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STRONG LAW OF LARGE NUMBERS FOR MARKOV CHAINS INDEXED BY SPHERICALLY SYMMETRIC TREES

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In this paper, we main consider spherically symmetric tree T. First, under the condition $\limsup_{n\to\infty} |T^{(n)}|/|L_n| < \infty$, we investigate the strong law of large numbers (SLLNs) for T-indexed Markov chains on the nth level of T. Then, combining the Stolz theorem, we obtain the SLLNs on T. Finally, we get Shannon–McMillan theorem for T-indexed Markov chains. The obtained theorems are generalizations of some known results on Cayley tree $T_{\mathrm{C},N}$ and Bethe tree $T_{\mathrm{B},N}$.

1. INTRODUCTION

By a tree T we mean an infinite, locally finite, connected graph with a distinguished vertex o called the *root* and without loops or cycles. We only consider trees without leaves. That is, the degree (the number of neighboring vertices) of each vertex (except o) is required to be at least 2.

The set of all vertices with distance $n(= 0, 1, 2, \ldots)$ from the root, denote by L_n, is called the nth level of T. We denote by $T^{(n)}_{(m)}$ the union from the mth to nth level of T, specially by $T^{(n)}$ the union from the root to nth level of T. For each vertex t, there is a unique path from o to t, and $|t|$ for the number of edges on this path. For any two vertices s and t, denote by $s \le t$, if s is on the unique path from the root o to t, denote by $s \wedge t$ the vertex farthest from o satisfying $s \wedge t \le s$ and $s \wedge t \le t$. If $s \le t$ and $|s| = |t| + k$, then we say t is the kth predecessor of s. In this paper, we denote the first predecessor of t by 1_t, the second predecessor of t by 2_t, $\ldots$, and by n_t the nth predecessor of t. We also call t as a son of 1_t. We set $X^A = \{X_t, t \in A\}$ and denote by $|A|$ the number of vertices of A.

If the degree of each vertex on a tree is $N + 1$, we call it a Bethe tree, denote by $T_{\mathrm{B},N}$; If each vertex on a tree has N sons, we call it a Cayley tree, denote by $T_{\mathrm{C},N}$. Both the

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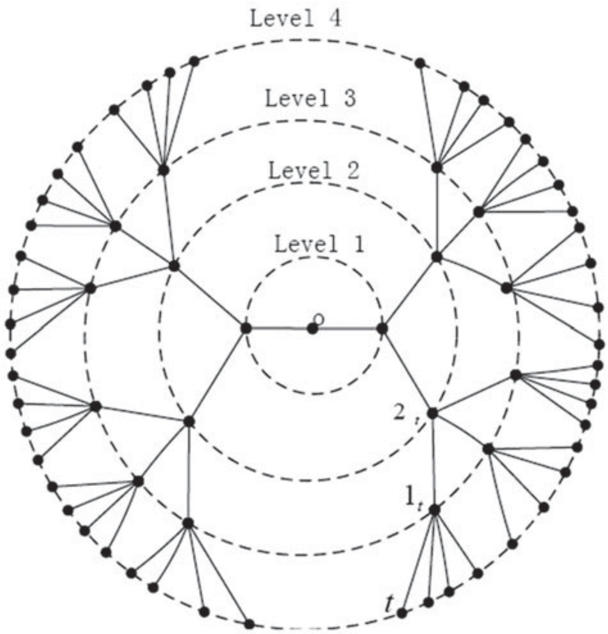

FIGURE 1. The $T^{(4)}$ of a spherically symmetric tree (solid line).

Bethe tree and the Cayley tree are called homogeneous trees. If the degrees of any vertices on a tree T are uniformly bounded, then we call T a uniformly bounded degree tree (see [3,6]). If every vertex at the nth level of T has the same degree (may depend on n), then we call T a spherically symmetric tree(see [1]). A spherically symmetric tree example is given below (see Figure 1).

Remark 1: From the definitions we know that the homogeneous tree model is a special case of the uniformly bounded degree tree or spherically symmetric tree.

DEFINITION 1 (see [1]): *Let S be a finite state space, $\{X_t, t \in T\}$ be a collection of S-valued random variables defined on the probability space $(\Omega, \mathbf{F}, \mathbf{P})$. Let*

$$\{p(x), x \in S\} \tag{1}$$

be a distribution on S, and

$$(P(y|x)), \quad x, y \in S \tag{2}$$

be a stochastic matrix on S^2. If for any vertex t,

$$\begin{aligned}\mathbf{P}(X_t = y|X_{1_t} = x \quad \textit{and} \quad X_s \quad \textit{for} \quad t \wedge s \le 1_t) \\ = \mathbf{P}(X_t = y|X_{1_t} = x) = P(y|x) \quad \forall x, y \in S,\end{aligned} \tag{3}$$

and

$$\mathbf{P}(X_0 = x) = p(x) \quad \forall x \in S,$$

$\{X_t, t \in T\}$ will be called S-valued Markov chains indexed by an infinite tree T with the initial distribution (1) and transition matrix (2), or called T-indexed Markov chains with state-space S.

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Since Benjamini and Peres [1] have given the notion of the tree-indexed Markov chains and studied the recurrence and ray-recurrence for them. Berger and Ye (see [2,4,12]) studied the entropy properties for some stationary random fields on homogeneous tree. Some papers, such as [9–11], studied the strong limit properties for Markov chains field on homogeneous trees. Some literatures, such as [3,6,8], studied the strong limit properties for Markov chains indexed by an infinite tree with uniformly bounded degree.

However, in [2,3,5,6,8–12], the degrees of the vertices in the tree models are uniformly bounded. What if the degree is not uniformly bounded? Many authors tried to study the limit properties of Markov chains indexed by such trees. Liu Wen and Yang Weiguo [5] studied the deviation theorems for Markov chains field on a generalized Bethe tree or a generalized Cayley tree, in fact the tree model is a special case of uniformly bounded degree tree. Wang Kangkang and Zong Decai [7] tried to establish some Shannon–McMillan approximation theorems for Markov chain field on the generalized Bethe tree, but the results seem crude. In this paper, we drop the uniformly bounded restriction. We mainly consider spherically symmetric trees. We arrange the rest of this paper as following. In part 2, we give some notations and lemmas; In part 3, we state our main results, under the condition $\limsup_{n\to\infty} |T^{(n)}|/|L_n| < \infty$, the strong law of large numbers and AEP with a.e. convergence for finite Markov chains indexed by a spherically symmetric tree. Our present outcomes can imply the results on Cayley tree $T_{\mathrm{C},N}$ and Bethe tree $T_{\mathrm{B},N}$ with degree $N \geq 2$ (see [11]), but cannot be implied by [3], and the technique is different from the above articles.

The following examples are used to explain the condition $\limsup_{n\to\infty} |T^{(n)}|/|L_n| < \infty$.

Example 1: A Cayley tree $T_{\mathrm{C},N}$ $(N \geq 2)$ satisfies this condition. In fact, in such a tree, $|L_n| = N^n$ and $|T^{(n)}| = 1 + N + N^2 + \cdots + N^n$, hence $\limsup_{n\to\infty} |T^{(n)}|/|L_n| = N/(N-1) < \infty$.

Example 2: A kind of spherically symmetric tree (see Figure 1), on which the degree of the root o is 2, and for any integer $n \geq 1$, the degree of the vertices on the nth level is $n+2$. Actually, in such a tree, $|L_n| = 2 \times n!$ and $|T^{(n)}| = 1 + 2 + 2 \times 2! + 2 \times 3! + \cdots + 2 \times n!$, where $n! = 1 \times 2 \times \cdots \times (n-1) \times n$. It is easy to verify that $\limsup_{n\to\infty} |T^{(n)}|/|L_n| = 1 < \infty$.

Example 3: If all the vertices have the same degree 2 in a spherically symmetric tree, then $\limsup_{n\to\infty} |T^{(n)}|/|L_n| = \infty$. Actually, in this case, the tree becomes the set of integers. We will not consider this case in this work.

2. SOME NOTATIONS AND LEMMAS

Let $k \in S$, denote

$$S_n(k) = \sum_{t \in T^{(n)}} \delta_k(X_t), \tag{4}$$

$$S_n^1(k) = \sum_{t \in T^{(n)}_{(1)}} \delta_k(X_{1_t}), \tag{5}$$

where

$$\delta_k(i) = \begin{cases} 1, & i = k, \\ 0, & i \neq k. \end{cases}$$

$S_n(k)$ can be considered as the number of k in the set of random variables $X^{T^{(n)}} = \{X_t : t \in T^{(n)}\}$, $S_n^1(k)$ can be considered as the number of k's among the variables in $T^{(n-1)}$, weighted according to the number of sons. Denote

$$L_n(k) = \sum_{t \in L_n} \delta_k(X_t) = S_n(k) - S_{n-1}(k), \tag{6}$$

$$L_n^1(k) = \sum_{t \in L_n} \delta_k(X_{1_t}) = S_n^1(k) - S_{n-1}^1(k). \tag{7}$$

Remark 2: By (4) and (5), we have $\sum_{k \in S} S_n(k) = |T^{(n)}|$, $\sum_{k \in S} S_n^1(k) = |T^{(n)}| - 1$.

Lemma 1 (Huang and Yang [3, Theorem 1]): *Let T be an infinite tree. Let $(X_t)_{t \in T}$ be a T-indexed Markov chain with finite states space S defined as before, $\{g_t(x, y), t \in T\}$ be functions defined on S^2. Let $L_o = \{o\}$, and*

$$G_n(\omega) = \sum_{t \in T_{(1)}^{(n)}} E[g_t(X_{1_t}, X_t)|X_{1_t}], \tag{8}$$

$\{a_n, n \geq 1\}$ be a sequence of nonnegative random variables. Let $\alpha > 0$. Set

$$B = \{\lim_{n \to \infty} a_n = \infty\}, \tag{9}$$

and

$$D(\alpha) = \left\{ \limsup_{n \to \infty} \frac{1}{a_n} \sum_{t \in T_{(1)}^{(n)}} E[g_t^2(X_{1_t}, X_t) e^{\alpha |g_t(X_{1_t}, X_t)|} |X_{1_t}] = M(\omega) < \infty \right\} \cap B, \tag{10}$$

$$H_n(\omega) = \sum_{t \in T_{(1)}^{(n)}} g_t(X_{1_t}, X_t). \tag{11}$$

Then

$$\lim_{n \to \infty} \frac{H_n(\omega) - G_n(\omega)}{a_n} = 0 \qquad a.e. \qquad on \qquad D(\alpha). \tag{12}$$

Lemma 2 (Yang [11, Theorem 2]): *Let T be a homogeneous tree, $(X_t)_{t \in T}$ be a T-indexed Markov chain with finite states space S defined by Definition 1, $S_n(k)$ be defined by (4). Let $(P(y|x))_{x,y \in S}$ be a ergodic stochastic matrix with unique stationary distribution π. Then*

$$\lim_{n \to \infty} \frac{S_n(k)}{|T^{(n)}|} = \pi(k) \qquad a.e.. \tag{13}$$

Lemma 3: *Let T be a spherically symmetric tree, $(X_t)_{t \in T}$ be a T-indexed Markov chain with finite states space S defined by Definition 1, $S_n(k)$ and $S_n^1(l)$ be defined by (4) and (5), respectively. If*

$$\limsup_{n \to \infty} \frac{|T^{(n)}|}{|L_n|} < \infty, \tag{14}$$

then we have

$$\lim_{n \to \infty} \left\{ \frac{S_n(k)}{|L_n|} - \frac{\sum_{l \in S} P(k|l) S_n^1(l)}{|L_n|} \right\} = 0 \quad a.e.. \tag{15}$$

PROOF: Let $a_n = |L_n|$ and $g_t(X_{1_t}, X_t) = \delta_k(X_t)$ in Lemma 1, we can find that

$$\begin{aligned} H_n(\omega) &= S_n(k) - \delta_k(X_o), \\ G_n(\omega) &= \sum_{t\in T_{(1)}^{(n)}} E[g_t(X_{1_t}, X_t)|X_{1_t}] = \sum_{t\in T_{(1)}^{(n)}} E[\delta_k(X_t)|X_{1_t}] \\ &= \sum_{t\in T_{(1)}^{(n)}} P(k|X_{1_t}) = \sum_{l\in S} P(k|l) \sum_{t\in T_{(1)}^{(n)}} \delta_l(X_{1_t}) = \sum_{l\in S} P(k|l) S_n^1(l). \end{aligned}$$

In order to show (15), we only need to verify $D(\alpha) = \Omega$ in Lemma 1. In fact, by (10) and (14),

$$\begin{aligned} &\limsup_{n\to\infty} \frac{1}{a_n} \sum_{t\in T_{(1)}^{(n)}} E[g_t^2(X_{1_t}, X_t) e^{\alpha|g_t(X_{1_t}, X_t)|}|X_{1_t}] \\ &= \limsup_{n\to\infty} \frac{1}{|L_n|} e^{\alpha} \sum_{l\in S} P(k|l) S_n^1(l) \le e^{\alpha} \limsup_{n\to\infty} \frac{|T^{(n)}|}{|L_n|} < \infty. \end{aligned} \tag{16}$$

By (9), (10) and (16), we have $D(\alpha) = \Omega$. Hence (15) follows by Lemma 1. ■

LEMMA 4: *Let $L_n(k)$ and $L_n^1(l)$ be defined by (6) and (7), respectively. Under the assumption of Lemma 2, we have*

$$\lim_{n\to\infty} \left\{ \frac{L_n(k)}{|L_n|} - \frac{\sum_{l\in S} P(k|l) L_n^1(l)}{|L_n|} \right\} = 0 \quad a.e., \tag{17}$$

$$\lim_{n\to\infty} \left\{ \frac{L_n(k)}{|L_n|} - \frac{\sum_{l\in S} P(k|l) L_{n-1}(l)}{|L_{n-1}|} \right\} = 0 \quad a.e.. \tag{18}$$

PROOF: Since all the vertices on the same level have the same degree, hence for any $n \ge 1$, by (6) and (7),

$$L_n^1(l) = \sum_{t\in L_n} \delta_l(X_{1_t}) = \sum_{\sigma\in L_{n-1}} \frac{|L_n|}{|L_{n-1}|} \delta_l(X_\sigma) = \frac{|L_n|}{|L_{n-1}|} L_{n-1}(l), \tag{19}$$

by (17)–(19) holds. Now we only need to show (17). In fact, by (6), (7), and (15) we have,

$$\begin{aligned} \left| \frac{L_n(k)}{|L_n|} - \frac{\sum_{l\in S} P(k|l) L_n^1(l)}{|L_n|} \right| &= \left| \frac{S_n(k) - S_{n-1}(k)}{|L_n|} - \frac{\sum_{l\in S} P(k|l)[S_n^1(l) - S_{n-1}^1(l)]}{|L_n|} \right| \\ &\le \left| \frac{S_n(k)}{|L_n|} - \frac{\sum_{l\in S} P(k|l) S_n^1(l)}{|L_n|} \right| + \left| \frac{S_{n-1}(k)}{|L_n|} - \frac{\sum_{l\in S} P(k|l) S_{n-1}^1(l)}{|L_n|} \right| \end{aligned}$$

$$= \left| \frac{S_n(k)}{|L_n|} - \frac{\sum_{l\in S} P(k|l) S_n^1(l)}{|L_n|} \right| + \frac{|L_{n-1}|}{|L_n|} \left| \frac{S_{n-1}(k)}{|L_{n-1}|} - \frac{\sum_{l\in S} P(k|l) S_{n-1}^1(l)}{|L_{n-1}|} \right| \to 0 \quad a.e. \qquad (n \to \infty),$$

which implies (17) directly. ■

3. STRONG LAW OF LARGE NUMBERS AND SHANNON–MCMILLAN THEROEM

THEOREM 1: *If T is a spherically symmetric tree, $L_n(k)$ is defined by (6), under the assumption of Lemma 2*

$$\lim_{n\to\infty} \frac{L_n(k)}{|L_n|} = \pi(k) \quad a.e.. \tag{20}$$

PROOF: The following method is similarly with [11], we give the details in order to make the reading clearly. Multiplying the kth equality of (17) by $P(i|k)$, adding them together, and using (18) once again, we have

$$\begin{aligned} 0 &= \lim_{n\to\infty} \left\{ \frac{\sum_{k\in S} P(i|k) L_n(k)}{|L_n|} - \frac{\sum_{l\in S}\sum_{k\in S} P(i|k)P(k|l)L_{n-1}(l)}{|L_{n-1}|} \right\} \\ &= \lim_{n\to\infty} \left\{ \frac{\sum_{k\in S} P(i|k) L_n(k)}{|L_n|} - \frac{L_{n+1}(i)}{|L_{n+1}|} + \frac{L_{n+1}(i)}{|L_{n+1}|} - \frac{\sum_{l\in S}\sum_{k\in S} P(i|k)P(k|l)L_{n-1}(l)}{|L_{n-1}|} \right\} \\ &= \lim_{n\to\infty} \left\{ \frac{L_{n+1}(i)}{|L_{n+1}|} - \frac{\sum_{l\in S} P^{(2)}(i|l) L_{n-1}(l)}{|L_{n-1}|} \right\} \quad a.e., \end{aligned}$$

where $P^{(m)}(l|j)$ is the m-step transition probability determined by the transition matrix $(P(y|x))_{x,y\in S}$. By induction, we have

$$\lim_{n\to\infty} \left\{ \frac{L_{n+m}(k)}{|L_{n+m}|} - \frac{\sum_{l\in S} P^{(m+1)}(k|l) L_{n-1}(l)}{|L_{n-1}|} \right\} = 0 \quad a.e. \tag{21}$$

Since

$$\lim_{m\to\infty} P^{(m+1)}(k|l) = \pi(k) \quad a.e. \quad k \in S, \tag{22}$$

and

$$\frac{\sum_{l\in S} L_{n-1}(l)}{|L_{n-1}|} = 1 \quad a.e., \tag{23}$$

by (21), (22) and (23), (20) holds. ■

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THEOREM 2: *If T is a spherically symmetric tree, $S_n(k)$ and $S_n^1(k)$ is defined by (4) and (5) respectively, under the assumption of Lemma 2, (13) holds. Furthermore, we have*

$$\lim_{n\to\infty}\frac{S_n^1(k)}{|T^{(n)}|}=\pi(k)\quad a.e. \tag{24}$$

PROOF: Noticing, by (4) and (20) is equivalent to

$$\lim_{n\to\infty}\frac{S_n(k)-S_{n-1}(k)}{|T^{(n)}|-|T^{(n-1)}|}=\pi(k)\quad a.e., \tag{25}$$

by (25) and Stolz theorem, (13) holds. By (19) and (20), we have

$$\lim_{n\to\infty}\frac{L_n^1(k)}{|L_n|}=\lim_{n\to\infty}\frac{L_{n-1}(k)}{|L_{n-1}|}=\pi(k)\quad a.e.,$$

by (5) we have

$$\lim_{n\to\infty}\frac{S_n^1(k)-S_{n-1}^1(k)}{|T^{(n)}|-|T^{(n-1)}|}=\pi(k)\quad a.e., \tag{26}$$

(24) follows by (26) and Stolz theorem.

According to the Theorem 2, we can establish the Shannon–McMillan theorem with a.e. convergence for Markov chain fields on a spherically symmetric tree.

Let T be a tree, $(X_t)_{t\in T}$ be a stochastic process indexed by tree T with state space S. Denote

$$\mathbf{P}(x^{T^{(n)}})=\mathbf{P}(X^{T^{(n)}}=x^{T^{(n)}}).$$

Let

$$f_n(\omega)=-\frac{1}{|T^{(n)}|}\ln\mathbf{P}(X^{T^{(n)}}), \tag{27}$$

$f_n(\omega)$ will be called the entropy density of $X^{T^{(n)}}$. If $(X_t)_{t\in T}$ is a T-indexed Markov chain with state space S defined by Definition 1, we have

$$\mathbf{P}(x^{T^{(n)}})=\mathbf{P}(X^{T^{(n)}}=x^{T^{(n)}})=p(x_o)\prod_{t\in T_{(1)}^{(n)}}P(x_t|x_{1_t}),$$

and

$$f_n(\omega)=-\frac{1}{|T^{(n)}|}[\ln P(X_o)+\sum_{t\in T_{(1)}^{(n)}}\ln P(X_t|X_{1_t})]. \tag{28}$$

The convergence of $f_n(\omega)$ to a constant in a sense (L_1 convergence, convergence in probability, a.e. convergence) is called the Shannon–McMillan theorem or the entropy theorem or the AEP in information theory. ■

LEMMA 5 (Yang, [11, Theorem 2]): *Let T be a homogeneous tree, under the same assumption of lemma 2, then*

$$\lim_{n\to\infty}f_n(\omega)=-\sum_{l\in S}\sum_{k\in S}\pi(l)P(k|l)\ln P(k|l)\quad a.e.. \tag{29}$$

THEOREM 3: *If T is a spherically symmetric tree, under the assumption of lemma 2, (29) holds.*

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PROOF: Let $a_n = |T^{(n)}|$ and $g_t(X_{1_t}, X_t) = -\ln P(X_t|X_{1_t})$ in Lemma 1. First, we say $D(\alpha) = \Omega$ for any $0 < \alpha < 1$. In fact, by (10),

$$\limsup_{n\to\infty} \frac{1}{a_n} \sum_{t\in T_{(1)}^{(n)}} E[g_t^2(X_{1_t}, X_t)e^{\alpha|g_t(X_{1_t},X_t)|}|X_{1_t}]$$

$$= \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{t\in T_{(1)}^{(n)}} E[(-\ln P(X_t|X_{1_t}))^2 e^{\alpha|-\ln P(X_t|X_{1_t})|}|X_{1_t}]$$

$$= \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{t\in T_{(1)}^{(n)}} \sum_{x_t\in S} (\ln P(x_t|X_{1_t}))^2 e^{-\alpha \ln P(x_t|X_{1_t})} P(x_t|X_{1_t}).$$

$$\leq \frac{4|S|}{e^2(\alpha-1)^2} < \infty,$$

where $|S|$ denotes the number of the states in S, and $(\ln P(x_t|X_{1_t}))^2 e^{-\alpha \ln P(x_t|X_{1_t})} P(x_t|X_{1_t})$ largest value is $4|S|/[e(\alpha-1)]^2$ for any $0 < \alpha < 1$.

By (8), (11), (24) and (28),

$$\frac{H_n(\omega)}{|T^{(n)}|} = -\frac{1}{|T^{(n)}|} \sum_{t\in T_{(1)}^{(n)}} \ln P(X_t|X_{1_t}) = f_n(\omega) + \frac{\ln p(X_o)}{|T^{(n)}|}, \tag{30}$$

and

$$\frac{G_n(\omega)}{|T^{(n)}|} = -\sum_{k,l\in S} P(k|l)\ln P(k|l)\frac{S_n^1(l)}{|T^{(n)}|} \to -\sum_{k,l\in S} \pi(l)P(k|l)\ln P(k|l) \quad a.e. \quad (n\to\infty). \tag{31}$$

By (30), (31), and Lemma 1, (29) holds. We complete the proof. ∎

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