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Upper bounds on the resistivity of a turbulent current layer

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Abstract. The basic equations for a layer of an electrically conducting fluid are considered in the magnetohydrodynamic approximation. An electric field is applied parallel to the layer. Because of various instabilities, turbulent fluid motions will be generated in general. The goal of the analysis presented in this paper is the derivation of upper bounds for the average turbulent resistivity of the layer. In order to demonstrate the concept of the upper bound theory, only the simplest case will be treated and the boundary conditions will be assumed in such a way that the analogy with the problem of bounds on the momentum transport in a turbulent Couette layer can be utilized.

1. Introduction

Upper bounds on the properties of turbulent fluid systems provide rigorous results when direct computations are not possible and theoretical estimates must rely on uncertain assumptions. Following an earlier proposal by Malkus (1954), Howard (1963) derived a rigorous bound for the heat transport by turbulent convection in a layer heated from below. The bound was later improved by Busse (1969a) through the use of the extremalizing multi-α-vector-fields in the case when the equation of continuity is imposed. The bounding method was also applied to turbulent shear flows (Busse 1969b, 1970) and to magnetohydrodynamic problems (Soward 1980; Krommes and Smith 1987; Kim and Krommes 1990; Wang et al. 1991).

More recently, the background-field-approach was introduced by Doering and Constantin (1992) and bounds similar to those found by Howard and Busse have been derived. In fact, Kerswell (1998) has been able to demonstrate the complementary nature of the bounds derived by the two methods when optimized in appropriate ways. Little progress has been achieved so far towards the goal of improving bounds through the imposition of additional constraints derived from the basic equations of motion. Computational solutions of the Euler–Lagrange equations of the variational problems can help us to reach this goal (Vitanov and Busse 2001).

In this brief report, we intend to demonstrate the close relationship between bounds on shear flow turbulence and on magnetohydrodynamic turbulence. It is expected that through similar analogies a wide spectrum of magnetohydrodynamic bounding problems can be solved.

2. Mathematical formulation of the problem

We consider a layer of thickness d of an electrically conducting incompressible fluid with kinetic viscosity ν and magnetic diffusivity λ, where the latter is defined as the inverse of the product of electrical conductivity σ and magnetic permeability μ. Using d as the length scale, d^2/λ as the time scale and $(\mu\varrho)^{1/2}\lambda/d$ as the scale of the magnetic field, we can write the equation of motion and the equation of magnetic induction in dimensionless form,

$$\left(\frac{\partial}{\partial t} + \boldsymbol{V}\cdot\nabla\right)\boldsymbol{V} = \boldsymbol{B}\cdot\nabla\boldsymbol{B} - \nabla\pi + P_m\nabla^2\boldsymbol{V} \tag{1a}$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{V}\cdot\nabla\right)\boldsymbol{B} = \boldsymbol{B}\cdot\nabla\boldsymbol{V} + \nabla^2\boldsymbol{B} \tag{1b}$$

$$\nabla\cdot\boldsymbol{B} = 0 \tag{1c}$$

$$\nabla\cdot\boldsymbol{V} = 0 \tag{1d}$$

where $P_m = \nu/\kappa$ and all the terms that can be written as gradients in the equation of motion have been combined into $\nabla\pi$. We introduce Cartesian coordinates with the z-axis normal to the layer and with the y-axis in the direction of the electric field applied to the layer. Indicating the average over the x, y-plane by a bar, we separate the velocity field and the magnetic field into averaged and fluctuating parts,

$$\boldsymbol{V} = \boldsymbol{U} + \boldsymbol{v}, \quad \boldsymbol{B} = \bar{\boldsymbol{B}} + \boldsymbol{b} \quad \text{with} \quad \bar{\boldsymbol{v}} = \bar{\boldsymbol{b}} = 0. \tag{2a}$$

We further separate the fluctuating fields into components parallel and perpendicular to the layer,

$$\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{k}w, \quad \boldsymbol{b} = \hat{\boldsymbol{b}} + \boldsymbol{k}b_z \quad \text{with} \quad \boldsymbol{u}\cdot\boldsymbol{k} = \hat{\boldsymbol{b}}\cdot\boldsymbol{k} = 0 \tag{2b}$$

where $\boldsymbol{k}$ is the unit vector in the direction of the z-axis.

We are interested in the turbulent fluid state under stationary conditions and we define this state by requiring that the time derivative of an average quantity vanishes. After taking the x, y-average of (1a) and (1b) we thus obtain

$$\frac{d^2}{dz^2}\bar{\boldsymbol{B}} = \frac{d}{dz}(\overline{w\hat{\boldsymbol{b}}} - \overline{b_z\boldsymbol{u}}) \tag{3a}$$

$$\frac{d^2}{dz^2}\bar{\boldsymbol{U}} = \frac{d}{dz}(\overline{w\boldsymbol{u}} - \overline{b_z\hat{\boldsymbol{b}}}) \tag{3b}$$

where the property has been used that the fields $\bar{\boldsymbol{B}}$ and $\bar{\boldsymbol{U}}$ possess only components parallel to the boundaries of the layer. Integration of (3a) and (3b) with respect to z yields

$$\frac{d}{dz}\bar{\boldsymbol{B}} = \overline{w\hat{\boldsymbol{b}}} - \overline{b_z\boldsymbol{u}} - \langle w\hat{\boldsymbol{b}} - b_z\boldsymbol{u}\rangle - \boldsymbol{k}\times J\boldsymbol{j} \tag{4a}$$

$$\frac{d}{dz}\bar{\boldsymbol{U}} = \overline{w\boldsymbol{u}} - \overline{b_z\hat{\boldsymbol{b}}} - \langle w\boldsymbol{u} - b_z\boldsymbol{b}\rangle \tag{4b}$$

where the angular brackets indicate the average over the entire layer. Here the boundary conditions

$$\boldsymbol{U} = 0 \quad \text{at } z = \pm\tfrac{1}{2}, \quad \boldsymbol{k}\times\left[\bar{\boldsymbol{B}}\left(\tfrac{1}{2}\right) - \bar{\boldsymbol{B}}\left(-\tfrac{1}{2}\right)\right] = J\boldsymbol{j} \tag{5}$$

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have been used where $J\boldsymbol{j}$ is the average current density of the layer, which like the applied electric field $E\boldsymbol{j}$ is directed in the y-direction indicated by the unit vector $\boldsymbol{j}$.

Ohm's law for the current density averaged over the layer yields the relationship

$$J = \boldsymbol{j} \cdot \langle \boldsymbol{v} \times \boldsymbol{b} \rangle + E = \langle b_x w - u_x b_z \rangle + E. \tag{6}$$

We define the dimensionless resistivity S (per unit length) by $S = E/J$ or

$$S = \frac{J + \langle b_z u_x - w b_z \rangle}{J} \equiv \frac{J + \breve{\mu}(J)}{J}. \tag{7}$$

Since the expression $\breve{\mu}(J)$ is expected to be positive (see (10) below), our goal is to find an upper bound μ for $\breve{\mu}$ at a given value of J subject to certain constraints derived from the basic equations (1).

After subtracting (3a) and (3b) from (1a) and (1b), multiplying the equation by $\boldsymbol{v}$ and $\boldsymbol{b}$, respectively, and averaging the result over the fluid layer, we obtain the following two equations,

$$\langle |\nabla \boldsymbol{b}|^2 \rangle = \langle \boldsymbol{b} \cdot (\boldsymbol{b} + \bar{\boldsymbol{B}}) \cdot \nabla \boldsymbol{v} \rangle + \left\langle \hat{\boldsymbol{b}} \cdot b_z \frac{d}{dz} \boldsymbol{U} \right\rangle - \left\langle \hat{\boldsymbol{b}} w \frac{d}{dz} \bar{\boldsymbol{B}} \right\rangle \tag{8a}$$

$$P_m \langle |\nabla \boldsymbol{u}|^2 \rangle = \langle \boldsymbol{v} \cdot (\boldsymbol{b} + \bar{\boldsymbol{B}}) \cdot \nabla \boldsymbol{b} \rangle - \left\langle \boldsymbol{u} \cdot \left(w \frac{d}{dz} \boldsymbol{U} \right) \right\rangle + \left\langle \boldsymbol{u} \cdot \left(b_z \frac{d}{dz} \bar{\boldsymbol{B}} \right) \right\rangle. \tag{8b}$$

The partial integrations used in the derivation of these equations are possible for very general boundary conditions. In the following, we consider two special kinds of boundary conditions which apply to the fluctuating parts of the velocity field and of the magnetic field in analogous ways,

$$\boldsymbol{v} = \boldsymbol{b} = 0 \quad \text{at } z = \pm\tfrac{1}{2} \tag{9a}$$

or

$$\boldsymbol{v} \cdot \boldsymbol{n} = \boldsymbol{b} \cdot \boldsymbol{n} = \boldsymbol{k} \times \frac{\partial}{\partial z} \boldsymbol{v} = \boldsymbol{k} \times \frac{\partial}{\partial z} \boldsymbol{b} = 0 \quad \text{at } z = \pm \frac{1}{2}. \tag{9b}$$

Boundary conditions (9a) are applicable at a solid wall of infinite electrical conductivity, but also covered with an insulating coating which prevents any electric current from entering. This boundary condition is also known as the line tied boundary condition (Morrison 2002; see also Wang et al. 1991). Boundary condition (9b) has a similar character for the magnetic field except that the highly conducting surface is in direct contact with the fluid. The property that no viscous stresses are exerted on the fluid may be more difficult to realize. We refer to the extensive discussion of Seehafer et al. (1996) for this case. After (8a) and (8b) have been added, the first terms on the right-hand sides cancel. Using relationship (4) we obtain

$$\begin{aligned} \langle |\nabla \boldsymbol{b}|^2 \rangle + P_m \langle |\nabla \boldsymbol{v}|^2 \rangle + \langle (\overline{b_z \boldsymbol{u}} - \overline{w \hat{\boldsymbol{b}}} - \langle b_z \boldsymbol{u} - w \hat{\boldsymbol{b}} \rangle)^2 \rangle \\ + \langle (\overline{w \boldsymbol{u}} - \langle w \boldsymbol{u} \rangle - \overline{\hat{\boldsymbol{b}} b_z} + \langle \hat{\boldsymbol{b}} b_z \rangle)^2 \rangle = J \langle b_z u_x - w b_x \rangle \end{aligned} \tag{10}$$

where we have used the identity $\langle f(f - \langle f \rangle) \rangle = \langle (f - \langle f \rangle)^2 \rangle$.

Anticipating that the upper bound μ for $\hat{\mu}(J)$ is a monotonously increasing function of J, we prefer to determine the lower bound J for the current density

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J at a given value μ of $\langle b_z u_x - w b_x \rangle$. We are thus led to the formulation of the following variational problem.

VARIATIONAL PROBLEM. *For a given value of μ determine the minimum J of the functional*

$$\mathscr{J} \equiv \frac{\langle |\nabla \boldsymbol{b}|^2 \rangle + P_m \langle |\nabla \boldsymbol{v}|^2 \rangle}{\langle b_z u_x - w b_x \rangle} + \mu \frac{\langle (\overline{b_z \boldsymbol{u}} - \overline{w \hat{\boldsymbol{b}}} - \langle b_z \boldsymbol{u} - w \hat{\boldsymbol{b}} \rangle)^2 \rangle}{\langle w b_x - b_z u_x \rangle^2} \tag{11}$$

among all the vector fields $\boldsymbol{v}$ and $\boldsymbol{b}$ which satisfy (1c), (1d) and boundary conditions (9) and for which $\langle b_z u_x - w b_x \rangle$ is positive.

The formulation of the functional (11) has the advantage of homogeneity, i.e. for any solution $\boldsymbol{v}, \boldsymbol{b}$ the fields $A\boldsymbol{v}, A\boldsymbol{b}$ also solve the variational problem. By fixing the amplitude A in such a way that

$$\mu = \langle b_z u_x - w b_x \rangle$$

is satisfied, we recover relationship (10) except for the term

$$\mu \frac{\langle (\overline{w \boldsymbol{u}} - \langle w \boldsymbol{u} \rangle - \overline{\hat{\boldsymbol{b}} b_z} - \langle \hat{b} b_z \rangle)^2 \rangle}{\langle w b_x - b_z u_x \rangle^2} \tag{12}$$

which vanishes, however, for the extremalizing solution of the variational problem (11) as will be demonstrated below. We thus could have included the positive definite term (12) in the formulation of the functional (11) without changing the result. Similarly, it can be seen that the y-component of the vector in the numerator of the second term on the right-hand side of (11) only makes a positive contribution to the functional. Anticipating that $\overline{b_z u_y} - \overline{w b_y}$ vanishes identically for the extremalizing solution, we neglect this term in the following.

3. Solutions for the variational problem

Before starting with the analysis of the variational problem (11), we find it convenient to eliminate the dependence of the functional on P_m. By introducing the transformation

$$\boldsymbol{v}^* = P_m^{1/4} \boldsymbol{v}, \quad \boldsymbol{b}^* = P_m^{-1/4} \boldsymbol{b}, \quad \mu^* = P_m^{-1/2} \mu, \quad \mathscr{J}^* = P_m^{-1/2} \mathscr{J} \tag{13}$$

we have eliminated the parameter P_m. The constraint of (1c) and (1d) can easily be accommodated by the introduction of the general representation

$$\boldsymbol{v}^* = \nabla \times (\nabla \times \boldsymbol{k} \varphi) + \nabla \times \boldsymbol{k} \psi, \quad \boldsymbol{b}^* = \nabla \times (\nabla \times \boldsymbol{k} h) + \nabla \times \boldsymbol{k} y. \tag{14}$$

We thus rewrite the functional (11) in the form

$$\mathscr{J}^* = \frac{\langle |\nabla_2 \nabla^2 h|^2 \rangle + \langle g \nabla^2 \Delta_2 g \rangle + \langle |\nabla_2 \nabla^2 \varphi|^2 \rangle + \langle \psi \nabla^2 \Delta_2 \psi \rangle}{\langle + \Delta_2 \varphi \partial_y g - \Delta_2 h \partial_y \psi \rangle}$$
$$+ \mu^* \frac{\langle (\overline{\Delta_2 h \partial_y \psi} - \overline{\Delta_2 \varphi \partial_y g} - \langle \Delta_2 h \partial_y \psi \rangle + \langle \Delta_2 \varphi \partial_y y \rangle)^2 \rangle}{\langle + \Delta_2 \varphi \partial_y y - \Delta_2 h \partial_y \psi \rangle^2} \tag{15}$$

where the symbols ∇_2 and Δ_2 are defined by $\nabla_2 = \nabla - \boldsymbol{k}\boldsymbol{k} \cdot \nabla$, $\Delta_2 = \nabla^2 - (\boldsymbol{k} \cdot \nabla)^2$ and where additional terms of the form $\overline{\Delta_2 \varphi \partial_{xz}^2 \varphi}$, $\overline{\Delta_2 h \partial_{xz}^2 h}$ have been neglected since we shall use the assumption that the minimizing solution of the variational functional is independent of x. This assumption has been justified to some extent by Busse (1970). It should be added that a rigorous proof for the x-independence of the extremalizing solution of the functional (11) can be obtained in the limit $\mu \to 0$ by the same method as used by Busse (1972).

From the form of the functional (15), it is obvious that extremalizing solutions can be found in the form

$$\varphi = \varphi_0(y,z), \quad g = g_0(y,z) \quad \psi \equiv h \equiv 0 \tag{16a}$$

or

$$h = h_0(y,z), \quad \psi = \psi_0(y,z) \quad \varphi \equiv g \equiv 0. \tag{16b}$$

Since the boundary conditions (9) correspond to either

$$\varphi = \frac{\partial}{\partial z}\varphi = h = \frac{\partial}{\partial z}h = \psi = g = 0 \quad \text{at } z = \pm\frac{1}{2} \tag{17a}$$

or

$$\varphi = \frac{\partial^2}{\partial z^2}\varphi = h = \frac{\partial^2}{\partial z^2}h = \frac{\partial \psi}{\partial z} = \frac{\partial g}{\partial z} = 0 \quad \text{at } z = \pm\frac{1}{2} \tag{17b}$$

the minimum $J^*(\mu^*)$ of the functional (15) is the same in both cases (16a) and (16b). The sum of the minimizing solutions (16a) and (16b) with arbitrary amplitudes also yields the minimum $J^*(\mu)$ because of the homogeneity of the functional (15). In this case, it must be ensured through appropriate phase shifts that terms such as $\overline{\Delta_2\varphi\partial_y\psi}$ and $\overline{\Delta_2 h\partial_y g}$ vanish identically such that no contribution arises from the neglected term (12). We thus restrict attention to either of the cases (16a) and (16b). Fortunately, there is no need to solve the variational problem because in the cases (16a) and (16b) the functional (15) becomes identical to the functional derived in Busse (1971) for the case of turbulent plane Couette flow. The asymptotic solution for large values of μ^* can thus be written in the form

$$J^*(\mu^*) = 4^{7/3}(\sigma^3\beta)^{1/4}\mu^{*\,1/2} \tag{18a}$$

$$J(\mu) = P_m^{1/4}4^{7/3}(\sigma^3\beta)^{1/4}\mu^{1/2} \tag{18b}$$

where σ and β are numerical constants of the order unity which assume the values

$$\sigma = 0.337, \quad \beta = 0.624 \tag{19a}$$

in the case of boundary conditions (16a) (see Busse 1978) and

$$\sigma = 0.207, \quad \beta = 0.51 \tag{19b}$$

in the case of boundary conditions (16b) (Straus 1973; β has been determined as in Busse 1969). From this result, we conclude that the turbulent resistivity S for the electrically conducting fluid is bounded from above by

$$S \leq 1 + P_m^{-1/2}(\sigma^3\beta)^{-1/2}4^{-4/3}J. \tag{20}$$

4. Concluding remarks

The goal of this paper has been to demonstrate that upper bounds on properties of turbulent magnetohydrodynamic systems can be derived rather simply if analogies to hydrodynamic systems can be used. It will be of interest to see how the bounds on the turbulent resistivity compare with experimental measurements. Since the bounds obtained for two different conditions do not differ very much, it is expected that the exact nature of the boundaries is not important as long as their electrical conductivity is much larger than that of the fluid.

In their analysis of the configuration considered in this paper, Seehafer et al. (1996) find that the basic motionless state is stable and other asymptotic solutions corresponding to turbulent states do not seem to exist. This unexpected result is highly surprising since this seems to be the first nonlinear system in which the excitation of a continuum of degrees of freedom does not occur even though it is energetically possible. The bound derived in this paper outlines the region in the parameter space where it might still be possible to eventually find time-asymptotic states different from the static state.

The functional (11) includes the energy stability limit obtained in the particular case $\mu = 0$. A comparison with the earlier work cited in Sec. 3 indicates that the static state is absolutely stable, i.e. all finite amplitude disturbances must decay for

$$E < P_m^{-1/2} J_E^* \tag{21}$$

where J_E^* assumes the values $2\sqrt{1708}$ and $\sqrt{27}\pi^2$ in the cases of boundary conditions (9a) and (9b), respectively.

The actual bounds may not be the most important results of analyses such as that presented in this paper. The identity of extremalizing vector fields of variational problems for different physical situations suggests similarities between the corresponding physical fields. Such a similarity between streamwise fluctuating velocity components in shear flow turbulence and the fluctuating temperature in turbulent convection has been pointed out by Busse (1970) and has been introduced in a model of turbulent boundary layers by Deardorff (1970). The extremalizing vector fields of the variational problem (1) suggest analogous similarities between problems of magnetohydrodynamic turbulence, shear flow turbulence and problems of turbulent convection, which could be helpful in the interpretation of the measured profiles of root-mean-square values of turbulent fields.

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