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On soluble groups which admit the dihedral group of order eight fixed-point-freely

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If the finite soluble group G admits the dihedral group of order eight as a fixed-point-free group of automorphisms then the nilpotent length of G is at most three.

A theorem of Berger [2] has substantially enlarged the class of nilpotent groups A for which the following statement holds.

(*) *If the soluble group G admits the group A as a fixed-point-free group of automorphisms and* $(|G|, |A|) = 1$, *then the nilpotent length of G is bounded by the number of primes, including multiplicities, which divide* $|A|$.

The smallest group A not covered by Berger's result is D_8 , the dihedral group of order 8 . It is our object here to establish (*) when $A = D_8$. In [5] Gross shows that, in this case, 4 is a bound, and in this paper he provides an important step in our argument.

$F_1(G), F_2(G), \ldots$ (or often just $F_1, F_2, \ldots$ when no confusion arises) will denote the successive terms of the upper nilpotent series of the soluble group G , and f.p.f. will be used to abbreviate both "fixed-point-free" and "fixed-point-freely". $\Phi(H)$ will denote the Frattini subgroup of H . All groups considered will be finite.

THEOREM. *If the soluble group G admits D_8 as a fixed-point-free*

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group of automorphisms then the nilpotent length of G *is at most* 3 .

Proof. Let G be a minimal counterexample to the theorem, so G has nilpotent length 4 and each D_8-admissible proper section of G has nilpotent length 3 . Let $D_8 = \langle \tau, \eta : \tau^4 = 1 = \eta^2, \tau^\eta = \tau^{-1} \rangle$ and put $\sigma = \tau^2$, the central involution. The hypothesis of f.p.f. action implies that G has odd order.

We first apply Theorem 2.4, Corollary 2.5 and Lemma 2.6 of Gross [6] to achieve a major part of the reduction.

(1) *$G = SRQP$ where S, R, Q and P are D_8-admissible subgroups of G and:*

- (a) *S is an s-group, R is an r-group, Q is a q-group and P is a p-group;*
- (b) *s, r, q and p are primes with $s \neq r \neq q \neq p$;*
- (c) *P normalizes Q, R and S ; Q normalizes R and S ; and R normalizes S ;*
- (d) $S \leq F_1(G)$, $R \leq F_2(G)$, $R \nleq F_1(G)$, $Q \leq F_3(G)$, $Q \nleq F_2(G)$, $P \nleq F_3(G)$;
- (e) $[Q, P] = Q$, $[R, Q] = R$ *and* $[S, R] = S$;
- (f) *each proper D_8-admissible subgroup of P lies in F_3 ; $P/P \cap F_3$ is elementary abelian and D_8-irreducible;*
- (g) *for each proper PD_8-admissible subgroup Q_1 of Q , $[Q_1, P] \leq F_2$ and $Q/Q \cap F$ is a special q-group with PD_8-irreducible Frattini quotient;*
- (h) *for each proper QPD_8-admissible subgroup R_1 of R , $[R_1, Q] \leq F_1$ and $R/R \cap F_1$ is a special r-group with QPD_8-irreducible Frattini quotient;*
- (i) *R centralizes each proper $RQPD_8$-admissible subgroup of S and S is a special s-group with $RQPD_8$-irreducible Frattini*

quotient.

(2) $S = F_1(G)$ *and is a faithful irreducible* $RQPD_8$*-module.*

If $\Phi(S) \neq 1$, $G/\Phi(S)$ has nilpotent length 3 by the minimality of G. Now $\Phi(S) \leq \Phi(G)$, so in this case $G/\Phi(G)$ has nilpotent length 3, from which it follows that G does too, a contradiction. Therefore by (1i), S is an elementary abelian s-group irreducible under the action of $RQPD_8$. Clearly the minimality of G implies that GD_8 has a unique minimal normal $2'$-subgroup, which must therefore be S. In particular $R \cap F_1 = 1$. Certainly GD_8 can have no normal 2-subgroup for otherwise G would admit the four-group $D_8/\langle\sigma\rangle$ f.p.f. contrary to a theorem of Bauman [1], which states that such groups have nilpotent derived group. It is now sufficient to prove that RQP complements S in G (for then GD_8 is a primitive soluble group with self-centralizing unique minimal normal subgroup S) and this will hold if $Q \cap S = 1 = P \cap S$. By (1c), $[Q \cap S, R] \leq R \cap S = 1$ but $[S, R] = S$ by (1e), so the irreducibility of S forces $Q \cap S = 1$. Similarly $[P \cap S, R] \leq R \cap S = 1$ implies $P \cap S = 1$.

(3) σ *centralizes* QP.

$P/\Phi(P)$ is a completely reducible D_8-module. If it were not D_8-irreducible (1f) would force $P \leq F_3$, against (1d). So $P/\Phi(P)$ is D_8-irreducible and therefore $P \cap F_3 \leq \Phi(P)$. Since $p \neq q$, $Q/\Phi(Q)$ is a completely reducible PD_8-module. If it were not irreducible, say $Q/\Phi(Q) = Q_1/\Phi(Q) + Q_2/\Phi(Q)$ where $Q \neq Q_1, Q_2$, then (1e) and (1f) would imply $Q = [Q_1Q_2, P] = [Q_1, P][Q_2, P] \leq F_2$, against (1d). So $Q/\Phi(Q)$ is PD_8-irreducible and therefore $Q \cap F_2 \leq \Phi(Q)$.

We may apply Theorem 1 of Gross [5] to the group $RQPD_8$, which by (2) acts faithfully and irreducibly on S, to deduce that σ centralizes F_3/F_2. Since G is a $2'$-group it follows immediately that σ centralizes G/F_2 and hence that σ centralizes $P/P \cap F_3$ and $Q/Q \cap F_2$. But then, in view of the inclusions proved above, σ centralizes $P/\Phi(P)$ and $Q/\Phi(Q)$ yielding the statement (3).

(4) *R is a special group of exponent r, σ inverts $R/\Phi(R)$ and centralizes $\Phi(R)$.*

By (2), $R \cap F_1 = 1$, so (1h) says that R is a special group whose Frattini quotient is isomorphic to a chief factor of GD_8. If $\Omega_1(R) = \langle x \in R : x^r = 1 \rangle$ were a proper subgroup of R then $[\Omega_1(R), Q] = 1$ by (1h), therefore by 5.3.10 of [4], Q would centralize R, contrary to (1e). Thus R is a special group generated by elements of order r, so it has exponent r. σ does not centralize $R/\Phi(R)$, otherwise the group RQP of nilpotent length 3 admits the four-group $D_8/\langle\sigma\rangle$ f.p.f. again contrary to Bauman's Theorem. Now by (3), σ is central in QPD_8 so $C_{R/\Phi(R)}(\sigma)$ is normalized by QPD_8, so by the irreducibility of $R/\Phi(R)$ this group is trivial. Thus σ inverts each element of $R/\Phi(R)$. R has class 2 so if $x, y \in R$ then $[x, y]^\sigma = [x^\sigma, y^\sigma] = \left[x^{-1}z_1, y^{-1}z_2\right] = [x^{-1}, y^{-1}] = [x, y]$ (for some $z_1, z_2 \in \Phi(R)$), that is, σ centralizes $R' = \Phi(R)$.

(5) *QP centralizes $\Phi(R)$.*

By (3) and (4), $[\Phi(R), QP] \le \left(C_G(\sigma)\right)'$. Now $C_G(\sigma)$ admits $D_8/\langle\sigma\rangle$ f.p.f. so Bauman's Theorem tells us that $[\Phi(R), QP] \le F_1\left(C_G(\sigma)\right)$. $C_S(\sigma)$ is non-trivial, for otherwise σ would invert S and therefore commute with the automorphisms of S induced by RQP, against (4). So $C_S(\sigma)$ is non-trivial and lies, with $[\Phi(R), QP]$, in $F_1\left(C_G(\sigma)\right)$. Since $r \ne s$ we deduce that $[\Phi(R), QP]$ centralizes $C_S(\sigma)$. But $[\Phi(R), QP] \triangleleft RQPD_8$ so the irreducibility of S implies that $[\Phi(R), QP]$ centralizes S, contrary to (2) unless $[\Phi(R), QP] = 1$.

At this point it is convenient to pass to a finite splitting field F for $RQPD_8$ and its subgroups, of characteristic s; and to a faithful irreducible $RQPD_8$-submodule S^* say, of $S \otimes_{GF(s)} F$. The condition that D_8 act f.p.f. on S, namely that $\sum_{\alpha \in D_8} \alpha$ be the zero transformation,

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remains invariant under these manoeuvres, so D_8 acts f.p.f. on S^* .

(6) *R is not elementary abelian.*

Let W be an RQP-homogeneous component of S^* and D_1 the stabilizer of W in D_8 . So W is an irreducible $RQPD_1$-module and $W = W_1 + \ldots + W_n$ where the W_i are isomorphic irreducible RQP-modules. The number of isomorphism types of irreducible R-submodules of W_i is prime to 2 , so D_1 stabilizes an R-homogeneous component V say, of W . S^* is a faithful R-module, irreducible for $RQPD_8$, therefore R acts f.p.f. on S^* and so R acts non-trivially on V .

If $D_1 = 1$, then for any non-trivial element $w \in W$, $\sum_{\alpha \in D_8} w\alpha$ is a non-trivial fixed-point of D_8 in S^* , contrary to our initial assumption.

Now suppose $\sigma \notin D_1$, so we may assume without loss of generality that $D_1 = \langle \eta \rangle$. Then a non-trivial fixed-point $w \in W$ of η would yield a non-trivial fixed-point $w + w\tau + w\sigma + w\sigma\tau \in S^*$ of D_8 , so η must act f.p.f. on W , therefore η inverts W and hence centralizes $RQP/\ker(RQP \text{ on } W)$. Therefore η centralizes $QP/\ker(QP \text{ on } W)$, $\eta^\tau = \eta\sigma$ centralizes $QP/\ker(QP \text{ on } W\tau)$, $\eta^\sigma = \eta$ centralizes $QP/\ker(QP \text{ on } W\sigma)$ and $\eta^{\sigma\tau} = \eta\sigma$ centralizes $QP/\ker(QP \text{ on } W\sigma\tau)$. However, by (3), σ centralizes QP so η itself centralizes these quotients. Therefore η centralizes QP (because $S^* = W + W\tau + W\sigma + W\sigma\tau$ is a faithful QP-module) and so QP admits $D_8/\langle \sigma, \eta \rangle$ f.p.f. which is impossible since QP is not abelian.

We have thus shown that $\sigma \in D_1$. Suppose R is elementary abelian. V is a homogeneous R-module , non-trivial for R , so $1 \neq R/\ker(R \text{ on } V)$ is cyclic and represented by scalar transformations. Therefore σ centralizes this quotient (whether σ is trivial on V or not) against (4).

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(7) *τ centralizes $\Phi(R)$ and R is extraspecial.*

Our aim is to show that η and $\eta\tau$ act f.p.f. on $\Phi(R)$, from which (7) follows readily. By (4), $C_{\Phi(R)}(\eta) = C_{\Phi(R)}(\langle\eta, \sigma\rangle) \triangleleft R$, so by (5), $C_{\Phi(R)}(\langle\eta, \sigma\rangle) \triangleleft RQPD_8$. If the four-group $\langle\eta, \sigma\rangle$ acts f.p.f. on S^* then it acts f.p.f. on an $RQP\langle\eta, \sigma\rangle$-homogeneous component, U say, of S^*. Now we may apply Theorem 4.1 of Shult [7] to deduce that some element, ω say, of $\langle\eta, \sigma\rangle$ centralizes $RQP/\ker(RQP \text{ on } U)$. If $\omega = \sigma$ then $\sigma = \sigma^\tau$ also centralizes $RQP/\ker(RQP \text{ on } U\tau)$, so σ centralizes RQP (because $S^* = U + U\tau$ is a faithful RQP-module) against (4). If $\omega = \eta$ or $\eta\sigma$ then an argument like that used in the proof of (6) yields a contradiction. Thus $C_{S^*}(\langle\eta, \sigma\rangle)$ is non-trivial.

Now $C_{S^*RQP}(\langle\eta, \sigma\rangle)$ admits $D_8/\langle\eta, \sigma\rangle$ f.p.f. so it is abelian. Therefore $C_{\Phi(R)}(\langle\eta, \sigma\rangle)$ centralizes $C_{S^*}(\langle\eta, \sigma\rangle)$. In view of the normality $C_{\Phi(R)}(\langle\eta, \sigma\rangle) \triangleleft RQPD_8$ and the irreducibility of S^*, we must have $C_{\Phi(R)}(\langle\eta, \sigma\rangle) = 1$, so η acts f.p.f. on $\Phi(R)$. Thus η inverts each element of $\Phi(R)$. But by the same argument so does $\eta\tau$, therefore τ centralizes $\Phi(R)$ as we require. This means that D_8 inverts $\Phi(R)$ so, in view of (4) and (5), it follows that each subgroup of $\Phi(R)$ is normal in $RQPD_8$. Therefore each element of $\Phi(R)$ acts f.p.f. on S^*, that is, $\Phi(R)$ acts regularly on S^*. (4) and (6) establish that R is a non-abelian special group, so by 5.3.14 of [4], $\Phi(R)$ is cyclic of order r.

(8) *S^* is the sum of 2 homogeneous components, S_1^* and S_2^* say, under $\Phi(R)$ and τ acts f.p.f. on S^*.*

Since, by (4), (5) and (7), $RQP\langle\tau\rangle$ centralizes $\Phi(R)$, S^* is either a homogeneous $\Phi(R)$-module or is the sum of 2 homogeneous components. In the first case $\Phi(R)$ acts as scalar transformations of S^*, so the transformations representing $\Phi(R)$ commute with those representing D_8, that is, D_8 centralizes $\Phi(R)$, contradicting the f.p.f. action of D_8 on G. Therefore $S^* = S_1^* + S_2^*$ say, the

$\Phi(R)$-homogeneous components S_1^* and S_2^* stabilized by and irreducible under $RQP\langle\tau\rangle$, and interchanged by η. If v is a non-trivial element of S_1^* centralized by τ then $v + v\eta$ is a non-trivial element of S^* centralized by D_8 again contrary to assumption. Similarly τ acts f.p.f. on S_2^* and so (8) is established.

Our final contradiction follows from

(9) *R has order* 3^3.

Let S_0^* be an irreducible $R\langle\tau\rangle$-submodule of S_1^*. Since S^* is an irreducible $RQPD_8$-module, $\Phi(R)$ acts f.p.f. on S^*, so S_0^* is a faithful R-module and hence, by (4), also faithful for $R\langle\tau\rangle$. Because S_0^* is homogeneous for $\Phi(R)$ and R is extraspecial, it follows that S_0^* is a homogeneous R-module. (The $r - 1$ faithful irreducible representations of R are characterized by the actions of $\Phi(R)$.) Since an irreducible projective representation of a cyclic group is 1-dimensional, the analogue of Theorem 51.7 of [3] in characteristic s shows that S_0^* is actually an irreducible R-module. By (4), τ acts regularly on the non-trivial elements of $R/\Phi(R)$ and by (8), τ acts f.p.f. on S_0^*. These last three facts enable us to use the Hall-Higman type argument of Shult [7] in his proof of Theorem 3.1 to deduce that R has order 3^3.

Now by (9) the chief factor $R/\Phi(R)$ of GD_8 has order 3^2 and must therefore be centralized by Q, against (1e). This final contradiction establishes the theorem.

References

[1] S.F. Bauman, "The Klein group as an automorphism group without fixed point", *Pacific J. Math.* 18 (1966), 9-13.

[2] T.R. Berger, "Nilpotent fixed point free automorphism groups of solvable groups", (to appear).

[3] Charles W. Curtis, Irving Reiner, *Representation theory of finite groups and associative algebras* (Pure and Applied Mathematics, XI. Interscience [John Wiley & Sons], New York, London, 1962).

[4] Daniel Gorenstein, *Finite groups* (Harper and Row, New York, Evanston, London, 1968).

[5] Fletcher Gross, "Fixed-point-free operator groups of order 8 ", *Pacific J. Math.* 28 (1969), 357-361.

[6] Fletcher Gross, "Elementary abelian operator groups", *Bull. Austral. Math. Soc.* 7 (1972), 91-100.

[7] Ernest E. Shult, "On groups admitting fixed point free abelian operator groups", *Illinois J. Math.* 9 (1965), 701-720.

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