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THE EXTRAORDINARY HIGHER TANGENT SPACES OF CERTAIN QUADRIC INTERSECTIONS

by R. H. DYE

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Let C_r be the intersection of $n-r$ quadrics with a common self-polar simplex S in projective n-space $[n]$. Let Γ_r be a C_r that can be taken in coordinate form as $\sum_{i=0}^{n} a_i^j x_i^2 = 0$, for $j=0,1,\ldots,n-r-1$. Every C_1 is a Γ_1 and its points of hyperosculation have special properties: they are the points of intersection of C_1 with the faces of S each counting $(n-1)(n-2)/2$ times, and the osculating $[s]$, for $s \leqq n-1$, has $2s$-point contact. Here we show that if $r \geqq 2$ and $n > 2r$ then *every* point of Γ_r has exceptional higher tangent spaces: the s-tangent space at a point P of an r-dimensional variety V is the intersection of all primes that cut V in a variety having an $(s+1)$-fold point (at least) at P, and normally has dimension $\binom{r+s}{s}-1$ if this is less than n. The s-tangent space to Γ_r at a point not in a face of S is an $[rs]$ (provided $rs < n$). Usually it is the existence of lines on V through P that cause a lower than expected s-tangent dimension. Not so on Γ_r, since its lines form a subvariety. If $n \geqq 5$ not every C_2 is a Γ_2. Take $n \geqq 5$. We show that C_2 is Γ_2 if and only if C_2 contains a line. Also C_2 is a Γ_2 if and only if at some one point of C_2 off the faces of S the second-tangent space is a $[4]$. Thus, unexpectedly, we have: if one point of C_2 off the faces of S has a $[4]$ for second-tangent space, then so do all such points of C_2. We obtain results for points of Γ_r in the faces of S.

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1. Introduction

We shall be concerned with the r-dimensional variety C_r that is the intersection of $n-r$ quadrics with a common self-polar simplex S in projective n-space $[n]$. We shall always assume that the linear system of quadrics through C_r has no cone with vertex of dimension more than $n-r-2$. This is the general case, and is (see Section 2.1) the condition that C_r is nonsingular The harmonic inversions with respect to a vertex of S and the opposite hyperface generate an Abelian group G of order 2^n fixing each quadric through C_r: if S is taken as the simplex of reference then each element of G corresponds to changing the signs of some of the coordinates.

We shall denote by Γ_r the particular C_r given by

$$\sum_i a_i^j x_i^2 = 0, \qquad j=0,1,\ldots,n-r-1. \tag{1}$$

Summations will always run over $0,1,2,\ldots,n$ unless otherwise indicated. The condition that no cone through Γ_r has vertex of dimension more than $n-r-2$ is that (see Section 2.1) $a_0, a_1, \ldots, a_n$ are distinct. With a slight abuse of language we shall say that "a C_r is a Γ_r" if coordinates can be found so that the C_r in question is given by (1) for some choice of (distinct) $a_0, a_1, \dots, a_n$.

Every C_1 is a Γ_1 [**2**, p. 334] and has, for $n \geqq 4$, exceptional osculating properties. These were discovered in [**2**, p. 334, 335], [**6**, p. 43], [**3**, p. 304]: [**2**] and [**3**] deal, more generally, with the intersection of $n-1$ primnals of order $m \geqq 2$ with a common self-polar simplex. A nonsingular curve of order N and genus g in $[n]$ has [**1**, p. 200], [**9**, p. 389], $(n+1)\{N+n(g-1)\}$ points of hyperosculation, i.e., points where the osculating $[n-1]$ has more than its statutory minimum n-point contact (intersection). Usually these points are distinct, and their osculating $[s]$ for $s<n-1$ have exactly $(s+1)$-point contact. For Γ_1 [**2**, p. 336], [**6**, p. 42] we have $N=2^{n-1}$ and $g=2^{n-2}(n-3)+1$, so there are $(n-1)(n-2)(n+1)2^{n-2}$ points of hyperosculation. These [**2**, p. 335] are the $(n+1)2^{n-1}$ points of intersection of Γ_1 with the hyperfaces of S, each counted $(n-1)(n-2)/2$ times. This confluence produces exceptionally high order of contact for the osculating spaces: at a point of hyperosculation the osculating $[s]$ for $s=0,1,2,\dots,n-1$ has $2s$-point contact. The first part of this paper exhibits the even more surprising tangent properties of Γ_r when $r \geqq 2$: provided $n>2r$ *every* point of Γ_r has exceptional higher tangent spaces.

First a reminder about the second, third,..., tangent spaces at a simple point P on an irreducible r-dimensional variety V_r in $[n]$. The first, or ordinary, tangent space to V_r at P is the $[r]$ that is the intersection of all the $[n-1]$ through P which cut V_r in a variety having a double-point (at least) at P. More generally, the s-tangent space to V_r at P is the intersection of all $[n-1]$ that cut V_r in a variety having an $(s+1)$-fold point (at least) at P: here we take an empty intersection of hyperplanes to be the whole space $[n]$. In general, the s-tangent space is $[n]$ if $\binom{r+s}{s} \geqq n+1$, and is an $[\binom{r+s}{s}-1]$ if $\binom{r+s}{s}<n+1$. However, the occurrence of lines or higher spaces on V_r and through P reduces the dimension: if P is on a $[k]$ on V_r then the second tangent space has dimension at most $\binom{r+2}{2}-\binom{k+1}{2}-1$. For more details see [**9**, pp. 402, 403], [**8**, pp. 905, 906, 922, 923], [**7**, pp. 20, 21]: Segre's $(s+1)$-tangent space is our, and Room's, s-tangent space.

We show (Theorem 1) that at each point of Γ_r not in a hyperface of S the s-tangent space is an $[rs]$ provided $rs<n$: if $rs \geqq n$ then it is $[n]$ itself. This is a remarkably small dimension for all but the smallest r and s. In fact, as an easy induction argument on r shows, if $s, r \geqq 2$ and $rs<n$ then the s-tangent space has smaller dimension than expected from general theory. This phenomenon is not accounted for by linear subspaces on Γ_r: if $r \geqq 2$ and $n \geqq 2r$ then the points of Γ_r that lie on lines on Γ_r form (Proposition 1) a subvariety of dimension $r-1$: some points of Γ_r not in hyperfaces of S are on this subvariety but most are not, and all have the same type of nest of tangent spaces. Our proof uses the explicit parameterisation of Γ_r, and allows (Theorem 1) the equations for the tangent spaces to be given explicitly. It is possible to deal with the points of Γ_r in one or more hyperfaces of S, but the algebra is much more involved, and the labour needed to deal with all possibilities would be disproportionate to the rewards. We content ourselves (Theorem 2) with the following: if P is a point on Γ_r in exactly t hyperfaces of S, where $1 \leqq t \leqq r$, and P is not on a line on Γ_r then the second-tangent space at P is a $[2r]$ (provided $2r<n$), but the third-tangent space is a $[3r-t]$ (provided $3r-t<n$). Thus if $t=r$, when P cannot be on a line on Γ_r, the second and third-tangent

spaces are the same $[2r]$. If P is on a line on Γ_r then the second-tangent space can be a $[2r]$ but may be of lower dimension: if $r=2$ we always have the second possibility and the second and third tangent spaces are $[3]$ and $[4]$ respectively. Thus we have complete results for Γ_2.

The second part of the paper obtains some unexpected results for C_2 and Γ_2. When $n=3$ or 4 every C_2 is a Γ_2; we have respectively the ordinary quadric in $[3]$ and the general intersection of two quadrics in $[4]$. When $n=5$ then Γ_2 is the nonsingular model of the Kummer surface, and its second-tangent spaces were found in [**4**, pp. 210, 211] by methods very different to those of this paper. Using a criterion by M. Reiss for 6 points to lie on a conic, Edge proves [**5**, pp. 954, 955] that if the surface of intersection of three linearly independent quadrics in $[5]$ with a common self-polar simplex has a line in general position (i.e., not meeting a plane face of the simplex), then the surface is a Γ_2. Suppose now that $n \geqq 5$. We prove, by a direct general method, (Theorem 3) that C_2 is a Γ_2 if and only if C_2 has a line. In fact Γ_2 has exactly 2^n lines forming a single orbit under G. More surprisingly we have (Theorem 4): C_2 is a Γ_2 if and only if at some point of C_2 not in a hyperface of S the second-tangent space is a $[4]$. We deduce (Theorem 5) that C_2 contains a line if and only if at some point of C_2 not in a hyperface of S the second-tangent space is a $[4]$, and also the unexpected result (Theorem 6) that if one point of C_2 not in a hyperface of S has a $[4]$ for second-tangent space, then so do all such points of C_2.

Γ_2 has, if $n \geqq 4$, only 2^n lines. It is interesting to contrast our results with Segre's claim [**8**, p. 906], quoting earlier references, and Room's assertion [**7**, p. 21] that if at all points of a surface the second-tangent space is a $[4]$, then the surface is ruled.

2. The tangent spaces of Γ_r

2.1. First a few remarks about C_r. Choosing S as the simplex of reference we may take C_r to be given by

$$\sum_i c_i^{(j)} x_i^2 = 0, \qquad j = 0, 1, \ldots, n-r-1. \tag{2}$$

Since the vertex of any cone $\sum_i \alpha_i x_i^2 = 0$ has dimension one less than the number of zero α_i, we see that there is no cone through C_r with vertex of dimension more than $n-r-2$ if and only if no non-zero linear combination of the $n-r$ quadratic forms of (2) has $n-r$ or more zero coefficients. This is equivalent to demanding that each $(n-r)\times(n-r)$ submatrix of $\mathbf{C}=(c_i^{(j)})$ is nonsingular. It follows from (2) that no point of C_r has more than r zero coordinates. Hence, if $\xi=(\xi_0, \xi_1, \ldots, \xi_n)$ is a point of C_r with say, $\xi_{i_0}\xi_{i_1}\ldots\xi_{i_{n-r}} \neq 0$ then the coefficients of $x_{i_0}x_{i_1}\ldots x_{i_{n-r}}$ in the $n-r$ hyperplanes

$$\sum_i c_i^{(j)} \xi_i x_i = 0 \tag{3}$$

form an $(n-r)\times(n-r)$ matrix whose determinant is $\xi_{i_0}\xi_{i_1}\ldots\xi_{i_{n-r}}$ times the determinant of an $(n-r)\times(n-r)$ submatrix of $\mathbf{C}$, and so is non-zero. Hence the hyperplanes (3) are linearly independent and meet in an $[r]$; ξ is a simple point on C_r.

For Γ_r given by (1) the appropriate $\mathbf{C}$ is $\mathbf{A}=(a_i^j)$. By considering its rows we see that an $(n-r)\times(n-r)$ submatrix of $\mathbf{A}$ is singular if and only if there is a non-zero polynomial of degree at most $n-r-1$ having $n-r$ of the a_i for roots: this can happen if and only if two of the a_i are equal. Hence if no two a_i are equal then Γ_r is a C_r and is nonsingular, as was claimed in Section 1.

2.2. Let

$$f(\theta)=(\theta-a_0)(\theta-a_1)(\theta-a_2)\dots(\theta-a_n). \tag{4}$$

By standard partial fraction theory, if $l<n+1$ then

$$\frac{\theta^l}{f(\theta)}=\sum_i \frac{a_i^l}{(\theta-a_i)f'(a_i)}.$$

Hence, on taking $\theta=0$ for $l=1,2,\dots,n$ we obtain

$$\sum_i \frac{a_i^j}{f'(a_i)}=0 \qquad j=0,1,\dots,n-1. \tag{5}$$

Thus $r+1$ solutions of (1), considered as linear equations in the x_i^2 are given by $x_i^2=a_i^k/f'(a_i)$ for $k=0,1,2,\dots,r$. Since the a_i are distinct these solutions are linearly independent: no non-zero polynomial of degree at most $r<n$ has $n+1$ distinct roots. Adopting the convention that (y_i) denotes the point $(y_0,y_1,y_2,\dots,y_n)$, we see that if (ξ_i) is a point on Γ_r then there are $p_0,p_1,\dots,p_r$ such that

$$f'(a_i)\xi_i^2=p(a_i) \tag{6}$$

where

$$p(\theta)=p_0+p_1\theta+\dots+p_r\theta^r. \tag{7}$$

Suppose, until further notice, that $p_0\neq 0$. Then (ξ_i^2) and the $(a_i^k/f'(a_i))$ for $k=1,2,\dots,r$ are linearly independent solutions of (1) in x_i^2. Hence, since our coordinates are homogeneous, if (x_i) is any point on Γ_r near (ξ_i) then it can be taken as

$$x_i^2=\xi_i^2+\frac{u_1a_i+u_2a_i^2+\dots+u_ra_i^r}{f'(a_i)} \tag{8}$$

for some (small) $u_1,u_2,\dots,u_r$. Hence, if no ξ_i is zero, then the analytic parametrisation of Γ_r near (ξ_i) is given by

$$x_i=\xi_i\left[1+\frac{u_1a_i+u_2a_i^2+\dots+u_ra_i^r}{p(a_i)}\right]^{1/2}$$

i.e., on expansion by the binomial series,

$$x_i=\xi_i\left\{1+\sum_{l=1}^{\infty} K_l \frac{(u_1a_i+u_2a_i^2+\cdots+u_ra_i^r)^l}{[p(a_i)]^l}\right\}, \tag{9}$$

where the K_l are non-zero numerical constants. A hyperplane $\sum_i h_i y_i=0$ meets Γ_r in a variety with at least an s-fold point at (ξ_i) if and only if, with x_i as in (9), $\sum_i h_i x_i$ has no monomial terms in $u_1,u_2,\dots,u_r$ appearing with total degree s or less. From (9) we see that this is the condition that the hyperplane contains all the points $(a_i^m\xi_i/[p(a_i)]^l)$ for $l\leqq m\leqq rl$ and $l=s,s-1,\dots,1,0$. These points thus span the s-tangent space T_s at (ξ_i). Writing $\lambda_i=\xi_i/[p(a_i)]^s\neq 0$, we see that T_s is spanned by the points of the array

$$\begin{array}{ll}
(a_i^s\lambda_i),(a_i^{s+1}\lambda_i),\dots, & (a_i^{rs}\lambda_i)\\[2mm]
(a_i^{s-1}p(a_i)\lambda_i),(a_i^sp(a_i)\lambda_i),\dots, & (a_i^{r(s-1)}p(a_i)\lambda_i)\\[2mm]
(a_i^{s-2}[p(a_i)]^2\lambda_i),(a_i^{s-1}[p(a_i)]^2\lambda_i),\dots, & (a_i^{r(s-2)}[p(a_i)]^2\lambda_i)\\
\vdots & \vdots\\
(a_i[p(a_i)]^{s-1}\lambda_i),\dots, & (a_i^r[p(a_i)]^{s-1}\lambda_i)\\[2mm]
([p(a_i)]^s\lambda_i). &
\end{array}$$

Since $p_0\neq 0$, we see from (7) that each point of the second row apart from the first is linearly dependent on those of the first row, while the first point of the second row is dependent on these and $(a_i^{s-1}\lambda_i)$. Similarly the points of the third row are dependent on those of the higher placed rows and $(a_i^{s-2}\lambda_i)$. And so on. Thus T_s is spanned by the $rs+1$ points $(\xi_ia_i^k/[p(a_i)]^s)$ for $k=0,1,\dots,rs$. Because the a_i are distinct the $n+1$ vectors $(a_i^0),(a_i^1),\dots,(a_i^n)$ are linearly independent; and (a_i^{n+1}), $(a_i^{n+2})\dots$ are linearly dependent on them. Hence the matrix with the spanning points for T_s for columns has $rs+1$ linearly independent rows if $rs<n$ and $n+1$ independent rows if $rs\geqq n$. Hence T_s is an $[rs]$ if $rs<n$ and the whole $[n]$ if $rs\geqq n$.

If $rs<n$ then, by (6), T_s is spanned by the points $(\xi_ia_i^k/[f'(a_i)]^s\xi_i^{2s})$ for $k=0,1,\dots,rs$. The value of $\sum_i a_i^j[f'(a_i)]^{s-1}\xi_i^{2s-1}x_i$ at the $(k+1)$th point is $\sum_i[a_i^{j+k}/f'(a_i)]$, which, by (5), is zero provided $0\leqq j+k\leqq n-1$. Hence T_s is in each of the $n-rs$ hyperplanes

$$\sum_i a_i^j[f'(a_i)]^{s-1}\xi_i^{2s-1}x_i=0, \qquad j=0,1,\dots,n-rs-1. \tag{10}$$

Again, the a_i being distinct means these hyperplanes are independent. Thus T_s is given by (10).

2.3. Suppose now that no ξ_i is zero, but $p_0=0$ in (7). If, for any α, we let $b_i=a_i+\alpha$ then, from (1), Γ_r can be taken as

$$\sum_i b_i^j x_i^2 = 0, \qquad j = 0, 1, \ldots, n-r-1.$$

Further, the b_i are distinct, and if $g(\theta)$ has the b_i for roots then

$$g'(b_i) = \prod_{j \neq i} (b_i - b_j) = \prod_{j \neq i} (a_i - a_j) = f'(a_i).$$

Also $p(a_i)$ is a polynomial $\hat{p}(b_i)$ in b_i whose constant term is $p(-\alpha)$. We may choose α so that $p(-\alpha) \neq 0$. Then, writing (6) as

$$f'(a_i)\xi_i^2 = g'(b_i)\xi_i^2 = \hat{p}(b_i)$$

the argument of Section 2.2. shows that if $rs < n$ then the s-tangent space at (ξ_i) is an $[rs]$ given by

$$\sum_i b_i^j [f'(a_i)]^{s-1} \xi_i^{2s-1} x_i = 0, \qquad j = 0, 1, \ldots, n - rs - 1.$$

This is just the $[rs]$ given by (10). We have proved

Theorem 1. *Let (ξ_i) be a point of Γ_r of $[n]$ that is not in any hyperface of the common self-polar simplex S. If $rs < n$ then the s-tangent space to Γ_r at (ξ_i) is the $[rs]$ given by*

$$\sum_i a_i^j [f'(a_i)]^{s-1} \xi_i^{2s-1} x_i = 0, \qquad j = 0, 1, \ldots, n - rs - 1.$$

If $rs \geqq n$ then the s-tangent space is the whole $[n]$.

2.4. To show that the low dimension of the tangent spaces at (ξ_i) is not due to the occurrence of line's or higher spaces through (ξ_i) and on Γ_r, and for latter use, we quickly prove:

Proposition 1. *If $4 \leqq 2r \leqq n$ then the points of Γ_r that lie on lines on Γ_r form a subvariety of dimension $r-1$.*

Proof. Let (ξ_i) be on a line of Γ_r and be given by (6). If (η_i) is another point on the line then, by (6), (7)

$$f'(a_i)\eta_i^2 = q(a_i) \tag{11}$$

where $q(\theta)$ is a polynomial of degree at most r. Since (ξ_i) has an orbit of size at most 2^n under G we may assume that (η_i) is not in this orbit and thus that $q(\theta)$ is not a constant multiple of $p(\theta)$. The line $(\xi_i)(\eta_i)$ is on Γ_r if and only if, from (1)

$$\sum_i a_i^j \xi_i \eta_i = 0, \qquad j = 0, 1, \ldots, n-r-1.$$

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Consequently

$$f'(a_i)\xi_i\eta_i = l(a_i) \tag{12}$$

where $l(\theta)$ has degree at most r. From (6), (11) and (12) $p(\theta)q(\theta)-[l(\theta)]^2$ has every a_i for a root, and degree at most $2r$. Hence it is the zero polynomial. If $\theta-\beta$ is a linear factor of $p(\theta)$ and not $q(\theta)$ it must occur to even power in $p(\theta)$, i.e.

$$p(\theta)=(\theta-\beta)^2 d(\theta) \tag{14}$$

for some polynomial $d(\theta)$ of degree at most $r-2$. The (ξ_i) given by (6) with $p(\theta)$ as in (14) form a subvariety of dimension $r-1$ on Γ_r. Notice that such a (ξ_i) lies on one of the lines given by letting λ vary in

$$f'(a_i)x_i^2=(a_i+\lambda)^2 d(a_i) \tag{15}$$

and extracting various combinations of square roots.

2.5. We now consider the case when (ξ_i) is in precisely t hyperfaces of S, where $1 \leqq t \leqq r$ (Section 2.1). We may deal with the case $\xi_0=\xi_1=\cdots=\xi_{t-1}=0$. Then (ξ_i) is given by (6) where

$$p(\theta)=(\theta-a_0)(\theta-a_1)\dots(\theta-a_{t-1})q(\theta) \tag{16}$$

and $q(\theta)$ is an $(r-t)-ic$. We shall assume that $p_0 \neq 0$, leaving the reader to apply the technique of Section 2.3 to the case $p_0=0$. Thus, by (16), no a_l for $l<t$ is zero. Then, if (x_i) is a point on Γ_r near (ξ_i) it can be taken as in (8). If

$$U(\theta)=u_1+u_2\theta+\cdots+u_r\theta^{r-1}$$

then, by considering partial fractions,

$$\frac{U(\theta)}{(\theta-a_0)(\theta-a_1)\dots(\theta-a_{t-1})}=v_1+v_2\theta+\cdots+v_{r-t}\theta^{r-t-1}+\sum_{l=0}^{t-1}\frac{w_l^2}{\theta-a_l}.$$

Then, by (8) we have

$$x_l=\kappa_l w_l \qquad l=0,1,\dots,t-1, \tag{17}$$

where

$$f'(a_l)\kappa_l^2=a_l\prod_{\substack{i=0\\ i\neq l}}^{t-1}(a_l-a_i) \tag{18}$$

and

$$x_i^2=\xi_i^2\left[1+\frac{v_1a_i+v_2a_i^2+\cdots+v_{r-t}a_i^{r-t}}{q(a_i)}+\sum_{l=0}^{t-1}\frac{a_iw_l^2}{q(a_l)(a_i-a_l)}\right] \quad \text{if } i\geqq t. \tag{19}$$

No κ_l is zero so the s-tangent space T_s at (ξ_i) contains the "first" t vertices of S. Call this set of vertices S_t. If we extract square roots of (19) by the binomial theorem and apply the arguments of Section 2.3 to the monomial terms involving no w_l, we see that T_2 is spanned by S_t, the $(\xi_ia_i^k/[q(a_i)]^2)$ for $k=0.1,\ldots,2(r-t)$, and the points $(\xi_ia_i/(a_i-a_l)q(a_i))^*$ for $l=0,1,\ldots,t-1$: the asterisk means that the coordinate representative is shown for $i\geqq t$; it is zero when $i<t$. Since $\theta q(\theta)$ has degree at most $r-t+1$ and leaves remainder $a_lq(a_l)$ on division by $\theta-a_l$ we see that $(\xi_ia_i/(a_i-a_l)q(a_i))^*$ is a linear combination of the points $(\xi_ia_i^k/[q(a_i)]^2)$ and $(\xi_ia_lq(a_l)/(a_i-a_l)[q(a_i)]^2)^*$. If $2r<n$ then the former set of $2(r-t)+1$ points together with the t points $(\xi_i(a_i-a_l)^{-1}/[q(a_i)]^2)^*$ are independent points in $x_0=x_1=\cdots=x_{t-1}=0$, since no polynomial of degree at most $2(r-t)+1+t=2r-t+1<(n+1)-t$ can have each of $a_t,a_{t+1},\ldots,a_n$ as a root. Hence, provided, $q(a_l)\neq 0$ for $l=0,\ldots,t-1, T_2$ is spanned by these points and the points of S_t: thus T_2 is a $[2r]$.

Notice that if $q(a_0)=0$, then by (16), $(\theta-a_0)^2$ is a factor of $p(\theta)$ and so, by Section 2.4 and (14), (ξ_i) is on a line on Γ_r. If $r=2$, $t=1$ and (ξ_i) with $\xi_0=0$ is on a line of Γ_r then, by (14), $(\theta-a_0)^2$ must be a factor of $p(\theta)$ so $q(a_0)=0$. Then T_2 is spanned by the first vertex of S and the points $(\xi_ia_i^k/[q(a_i)]^2)$: T_2 is a $[3]$. If $r>2$, $t=1$, and (ξ_i) with $\xi_0=0$ is on a line on Γ_r then, by (14), $(\theta-a_0)^2$ need not be a factor of $p(\theta)$, though it could be: different dimensions for T_2 can occur.

A similar discussion with the cubic terms in the expansion form (19) shows that T_3 is spanned, if no $q(a_l)$ is 0 and $3r-t<n$, by the t independent points, S_t, the $(\xi_ia_i^k/[q(a_i)]^3)$ for $k=0,1,\ldots,3(r-t)$ and the $(\xi_i(a_i-a_l)^{-1}/[q(a_i)]^3)^*$: thus T_3 is a $[3r-t]$. If $r=2$, $t=1$ and $q(a_0)=0$ then no asterisked point occurs and T_3 is a $[4]$. We have:

Theorem 2. *Suppose that (ξ_i) is a point on Γ_r in exactly t hyperfaces of the common self-polar simplex. Then:*

(i) *if (ξ_i) is not on a line on Γ_r its second-tangent space is a $[2r]$ (provided $2r<n$) and its third-tangent space is a $[3r-t]$ (provided $3r-t<n$);*

(ii) *if $r=2$, $t=1$, and (ξ_i) is on a line on Γ_2 then its second-tangent space is a $[3]$ (provided $n>3$) and its third-tangent space is a $[4]$ (provided $n>4$).*

3. C_2, Γ_2 Lines and osculating [4]

3.1. We first prove:

Theorem 3. *If $n\geqq 5$ then C_2 is a Γ_2 if and only if it contains a line.*

Proof. By Section 2.4 Γ_2 contains 2^n lines given by (15) with $d(\theta)=1$. Thus we need only prove that if C_2, given by (2) with $r=2$, contains a line L then C_2 is a Γ_2. The matrix condition of Section 2.1 on $(c_i^{(j)})$ shows that C_2 meets any hyperface $x_l=0$ of S in a C_1 which is, by Section 1, an irreducible curve Γ_1 of order 2^{n-1}. Hence L cannot lie in a hyperface of S. Let (ξ_i), with no ξ_i zero, be one point of L, and let (η_i) be another point of L not in the (finite) orbit of (ξ_i) under G. Thus (η_i^2) is not a scalar multiple of (ξ_i^2).

If (ξ_i^2), $(\xi_i\eta_i)$ and (η_i^2) were linearly dependent we should have constants λ, μ, ν, not all zero such that, if $a_i=\eta_i/\xi_i$ then

$$\lambda a_i^2+\mu a_i+\nu=0 \qquad \text{for } i=0,1,\ldots,n.$$

Hence each a_i must take one of two possible values. Since $n+1\geqq 6$ there is a set of three a_i with the same value, say κ. Not all the a_i can equal κ since the points (ξ_i) and (η_i) are distinct. Hence the point $(\eta_i-\kappa\xi_i)$ is on L and C_2, and has at least three zero coordinates: contradicting Section 2.1. Since L is on each quadric of (2) it follows that (ξ_i^2), $(\xi_i\eta_i)$ and (η_i^2) span the solution space of the equations (2) regarded as linear in the x_i^2. Hence the points of C_2 are given, as p, q, r, vary, by

$$x_i^2=p\xi_i^2+q\xi_i\eta_i+r\eta_i^2.$$

Taking new coordinates (y_i) with $y_i=x_i/\xi_i$, C_2 is given by

$$y_i^2=p+qa_i+ra_i^2.$$

Were two a_i to be equal, say $a_0=a_1$, then C_2 would lie in the cone (plane-pair) $y_0^2-y_1^2=0$, whose vertex has dimension $n-2$, which is greater than the permitted minimum (Section 1) of $n-4$. Hence if $f(\theta)$ is defined by (4) no $f'(a_i)$ is zero. So if new coordinates (z_i) are taken with $z_i=[f'(a_i)]^{1/2}y_i$, for some choice of square roots, then C_2 is given by

$$f'(a_i)z_i^2=p+qa_i+ra_i^2. \tag{20}$$

Thus, by (6) with $r=2$, we see that, in this coordinate system, C_2 is Γ_2, and we are done.

3.2. A much more remarkable result is:

Theorem 4. *If $n\geqq 5$ and if at one point of C_2 not in a hyperface of S the second-tangent space is a* [4], *then C_2 is a Γ_2.*

Proof. Suppose that the second-tangent space T_2 at (ξ_i) on C_2, with no ξ_i zero, is a [4]. Let (η_i^2) and (ζ_i^2), together with (ξ_i^2) span the solution space of (2). Then the parametrisation of C_2 near (ξ_i) is given by

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$$x_i^2=\xi_i^2+u\eta_i^2+v\zeta_i^2.$$

Writing $b_i=\eta_i^2/\xi_i^2$ and $c_i=\zeta_i^2/\xi_i^2$ we see that this is

$$x_i=\xi_i[1+ub_i+vc_i]^{1/2}=\xi_i\{1+K_1(b_iu+c_iv)+K_2(b_iu+c_iv)^2+\ldots\},$$

where K_1, K_2 are non-zero constants. Hence T_2 is spanned by the six points (ξ_i), (ξ_ib_i), (ξ_ic_i), $(\xi_ib_i^2)$, $(\xi_ib_ic_i)$, $(\xi_ic_i^2)$. Since T_2 is a [4] these points are linearly dependent, i.e. there are constants A, B, C, D, E, F, not all zero, such that

$$A+Bb_i+Cc_i+Db_i^2+Eb_ic_i+Fc_i^2=0, \text{ for } i=0,1,\ldots,n.$$

Hence the points $(1,b_i,c_i)$ all lie on a conic $\mathscr{C}$. No 3 of these points are linearly dependent. For if, say, $(1,b_0,c_0)$ $(1,b_1,c_1)$ $(1,b_2,c_2)$ where dependent then there are constants α, β, γ, not all zero such that

$$\alpha+\beta b_0+\gamma c_0=\alpha+\beta b_1+\gamma c_1=\alpha+\beta b_2+\gamma c_2=0.$$

Hence the points of C_2 given by

$$x_i^2=\alpha\xi_i^2+\beta\eta_i^2+\gamma\zeta_i^2=\xi_i^2(\alpha+\beta b_i+\gamma c_i)$$

would have 3, at least, zero coordinates, contradicting Section 2.1. Hence the $n+1$ points $(1,b_i,c_i)$ are distinct and $\mathscr{C}$ is a nonsingular conic. $\mathscr{C}$ may, in some coordinate system, be taken as $(1,t,t^2)$. Hence, using primes to denote transposes, there is a nonsingular 3×3 matrix B, and distinct values $a_0,a_1,\ldots,a_n$ such that

$$(1,b_i,c_i)'=B(1,a_i,a_i^2)', \qquad i=0,\ldots,n+1. \tag{21}$$

Any point (x_i) on C_2 is given by

$$x_i^2=\hat{p}\xi_i^2+\hat{q}\eta_i^2+\hat{r}\zeta_i^2$$

for some $\hat{p}$, $\hat{q}$, $\hat{r}$. Using (21) we see that in the new coordinate system (y_i) given by $y_i=x_i/\xi_i$ C_2 is given by

$$y_i^2=p+qa_i+ra_i^2$$

where $(p,q,r)=(\hat{p},\hat{q},\hat{r})B$. Now taking $z_i=\sqrt{f'(a_i)}\ y_i$, with $f(\theta)$ given by (4) shows us that C_2 is given by (20), and thus is Γ_2.

3.3. From Theorems 1, 3 we deduce:

Theorem 5. *If $n\geqq 5$ then C_2 contains a line if and only if at some point of C_2 not in a hyperface of S the second-tangent space is a* [4].

From Theorems 1, 4 we deduce the unexpected:

Theorem 6. *If* $n \geqq 5$ *and at one point of* C_2 *off the hyperfaces of* S *the second-tangent space is a* [4], *then this is so at all such points of* C_2.

3.4. We conclude by mentioning that some of the theorems can be generalised immediately to the varieties given by (1) with x_i^2 replaced by $x_i^m (m \geqq 2)$ for all i. We mention also that (6) and (7) imply that there is a finite morphism from Γ_r to $[T]$. A description of the degree ramification etc. of this map has been given by Terasoma **[10]**.

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Department of Mathematics and Statistics
The University
Newcastle upon Tyne, NE1 7RU
England