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FRACTALS IN THE LARGE

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ABSTRACT. A *reverse iterated function system* (r.i.f.s.) is defined to be a set of expansive maps $\{T_1, \ldots, T_m\}$ on a discrete metric space M. An invariant set F is defined to be a set satisfying $F = \bigcup_{j=1}^m T_j F$, and an invariant measure μ is defined to be a solution of $\mu = \sum_{j=1}^m p_j \mu \circ T_j^{-1}$ for positive weights p_j. The structure and basic properties of such invariant sets and measures is described, and some examples are given. A *blowup* $\mathcal{F}$ of a self-similar set F in $\mathbb{R}^n$ is defined to be the union of an increasing sequence of sets, each similar to F. We give a general construction of blowups, and show that under certain hypotheses a blowup is the sum set of F with an invariant set for a r.i.f.s. Some examples of blowups of familiar fractals are described. If μ is an invariant measure on $\mathbb{Z}^+$ for a linear r.i.f.s., we describe the behavior of its *analytic* transform, the power series $\sum_{n=0}^\infty \mu(n) z^n$ on the unit disc.

1. **Introduction.** Fractal structure is characterized by repetition of detail at all small scales. Why not large scales as well? In this paper we explore two ways to carry this out.

In the first, we work with a discrete metric space, and a set of expansive mappings that we call a *reverse iterated function system* (r.i.f.s). Related ideas are discussed in Bandt [Ba]. In contrast to the case of a contractive i.f.s. ([B], [F], [Hu]) there is neither existence nor uniqueness in general for invariant sets. Nevertheless we are able to give a satisfactory description of invariant sets as unions of forward orbits of fixed points of iterated maps from the r.i.f.s. We also define the notion of an invariant measure, which is the analogue of a self-similar measure for an i.f.s. A simple example of an invariant set for a r.i.f.s. is the integer Cantor set (all positive integers expressible base 3 using only 0's and 2's as digits) discussed by Bedford and Fisher [BF]. We prove a dimension and density theorem for this and related examples. Another interesting example, which we call the *Fibonacci hibachi*, is the set of integer lattice points in the plane lying between the lines $y = \rho x$ and $y = \rho x + 1$, where $\rho = (\sqrt{5} - 1)/2$ is the reciprocal Golden Ratio. This examples yields by projection onto the line $y = \rho x$ the *Fibonacci comb*, which is a well-known example of a quasi-periodic tiling, a primitive version of the Penrose tilings (see Senechal [Se]). We show that by projection on the orthogonal direction of an invariant measure on the Fibonacci hibachi it is possible to obtain, after renormalizing, the *golden measure*, which is the self-similar measure on [0, 1] associated to the overlapping i.f.s. $S_1 x = \rho x$, $S_2 x = \rho x + 1 - \rho$, with equal probabilities ([L1], [L2], [LN], [STZ]).

The second method we discuss is called *fractal blowups*. We start with a self-similar fractal F in $\mathbb{R}^n$ that is the attractor of an i.f.s. of contractive similarities, and define a

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blowup $\mathcal{F}$ to be the union of an increasing sequence $F = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots$ where each F_j is similar to F. We give a general method of construction that produces a family of blowups in one-to-one correspondence with points of F, and generically produces an uncountable number of blowups that are not similar. This method is essentially the same as one used in the theory of self-similar tilings [Se]. There is a special class of blowups, which we call *periodic*, possessing a global symmetry, so the whole blowup is similar to itself on all scales. Under additional hypotheses we show that a periodic blowup is the sum set of F with an invariant set for a r.i.f.s., thus tying together the two concepts. We give some examples of blowups of familiar fractals such as the Cantor set, the Sierpinski gasket, and the von Koch curve. The von Koch curve is interesting because, although it is usually described as the attractor of a 4 element nonoverlapping i.f.s., it has an additional symmetry (contracting the curve by a factor 1/3 with fixed point at the summit) which can be used to create different blowups. Some of these blowups reveal a spiral structure which is not usually noticeable in drawings of the von Koch curve since it involves comparing different scales with a dilation factor of 729.

The Fourier transform of a self-similar measure reveals an intricate structure that encodes the self-similarity of the measure. This is discussed in detail in a number of recent papers ([Ho1], [JRS], [LW], [S1], [S2], [S3], [S4]). Here we study a related idea for invariant measures of a r.i.f.s. on the nonnegative integers. To every measure μ on $\mathbb{Z}^+$ (of reasonable growth) we can associate an analytic function $\sum_{n=0}^{\infty} \mu(n)z^n$ on the unit disc, which we call the *analytic transform* of μ. We show under certain hypotheses that the L^2 norm on circles of radius r about the origin of the analytic transform of an invariant measure has a predictable growth rate as $r \longrightarrow 1$.

The reader is referred to [B], [F], [M] or other books on fractals for the general theory, and to the survey article [S4] for some of the specific work on "fractals in the small" that motivated this work. Also, the recent work of Holschneider [Ho1] gives a different perspective on the relationship between small scale and large scale fractal behavior.

2. Reverse iterated function systems.

Let M be a locally compact metric space that is complete and discrete (every point is isolated). A mapping $T: M \longrightarrow M$ is said to be *expansive* if there exists a constant $r > 1$ (called the *expansive factor*) such that

$$d(Tx, Ty) \geq r\, d(x,y) \tag{2.1}$$

for all x and y in M. An expansive map is automatically one-to-one, and has at most one fixed point. A *reverse iterated function system* (r.i.f.s.) is a set of $m \geq 2$ expansive maps, $\{T_1, \ldots, T_m\}$. For any multi-index $J = (j_1, \ldots, j_N)$ of length $N (= |J|)$ we denote by T_J the composition $T_{j_1} \circ T_{j_2} \circ \cdots \circ T_{j_N}$, and refer to such maps as *iterated maps*. If $r_1, \ldots, r_m$ are the expansive factors for the r.i.f.s. (we can, for convenience, assume $r_1 \leq r_2 \leq \cdots \leq r_m$) then $r_J = r_{j_1} \cdots r_{j_N}$ will serve as an expansive factor for T_J. In particular, we have the lower bound

$$r_J \geq (r_1)^{|J|}. \tag{2.2}$$

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If x is any point in M, the *forward orbit of* x, denoted F_x, is the set $\{T_J x\}$. Note that we do not allow J to be the empty set, so x is not necessarily an element of its forward orbit; this happens if and only if x is a fixed point of one of the iterated maps. We let P denote this set of fixed points.

A set F is said to be *invariant* under the r.i.f.s. if it is nonempty and satisfies

$$F = \bigcup_{j=1}^{m} T_j F. \tag{2.3}$$

We say that the invariant set is *non-overlapping* if the union in (2.3) is disjoint. Examples of invariant sets are the forward orbits F_x where x is in P. We will show that all invariant sets are unions of such forward orbits.

LEMMA 2.1. *For any z in M, there exists $R > 0$ and $\rho > 1$ such that $d(x,z) \geq R$ implies $d(T_j x, z) \geq \rho\, d(x,z)$ for $j = 1, \ldots, m$.*

PROOF. Let $a_j = d(T_j z, z)$ and choose $R > 0$ and $\rho > 1$ such that $(r_j - \rho)R \geq a_j$. Then if $d(x,z) \geq R$ we have $(r_j - \rho)\, d(x,z) \geq a_j$ hence

$$\begin{aligned} d(T_j x, z) &\geq d(T_j x, T_j z) - d(T_j z, z) \\ &\geq r_j\, d(x,z) - a_j \\ &\geq \rho\, d(x,z). \end{aligned}$$ ∎

COROLLARY 2.2. *The set P is finite. In fact, it is contained in $B_R(z)$.*

PROOF. Note that if $d(x,z) \geq R$ then $d(T_j x, z) \geq \rho R \geq R$. Thus we may iterate the lemma to obtain

$$d(T_J x, z) \geq \rho^{|J|}\, d(x,z),$$

which is clearly impossible if $T_J x = x$. Thus the fixed-points all lie in $B_R(z)$, which is a finite set since the metric is discrete and complete. ∎

Note that if a is a fixed-point of one iterated map T_J, it is also a fixed-point of infinitely many iterated maps, namely the powers of T_J.

THEOREM 2.3. *A set F is invariant if and only if it is a finite union of forward orbits of points in P. In particular, invariant sets exist if and only if P is nonempty, and there are at most a finite number of invariant sets.*

PROOF. First we show that F_a is an invariant set if $T_{J'} a = a$ for some J'. Of course any forward orbit satisfies $F_a \supseteq \bigcup_{j=1}^{m} T_j F_a$, so we need only show the reverse containment: $x \in F_a$ implies there exists j and $y \in F_a$ such that $x = T_j y$. But $x \in F_a$ means $x = T_J a$ for some J. If $|J| \geq 2$ then $x = T_{j_1} y$ for $y = T_{j_2} \circ \cdots \circ T_{j_N} a \in F_a$. Finally, if $|J| = 1$, then $x = T_{j_1} a$ and we may take $y = a$ since $a = T_{J'} a$ belongs to F_a.

It is clear that unions of invariant sets are invariant. It remains to show that every invariant set F is a union of forward orbits of points of P (by Corollary 2.2 P is finite). To do this we show that every $x \in F$ belongs to F_a for some point a in P. Now from the definition of invariant set there exists an infinite sequence of points $x_k \in F$ and indices j_k such that $x = x_1$, and $x_{k-1} = T_{j_k} x_k$ for $k \geq 2$. Thus x belongs to the forward orbit of each of these points. Now Lemma 2.1 tells us that if x_k does not belong to $B_R(z)$, then $d(x_k, z) \geq \rho^{-1} d(x_{k-1}, z)$. Since $\rho^{-1} < 1$ this means that x_k belongs to $B_R(z)$ for all sufficiently large k. Since $B_R(z)$ is finite this means $x_k = x_{k'}$ for $k' \neq k$, and this is our fixed point of an iterated map. ∎

Given an r.i.f.s., we define an *invariant measure* with weights $p_1, \ldots, p_m$ (all $p_j > 0$) to be a positive measure satisfying

$$\mu(A) = \sum_{j=1}^{m} p_j \mu(T_j^{-1} A) \text{ for all } A \subseteq M. \tag{2.4}$$

Since M is discrete, the measure is determined by the measure of singleton sets, which we write $\mu(x)$ rather than $\mu(\{x\})$. We make the convention that $\mu(T_j^{-1}x) = 0$ if $T_j^{-1}x$ does not exist. Then an invariant measure is determined by a nonnegative solution to the equation

$$\mu(x) = \sum_{j=1}^{m} p_j \mu(T_j^{-1} x) \text{ for all } x \in M. \tag{2.5}$$

It is easy to see that the support of an invariant measure is an invariant set. If the support happens to be a nonoverlapping invariant set, then at most one preimage $T_j^{-1}x$ can exist, so the right side of (2.5) is not really a sum, and we can rewrite it more simply as

$$\mu(T_j x) = p_j \mu(x). \tag{2.6}$$

Under this assumption, if the support of μ is the forward orbit of the fixed point a of, say, T_1, then a necessary and sufficient condition for the existence of a solution is that $p_1 = 1$, for then we can choose $\mu(a)$ to be an arbitrary positive value for the fixed point, and

$$\mu(T_J a) = p_{j_1} \cdots p_{j_N} \mu(a) \tag{2.7}$$

(more generally, if the support is F_a for $a = T_{k_1} \cdots T_{k_M} a$, then the condition is $p_{k_1} \cdots p_{k_M} = 1$ and we can use (2.7)). It is also easy to see that a nonoverlapping invariant set is a disjoint union of forward orbits of points in P, so we have a simple description of all invariant measures in this case. Without the nonoverlapping assumption we can't give such a simple description. However, Lemma 2.1 implies that it suffices to solve (2.5) for x in $B_R(z)$, which is just a Perron-Frobenius type eigenvalue problem, and then (2.5) enables us to extend the solution to all of M. In particular, for each choice of weights, the space of invariant measures is finite dimensional, and there always exist weights (in fact we can take $p_j = \lambda q_j$ for any given $\{q_j\}$ and some λ) for which nontrivial invariant measures exist.

We consider next some simple examples, with $M = \mathbb{Z}$, the integers, and the r.i.f.s. consists of linear transformations

$$T_j x = r_j x + b_j, j = 1, \ldots, m \tag{2.8}$$

with r_j and b_j integers, $|r_j| \geq 2$. If we take the two transformations $T_1x = 3x$ and $T_2x = 3x+2$, then the forward orbit of 0 is the integer Cantor set (see Figure 2.1) of all positive integers whose base 3 representation has only 0's and 2's [BF]. But there is another invariant set, the forward orbit of -1, which consists entirely of negative numbers. In fact, it is easy to see that this invariant set is just the image of the integer Cantor set under the map $n \longrightarrow -(n+1)$, since conjugation with this map permutes T_1 and T_2. Both invariant sets are nonoverlapping.

FIGURE 2.1: An initial segment of the integer Cantor set.

A number in the integer Cantor set can be written uniquely as $2\sum_{k\in A} 3^k$, where A is a finite subset of the nonnegative integers. The invariant measures supported on the integer Cantor set are determined by two positive parameters, $\mu(0)$ and p_2 (we must have $p_1 = 1$). Once these values are chosen we have

$$\mu\Big(2\sum_{k\in A} 3^k\Big) = p_2^{|A|}\mu(0) \tag{2.9}$$

where $|A|$ denotes the cardinality of A. Note that we never obtain a finite measure, even if we choose $p_2 < 1$, since there are infinitely many sets A with the same cardinality.

More generally, if we take

$$T_jx = rx + b_j,\ j = 1,\dots,m \tag{2.10}$$

for the same integer r, and b_j distinct modulo r, then we obtain a nonoverlapping r.i.f.s. since the images of $\mathbb{Z}$ under T_j are disjoint. This gives rise to the following interesting questions we have not been able to answer:

1) Does there exist a r.i.f.s. of the form (2.8) with a nonoverlapping invariant set but such that the images of $\mathbb{Z}$ under T_j overlap?
2) Is it possible for such r.i.f.s. to have both overlapping and nonoverlapping invariant sets?
3) Is it possible for such r.i.f.s. to have an overlapping invariant set but with just a finite number of overlaps?

We now give a dimension and density theorem for the type of r.i.f.s. just considered. For a subset F of $\mathbb{Z}$, define the dimension by

$$\dim F = \lim_{N\to\infty} \log \#\{F\cap[-N,N]\}/\log N \tag{2.11}$$

if the limit exists. Similarly, we define the dimension of a measure μ on $\mathbb{Z}$ by

$$\dim \mu = \lim_{N\to\infty} \log \mu([-N,N])/\log N. \tag{2.12}$$

Given that the dimension is α, we can investigate the limiting behavior of

$$\begin{cases} \#\{F\cap[-N,N]\}/(2N+1)^\alpha \text{ or} \\ \mu([-N,N])/(2N+1)^\alpha, \end{cases} \tag{2.13}$$

and we refer to such quantities as α-densities (usually the limit does not exist in the usual sense).

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THEOREM 2.4. *Let F be an invariant set for a nonoverlapping r.i.f.s. on $\mathbb{Z}$ of the form (2.10). Then F has dimension*

$$\alpha = \log m / \log r \tag{2.14}$$

and the α-density (2.13) is asymptotically multiplicatively periodic in the following sense: there exists a continuous function $g(x)$, bounded and bounded away from zero, satisfying

$$g(x) = g(rx), \tag{2.15}$$

such that the difference of (2.13) and $g(N)$ tends to zero as $N \longrightarrow \infty$. Similarly, the same is true for an invariant measure for

$$\alpha = \log\Big(\sum_{j=1}^{m} p_j\Big) / \log r. \tag{2.16}$$

PROOF. We prove the result for invariant measures, since the statement for the set F is just the special case when all $p_j = 1$, by the nonoverlapping hypothesis. Let $h(N)$ denote the quantity defined by (2.13) for α given by (2.16). Let $b = \max_j |b_j|/r$. The key estimate is

$$\Big(1 - \frac{c}{N}\Big)^{\alpha} h(N-b) \le h(rN) \le \Big(1 + \frac{c}{N}\Big)^{\alpha} h(N+b) \tag{2.17}$$

for some positive constant c and all sufficiently large N. To see this we observe first that

$$[-N+b, N-b] \subseteq T_j^{-1}[-rN, rN] \subseteq [-N-b, N+b]$$

so that

$$\begin{aligned} \Big(\sum_{j=1}^{m} p_j\Big)\mu([-N+b, N-b]) &\le \mu([-rN, rN]) \\ &\le \Big(\sum_{j=1}^{m} p_j\Big)\mu([-N-b, N+b]) \end{aligned}$$

by (2.4). We can replace $\sum_{j=1}^{m} p_j$ by r^{α} by (2.16), and then divide by $(2rN+1)^{\alpha}$ to obtain

$$\Big(\frac{r(2N-2b+1)}{2rN+1}\Big)^{\alpha} h(N-b) \le h(rN) \le \Big(\frac{r(2N+2b+1)}{2rN+1}\Big)^{\alpha} h(N+b)$$

which is of the form (2.17).

Having established (2.17), the rest of the proof is routine, based on the convergence of the infinite products $\prod_{k=0}^{\infty}\big(1 \pm \frac{c}{Nr^k}\big)^{\alpha}$ for sufficiently large N. We take

$$g(x) = \lim_{k \to \infty} h([r^k x]) \tag{2.18}$$

with the limit existing by (2.17), and a similar reasoning shows $\lim_{N\to\infty} g(N) - h(N) = 0$. Note that (2.15) is obvious from the definition. To show that h (and hence g) is bounded away from zero for all $N \ge N_0$ we let

$$\lambda_k = \inf\{h(N) : N_0 \le N \le N_0 r^{k+1}\}.$$

Then the first inequality in (2.17) implies

$$\lambda_k \geq \left(1 - \frac{c}{N_0 r^k}\right)^\alpha \lambda_{k-1}$$

hence

$$h(N) \geq \lambda_0 \prod_{k=1}^{\infty} \left(1 - \frac{c}{N_0 r^k}\right)^\alpha$$

for all $N \geq N_0$, and this is bounded away from zero for N_0 sufficiently large. A similar argument shows h is bounded. ∎

This result implies a second order density theorem in Section 3 of Bedford and Fisher [BF] (this paper also contains results of this nature for various Cantor sets) simply because a periodic function has a mean value. By using renewal theory methods (as in [L1]) it is possible to show for the general r.i.f.s. of the form (2.8) that an invariant measure has dimension α given by

$$\sum_{j=1}^{m} p_j r_j^{-\alpha} = 1, \tag{2.19}$$

and furthermore we have the dichotomy that either $r_j = r^{k_j}$ for some integers r and k_j (the arithmetic case), in which case the asymptotic periodicity of $h(N)$ holds as in the theorem, or in the contrary (nonarithmetic) case the limit of $h(N)$ as $N \longrightarrow \infty$ actually exists. It is not necessary to assume that the r.i.f.s. is nonoverlapping to obtain this result for invariant measures, but it is important to realize that the counting measure on F is invariant only in the nonoverlapping case, so we do not obtain any results about F without the nonoverlapping hypotheses. In particular, we do not know any interesting examples of invariant measures in the nonarithmetic case. These results also carry over to r.i.f.s. of similarity transformations on $\mathbb{Z}^n$.

We turn now to a more complicated example. Let $\rho = (\sqrt{5} - 1)/2$, the reciprocal Golden Ratio, and let FH (the *Fibonacci hibachi*) denote the subset of $\mathbb{Z}^2$ of lattice points in the plane (n, m) satisfying

$$0 \leq m - \rho n \leq 1. \tag{2.20}$$

(See Figure 2.2.) Except for $n = 0$, there is one such point for every integer n, namely $(n, [\rho n] + 1)$. Consider the r.i.f.s. consisting of

$$\begin{cases} T_0(n, m) = (-n - m, -n) \\ T_1(n, m) = (1 - n - m, 1 - n). \end{cases} \tag{2.21}$$

Note that these are affine transformations, but not similarities. Strictly speaking, they are not expansive on all of $\mathbb{Z}^2$, but they have an expansive ratio of ρ^{-1} on FH. We will show that FH is an invariant set for this r.i.f.s. To see this, define the "projection" operator

$$P(n, m) = m - \rho n \tag{2.22}$$

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(up to a constant multiple, this is the projection onto the line perpendicular to $y = \rho x$), so that FH is just the inverse image of $[0, 1]$ under P. A simple computation, using the identity $\rho^2 = 1 - \rho$, shows the intertwining property

$$PT_0 = S_0 P \text{ and } PT_1 = S_1 P \tag{2.23}$$

with the i.f.s.

$$S_0 x = \rho x, \quad S_1 x = \rho x + 1 - \rho \tag{2.24}$$

on $[0, 1]$. This immediately implies that T_0 and T_1 map FH to itself. To show that we get all of FH from the union of $T_0(FH)$ and $T_1(FH)$ we observe first that the inverses T_0^{-1} and T_1^{-1} are also maps of the $\mathbb{Z}^2$ lattice:

$$T_0^{-1}(n, m) = (-m, m - n)$$

and

$$T_1^{-1}(n, m) = (1 - m, m - n - 1).$$

If we start with any (n, m) in FH, then $P(n, m)$ must lie in either $[0, \rho]$ or $[1 - \rho, 1]$ (both can happen), so in the first case $T_0^{-1}(n, m) \in FH$ and $(n, m) = T_0\big(T_0^{-1}(n, m)\big)$ while in the second case $T_1^{-1}(n, m) \in FH$ and $(n, m) = T_1\big(T_1^{-1}(n, m)\big)$.

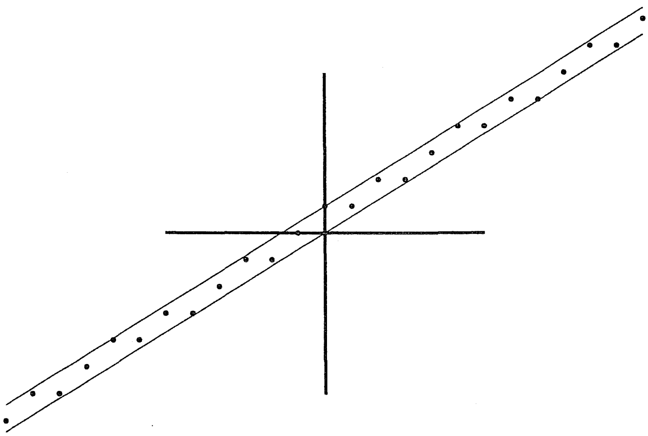

FIGURE 2.2: A portion of the Fibonacci hibachi FH, whown with the lines $y = \rho x$ and $y = \rho x + 1$. Note the 180° rotational symmetry about the point $(0, \frac{1}{2})$.

Of course, both the r.i.f.s T_0, T_1 and the i.f.s. S_0, S_1 are overlapping. It is not hard to analyze the structure of FH in terms of forward orbits of fixed points in P. In fact P consists of just 4 points:

$(0, 0)$, fixed point of T_0,
$(0, 1)$, fixed point of T_1,
$(-1, 0)$, fixed point of $T_1 T_0$,
$(1, 1)$, fixed point of $T_0 T_1$.

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The forward orbit of either $(-1,0)$ or $(1,1)$ is FH with $(0,0)$ and $(0,1)$ removed. The forward orbit of $(0,0)$ is FH with $(0,1)$ removed, and vice versa. Any of the 4 invariant subsets of FH (remove, or not, $(0,0)$ and $(0,1)$) is described by choosing strict or nonstrict inequalities in (2.20). It is easy to show that invariant measures cannot assign positive values to either $(0,0)$ or $(0,1)$, and so must be supported on FH with $(0,0)$ and $(0,1)$ removed. The simplest example has $p_0 = p_1 = 1$ and assigns measure one to the points $(-1,0)$ and $(1,1)$:

$$\begin{cases} \mu(A) = \mu(T_0^{-1}A) + \mu(T_1^{-1}A) \\ \mu\big((-1,0)\big) = \mu\big((1,1)\big) = 1. \end{cases} \tag{2.25}$$

This measure is completely determined by the sequence of positive integers

$$c_n = \mu\big((n,[\rho n]+1)\big) \text{ for } n \neq 0 \tag{2.26}$$

(by symmetry $c_n = c_{-n}$, so we may take $n > 0$). It is easy to see that c_n satisfy the recursion relations

$$c_n = \begin{cases} c_{[\rho n]+1} & \text{if } \rho < \langle \rho n\rangle < 1 \\ c_{[\rho n]} & \text{if } 0 < \langle \rho n\rangle < 1-\rho \\ c_{[\rho n]+1} + c_{[\rho n]} & \text{if } 1-\rho < \langle \rho n\rangle < \rho \end{cases} \tag{2.27}$$

starting with $c_0 = 0$, $c_1 = 1$. The structure of the sequence c_n is quite intriguing; for example, it is easy to show that $c_n = 1$ if and only if n belongs to the Fibonacci sequence.

Now we show that by projection and renormalization we can pass from μ to the *golden measure* ν, which is defined to be the probability measure on [0, 1] satisfying the self-similar identity

$$\nu = \frac{1}{2}\nu \circ S_0^{-1} + \frac{1}{2}\nu \circ S_1^{-1} \tag{2.28}$$

(see [L1], [L2], [LN], [STZ] for properties of this measure). The renormalization process is quite natural, since μ is an infinite discrete measure, and ν is a finite continuous (but singular) measure. Let μ_N denote the restriction of μ to the region $-N \le n \le N$, and let $\tilde{\mu}_N$ denote the probability measure obtained by normalizing μ_N by dividing by $\|\mu_N\|$. It follows from (2.5) that we have

$$\tilde{\mu}_N \approx \frac{1}{2}\tilde{\mu}_{[\rho N]} \circ T_0^{-1} + \frac{1}{2}\tilde{\mu}_{[\rho N]} \circ T_1^{-1} \tag{2.29}$$

(the equality is only approximate since the point masses at the endpoints of the interval may be split differently, but this does not matter in the limit). If we write $\nu_N = \tilde{\mu}_N \circ P^{-1}$ then (2.29) translates to

$$\nu_N \approx \frac{1}{2}\nu_{[\rho N]} \circ S_0^{-1} + \frac{1}{2}\nu_{[\rho N]} \circ S_1^{-1} \tag{2.30}$$

by the intertwining property (2.23). This implies that $\nu_N \longrightarrow \nu$ as $N \longrightarrow \infty$ in the Hutchinson metric since the mapping $\mu \longrightarrow \frac{1}{2}\mu \circ S_0^{-1} + \frac{1}{2}\mu \circ S_1^{-1}$ on probability measures on [0, 1] is contractive in this metric. We have the explicit representation

$$\nu_N = (2C_N)^{-1} \sum_{n=1}^{N} c_n\big(\delta(x - \langle \rho n\rangle) + \delta(x - 1 + \langle \rho n\rangle)\big) \tag{2.31}$$

for $C_N = \sum_{n=1}^{N} c_n$ and c_n given by (2.27). We note that it is really necessary to use the invariant measure μ; if we were to start from counting measure on FH and perform the same renormalization and projection process, we would end up with Lebesgue measure on $[0, 1]$. This is a simple consequence of the uniform distribution of $\langle \rho n \rangle$ on $[0, 1]$.

3. **Fractal blowups.** Let F be a self-similar fractal in $\mathbb{R}^n$, the attractor of an i.f.s. $S_1, \ldots, S_m$ of similarity transformations,

$$S_j x = r_j R_j x + b_j$$

where $0 < r_j < 1$, $b_j \in \mathbb{R}^n$ and R_j is an orthogonal transformation. We say that $\mathcal{F}$ is a *blowup* of F if $\mathcal{F}$ is the union of an increasing sequence of sets $F = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots$ where each F_j is similar to F. We say the blowup is *proper* if the similarities mapping F_j to F belong to the set $\mathcal{S}$ of iterated maps of the i.f.s. We will confine our attention to proper blowups (we are not aware of any interesting examples of blowups that are not proper, but it is easy to construct trivial examples starting with F equal to an interval).

There is a simple construction of proper blowups. Given any infinite sequence $S_{k_1}, S_{k_2}, \ldots$ from the i.f.s., set

$$F_j = S_{k_1}^{-1} \circ S_{k_2}^{-1} \circ \cdots \circ S_{k_j}^{-1} F. \tag{3.1}$$

Since $F \subseteq S_{k_j}^{-1} F$ it follows that $F_{j-1} \subseteq F_j$. In a generic sense this is the most general blowup. Suppose we assume, for example, that every similarity mapping F into itself belongs to $\mathcal{S}$. Then $F_1 = S_{j_1}^{-1} \circ \cdots \circ S_{j_N}^{-1} F$ and $F_2 = S_{k_1}^{-1} \circ \cdots \circ S_{k_M}^{-1} F$. But $F_1 \subseteq F_2$ means $S_{k_M} \circ \cdots \circ S_{k_1} \circ S_{j_1}^{-1} \circ \cdots \circ S_{j_N}^{-1}$ maps F into itself, hence belongs to $\mathcal{S}$. Thus $S_{k_M} \circ \cdots \circ S_{k_1} = S_{\ell_L} \circ \cdots \circ S_{\ell_1} \circ S_{j_N} \circ \cdots \circ S_{j_1}$ and so we can also write $F_2 = S_{j_1}^{-1} \circ \cdots \circ S_{j_N}^{-1} \circ S_{\ell_1}^{-1} \circ \cdots \circ S_{\ell_L}^{-1} F$. Thus we can replace the containments $F = F_0 \subseteq F_1 \subseteq F_2$ with a sequence of N_L containments of the form (3.1), and by iterating this argument we obtain the entire blowup in the form (3.1).

Actually, many familiar examples of self-similar fractals do not satisfy this genericity hypothesis because they have symmetries. The following hypothesis will handle these examples: assume there is a finite group Σ of isometries of F such that conjugation by each $\sigma \in \Sigma$ permutes the i.f.s., and every similarity mapping F into F is the composition of an element of $\mathcal{S}$ with an element of Σ. We can essentially repeat the same argument under this hypothesis, conjugating all the isometries $\sigma \in \Sigma$ to the right, and since $\sigma F = F$ we can then get rid of them.

For the rest of this section we will only consider blowups of the form (3.1). Note that we are not asserting in general that the representation (3.1) is unique, since the same mapping can be represented in different ways as an iterated map. This does not happen, however, if we assume the open set condition.

In general, a blowup looks like a countable union of similar copies of F, all of comparable size, but they do not have to be spatially separate. This will be the case if we assume F satisfies the open set condition (there exists an open set U with the sets $S_j U$ disjoint and contained in U).

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THEOREM 3.1. *Let $\mathcal{F}$ be a blowup of F of the form (3.1), and assume F satisfies the open set condition. Then $\mathcal{F}$ is the union of sets G_k which are similar to F with contraction ratios bounded from above and below, and the number of sets G_k that intersect any ball of radius R is at most a multiple of R^n. In particular, the union $\mathcal{F} = \bigcup_{k=1}^{\infty} G_k$ is locally finite, and furthermore the intersection of $\mathcal{F}$ with any compact set is equal to the intersection of F_N with that compact set for a sufficiently large N.*

PROOF. There is a standard "stopping time" argument that decomposes F, for every scale size $\delta \leq 1$, into a union

$$F = \bigcup_{J \in \mathcal{J}_\delta} S_J F \tag{3.2}$$

where $\mathcal{J}_\delta$ is set of multiindices, such that the contraction ratio of S_J satisfies

$$c\delta < r_J \leq \delta \tag{3.3}$$

for a constant c that depends on the i.f.s. (the minimum quotient of two contraction ratios). It is obtained by iterating the original decomposition $F = \bigcup_{j=1}^{m} S_j F$, and stopping when the contraction ratio first satisfies (3.3). Under the open set condition we also have the fact that the corresponding open sets $S_J U$ are disjoint. We can also obtain an analogous decomposition for a piece of F.

Now for each N, choose δ to be the contraction ratio $r_{k_1} \cdots r_{k_N}$ of the mapping $S_K = S_{k_N} \circ \cdots \circ S_{k_1}$. Now blowup the decomposition (3.2) by the inverse of this mapping, giving

$$F_N = \bigcup_{J \in \mathcal{J}_\delta} S_K^{-1} \circ S_J F, \tag{3.4}$$

in view of (3.1). Note that the contraction ratios of the similarities $S_K^{-1} \circ S_J$ in (3.4) all lie between c and 1, and the open sets $S_K^{-1} \circ S_J U$ are disjoint. Essentially, the sets G_k will just be a listing of all the sets $S_K^{-1} \circ S_J F$ obtained for all N. However, this is not quite correct because the list may not be consistent when we pass from N to $N+1$. To insure this consistency we note that every set $S_{k_1}^{-1} \circ \cdots \circ S_{k_N}^{-1} \circ S_J F$ can also be written in the form $S_{k_1}^{-1} \circ \cdots \circ S_{k_{N+1}}^{-1} \circ S_{J'} F$ for some J', namely $J' = (k_{N+1}, j_1, j_2, \ldots)$. Thus, when we perform the decomposition (3.2) at stage $N+1$, we can first reserve the images under $S_{k_{N+1}}$ of the sets obtained at stage N, do the stopping time argument on the rest of F, and combine the two. When we blowup to (3.4) at stage $N+1$, all the sets in (3.4) at stage N will reappear. With this modification, we have a well defined sequence G_k whose union is $\mathcal{F}$.

The construction clearly produces sets which are similar to F with contraction ratios bounded between c and 1. Also, each G_k lies in the closure of a similar copy U_k of U with the same contraction ratio, and all these open sets are disjoint. This is enough to give us the desired local finiteness of the union. Indeed, we may assume without loss of generality that U is bounded, so if G_k intersects a ball of radius R, then U_k must lie in the concentric ball of radius $R + R_0$ for a suitable constant R_0. Since U has positive volume, at most a fixed multiple of $(R+R_0)^n$ gives an upper bound for the number of disjoint sets U_k in this ball. ∎

A blowup of the form (3.1) is associated to an infinite sequence of maps from the i.f.s. There is also a point in F associated with such a sequence, namely $z = \lim_{j\to\infty} S_{k_1} \circ S_{k_2} \circ \cdots \circ S_{k_j} x$ (for any initial point x). Under the open set condition, the correspondence is essentially one-to-one. Thus there is a correspondence between points of F and blowups (in the case where the closures of the images $S_j U$ intersect, there may be more than one blowup associated to points in a set of Hausdorff measure zero). We will say that $\mathcal{F}$ is the *blowup about* z if z is the corresponding point (with the possibility of ambiguity mentioned above). Two blowups are similar if the corresponding points lie in the same two sided orbit under the i.f.s. ($z_1 = S_J S_{J'}^{-1} z_2$), and this is equivalent to the two sequences having equal tails.

In the blowup of F about z, the relative position of the original fractal in F_N mirrors the relative location of z in a corresponding small neighborhood, but only on a crude level. More precisely, we can write $F_N = \bigcup_{j=1}^m F_{N,j}$ where $F_{N,j} = S_{k_1}^{-1} \circ \cdots \circ S_{k_N}^{-1} \circ S_j F$, and this decomposition is similar to the decomposition $F = \bigcup_{j=1}^m S_j F$ of F. In which of the pieces does F lie? The answer is easily seen to be F_{N,k_N}, for this set is exactly F_{N-1}. On the other hand, z lies in the neighborhood $\tilde{F}_{N-1} = S_{k_1} \circ \cdots \circ S_{k_{N-1}} F$, which has a similar decomposition

$$\tilde{F}_{N-1} = \bigcup_{j=1}^{m} \tilde{F}_{N-1,j}$$

for $\tilde{F}_{N-1,j} = S_{k_1} \circ \cdots \circ S_{k_{N-1}} \circ S_j F$, and z lies in the corresponding piece $\tilde{F}_{N-1,k_N}$. It is on this crude level that $F \subseteq F_N$ mirrors $z \in \tilde{F}_{N-1}$. However, we could not continue the correspondence to involve more than one level, because the orders of the transformations are reversed.

Next we consider a class of blowups whose structure is more closely related with that of the original fractal. We say that a blowup is *periodic* if the sequence $S_{k_1}, S_{k_2}, \ldots$ is periodic. In that case we can simply write

$$\mathcal{F} = \bigcup_{n=1}^{\infty} (S_J^{-1})^n F$$

for some iterated map S_J. The periodic blowups are the same as blowups about a fixed point of some iterated map. It is obvious that S_J, or S_J^{-1}, is a symmetry of $\mathcal{F}$. In this sense, the whole blowup is similar to itself on all scales. Of course there are only a countable number of periodic blowups, as compared with the uncountable number of nonperiodic blowups. Note that periodic blowups based on iterated maps S_J and $S_{J'}$ will be similar if J' is a cyclic permutation of J.

Next we will show that in some circumstances the discrete structure of the decomposition given in Theorem 3.1 can be described by an invariant set for a r.i.f.s. We assume the i.f.s. has the special form

$$S_j x = rx + b_j,\ j = 1, \ldots, m \tag{3.5}$$

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for a single contraction ratio $r < 1$. Consider the blowup

$$\mathcal{F} = \bigcup_{n=1}^{\infty} (S_1^{-1})^n F. \tag{3.6}$$

THEOREM 3.2. *Assume the i.f.s. (3.5) satisfies the open set condition, and the blowup $\mathcal{F}$ is given by (3.6). Then $\mathcal{F} = F + D$ where D is an invariant set for the r.i.f.s.*

$$T_j x = r^{-1}(x + b_j - b_1), j = 1, \ldots, m. \tag{3.7}$$

Specifically, D is the forward orbit of 0, the fixed point set of T_1.

PROOF. It is easy to see that the decomposition of F_n given by Theorem 3.1 is just a union of translates of F by

$$r^{-n}(b_{j_1} - b_1) + r^{-(n-1)}(b_{j_2} - b_1) + \cdots + r^{-1}(b_{j_n} - b_1) \tag{3.8}$$

as each of the indices j_k varies over $1, 2, \ldots, m$. But the set of points of the form (3.8) is exactly the forward orbit of 0 under the r.i.f.s. (3.7), and the open set condition implies this set is discrete. ∎

The correct context for this theorem, and presumably generalizations of it under weaker hypotheses, is to consider the group $\mathrm{Sim}(\mathbb{R}^n)$ of similarities of $\mathbb{R}^n$ and a suitable invariant set Φ under a r.i.f.s. on $\mathrm{Sim}(\mathbb{R}^n)$, so that $\mathcal{F} = \bigcup_{\varphi \in \Phi} \varphi(F)$. The main obstacle to carrying out this program is to find a metric on $\mathrm{Sim}(\mathbb{R}^n)$ that makes the required mappings expansive.

We now look at some examples. Suppose F is the usual Cantor set, given by the i.f.s. $S_1 x = \frac{1}{3}x$, $S_2 x = \frac{1}{3}x + \frac{2}{3}$. Then (3.6) defines the blowup about 0, which we call the *large Cantor set*. It is clear in this case that D is exactly the integer Cantor set, so the large Cantor set is just the sum set of the usual Cantor set and the integer Cantor set, and it can be described as the set of nonnegative reals that can be represented base 3 using only the digits 0 and 2. It is also easy to see that the only blowups of the Cantor set contained in the positive half line are translates of the large Cantor set.

There is an analogous story for the Sierpinski gasket. For simplicity we take the gasket based on the half square triangle, so F is generated by the i.f.s.

$$S_1(x,y) = \left(\frac{1}{2}x, \frac{1}{2}y\right)$$
$$S_2(x,y) = \left(\frac{1}{2}(x+1), \frac{1}{2}y\right)$$
$$S_3(x,y) = \left(\frac{1}{2}(x+1), \frac{1}{2}(y+1)\right)$$

The *large Sierpinski gasket* (see Figure 3.1) is the blowup about $(0,0)$ given by (3.6). It is the sum set of the Sierpinski gasket and the integer Sierpinski gasket given as the forward orbit of $(0,0)$ under the r.i.f.s. on $\mathbb{Z}^2$

$$T_1(x,y) = (2x, 2y)$$
$$T_2(x,y) = (2x+1, 2y)$$
$$T_3(x,y) = (2x+1, 2y+1).$$

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The large Sierpinski gasket is contained in the wedge $0 \le y \le x$. Another interesting class of blowups is obtained by using only S_1 and S_2 in (3.1) (or, equivalently, blowing up about a point on the x axis). These blowups are all contained in the upper half plane $y \ge 0$. A generic blowup of the Sierpinski gasket is not contained in any half plane.

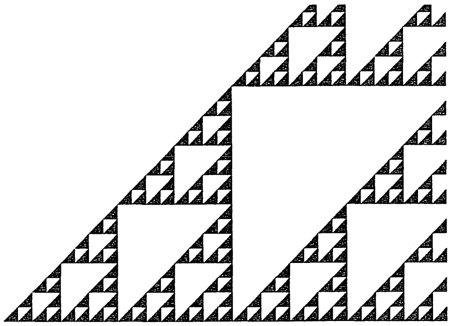

FIGURE 3.1: A portion of the large Sierpinski gasket, shown schematically (each triangle represents and ordianry Sierpinski gasket).

Next suppose F is the von Koch curve based on an equilateral triangle, defined by the i.f.s. on $\mathbb{C}$ given in complex notation by

$$\begin{aligned} S_1 z &= \frac{1}{3} z \\ S_2 z &= \frac{1}{3} e^{\pi i/3} z + \frac{1}{3} \\ S_3 z &= \frac{1}{3} e^{-\pi i/3} z + \frac{1}{3} + \frac{1}{3} e^{\pi i/3} \\ S_4 z &= \frac{1}{3} z + \frac{2}{3}. \end{aligned}$$

(It is also possible to describe the von Koch curve by the two element i.f.s.

$$\begin{aligned} \tilde{S}_1 z &= \frac{1}{\sqrt{3}} e^{\pi i/6} (1 - z) \\ \tilde{S}_2 z &= 1 - \frac{1}{\sqrt{3}} e^{-\pi i/6} z, \end{aligned}$$

which yields (S_1, S_2, S_3, S_4) by iteration.) Aside from reflection symmetries, which do not produce different blowups, there is a fifth symmetry of F, namely

$$S_5 z = \frac{1}{3} z + \frac{2}{9} + \frac{2}{9} e^{\pi i/3}$$

(note that $S_5 = S_3 S_1 S_3^{-1} = S_2 S_4^{-1} S_2$) which contracts F onto its "summit". That is, F is also the attractor of the i.f.s. $S_1, \ldots, S_5$. By using this larger i.f.s. we lose the nonoverlapping property (open set condition), but we obtain new blowups. We denote by z_j the

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fixed points of each of the mappings S_j, so

$$z_1 = 0$$
$$z_2 = (2 + e^{\pi i/3})/7$$
$$z_3 = (4 + e^{\pi i/3})/7$$
$$z_4 = 1$$
$$z_5 = (1 + e^{\pi i/3})/3.$$

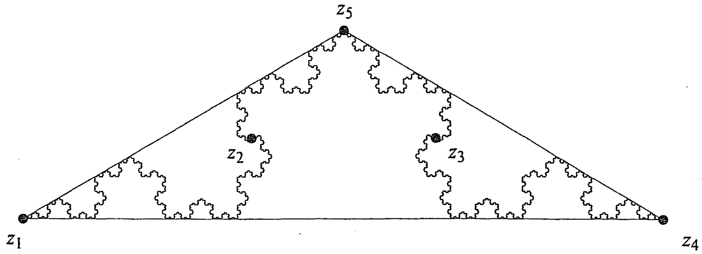

FIGURE 3.2: The von Koch curve, in its containing triangle, with the points z_1, z_2, z_3, z_4, z_5 labelled.

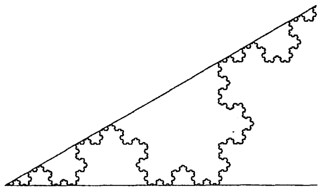

FIGURE 3.3: A portion of the von Koch wedge.

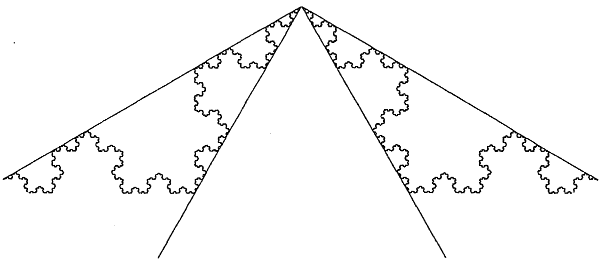

FIGURE 3.4: A portion of the von Koch double wedge.

The ends z_1, z_4 and the summit z_5 form a 30°-120°-30° triangle containing F. (See Figure 3.2.) The blowup about z_1 may be called a *von Koch wedge* (Figure 3.3), as it lies in the 30° wedge $0 \leq \theta \leq \pi/6$ and touches the boundary rays infinitely many times.

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Its mirror image is the blowup about z_4. The blowup about the summit z_5 (using S_5) is a *double wedge* (Figure 3.4), as it is just the union of two wedges joined together at their ends. More precisely, take a left wedge (the blowup about z_1), translate it so z_1 moves to the summit z_5, and rotate it 60° to the right, together with a right wedge translated so z_4 moves to the summit and rotated 60° to the left. The double wedge has no ends, and has the same left-right reflection symmetry as F.

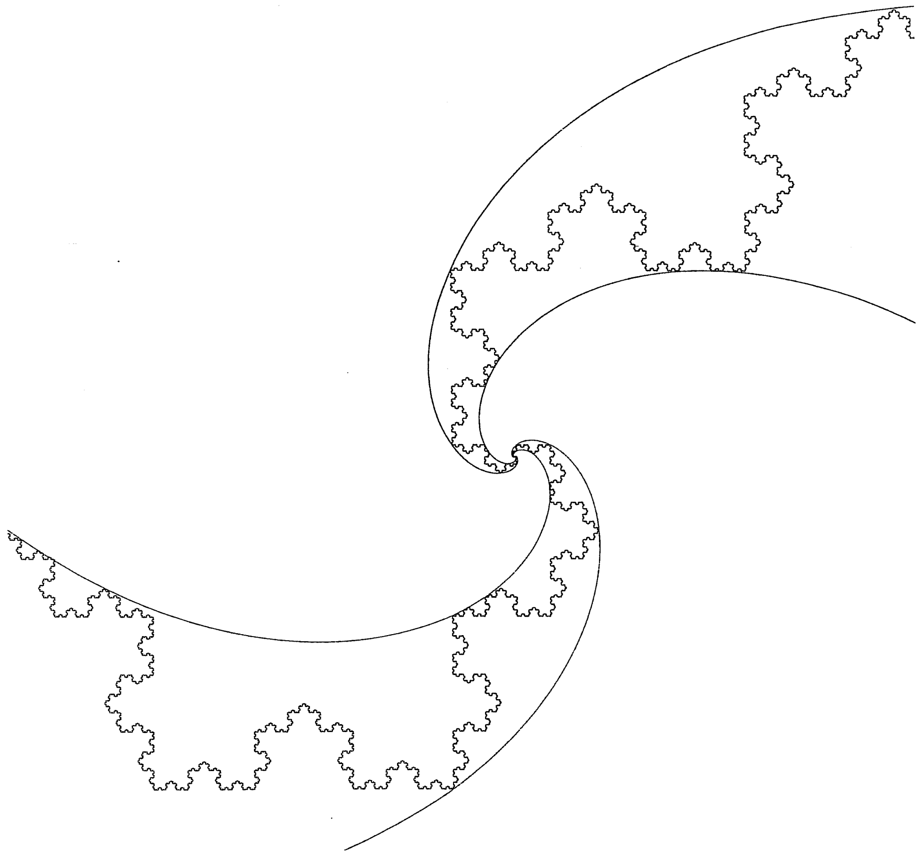

FIGURE 3.5: A portion of the von Koch double spiral, with bounding exponential spirals.

The blowup about z_2 may be described as a *von Koch double spiral*, (Figure 3.5). Since S_2 involves a 60° rotation, it is perhaps easier to imagine this blowup as $\bigcup_{k=0}^{\infty} S_2^{-6k}F$, since S_2^{-6} is a dilation (by a factor of $3^6 = 729$) centered at z_2, although a more accurate description of S_2^{-6} involves also a 360° rotation. The spiral structure is not surprising, given the combination of rotation and dilation, but is usually not too apparent in pictures of the von Koch curve, because the large value 729 of the dilation factor makes it difficult for the eye to compare features at such disparate scales. The two arms of the spiral correspond to blowups of the left third of F, from z_1 to z_2, and the right two-thirds of F, from z_2 to z_4. They are not similar curves, although they follow the outlines of similar exponential spirals.

The left arm of the spiral can best be visualized by considering the sequence of points $S_2^k z_1$ in F, $k = 0, 1, 2, \ldots$ which converge to z_2 as $k \longrightarrow \infty$. If you "join the dots" and erect a suitably scaled von Koch curve on each segment, you will obtain exactly the left third of F. By extending the values of k to the negative integers as well, you obtain the left arm of the von Koch double spiral. Since

$$z_2 = \sum_{j=0}^{\infty} 3^{-j-1} e^{\pi i j/3}$$

and

$$S_2^k z_1 = \sum_{j=0}^{k-1} 3^{-j-1} e^{\pi i j/3}$$

we find

$$S_2^k z_1 = z_2 + \frac{(\frac{1}{3} e^{\pi i/3})^k}{e^{\pi i/3} - 3},$$

so all these points lie on an exponential spiral centered at z_2 of the type $r = ce^{-\lambda\theta}$ for $\lambda = \frac{3}{\pi} \log 3$. A similar exponential spiral obtained by connecting the points $S_2^k S_1 z_5$ forms an upper cap to the left arm of the von Koch double spiral. To describe the right arm we consider the spiral obtained by connecting the points $S_2^k z_4$ for all integer k, and we suspend a suitably scaled image of a half von Koch curve under consecutive points, the image under S_2^k of the right half curve joining $z_5 = S_2 z_4$ to z_4. The points $S_2^k S_3 z_4$ generate a lower bound spiral for the right arm.

4. **The analytic transform.** Let μ be an invariant measure for a linear r.i.f.s. on $\mathbb{Z}$ of the form

(4.1) $$T_j x = r_j x + b_j, j = 1, \ldots, m,$$

for integers $r_j \geq 2$ and b_j. Then we can always form the Fourier series

(4.2) $$\sum_{n=-\infty}^{\infty} \mu(n) e^{in\theta},$$

and this makes sense as a tempered distribution, since we always have polynomial growth estimates

(4.3) $$\sum_{-R}^{R} \mu(n) \leq cR^a \text{ for some } c \text{ and } a.$$

Indeed, by Lemma 2.1 we know $|n| \geq \rho |T_j^{-1} n|$ for some $\rho > 1$ and all sufficiently large n, hence we have

$$\begin{aligned}\sum_{-R}^{R} \mu(n) &= \sum_{j=1}^{m} \sum_{n=-R}^{R} p_j \mu(T_j^{-1} n)\\ &\leq \Big(\sum_{j=1}^{m} p_j\Big) \sum_{n=-R/\rho}^{R/\rho} \mu(n)\end{aligned}$$

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and (4.3) follows by a routine argument. Nevertheless, (4.2) is usually not a function, or even a measure, so it is not clear how to obtain interesting information from it.

If μ is supported in the nonnegative integers $\mathbb{Z}^+$, then we can form the power series

$$\varphi(z) = \sum_{n=0}^{\infty} \mu(n) z^n, \tag{4.4}$$

and it follows from (4.3) that this converges to an analytic function in the unit disc, which we call the *analytic transform* of μ. In some ways, this analytic transform plays the role of the Fourier transform of a self similar measure. For example, it is easy to see that the defining identity (2.5) is equivalent to the identity

$$\varphi(z) = \sum_{j=1}^{m} p_j z^{b_j} \varphi(z^{r_j}). \tag{4.5}$$

We will assume that the support of μ contains 0, since this can always be arranged by translation. But since we are assuming the support lies in $\mathbb{Z}^+$, it follows that $b_j \geq 0$, so none of the fixed points of T_j can be positive. Thus we must have at least one of the b_j equal to 0, say $b_1 = 0$, and the support of μ is the forward orbit of 0. We will normalize μ to have $\mu(0) = 1$, so also $\varphi(0) = 1$. Note that the sum of the p_j for those $b_j = 0$ must be 1. This is clearly required for the consistency of (4.5) at $z = 0$.

One special case we will examine in detail is when all r_j are equal, say $r_j = \rho$ for $\rho \geq 2$ an integer. Then (4.5) becomes

$$\varphi(z) = \Big(\sum_{j=1}^{m} p_j z^{b_j}\Big) \varphi(z^{\rho}) \tag{4.6}$$

and this leads to the infinite product representation

$$\varphi(z) = \prod_{k=0}^{\infty} g(z^{\rho^k}) \tag{4.7}$$

where

$$g(z) = \sum_{j=1}^{m} p_j z^{b_j}. \tag{4.8}$$

The condition $g(0) = 1$, already noted, shows that the infinite product converges.

We will need to assume the nonoverlapping condition on the support of μ, and it is easy to see that this holds if all b_j are distinct mod ρ, for then the images of $\mathbb{Z}$ under the r.i.f.s. mappings are disjoint. Under this assumption, the individual powers of z in the product $\varphi(z)$ occur at most once.

An analytic function on the unit disc belongs to the Hardy space H^2 if the L^2 averages

$$\frac{1}{2\pi} \int_0^{2\pi} |\varphi(re^{i\theta})|^2 \, d\theta = h(r) \tag{4.9}$$

are uniformly bounded on $r < 1$. This will not be the case for our analytic transforms, so we seek instead to give a precise statement about the behavior of $h(r)$ as $r \longrightarrow 1$.

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THEOREM 4.1. *Let $\varphi(z)$ be the analytic transform of an invariant measure supported on the forward orbit of 0 for the r.i.f.s. $T_j x = \rho x + b_j$, $j = 1, \ldots, m$, for $\rho \geq 2$ a positive integer and b_j distinct nonnegative integers mod ρ, with $b_1 = 0$. Then there exists a bounded continuous function $\psi(r)$, also bounded away from zero, with the asymptotic periodicity property*

$$\lim_{r \to 1^-} \psi(r)/\psi(r^\rho) = 1, \tag{4.10}$$

such that

$$h(r) = (1-r)^{-\alpha}\psi(r), \tag{4.11}$$

where α is the L^2 dimension defined by

$$\alpha = \log\Big(\sum_{j=1}^m p_j^2\Big) \Big/ \log \rho. \tag{4.12}$$

PROOF. By Parseval's identity we have

$$h(r) = \sum_{n=0}^\infty r^{2n}\mu(n)^2. \tag{4.13}$$

From the defining identity (2.5) and the nonoverlapping assumption we have

$$\mu(n)^2 = \Big(\sum_{j=1}^m p_j \mu(T_j^{-1}n)\Big)^2 = \sum_{j=1}^m p_j^2 \mu(T_j^{-1}n)^2 \tag{4.14}$$

since at most one term is nonzero. We substitute (4.14) into (4.13) and make a change of variable to obtain

$$h(r) = \Big(\sum_{j=1}^m p_j^2 r^{2b_j}\Big) h(r^\rho). \tag{4.15}$$

Use (4.11) to define the function $\psi(r)$. Then (4.15) becomes

$$\psi(r) = \Big(\sum_{j=1}^m p_j^2 r^{2b_j}\Big)(1-r)^\alpha (1-r^\rho)^{-\alpha}\psi(r^\rho),$$

which we can rewrite as

$$\frac{\psi(r)}{\psi(r^\rho)} = \Big(\frac{\rho(1-r)}{1-r^\rho}\Big)^\alpha \Big(\frac{\sum_{j=1}^m p_j^2 r^{2b_j}}{\sum_{j=1}^m p_j^2}\Big) \tag{4.16}$$

since $\rho^\alpha = \sum_{j=1}^m p_j^2$ by (4.12). The right side of (4.16) is an elementary function, and it is a simple exercise to verify that it approaches 1 as $r \longrightarrow 1^-$, proving (4.10). Since $\varphi(0) = 1$ we have that $\psi(r)$ is bounded and bounded away from 0 in a neighborhood of $r = 0$ and using (4.16) it is not hard to extend this to all of $r < 1$. ∎

It would also be possible to prove this result as a consequence of Theorem 2.4, applied to the square of the measure, which in the nonoverlapping case is also an invariant measure with weights p_j^2. See [LW] where this method is used in the context of Fourier asymptotics of self-similar measures. It would be interesting to know if the analytic transform has any pointwise behaviors analogous to those of the Fourier transform of self-similar measures described in [JRS].

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