : X_t = 0\}$ and assume that 0 is regular in the sense that $\mathrm{P}(\sigma_0 = 0) = 1$. By $L^{(a)}(X)$ we denote the local time of X at $a \in E$, normalized in line with the setup of excursion theory, that is,

$$\int_0^t h(X_s)\,\mathrm{d}s = \int_E h(a)L_t^{(a)}(X)m(\mathrm{d}a),$$

for any non-negative measurable function h (see [22, Section 1(v)]). We denote $L(X) := L^{(0)}(X)$. In all considerations below, $p_t(\cdot, \cdot)$ denotes a transition density with respect to the speed measure m. In particular, it is well known that

$$\mathrm{E}(L_t(X)) = \int_0^t p_u(0, 0)\,\mathrm{d}u \tag{2}$$

[3, Chapter II, Section 2(d)]. Since we will use the Tanaka formula we need to adjust a normalizing constant. Namely, if

$$\mathrm{d}|X_t| = \mathrm{sgn}(X_t)\mathrm{d}X_t + \mathrm{d}L_t^*(X), \tag{3}$$

then it is well known that there exists a strictly positive constant c^* such that $L^*(X) = c^*L(X)$. The notation c^* will be reserved explicitly for this constant. Below, it will be called a local

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time constant. If X is a standard Brownian motion, then $c^*=\frac{1}{2}$. In particular, the formula for c^* follows from (2) and (3):

$$c^* = \bigg(\mathrm{E}\bigg(|X_t| - \int_0^t \mathrm{sgn}(X_s)\,\mathrm{d}X_s\bigg)\bigg)\bigg(\int_0^t p_u(0, 0)\,\mathrm{d}u\bigg)^{-1},$$

where $t>0$, and it follows that the right-hand side of the last formula does not depend on t.

Let us sum up the problem of normalization of local time. If s is given then the speed measure is affected by this choice to preserve $\frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}}{\mathrm{d}s}=\mathcal{A}$ (and $\mathcal{A}$ denotes the generator of a diffusion). Since for any Borel A we have $\mathrm{P}_x(X_t\in A)=\int_A p_t(x, z)m(\mathrm{d}z)$, it is clear that the last choice affects p_t and, by the identity $\mathrm{E}_x(L_t)=\int_0^t p_u(x, 0)\,\mathrm{d}u$, it determines the normalizing constant of local time L. So, if we set L to satisfy (2), then we define a local time constant c^* by $L^*(X)=c^*L(X)$.

3. Brownian motion with drift reflecting at 0

Let (B_t) be a Brownian motion considered on $\mathcal{C}([0, \infty), \mathcal{B})$, the canonical space of all continuous functions on $[0, \infty)$. Let $X_t=B_t+\mu t$, $\mu\geq 0$. If $\mu=0$ the process is recurrent. If $\mu>0$ it is well known that under P_x the process X is a transient diffusion, and for any $x>0$ we have $\mathrm{P}_x(\sigma_0<\infty)=\mathrm{e}^{-2\mu x}$. This implies that the total local time at 0 of the process X, i.e. $L_\infty(X)$, is finite a.s., and the associated point process of excursions cannot be Poisson (see [2, Chapter III, Section (g)]). However, the setup for the excursion path space $(U, \mathcal{U})$ for the non-recurrent case is, except for the fact that R can now be infinite, the same as in the recurrent one. It turns out that the effective substitute of the PPP is obtained in this case by considering the conditional probability given $\{\tau_t<\infty\}$, where (τ_t) denotes the inverse of the local time $L(X)$, i.e. $\tau_t=\inf\{s>0 : L_s(X)>t\}$ (obviously $\tau_{t-}=\inf\{s>0 : L_s(X)\geq t\}$). Thus, the fundamental excursion formula [2, Chapter III, (3.27)] and the Markov property under the excursion measure remain valid (see [2, Chapter III, Section (g)]). In case the process is transient, the natural decomposition of the excursion measure is

$$\widehat{\mathrm{P}}_1(A)=\widehat{\mathrm{P}}(A\cap\{R=\infty\}), \quad \widehat{\mathrm{P}}_2(A)=\widehat{\mathrm{P}}(A\cap\{R<\infty\}), \quad A\in\mathcal{B}(\mathbb{R}_+).$$

If $\mu>0$ and $P_0(t, x, A)$ denotes the transition function of the Brownian motion with drift μ killed at 0, then it is well known that for the entrance law $\{\eta_t, t>0\}$ and $\Delta(x)=\mathrm{P}_x(\sigma_0=\infty)$, the coordinate process $(X_t, t>0)$ under the measure $\widehat{\mathrm{P}}_1$ is a homogeneous Markov process with transition function $Q_t^1(x, \mathrm{d}z)=P_0(t, x, \Delta(z)\mathrm{d}z)/\Delta(x)$ and entrance law $\Delta(z)\eta_t(\mathrm{d}z)$ (see [2, Chapter III, Theorem 3.29]). Relative to $\widehat{\mathrm{P}}_2$ the coordinate process is a time-homogeneous Markov process with transition function $Q_t^2(x, \mathrm{d}z)=P_0(t, x, (1-\Delta(z))\mathrm{d}z)/(1-\Delta(x))$ and entrance law $(1-\Delta(z))\eta_t(\mathrm{d}z)$.

Consider now functionals of Brownian motion with drift reflecting at 0. Since we consider excursions from 0 we will assume that the process starts from 0. It is well known that such a process is realized by $|X|$, where X is a strong solution of the stochastic differential equation

$$\mathrm{d}X_t=\mathrm{d}B_t+\mu\,\mathrm{sgn}(X_t)\mathrm{d}t, \qquad X_0=0, \tag{4}$$

and B is a standard Brownian motion (see [13]). We will consider functionals of $|X|$ under the excursion measure of X denoted by $\widehat{\mathrm{P}}$. We will see in Remark 3.2 that there is a simple relation between the excursion measures for X and $|X|$. If $\mu=0$ then $X=B$. Recall the distributional Lévy's theorem

$$(S-B, S)_{t\geq 0} \overset{\text{law}}{=} (|B|, L^*(B))_{t\geq 0},$$

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(see [20, Chapter VI, Theorem 2.3]). We will use the distributional Lévy's theorem given by Graversen and Shiryaev [9, Theorem 1] and extended by Peskir [13]. We have

$$(S^{(-\mu)} - B^{(-\mu)}, S^{(-\mu)})_{t\geq 0} \overset{\text{law}}{=} (|X|, L^*(X))_{t\geq 0}, \tag{5}$$

where B is a standard Brownian motion, $B_t^{(-\mu)} = B_t - \mu t$, $S_t^{(-\mu)} = \sup_{u\leq t} B_u^{(-\mu)}$, and L^* is a local time compatible with the Tanaka formula. We will further assume that $\mu \geq 0$.

We will now present a description of the integral functionals of reflecting Brownian motion with non-negative drift under the excursion measure $\widehat{\mathrm{P}}$, and for $\mu > 0$ under the restricted measures $\widehat{\mathrm{P}}_1$ and $\widehat{\mathrm{P}}_2$. Let $\sigma_z = \inf\{t > 0 : X_t = z\}$, $z \geq 0$. For a non-negative, Borel function f on $[0, \infty)$, $\lambda > 0$, and $x \geq 0$, define

$$h_{\lambda,f}^{(\mu)}(x) = \mathrm{E}_x \exp\left(-\lambda \int_0^{\sigma_0} f(|X_s|)\, \mathrm{d}s\right). \tag{6}$$

Notice that the generator of $|X|$ is $\mathcal{A}f = \frac{1}{2}\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + \mu \frac{\mathrm{d}f}{\mathrm{d}x}$, $x > 0$, and $\mathcal{A}f(0) = \frac{1}{2}f''(0) + \mu f'(0)$. The domain of $\mathcal{A}$ is $\{f : f, \mathcal{A}f \in \mathcal{C}_\mathrm{b}[0, \infty), f'(0+) = 0\}$, where $\mathcal{C}_\mathrm{b}(E)$ is a set of continuous and bounded functions on E (see [3, p. 129]). Thus, by considering the classical Dirichlet problem on a finite interval we find that whenever $h_{\lambda,f}^{(\mu)} \in \mathcal{C}^2(0, \infty)$ and f is locally integrable then $h_{\lambda,f}^{(\mu)}$ solves

$$\frac{1}{2}\frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} + \mu \frac{\mathrm{d}u(x)}{\mathrm{d}x} = \lambda f(x) u(x), \qquad x \in (0, \infty),$$
$$u(0) = 1.$$

Definition. [Hypothesis A] We say that a non-negative Borel function f on $[0, \infty)$ satisfies hypothesis A if

1. f is locally integrable,
2. the function $h_{\lambda,f}^{(\mu)}$ is $\mathcal{C}^2(0, \infty)$ and has the first right-hand side derivative in 0,
3. $(h_{\lambda,f}^{(\mu)})'$ is locally bounded.

By the above arguments $h_{\lambda,f}^{(\mu)}$ is $\mathcal{C}^2(0, \infty)$, so to have hypothesis A we need to verify that $h_{\lambda,f}^{(\mu)}$ has the first right-hand side derivative in 0 and $(h_{\lambda,f}^{(\mu)})'$ is locally bounded. Recall that $|X|$ is reflecting (at 0) Brownian motion with drift μ, and λ is fixed. We have the following result.

Proposition 3.1. *Let f be a non-negative Borel function on* $[0, \infty)$. *If*

$$\int_0^\infty \mathrm{e}^{-\mu y} f(y) m(\mathrm{d}y) < \infty$$

then $h_{\lambda,f}^{(\mu)}$ given by (6) has the first right-hand side derivative in 0 *and $(h_{\lambda,f}^{(\mu)})'$ is locally bounded on* $[0, \infty)$.

Proof. Clearly $x \mapsto h_{\lambda,f}^{(\mu)}(x)$ is bounded, so the assertion is true on $[\delta, \infty)$ for any fixed $\delta > 0$. Thus it is enough to prove the proposition in $(0, \delta)$. For x small and positive we have

$$\frac{h_{\lambda,f}^{(\mu)}(x) - 1}{x} \approx -\frac{\lambda}{x}\, \mathrm{E}_x \int_0^{\sigma_0} f(X_s)\, \mathrm{d}s. \tag{7}$$

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Observe that, by monotone convergence,

$$\mathrm{E}_x \int_0^{\sigma_0} f(X_s)\,\mathrm{d}s = \lim_{z\to\infty} \mathrm{E}_x \int_0^{\sigma_0\wedge\sigma_z} f(X_s)\,\mathrm{d}s.$$

Since $s(y) = (1-\mathrm{e}^{-2\mu y})/y$ is a scale function of $|X|$ (see [3, p. 129]), we have, from [20, Chapter VII, Corollary 3.8],

$$\lim_{z\to\infty} \mathrm{E}_x \int_0^{\sigma_0\wedge\sigma_z} f(X_s)\,\mathrm{d}s = 2\mu\, s(x) \int_0^{\infty} \mathrm{e}^{-\mu y} f(y) m(\mathrm{d}y).$$

We finish the proof by taking $x \to 0$ in (7). □

We define the integral functional

$$A_R^f(u) = \int_0^{R(u)} f(|u(s)|)\,\mathrm{d}s,$$

where f is some given non-negative, Borel function on $[0,\infty)$. Obviously, in the above setup $u \mapsto A_R^f(u)$ is measurable. The integral functional of $f(|X|)$ will be denoted in the same manner by A^f, that is, for any $t>0$,

$$A_t^f = \int_0^t f(|X_s|)\,\mathrm{d}s, \qquad A_\infty^f = \lim_{t\to\infty} A_t^f.$$

For $t>0$ we denote by g_t the last 0 of X before the moment t, that is, $g_t = \sup\{s \le t : X_s = 0\}$. Since $t \mapsto g_t$ is non-decreasing, there exists a limit $\widehat{g} = \lim_{t\to\infty} g_t$, and since X may be transient, $\widehat{g}$ may be finite a.s.

Theorem 3.1. *Under hypothesis A,*

$$\widehat{\mathrm{P}}\big(1-\mathrm{e}^{-\lambda A_R^f}\big) = -c^*(h_{\lambda,f}^{(\mu)})'(0), \tag{8}$$

where c^* is a local time constant. Moreover, for $\mu>0$,

$$\widehat{\mathrm{P}}_1\big(1-\mathrm{e}^{-\lambda A_R^f}\big) = -c^*(h_{\lambda,f}^{(\mu)})'(0)\frac{\mathrm{E}\big[\mathrm{e}^{-\lambda A_{\widehat{g}}^f} - \mathrm{e}^{-\lambda A_\infty^f}\big]}{\mathrm{E}\big[1-\mathrm{e}^{-\lambda A_\infty^f}\big]}, \tag{9}$$

$$\widehat{\mathrm{P}}_2\big(1-\mathrm{e}^{-\lambda A_R^f}\big) = -c^*(h_{\lambda,f}^{(\mu)})'(0)\frac{\mathrm{E}\big[1-\mathrm{e}^{-\lambda A_{\widehat{g}}^f}\big]}{\mathrm{E}\big[1-\mathrm{e}^{-\lambda A_\infty^f}\big]}. \tag{10}$$

Proof. Fix $t>0$. Let $d_t = \inf\{s \ge t : X_s = 0\}$. Note that we have $d_t = t + \sigma_0 \circ \theta_t$. For fixed $t>0$ and $\lambda>0$, we write the integral $\mathrm{E}\int_0^{d_t} \mathrm{e}^{-\lambda \int_0^s f(|X_v|)dv} f(|X_s|)\,\mathrm{d}s$ in terms of excursions of X. Recall that $L(X)$ is the process of the local time of X at 0, and $\tau_t = \inf\{s : L_s(X) > t\}$ is its inverse. Both processes are used to parametrize excursions of X from the point 0. We have

$$\mathrm{E}\big(1-\mathrm{e}^{-\lambda A_{d_t}^f}\big) = \lambda \mathrm{E}\int_0^{d_t} \mathrm{e}^{-\lambda A_u^f} f(|X_u|)\,\mathrm{d}u \tag{11}$$

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$$= \mathrm{E} \sum_{\tau_{s-} \le t} \int_{\tau_{s-}}^{\tau_s} \mathrm{e}^{-\lambda \int_0^r f(|X_v|)\mathrm{d}v} f(|X_r|)\, \mathrm{d}r$$

$$= \lambda \mathrm{E} \sum_{\tau_{s-} \le t} \mathrm{e}^{-\lambda A^f_{\tau_{s-}}} \left(\int_0^{\sigma_0} \mathrm{e}^{-\lambda A^f_u} f(|X_u|)\, \mathrm{d}u \right) \circ \theta_{\tau_{s-}}$$

$$= \mathrm{E} \sum_{\tau_{s-} \le t} \mathrm{e}^{-\lambda A^f_{\tau_{s-}}} \big(1 - \mathrm{e}^{-\lambda A^f_{\sigma_0}}\big) \circ \theta_{\tau_{s-}}.$$

The excursion formula [2, Chapter III, (3.27)] applied to the last expression yields

$$\mathrm{E}(1 - \mathrm{e}^{-\lambda A^f_{d_t}}) = \left(\mathrm{E} \int_0^t \mathrm{e}^{-\lambda A^f_s}\, \mathrm{d}L_s(X) \right) \widehat{\mathrm{P}}\big(1 - \mathrm{e}^{-\lambda A_R(f)}\big). \tag{12}$$

Recall that regardless of whether the process is recurrent or transient, there exists a σ-finite measure $\widehat{\mathrm{P}}$ on the excursion path space U such that (12) holds (see [2, Chapter III, Section (g)]). We now compute $\mathrm{E} \int_0^t \mathrm{e}^{-\lambda A^f_s}\, \mathrm{d}L_s(X)$. Using equality in law (5) we write

$$\mathrm{E} \int_0^t \mathrm{e}^{-\lambda A^f_s}\, \mathrm{d}L^*_s(X) = \mathrm{E} \int_0^t \mathrm{e}^{-\lambda \int_0^s f(S^{(-\mu)}_u - B^{(-\mu)}_u)\mathrm{d}u}\, \mathrm{d}S^{(-\mu)}_s.$$

It follows from (4) and Tanaka's formula that $\langle |X| \rangle_t = t$, so for $g \in \mathcal{C}^2([0, \infty))$,

$$\begin{aligned}
&\mathrm{E} g\big(S^{(-\mu)}_t - B^{(-\mu)}_t\big) \mathrm{e}^{-\lambda \int_0^t f\left(S^{(-\mu)}_u - B^{(-\mu)}_u\right)\mathrm{d}u} \\
&\quad = g(0) + \mathrm{E} \int_0^t g'(S^{(-\mu)}_u - B^{(-\mu)}_u) \mathrm{e}^{-\lambda \int_0^u f(S^{(-\mu)}_v - B^{(-\mu)}_v)\mathrm{d}v}\, \mathrm{d}\big(S^{(-\mu)}_u - B^{(-\mu)}_u\big) \\
&\qquad + \mathrm{E} \int_0^t \mathrm{e}^{-\lambda \int_0^u f\left(S^{(-\mu)}_v - B^{(-\mu)}_v\right)\mathrm{d}v} \left[\frac{1}{2} g''\big(S^{(-\mu)}_u - B^{(-\mu)}_u\big) - \lambda (fg)\big(S^{(-\mu)}_u - B^{(-\mu)}_u\big) \right] \mathrm{d}u,
\end{aligned}$$

which can be rewritten in the form

$$\begin{aligned}
&\mathrm{E} g\big(S^{(-\mu)}_t - B^{(-\mu)}_t\big) \mathrm{e}^{-\lambda \int_0^t f\left(S^{(-\mu)}_u - B^{(-\mu)}_u\right)\mathrm{d}u} \\
&\quad = g(0) + \mathrm{E} \int_0^t g'(S^{(-\mu)}_u - B^{(-\mu)}_u) \mathrm{e}^{-\lambda \int_0^u f(S^{(-\mu)}_v - B^{(-\mu)}_v)\mathrm{d}v}\, \mathrm{d}S^{(-\mu)}_u \\
&\qquad - \mathrm{E} \int_0^t g'(S^{(-\mu)}_u - B^{(-\mu)}_u) \mathrm{e}^{-\lambda \int_0^u f(S^{(-\mu)}_v - B^{(-\mu)}_v)\mathrm{d}v}\, \mathrm{d}B_u \\
&\qquad + \mathrm{E} \int_0^t \mathrm{e}^{-\lambda \int_0^u f\left(S^{(-\mu)}_v - B^{(-\mu)}_v\right)\mathrm{d}v} \left[\frac{1}{2} g''\big(S^{(-\mu)}_u - B^{(-\mu)}_u\big) \right. \\
&\qquad \left. + \mu g'\big(S^{(-\mu)}_u - B^{(-\mu)}_u\big) - \lambda (fg)\big(S^{(-\mu)}_u - B^{(-\mu)}_u\big) \right] \mathrm{d}u.
\end{aligned} \tag{13}$$

Since $f \ge 0$, to eliminate (on the right-hand side of the last equality) the expectation of the stochastic integral, it is enough to assume that g' is locally bounded and use some standard

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localization arguments. Since it is assumed that $(h_{\lambda,f}^{(\mu)})'$ is locally bounded, we can set $g = h_{\lambda,f}^{(\mu)}$ to obtain, from (13) and from the fact that $h_{\lambda,f}^{(\mu)}$ solves the underlying Dirichlet problem,

$$\mathrm{E}h_{\lambda,f}^{(\mu)}\big(S_t^{(-\mu)} - B_t^{(-\mu)}\big)\mathrm{e}^{-\lambda\int_0^t f(S_u^{(-\mu)}-B_u^{(-\mu)})\,\mathrm{d}u} = 1 + \mathrm{E}\int_0^t (h_{\lambda,f}^{(\mu)})'(S_u^{(-\mu)} - B_u^{(-\mu)})\mathrm{e}^{-\lambda\int_0^u f(S_v^{(-\mu)}-B_v^{(-\mu)})\,\mathrm{d}v}\,\mathrm{d}S_u^{(-\mu)}.$$

The measure $\mathrm{d}S_u^{(-\mu)}$ is non-zero on the set $\{u : S_u^{(-\mu)} - B_u^{(-\mu)} = 0\}$, so the last equality can be written in the form

$$\mathrm{E}h_{\lambda,f}^{(\mu)}\big(S_t^{(-\mu)} - B_t^{(-\mu)}\big)\mathrm{e}^{-\lambda\int_0^t f(S_u^{(-\mu)}-B_u^{(-\mu)})\,\mathrm{d}u} = 1 + (h_{\lambda,f}^{(\mu)})'(0)\mathrm{E}\int_0^t \mathrm{e}^{-\lambda\int_0^u f(S_v^{(-\mu)}-B_v^{(-\mu)})\,\mathrm{d}v}\,\mathrm{d}S_u^{(-\mu)}.$$

Hence, using (5) and $L^* = c^*L$, we have

$$\mathrm{E}h_{\lambda,f}^{(\mu)}(|X_t|)\mathrm{e}^{-\lambda\int_0^t f(|X_u|)\,\mathrm{d}u} = 1 + c^*(h_{\lambda,f}^{(\mu)})'(0)\mathrm{E}\int_0^t \mathrm{e}^{-\lambda\int_0^u f(|X_v|)\,\mathrm{d}v}\,\mathrm{d}L_u(X). \tag{14}$$

We compute the left-hand side of (12). Since $|X|$ is Markov,

$$\mathrm{E}(1 - \mathrm{e}^{-\lambda A_{d_t}^f}) = \mathrm{E}(1 - \mathrm{e}^{-\lambda A_t^f - \lambda(A_{\sigma_0}^f)\circ\theta_t}) = \mathrm{E}\big(1 - \mathrm{e}^{-\lambda A_t^f}h_{\lambda,f}^{(\mu)}(|X_t|)\big).$$

Inserting the last expression and (14) in (12) yields

$$-c^*(h_{\lambda,f}^{(\mu)})'(0)\mathrm{E}\big(1 - \mathrm{e}^{-\lambda A_t^f}h_{\lambda,f}^{(\mu)}(|X_t|)\big) = \big(1 - \mathrm{e}^{-\lambda A_t^f}h_{\lambda,f}^{(\mu)}(|X_t|)\big)\widehat{\mathrm{P}}\big(1 - \mathrm{e}^{-\lambda A_R(f)}\big),$$

which finishes the proof of (8). To prove (9), for $t > 0$ and $\mu > 0$ we consider the sum

$$I = \mathrm{E}\sum_{\tau_{s-}\le t}\mathbf{1}_{\{\sigma_0\circ\theta_{\tau_{s-}}=\infty\}}\mathrm{e}^{-\lambda A_{\tau_{s-}}^f}\big[1 - \mathrm{e}^{-\lambda A_u^f}\big]\circ\theta_{\tau_{s-}}, \tag{15}$$

which by the master excursion formula yields

$$I = \bigg(\mathrm{E}\int_0^t \mathrm{e}^{-\lambda A_s^X}\,\mathrm{d}L_s(X)\bigg)\widehat{\mathrm{P}}_1\big(1 - \mathrm{e}^{-\lambda A_R(f)}\big).$$

On the other side, following [2, Chapter III, Theorem 3.29], we observe that under the sum in (15) there can be at most one excursion starting at some $\tau_{s-} \in \{r : \tau_r > \tau_{r-}\}$ such that $\sigma_0 \circ \theta_{\tau_{s-}} = \infty$. Moreover, if this τ_{s-} appears after the moment t, then $d_t < \infty$ and the sum under expectation in I is zero. If $\tau_{s-} \le t$, then $d_t = \infty$, $\tau_{s-} = g_t$, and $\{\sigma_0 \circ \theta_{\tau_{s-}} = \infty\} = \{d_t = \infty\}$. As a result, we can rewrite I as

$$I = \mathrm{E}\mathbf{1}_{\{d_t=\infty\}}\mathrm{e}^{-\lambda A_{g_t}^f}\big[1 - \mathrm{e}^{-\lambda A_{\sigma_0}^f\circ\theta g_t}\big] = \mathrm{E}\mathbf{1}_{\{d_t=\infty\}}\big[\mathrm{e}^{-\lambda A_{g_t}^f} - \mathrm{e}^{-\lambda A_{d_t}^f}\big],$$

since $d_t = g_t + \sigma_0 \circ \theta g_t$. So, what we actually obtained from the two different forms of I is

$$\mathrm{E}\mathbf{1}_{\{d_t=\infty\}}\big[\mathrm{e}^{-\lambda A_{g_t}^f} - \mathrm{e}^{-\lambda A_{d_t}^f}\big] = \bigg(\mathrm{E}\int_0^t \mathrm{e}^{-\lambda A_s^X}\,\mathrm{d}L_s(X)\bigg)\widehat{\mathrm{P}}_1\big(1 - \mathrm{e}^{-\lambda A_R(f)}\big). \tag{16}$$

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By (8), we have

$$-c^*(h^{(\mu)}_{\lambda,f})'(0)\mathrm{E}\int_0^t \mathrm{e}^{-\lambda A^X_s}\,\mathrm{d}L_s(X) = \mathrm{E}\big[1-\mathrm{e}^{-\lambda A^f_{d_t}}\big],$$

so we can rewrite (16) as

$$-c^*(h^{(\mu)}_{\lambda,f})'(0)\mathrm{E}\mathbf{1}_{\{d_t=\infty\}}\big[\mathrm{e}^{-\lambda A^f_{g_t}} - \mathrm{e}^{-\lambda A^f_{d_t}}\big] = \mathrm{E}\big[1-\mathrm{e}^{-\lambda A^f_{d_t}}\big]\widehat{\mathrm{P}}_1\big(1-\mathrm{e}^{-\lambda A_R(f)}\big).$$

Since both $t\to g_t$ and $t\to d_t$ are monotone and $d_t\geq t$, we obtain from the last equality, by letting t tend to ∞,

$$-c^*(h^{(\mu)}_{\lambda,f})'(0)\mathrm{E}\big[\mathrm{e}^{-\lambda A^f_g} - \mathrm{e}^{-\lambda A^f_\infty}\big] = \mathrm{E}\big[1-\mathrm{e}^{-\lambda A^f_\infty}\big]\widehat{\mathrm{P}}_1\big(1-\mathrm{e}^{-\lambda A_R(f)}\big),$$

that is, (9). The equality (10) follows easily from (8) and (9), since

$$\widehat{\mathrm{P}}\big(1-\mathrm{e}^{-\lambda A_R(f)}\big) = \widehat{\mathrm{P}}_1\big(1-\mathrm{e}^{-\lambda A_R(f)}\big) + \widehat{\mathrm{P}}_2\big(1-\mathrm{e}^{-\lambda A_R(f)}\big).$$

This completes the proof. □

Remarks 3.1. The local time constant c^* in Theorem 3.1 may be computed as follows: for any $t>0$,

$$c^* = \frac{\phi(t)-\mu t}{\int_0^t p_u(0,0)\,\mathrm{d}u}, \quad \phi(t) = \int_0^\infty y\kappa(t,y)\,\mathrm{d}y - \mu\int_0^\infty\int_y^\infty \kappa(t,v)\,\mathrm{d}v\,\mathrm{d}y, \tag{17}$$

$$\kappa(t,y) = \frac{1}{\sqrt{2\pi t}}\mathrm{e}^{-\frac{(y+\mu t)^2}{2t}}, \quad p_t(0,0) = \frac{1}{\sqrt{2\pi t}}\mathrm{e}^{-\frac{\mu^2}{2}t} - \frac{\mu}{\sqrt{\pi}}\int_{\mu\sqrt{t/2}}^\infty \mathrm{e}^{-z^2}\,\mathrm{d}z.$$

Indeed, from the definition of the local time constant

$$c^* = \mathrm{E}L^*_t(X)\bigg(\int_0^t p_u(0,0)\,\mathrm{d}u\bigg)^{-1},$$

where from the Tanaka formula $\mathrm{E}L^*_t(X) = \mathrm{E}|X_t| - \mu t$. Directly from the form of the transition density of $|X_t|$ (see [3, p. 130]) we have the formula for $p_t(0,0)$ and $\mathrm{E}|X_t| = \phi(t)$, where ϕ is given by (17). Notice that c^* does not depend on t.

Remarks 3.2. Since $L(|X|) = 2L(X)$ it follows from the proof of Theorem 3.1 that the excursion measure of $|X|$ is equal to $\frac{1}{2}\widehat{\mathrm{P}}$. The formula in (8) corresponds to the fact that for a standard Brownian motion B, the solution of the corresponding Sturm–Liouville equation determines the Lévy measure of a subordinator associated with the functional $\int_0^t f(B_s)\,\mathrm{d}s$ (unlike in our considerations where the functional is $\int_0^t f(|X_s|)\,\mathrm{d}s$), and hence determines the distribution of A^f_R under the excursion measure (see [1, Proposition 9.3])

$$\widehat{\mathrm{P}}\big(1-\mathrm{e}^{-\lambda A^f_R}\big) = \frac{1}{2}\big((g_{\lambda,f})'(0-) - (g_{\lambda,f})'(0+)\big),$$

where $g_{\lambda,f}$ is the unique bounded solution of the associated Sturm–Liouville equation.

Corollary 3.1. *The density of R under the excursion measure $\widehat{\mathrm{P}}$ is given on $(0,\infty)$ by*

$$\widehat{\mathrm{P}}(R\in\mathrm{d}z) = \frac{c^*}{\sqrt{2\pi z^3}}\mathrm{e}^{-\frac{\mu^2}{2}z}\,\mathrm{d}z. \tag{18}$$

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Moreover,

$$\int_0^\infty (1-\mathrm{e}^{-\lambda z})\frac{1}{\sqrt{2\pi z^3}}\mathrm{e}^{-\frac{\mu^2 z}{2}}\,\mathrm{d}z = \mu + \sqrt{\mu^2+2\lambda}, \qquad \lambda > 0.$$

Proof. It follows from (1), and we will also prove it independently in Corollary 4.2, that

$$\widehat{\mathrm{P}}(R\in \mathrm{d}z) = \lim_{x\downarrow 0}\frac{c^*}{x} g_{x0}(z)\,\mathrm{d}z.$$

We fix $\lambda > 0$ and set $f(x) = \mathbf{1}_{(0,\infty)}(x)$, $x \geq 0$. To simplify the notation set $h_\lambda := h^{(\mu)}_{\lambda,f}$. It follows that, for $x \geq 0$,

$$h_\lambda(x) = \mathrm{E}_x \mathrm{e}^{-\lambda\sigma_0} = \mathrm{e}^{-\mu x - x\sqrt{\mu^2+2\lambda}}.$$

The formula in (18) follows since we may conclude by inverting the Laplace transform that

$$g_{x0}(z) = \frac{x}{\sqrt{2\pi z^3}}\mathrm{e}^{-\mu x - \frac{\mu^2 z}{2} - \frac{x^2}{2z}}.$$

The second assertion follows from Theorem 3.1, since we have $-h'_\lambda(0) = \mu + \sqrt{\mu^2+2\lambda}$. □

Theorem 3.1 enables us to describe the distribution of the occupation time of excursions under $\widehat{\mathrm{P}}$. Assume that $\mu = 0$, so $|X|$ is a recurrent process. For a given $\alpha > 0$ we define the occupation time for excursions on the path space U as $T_\alpha : U \to [0, \infty)$ given by

$$T_\alpha(u) = \int_0^{R(u)} \mathbf{1}_{[\alpha,\infty)}(u(s))\,\mathrm{d}s,$$

and $\widetilde{T}_\alpha : U \to [0, \infty)$ by

$$\widetilde{T}_\alpha(u) = \int_0^{R(u)} \mathbf{1}_{(0,\alpha]}(u(s))\,\mathrm{d}s.$$

Clearly, both T_α and $\widetilde{T}_\alpha$ are measurable. Moreover, $T_\alpha(u) = A_R(f_\alpha)(u)$ for $u \in U$ and $f_\alpha(x) = \mathbf{1}_{[\alpha,\infty)}(x)$, $x \geq 0$, and $\widetilde{T}_\alpha(u) = A_R(g_\alpha)(u)$ for $u \in U$ and $g_\alpha(x) = \mathbf{1}_{(0,\alpha]}(x)$, $x \geq 0$. Recall that

$$\Gamma(a, x) = \int_x^\infty t^{a-1}\mathrm{e}^{-t}\,\mathrm{d}t, \qquad a > 0, \quad x \geq 0.$$

Proposition 3.2. *For $\alpha > 0$, $x \geq 0$ we have, on $(0, \infty)$,*

$$\widehat{\mathrm{P}}(T_\alpha \in \mathrm{d}y) = \frac{1}{4\sqrt{\pi}\alpha^2}\mathrm{e}^{y/(2\alpha)^2}\bigg[\Gamma\bigg(\frac{1}{2}, \frac{y}{2\alpha^2}\bigg) - 2\frac{\alpha}{\sqrt{2y}}\Gamma\bigg(1, \frac{y}{2\alpha^2}\bigg)\bigg]\,\mathrm{d}y. \tag{19}$$

Moreover, if $\alpha \geq x$ then

$$\widehat{\mathrm{P}}(\widetilde{T}_\alpha > t) = \frac{1}{\sqrt{2\pi t}}\sum_{n=1}^\infty (-1)^n\big[\mathrm{e}^{-2\alpha^2 n^2/t} - \mathrm{e}^{-2\alpha^2(n-1)^2/t}\big], \qquad t > 0. \tag{20}$$

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Proof. Recall that $c^* = \frac{1}{2}$ for $\mu = 0$. We first prove (19). We have, from [3, Section 2.4.1, p. 200],

$$h_\lambda(x) = \mathrm{E}_x \mathrm{e}^{-\lambda \int_0^{\sigma_0} \mathbf{1}_{(\alpha,\infty)}(B_s)\mathrm{d}s} = \mathbf{1}_{[\alpha,\infty)}(x) \frac{\mathrm{e}^{-x\sqrt{2\lambda}}}{1+\alpha\sqrt{2\lambda}} + \mathbf{1}_{(0,\alpha]}(x) \frac{1+(\alpha - x)\sqrt{2\lambda}}{1+\alpha\sqrt{2\lambda}}.$$

Thus, $h'_\lambda(0) = -\frac{\sqrt{2\lambda}}{1+\alpha\sqrt{2\lambda}}$, and it follows from Theorem 3.1 that

$$\widehat{\mathrm{P}}\big(1 - \mathrm{e}^{-\lambda A_R(f_\alpha)}\big) = \frac{1}{\sqrt{2}} \frac{\sqrt{\lambda}}{1+\alpha\sqrt{2\lambda}}. \tag{21}$$

On the other side,

$$\begin{aligned}
\frac{\sqrt{\lambda}}{1+\alpha\sqrt{2\lambda}} &= \frac{1}{\sqrt{2}\alpha} \int_0^\infty \big(1 - \mathrm{e}^{-\alpha\sqrt{2\lambda}t}\big)\mathrm{e}^{-t}\,\mathrm{d}t \\
&= \frac{1}{\sqrt{2}\alpha} \int_0^\infty \int_0^\infty \big(1 - \mathrm{e}^{-\alpha^2\lambda t^2/(2v)}\big) \frac{1}{\sqrt{\pi v}} \mathrm{e}^{-v-t}\,\mathrm{d}v\,\mathrm{d}t \\
&= \int_0^\infty \big(1 - \mathrm{e}^{-\lambda y}\big) \frac{\alpha}{2\sqrt{\pi y^3}} \bigg[\int_0^\infty t\mathrm{e}^{-t-\frac{\alpha^2 t^2}{2y}}\,\mathrm{d}t\bigg]\mathrm{d}y =: I,
\end{aligned}$$

where in the last equality we used Fubini's theorem. We compute the inner integral in the last expression:

$$\begin{aligned}
I = \int_0^\infty \big(1 - \mathrm{e}^{-\lambda y}\big) \frac{\alpha}{2\sqrt{\pi y^3}} \frac{\mathrm{e}^{y/(2\alpha^2)}}{4(\alpha^2/(2y))^{\frac{3}{2}}} \\
\times \bigg[\Gamma\bigg(\frac{1}{2}, y/(2\alpha^2)\bigg) - 2\sqrt{\alpha^2/(2y)}\Gamma\big(1, y/(2\alpha^2)\big)\bigg]\mathrm{d}y.
\end{aligned} \tag{22}$$

Equation (19) follows now from inserting (22) into (21). To prove (20), we use the formula [3, Section 2.5.1, p. 201]

$$h_\lambda(x) = \mathrm{E}_x \mathrm{e}^{-\lambda \int_0^{\sigma_0} \mathbf{1}_{[0,\alpha]}(B_s)\mathrm{d}s} = \frac{\cosh\big(\sqrt{2\lambda}(\alpha - x)\big)}{\cosh\big(\sqrt{2\lambda}\alpha\big)}, \qquad x \geq 0, \quad \alpha \geq x.$$

Since $h'_\lambda(0) = -\tanh(\alpha\sqrt{2\lambda})\sqrt{2\lambda}$, we have, from Theorem 3.1,

$$\widehat{\mathrm{P}}\big(1 - \mathrm{e}^{-\lambda \widetilde{T}_\alpha}\big) = \frac{1}{2} \tanh(\alpha\sqrt{2\lambda})\sqrt{2\lambda}. \tag{23}$$

Observe that $\widetilde{T}_\alpha \leq R(u)$. It is well known that $\widehat{\mathrm{P}}(R > t) = \frac{1}{\sqrt{\pi t}}$ for any $t > 0$ (see [2, p. 112]). As a result, $\widehat{\mathrm{P}}(\widetilde{T}_\alpha > t) < \infty$, and from Fubini's theorem,

$$\widehat{\mathrm{P}}\big(1 - \mathrm{e}^{-\lambda \widetilde{T}_\alpha}\big) = \lambda \int_0^\infty \mathrm{e}^{-\lambda t} \widehat{\mathrm{P}}(\widetilde{T}_\alpha > t)\,\mathrm{d}t. \tag{24}$$

On the other side, it is well known that for any $p > 0$

$$\frac{\tanh(\alpha p)}{p} = \int_0^\infty \mathrm{e}^{-pt}\psi_\alpha(t)\mathrm{d}t, \tag{25}$$

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where $\psi_\alpha(t)=\sum_{n=1}^{\infty}(-1)^{n-1}\mathbf{1}_{[2\alpha(n-1),2\alpha n]}(t)$, (see [24, table of inverse Laplace transforms – hyperbolic functions, p. 11]. Using $\mathrm{E}\exp\left(-\frac{p^2}{4\gamma_{\frac{1}{2}}}\right)=\mathrm{e}^{-p}$ ($\gamma_{\frac{1}{2}}$ being the gamma random variable with parameter $\frac{1}{2}$), we obtain that

$$\int_0^\infty \mathrm{e}^{-pt}\psi_\alpha(t)\,\mathrm{d}t=\int_0^\infty \mathrm{e}^{-\frac{p^2}{2}t}\frac{1}{\sqrt{2\pi}\sqrt{t^3}}\left[\int_0^\infty \mathrm{e}^{-\frac{w^2}{2t}}\psi_\alpha(w)w\,\mathrm{d}w\right]\mathrm{d}t. \tag{26}$$

Joining (23), (24), (25), and (26) yields

$$\frac{1}{2}\int_0^\infty \mathrm{e}^{-\lambda t}\,\widehat{\mathrm{P}}(\widetilde{T}_\alpha>t)\,\mathrm{d}t=\frac{1}{2}\int_0^\infty \mathrm{e}^{-\lambda t}\frac{1}{\sqrt{2\pi}\sqrt{t^3}}\left[\int_0^\infty \mathrm{e}^{-\frac{w^2}{2t}}\psi_\alpha(w)w\,\mathrm{d}w\right]\mathrm{d}t.$$

Since $|\psi_\alpha(t)|\le\sum_{n=1}^{\infty}\mathbf{1}_{[2\alpha(n-1),2\alpha n]}(t)=1$, we have, from Fubini's theorem,

$$\begin{aligned}\widehat{\mathrm{P}}(\widetilde{T}_\alpha>t)&=\sqrt{1/(2\pi t^3)}\sum_{n=1}^{\infty}\int_{2\alpha(n-1)}^{2\alpha n}(-1)^{n-1}w\mathrm{e}^{-\frac{w^2}{2t}}\,\mathrm{d}w\\&=\sqrt{1/(2\pi t^3)}\sum_{n=1}^{\infty}(-1)^n t\left[\mathrm{e}^{-2\alpha^2n^2/t}-\mathrm{e}^{-2\alpha^2(n-1)^2/t}\right],\end{aligned}$$

which completes the proof. □

4. Itó diffusions

Under some additional assumptions we are able to extend the result of Theorem 3.1 on the two subclasses of Itó diffusions. Precisely, consider one-dimensional Itô diffusion X with values in $E\subseteq\mathbb{R}$ satisfying the stochastic differential equation

$$\mathrm{d}X_t=\sigma(X_t)\mathrm{d}B_t+\mu(X_t)\mathrm{d}t, \tag{27}$$

where B is a Brownian motion, $X_0=x\in E$ under P_x a.s., and σ and μ are locally bounded measurable functions on E such that (27) has a solution unique in probability. For this we may assume the Engelbert–Schmidt conditions, i.e. that $\sigma\neq 0$ and $\frac{1}{\sigma^2}$, $\frac{\mu}{\sigma^2}$ are locally integrable (see [12, Proposition 5.15]). We assume that $0\in E$, and if m denotes the speed measure of X we demand that $m(\{0\})=0$. We assume that X does not explode during lifetime. It is well known [3, Chapter II, pp. 4] that in the above setup X has a jointly continuous transition density $p_t(x,y)$ with respect to the speed measure m, i.e.

$$\mathrm{P}_x(X_t\in A)=\int_A p_t(x,y)m(\mathrm{d}y)$$

for every Borel subset A of E. Recall that $\sigma_z=\inf\{t>0:X_t=z\}$ for $z\in E$, and notice that it is a terminal stopping time, so on $\{\sigma_z>t\}$ it is equal to $\sigma_z=t+\sigma_z\circ\theta_t$. Since the excursion path space is a subspace of the canonical space, the same stopping time σ_z may be defined on the space of excursions (see the discussion in [20, p. 491]). Notice that R and σ_0 are terminal stopping times. $\widehat{\mathrm{P}}$ denotes the excursion measure on the path space, associated to excursions of X, and by $\widehat{X}$ we denote the coordinate process killed under $\widehat{\mathrm{P}}$ at R, so $\widehat{X}_t(u)=u(t)$ for $t<R$ (see [20, p. 482]). For a positive Borel function f and $z\in E$ define

$$g^{(z)}_{\lambda,f}(x)=\mathrm{E}_x\exp\left(-\lambda\int_0^{\sigma_0\wedge\sigma_z} f(|X_s|)\,\mathrm{d}s\right), \tag{28}$$
$$g^{(0)}_{\lambda,f}(x)=\mathrm{E}_x\exp\left(-\lambda\int_0^{\sigma_0} f(|X_s|)\,\mathrm{d}s\right).$$

Let $\mathcal{A}$ denote the generator of X. Consider the Dirichlet problem

$$\mathcal{A}g(x)=\lambda f(x)g(x), \qquad x\in(0,z), \tag{29}$$
$$g(0)=g(z)=1,$$

where $z>0$ and f is a given Borel and non-negative function on E. It is well known that under some mild conditions $g^{(z)}_{\lambda,f}$ solves the Dirichlet problem (29). It is enough to assume that f is non-negative and continuous (see [12, Proposition 7.2, Chapter 5] and [20, Proposition 3.1, Chapter VII]), but the assumptions on f can be relaxed. For instance, f may be chosen locally L^d for some natural d (see the discussion on sufficient assumptions in [4, 8, 23]). For a non-negative Borel function f on E, recall the definition of the integral functional $A^f_R(u)=\int_0^{R(u)} f(|u(s)|)\,\mathrm{d}s$. The assumptions which enable us to formulate the version of Theorem 3.1 for $E=[0,\infty)$ are gathered in the following hypothesis:

Definition. [Hypothesis B] We say that hypothesis B is satisfied for a non-negative Borel function f on E if

1. 0 is regular and instantaneously reflecting,
2. for every $z\in E$ the function $g^{(z)}_{\lambda,f}$ is $\mathcal{C}^2(0,\infty)$, solves (29), and has the first right derivative in 0,
3. the function $(g^{(z)}_{\lambda,f})'$ is locally bounded.

An example of a process satisfying the above conditions is a squared Bessel process with index $\mu>-1$ (see [20, Chapter XI]). Another example is a radial Ornstein–Uhlenbeck process [3, Appendix I, 26] or a Pearson diffusion ([7]). Alternatively, we may define a diffusion reflecting at 0 as a solution of

$$\mathrm{d}X_t=\sigma(X_t)\mathrm{d}B_t+\mu(X_t)\mathrm{d}t+\mathrm{d}L_t(X),$$

where σ and μ are some well measurable coefficients and $L(X)$ denotes the local time of X at 0. One may check that the last approach is analogous to what we will present in the following. If $E=\mathbb{R}$ we have the following set of assumptions:

Definition. [Hypothesis B′] We say that hypothesis B′ is satisfied for a non-negative Borel function f on E if

1. 0 is regular,
2. $|X|$ is a Markov process, $\mu(x)=-\mu(-x)$, and $\sigma^2(x)=\sigma^2(|x|)$ for every $x\in E$,

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3. for every $z \in E$ the function $g^{(z)}_{\lambda,f}$ is $\mathcal{C}^2(0,\infty)$, solves (29), and has the first right derivative in 0,
4. the function $(g^{(z)}_{\lambda,f})'$ is locally bounded.

An example of a process satisfying hypothesis B′ is an Ornstein–Uhlenbeck process [3, Appendix I, 24]. That $|X|$ is a Markov process may be verified with using the semi-group of the process and the conditions of [15, Theorem 2].

Theorem 4.1. *Let f be Borel and non-negative, $\lambda > 0$, and $z \in E$. Under either hypothesis B or B′ we have*

$$\widehat{\mathrm{P}}\big(1-\mathrm{e}^{-\lambda A^f_{R\wedge\sigma_z}}\big) = -c^*(g^{(z)}_{\lambda,f})'(0), \tag{30}$$

where c^* is a local time constant. In particular,

$$\widehat{\mathrm{P}}\big(1-\mathrm{e}^{-\lambda A^f_R}\big) = -c^*(g^{(0)}_{\lambda,f})'(0). \tag{31}$$

Proof. Let $z \in E$ ($E=[0,\infty)$ under B and $E=\mathbb{R}$ under B′). We follow the same idea as the proof of Theorem 3.1. Beside stopping at σ_z, the difference is the use of the Tanaka formula instead of extended Levy's theorem. The terminal property of σ_z enables the use of summation trick (11):

$$\begin{aligned}\mathrm{E}(1-\mathrm{e}^{-\lambda A^f_{d_t\wedge\sigma_z}}) &= \lambda \mathrm{E}\int_0^{d_t} \mathbf{1}_{\{u<\sigma_z\}}\mathrm{e}^{-\lambda A^f_u} f(|X_u|)\,\mathrm{d}u \\ &= \lambda \mathrm{E}\sum_{\tau_{s-}\le t\wedge\sigma_z} \mathrm{e}^{-\lambda A^f_{\tau_{s-}}}\bigg(\int_0^{\sigma_0}\mathbf{1}_{\{u<\sigma_z\}}\mathrm{e}^{-\lambda A^f_u} f(|X_u|)\,\mathrm{d}u\bigg)\circ\theta_{\tau_{s-}}.\end{aligned}$$

The master formula of excursions yields

$$\mathrm{E}(1-\mathrm{e}^{-\lambda A^f_{d_t\wedge\sigma_z}}) = \mathrm{E}\bigg(\int_0^{t\wedge\sigma_z}\mathrm{e}^{-\lambda A^f_s}\,\mathrm{d}L_s(X)\bigg)\widehat{\mathrm{P}}\big(1-\mathrm{e}^{-\lambda A^f_{R\wedge\sigma_z}}\big). \tag{32}$$

Let g be a $\mathcal{C}^2(E)$ function with both derivatives locally bounded. Let $\lambda > 0$. By the Itó and Tanaka formulas,

$$\begin{aligned} g(|X_t|)\mathrm{e}^{-\lambda\int_0^t f(|X_s|)\mathrm{d}s} &= g(0) \\ &\quad + \int_0^t \mathrm{e}^{-\lambda\int_0^s f(|X_u|)\mathrm{d}u}\bigg(g'(|X_s|)\,\mathrm{d}|X_s| + \frac{1}{2}g''(|X_s|)\sigma^2(X_s)\bigg)\,\mathrm{d}s \\ &\quad - \lambda\int_0^t f(|X_s|)g(|X_s|)\mathrm{e}^{-\lambda\int_0^s f(|X_u|)\mathrm{d}u}\,\mathrm{d}s \\ &= g(0) + \int_0^t \mathrm{e}^{-\lambda\int_0^s f(|X_u|)\mathrm{d}u} g'(|X_s|)\big(\mathrm{sgn}(X_s)(\sigma(X_s)\,\mathrm{d}B_s \\ &\quad + \mu(X_s)\,\mathrm{d}s) + \mathrm{d}L^*_s(X)\big) \\ &\quad + \frac{1}{2}\int_0^t \mathrm{e}^{-\lambda\int_0^s f(|X_u|)\mathrm{d}u} g''(|X_s|)\sigma^2(X_s)\,\mathrm{d}s \\ &\quad - \lambda\int_0^t f(|X_s|)g(|X_s|)\mathrm{e}^{-\lambda\int_0^s f(|X_u|)\mathrm{d}u}\,\mathrm{d}s,\end{aligned}$$

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where clearly $L^*(X)$ denotes the process of local time of X at 0 compatible with the Tanaka formula. A standard localization argument shows that under hypothesis B we have

$$\begin{aligned}
&\mathrm{E}g(|X_{t\wedge\sigma_z}|)\mathrm{e}^{-\lambda\int_0^{t\wedge\sigma_z} f(|X_s|)\mathrm{d}s} \\
&= g(0) + \mathrm{E}\int_0^{t\wedge\sigma_z} \mathrm{e}^{-\lambda\int_0^s f(|X_u|)\mathrm{d}u} g'(|X_s|)(\operatorname{sgn}(X_s)\mu(X_s)\,\mathrm{d}s) + \mathrm{d}L_s^*(X)) \\
&\quad + \frac{1}{2}\int_0^{t\wedge\sigma_z} \mathrm{e}^{-\lambda\int_0^s f(|X_u|)\mathrm{d}u} g''(|X_s|)\sigma^2(X_s)\,\mathrm{d}s \\
&\quad - \lambda\int_0^{t\wedge\sigma_z} f(|X_s|)g(|X_s|)\mathrm{e}^{-\lambda\int_0^s f(|X_u|)\mathrm{d}u}\,\mathrm{d}s,
\end{aligned} \tag{33}$$

so that

$$\begin{aligned}
\mathrm{E}g(|X_{t\wedge\sigma_z}|)\mathrm{e}^{-\lambda\int_0^{t\wedge\sigma_z} f(|X_s|)\mathrm{d}s} &= g(0) + g'(0)\mathrm{E}\int_0^{t\wedge\sigma_z} \mathrm{e}^{-\lambda\int_0^s f(|X_u|)\mathrm{d}u}\,\mathrm{d}L_s^*(X) \\
&+ \mathrm{E}\int_0^{t\wedge\sigma_z} \mathrm{e}^{-\lambda\int_0^s f(|X_u|)\mathrm{d}u}\Big(\frac{1}{2}g''(|X_s|)\sigma^2(|X_s|) + g'(|X_s|)\mu(|X_s|) - \lambda f(|X_s|)g(|X_s|)\Big)\,\mathrm{d}s \\
&+ \mathrm{E}\int_0^{t\wedge\sigma_z} \mathrm{e}^{-\lambda\int_0^s f(|X_u|)\mathrm{d}u} g'(|X_s|)\mu(|X_s|)(\operatorname{sgn}(X_s) - 1)\,\mathrm{d}s,
\end{aligned}$$

and the last term of the above equality is easily seen to be 0. We choose $g = g_{\lambda,f}^{(z)}$ defined by (28), and since it is a solution of the associated Dirichlet problem, we obtain

$$\mathrm{E}g_{\lambda,f}^{(z)}(|X_{t\wedge\sigma_z}|)\mathrm{e}^{-\lambda\int_0^{t\wedge\sigma_z} f(|X_s|)\mathrm{d}s} = 1 + (g_{\lambda,f}^{(z)})'(0)\mathrm{E}\int_0^{t\wedge\sigma_z} \mathrm{e}^{-\lambda\int_0^s f(|X_u|)\mathrm{d}u}\,\mathrm{d}L_s^*(X). \tag{34}$$

Under hypothesis B′ we rewrite (33) with $\sigma_z \wedge \sigma_{-k}$, $k > 0$, in place of σ_z and observe that the identities $\operatorname{sgn}(x)\mu(x) = \mu(x)$, $\sigma^2(x) = \sigma^2(|x|)$ and the choice of $g = g_{\lambda,f}^{(z)}$ yield

$$\begin{aligned}
&\mathrm{E}g_{\lambda,f}^{(z)}(|X_{t\wedge\sigma_z\wedge\sigma_{-k}}|)\mathrm{e}^{-\lambda\int_0^{t\wedge\sigma_z\wedge\sigma_{-k}} f(|X_s|)\,\mathrm{d}s} \\
&\quad = 1 + (g_{\lambda,f}^{(z)})'(0)\mathrm{E}\int_0^{t\wedge\sigma_z\wedge\sigma_{-k}} \mathrm{e}^{-\lambda\int_0^s f(|X_u|)\mathrm{d}u}\,\mathrm{d}L_s^*(X).
\end{aligned}$$

Letting $k \to \infty$ we obtain that the equality (34) holds under B′. Since $L^* = c^*L$, we finally have

$$\mathrm{E}g_{\lambda,f}^{(z)}(|X_{\sigma_z\wedge t}|)\mathrm{e}^{-\lambda\int_0^{\sigma_z\wedge t} f(|X_s|)\mathrm{d}s} = 1 + c^*(g_{\lambda,f}^{(z)})'(0)\mathrm{E}\int_0^{\sigma_z\wedge t} \mathrm{e}^{-\lambda\int_0^s f(|X_u|)\mathrm{d}u}\,\mathrm{d}L_s(X). \tag{35}$$

Using the Markov property ($|X|$ is a Markov process under B and B′) and the terminal property of σ_z, we write

$$\begin{aligned}
\mathrm{E}(1 - \mathrm{e}^{-\lambda A^f_{d_t\wedge\sigma_z}}) &= \mathrm{E}(1 - \mathrm{e}^{-\lambda A^f_{d_t\wedge\sigma_z}})(\mathbf{1}_{\{\sigma_z\ge t\}} + \mathbf{1}_{\{\sigma_z<t\}}) \\
&= \mathrm{E}(1 - \mathrm{e}^{-\lambda A^f_{t+\sigma_0\circ\theta_t}})\mathbf{1}_{\{\sigma_z\ge t\}}\mathbf{1}_{\{\sigma_z\circ\theta_t\ge\sigma_0\circ\theta_t\}} \\
&\quad + \mathrm{E}(1 - \mathrm{e}^{-\lambda A^f_{t+\sigma_z\circ\theta_t}})\mathbf{1}_{\{\sigma_z\ge t\}}\mathbf{1}_{\{\sigma_z\circ\theta_t<\sigma_0\circ\theta_t\}}
\end{aligned}$$

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$$
\begin{aligned}
&+\mathrm{E}(1-\mathrm{e}^{-\lambda A^f_{\sigma_z}} g^{(z)}_{\lambda,f}(z))\mathbf{1}_{\{\sigma_z<t\}}\\
&=\mathrm{E}\big(1-\mathrm{e}^{-\lambda A^f_t} g^{(z)}_{\lambda,f}(|X_t|)\big)\mathbf{1}_{\{\sigma_z\ge t\}}+\mathrm{E}\big(1-\mathrm{e}^{-\lambda A^f_{\sigma_z}} g^{(z)}_{\lambda,f}(z)\big)\mathbf{1}_{\{\sigma_z<t\}}\\
&=\mathrm{E}\big(1-\mathrm{e}^{-\lambda A^f_{t\wedge\sigma_z}} g^{(z)}_{\lambda,f}(|X_{t\wedge\sigma_z}|)\big),
\end{aligned}
$$

so that

$$\mathrm{E}\mathrm{e}^{-\lambda A^f_{d_t\wedge\sigma_z}}=\mathrm{E}\mathrm{e}^{-\lambda A^f_{t\wedge\sigma_z}} g^{(z)}_{\lambda,f}(|X_{t\wedge\sigma_z}|).$$

Inserting the last equality into (35) and comparing it with (32) finally yield

$$\widehat{\mathrm{P}}\big(1-\mathrm{e}^{-\lambda A^f_{R\wedge\sigma_z}}\big)=-c^*(g^{(z)}_{\lambda,f})'(0),$$

where we used $\mathrm{E}\int_0^t \mathrm{e}^{-\lambda\int_0^s f(|X_u|)\mathrm{d}u}\,\mathrm{d}L_s(X)>0$, which is true by (32). Formula (31) follows from (30) by the monotone convergence and Lebesgue theorems. □

Theorem 4.2. *Let $z\in E$. Assume hypothesis B or B′ holds for f. Then*

$$\widehat{\mathrm{P}}\big(A^f_{R\wedge\sigma_z}\in\mathrm{d}v\big)=\lim_{x\downarrow 0}\frac{c^*}{x}\mathrm{P}_x\bigg(\int_0^{\sigma_0\wedge\sigma_z} f(|X_u|)\,\mathrm{d}u\in\mathrm{d}v\bigg),$$

where the limit is in the sense of weak convergence.

Proof. Under either B or B′, for any $\lambda>0$ we have, by (30),

$$\int_0^\infty(1-\mathrm{e}^{-\lambda v})\widehat{\mathrm{P}}\big(A^f_{R\wedge\sigma_z}\in\mathrm{d}v\big)=\lim_{x\downarrow 0}\int_0^\infty(1-\mathrm{e}^{-\lambda v})\frac{c^*}{x}\mathrm{P}_x\bigg(\int_0^{\sigma_0\wedge\sigma_z} f(|X_u|)\,\mathrm{d}u\in\mathrm{d}v\bigg),$$

so the proof will be done if we show that we may apply the monotone convergence theorem. Recall that $c^*>0$. It is enough to conclude that the function $x\mapsto\frac{1}{x}\mathrm{P}_x\big(\int_0^{\sigma_0\wedge\sigma_z} f(|X_u|)\,\mathrm{d}u\in\mathrm{d}v\big)$ is monotone on some interval $(0,\epsilon)$. We will prove that $x\mapsto\frac{1}{x}\mathrm{E}_x\big(1-\mathrm{e}^{-\lambda\int_0^{\sigma_0\wedge\sigma_z} f(|X_u|)\mathrm{d}u}\big)$ is monotone. Due to the assumptions on $g^{(z)}_{\lambda,f}$ we may write, for $x>0$,

$$g^{(z)}_{\lambda,f}(x)=1+x(g^{(z)}_{\lambda,f})'(0)+\frac{1}{2}\int_0^x(x-v)(g^{(z)}_{\lambda,f})''(v)\,\mathrm{d}v,$$

so that

$$\frac{1}{x}\mathrm{E}_x\big(1-\mathrm{e}^{-\lambda\int_0^{\sigma_0\wedge\sigma_z} f(|X_u|)\mathrm{d}u}\big)=-(g^{(z)}_{\lambda,f})'(0)-\frac{1}{2}\int_0^x\bigg(1-\frac{v}{x}\bigg)(g^{(z)}_{\lambda,f})''(v)\,\mathrm{d}v.$$

If we show that $g^{(z)}_{\lambda,f}$ is convex on some $(0,\epsilon)$ then we will have there

$$\frac{\mathrm{d}}{\mathrm{d}x}\bigg(\int_0^x\bigg(1-\frac{v}{x}\bigg)(g^{(z)}_{\lambda,f})''(v)\,\mathrm{d}v\bigg)=\int_0^x\frac{v}{x^2}(g^{(z)}_{\lambda,f})''(v)\,\mathrm{d}v>0.$$

For this, let $0<a<b$ and $I=(a,b)$, and define $\sigma_I=\sigma_a\wedge\sigma_b$. Let $x\in I$ and define $g^{(z,I)}_{\lambda,f}(x)=\mathrm{E}_x\mathrm{e}^{-\lambda\int_0^{\sigma_I}\mathbf{1}_{\{u<\sigma_z\}}f(|X_u|)\mathrm{d}u}$. Let $\alpha\in(0,1)$. We choose $J=(c,d)$ such that $a<c<x<d<b$, and

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such that $\alpha = \mathrm{P}_x(\sigma_c < \sigma_d)$ (and define σ_J and $g^{(\sigma_z,J)}_{\lambda,f}$ analogously). We have, by the terminal property of σ_z and the strong Markov property,

$$\begin{aligned} g^{(z,I)}_{\lambda,f}(x) &= \mathrm{E}_x \exp\left(-\lambda\left(\int_0^{\sigma_J} \mathbf{1}_{\{s<\sigma_z\}} f(|X_s|)\,\mathrm{d}s + \left(\int_0^{\sigma_I} \mathbf{1}_{\{s<\sigma_z\}} f(|X_s|)\,\mathrm{d}s\right)\circ\theta_{\sigma_J}\right)\right) \\ &= \mathrm{E}_x\left[\exp\left(-\lambda\int_0^{\sigma_J} \mathbf{1}_{\{s<T\}} f(|X_s|)\,\mathrm{d}s\right) g^{(T,I)}_{\lambda,f}(X_{\sigma_J})\right] \\ &\le g^{(z,I)}_{\lambda,f}(c)\mathrm{P}_x(\sigma_c<\sigma_d) + g^{(z,I)}_{\lambda,f}(d)\mathrm{P}_x(\sigma_d<\sigma_c) = \alpha\, g^I_{\lambda,f}(c) + (1-\alpha)\, g^I_{\lambda,f}(d). \end{aligned}$$

So, we conclude that $g^{(z,I)}_{\lambda,f}$ is convex on every such I. So is $g^{(z)}_{\lambda,f}$, since it is an extension of $g^I_{\lambda,f}$ for $a \downarrow 0$ and $b \uparrow \infty$. □

Corollary 4.1. *Let $z \in E$ and assume that hypothesis B or B′ holds for both functions f and $h(x) = \mathbf{1}_{(0,\infty)}(x)\big(\lambda f(x) + \frac{\gamma}{\lambda}\big)$, where f is non-negative and Borel, and $\lambda > 0$, $\gamma > 0$. Then*

$$\widehat{\mathrm{P}}\big(A^f_{R\wedge\sigma_z} \in \mathrm{d}v,\, R\wedge\sigma_z \in \mathrm{d}v\big) = \lim_{x\downarrow 0} \frac{c^*}{x} \mathrm{P}_x\left(\int_0^{\sigma_0\wedge\sigma_z} f(|X_u|)\,\mathrm{d}u \in \mathrm{d}v,\ \sigma_0\wedge\sigma_z \in \mathrm{d}v\right).$$

Proof. It follows that for any non-negative Borel f_0 we have, from Theorem 4.2,

$$\widehat{\mathrm{P}} f_0\big(A^f_{R\wedge\sigma_z}\big) = \lim_{x\downarrow 0}\frac{c^*}{x}\mathrm{E}_x f_0\left(\int_0^{\sigma_0\wedge\sigma_z} f(|X_u|)\,\mathrm{d}u\right).$$

We set $f_0(x) = \exp\{-\lambda x\}$ and consider A^h_R. By a standard Laplace transform argument we conclude that $(A^f_{R\wedge\sigma_z}, R\wedge\sigma_z)$ has, under $\widehat{\mathrm{P}}$, the same distribution as $\big(\int_0^{\sigma_0\wedge\sigma_z} f(|X_s|)\,\mathrm{d}s,\ \sigma_0\wedge\sigma_z\big)$ under $\lim_{x\downarrow 0}\frac{c^*}{x}\mathrm{P}_x$. □

Remark 1. Recall that the limit describing $\widehat{\mathrm{P}}$ in [17] was taken with respect to the scale function. Observe that for Itó diffusion defined by (27) we can choose $s(x) = \int_0^x \exp\{-2\int_0^y \mu(z)\sigma^{-2}(z)\,\mathrm{d}z\}\,\mathrm{d}y$, so $s(0) = 0$ and $s'(0) = 1$. As a result, identity (1) may be rewritten as

$$\begin{aligned} \widehat{\mathrm{P}}(A) &= \lim_{x\downarrow 0}\frac{\mathrm{P}_x(A)}{s(x)} = \lim_{x\downarrow 0}\frac{\mathrm{P}_x(A)}{x}\,\frac{x}{s(x)} = \frac{1}{s'(0)}\lim_{x\downarrow 0}\frac{\mathrm{P}_x(A)}{x} \\ &= \lim_{x\downarrow 0}\frac{\mathrm{P}_x(A)}{x}. \end{aligned}$$

Notice that $\widehat{\mathrm{P}}$ depends on the choice of s.

Corollary 4.1 gives the explicit form of the distribution of R.

Corollary 4.2. *Under the assumptions of Corollary 4.1, we have, from Theorem 4.2,*

$$\widehat{\mathrm{P}}(R \in \mathrm{d}z) = \lim_{x\downarrow 0}\frac{c^*}{x} g_{x0}(z)\,\mathrm{d}z,$$

so the distribution of R under the excursion measure is retrieved from the density of the first hitting time of 0.

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Last, but not least, we present the absolute-continuity relationship between functionals on excursion spaces associated with local martingales. If X is a local martingale and satisfies (27), then $\mu(x) = 0$ for all $x \in E$.

Theorem 4.3. *Let $E = [0, \infty)$. Assume that we are given two different excursion measures $\widehat{\mathrm{P}}$ and $\widehat{\mathrm{Q}}$ associated with two reflected at 0 Markov processes denoted by $X^{(1)}$ and $X^{(2)}$ respectively. Assume that hypothesis B holds for both processes. If $X^{(1)}$ and $X^{(2)}$ are local martingales, and m_1', m_2' denote the densities with respect to the Lebesgue measure of their speed measures respectively, then for any $b > 0$ and $\widehat{X}$ denoting the coordinate process killed at R we have*

$$c_2^* \, \widehat{\mathrm{P}}\bigg(\int_0^{\sigma_b \wedge R} f(\widehat{X}_s) m_2'(\widehat{X}_s) \, \mathrm{d}s \bigg) = c_1^* \, \widehat{\mathrm{Q}}\bigg(\int_0^{\sigma_b \wedge R} f(\widehat{X}_s) m_1'(\widehat{X}_s) \, \mathrm{d}s \bigg),$$

where c_1^*, c_2^* are associated local time constants and f is a non-negative Borel measurable function.

Proof. It is clear that if $X^{(1)}$ and $X^{(2)}$ are local martingales, then they are in natural scale, i.e. their scale functions are up to multiplicative constants the identity functions. Consider the first process and the excursion measure $\widehat{\mathrm{P}}$. By Theorem 4.2,

$$\widehat{\mathrm{P}} \int_0^{\sigma_b \wedge R} f(\widehat{X}_s) \, \mathrm{d}s = \lim_{x \downarrow 0} \frac{c_1^*}{x} \mathrm{E}_x \int_0^{\sigma_b \wedge \sigma_0} f(X_s^{(1)}) \, \mathrm{d}s.$$

We use [20, Corollary 3.8, Chapter VII] to write, for $0 < a < x < b$ and $T = \sigma_a \wedge \sigma_b$,

$$\mathrm{E}_x \int_0^T f(X_s^{(1)}) \, \mathrm{d}s = \int_{(a,b)} G(x, y) f(y) m_1'(y) \, \mathrm{d}y,$$

where $G(x, y) = \frac{(x \wedge y - a)(b - x \vee y)}{b - a}$. Simple algebra shows that

$$\mathrm{E}_x \int_0^T f(X_s^{(1)}) \, \mathrm{d}s = \frac{b - x}{b - a} \int_a^x (y - a) f(y) m_1'(y) \, \mathrm{d}y + \frac{x - a}{b - a} \int_x^b (b - y) f(y) m_1'(y) \, \mathrm{d}y.$$

Letting $a \downarrow 0$ we obtain

$$\begin{aligned} \lim_{x \downarrow 0} \frac{1}{x} \mathrm{E}_x \int_0^{\sigma_b \wedge \sigma_0} f(X_s^{(1)}) \, \mathrm{d}s &= \lim_{x \downarrow 0} \left[\frac{b - x}{xb} \int_0^x y f(y) m_1'(y) \, \mathrm{d}y + \frac{1}{b} \int_x^b (b - y) f(y) m_1'(y) \, \mathrm{d}y \right] \\ &= \frac{1}{b} \int_0^b (b - y) f(y) m_1'(y) \, \mathrm{d}y. \end{aligned}$$

Repeating the same procedure for $\widehat{\mathrm{Q}}$ yields $\widehat{\mathrm{Q}} \int_0^{\sigma_b \wedge R} f(\widehat{X}_s) \, \mathrm{d}s = \frac{c_2^*}{b} \int_0^b (b - y) f(y) m_2'(y) \, \mathrm{d}y$, so taking $h_1 = f \, m_2'$ and $h_2 = f \, m_1'$ we obtain

$$\widehat{\mathrm{P}} \int_0^{\sigma_b \wedge R} h_1(\widehat{X}_s) \, \mathrm{d}s = \frac{c_1^*}{b} \int_0^b (b - y) f(y) m_1'(y) m_2'(y) \, \mathrm{d}y = \frac{c_1^*}{c_2^*} \widehat{\mathrm{Q}} \int_0^{\sigma_b \wedge R} h_2(\widehat{X}_s) \, \mathrm{d}s,$$

completing the proof. □

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