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ON THE PRIMITIVE IDEALS OF NEST ALGEBRAS

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Abstract We show that Ringrose's diagonal ideals are primitive ideals in a nest algebra (subject to the continuum hypothesis). This answers an old question of Lance and provides for the first time concrete descriptions of enough primitive ideals to obtain the Jacobson radical as their intersection. Separately, we provide a standard form for all left ideals of a nest algebra, which leads to insights into the maximal left ideals. In the case of atomic nest algebras, we show how primitive ideals can be categorized by their behaviour on the diagonal and provide concrete examples of all types.

Keywords: nest algebra; primitive ideals; nets; continuum hypothesis

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1. Introduction

The Jacobson radical has been a frequent object of study in non-self-adjoint algebras, and considerable effort has been expended to identify the radical in the context of various classes of non-self-adjoint algebras, e.g. [**4**, **5**, **8**, **9**, **12**, **16**, **22**, **23**]. Why is this? At first glance it might seem that since many non-self-adjoint algebras are modelled more or less on the algebra of finite-dimensional upper triangular matrices, the desire is to obtain Wedderburn-type structure theorems for the algebras. In fact, however, the Jacobson radical is rarely the correct ideal for such a decomposition, if it is even possible. The Jacobson radical is often too small, and indeed in some cases non-self-adjoint algebras are even semisimple [**8**, **9**, **16**]. Thus, knowledge about the Jacobson radical rather points towards more general structural information about the algebra and, in particular, when the radical is small, indicates the presence of a rich supply of irreducible representations, even in algebras which have a strong heuristic connection with the upper triangular matrix algebra.

The nest algebras are one such case. Indeed, the main result of Ringrose's paper [**23**], which introduced the class of nest algebras, was to describe the Jacobson radical $\mathcal{R}_{\mathcal{N}}$ of a nest algebra $\mathcal{T}(\mathcal{N})$ (see § 2 below for precise definitions of terms). However, except in the trivial case of a finite nest, there is no Wedderburn-type decomposition $\mathcal{T}(\mathcal{N}) = \mathcal{D}(\mathcal{N}) \oplus \mathcal{R}_{\mathcal{N}}$ as the sum of the diagonal algebra and the Jacobson radical. In fact, by [**18**, Theorem 4.1], a decomposition $\mathcal{T}(\mathcal{N}) = \mathcal{D}(\mathcal{N}) \oplus \mathcal{R}$ for some ideal $\mathcal{R}$ is only possible if $\mathcal{R}$ is Larson's ideal $\mathcal{R}_{\mathcal{N}}^{\infty}$ [**14**], and then only if the nest has no continuous part. At issue here is the fact that unless the nest is finite, $\mathcal{R}_{\mathcal{N}}^{\infty}$ is much bigger than the Jacobson radical. In the case of upper triangular matrixes on $\ell^2(\mathbb{N})$, $\mathcal{R}_{\mathcal{N}}^{\infty}$ is the collection of all strictly upper triangular operators, while $\mathcal{R}_{\mathcal{N}}$ is the set of *compact* strictly upper triangular operators. Thus, the comparatively small Jacobson radical in nest algebras indicates that there must be many irreducible representations other than the trivial ones obtained as the compression to an atom of the nest.

However, up to now, the only other primitive ideals which could be identified explicitly were the maximal two-sided ideals. (Maximal two-sided ideals are primitive; see Remark 3.1 for a review of this and other ring-theoretic facts.) In [**17**] we described the maximal two-sided ideals of a continuous nest algebra, and in [**20**] we extended the description to cover all nest algebras. (It should be noted these results rest on deep foundations; between them, they require the similarity theory of nests and the paving theorem.) Even so, however, these ideals alone do not account for the small Jacobson radical. Their intersection, called the *strong radical*, is similar in character to $\mathcal{R}_{\mathcal{N}}^{\infty}$, and in fact the two coincide when the nest is atomic.

The goal of this paper is to identify enough examples of primitive ideals of nest algebras to account for the small Jacobson radical, by which we mean that their intersection should equal the Jacobson radical. The key examples have been in plain view all along; they are the 'diagonal ideals' which Ringrose used in his original description of the radical [**23**, Theorem 5.3]. We shall show in Theorem 3.7 that the diagonal ideals are primitive. This answers an old open question of Chris Lance [**13**, § 5] (repeated by Davidson in [**2**]). Interestingly, this result relies on assuming a positive answer to the continuum hypothesis. See the excellent survey paper [**24**] for other recent results in operator algebras which make use of non-standard foundational considerations.

After this, we turn to an analysis of the left ideals of nest algebras in § 4. We establish a standard form for *all* left ideals, and also a stronger form which holds for many norm-closed left ideals, including the maximal left ideals. In § 5 we explore the primitive ideals of atomic nest algebras in more depth. We identify three classes of primitive ideals (the smallest, the largest and the intermediate ones), and we show that they are distinguished by their behaviour on the diagonal. Section 6 focuses on the infinite upper triangular matrices, where we can give concrete examples of all types of primitive ideals, and also applications to quasitriangular algebras.

2. Preliminaries

Throughout this paper, the underlying Hilbert spaces are always assumed to be separable. A *nest* is a set of projections on a Hilbert space which is linearly ordered, contains 0 and I, and is weakly closed (or, equivalently, order-complete). The *nest algebra*, $\mathcal{T}(\mathcal{N})$, of a nest $\mathcal{N}$ is the set of bounded operators leaving invariant the ranges of $\mathcal{N}$. The *diagonal algebra*, $\mathcal{D}(\mathcal{N})$, is the set of operators having the ranges of projections in $\mathcal{N}$ as reducing subspaces (equivalently, the commutant of $\mathcal{N}$). An *interval* of $\mathcal{N}$ is the difference $N - M$ of two projections $N > M$ in $\mathcal{N}$. Minimal intervals are called *atoms*, and the atoms (if there are any) are pairwise orthogonal. If the join of the atoms is I the nest is called *atomic*; if there are no atoms it is called *continuous*. For $N \in \mathcal{N}$, define

$$N^- := \bigvee\{M \in \mathcal{N} : M < N\} \quad \text{and} \quad N^+ := \bigwedge\{M \in \mathcal{N} : M > N\}.$$

Conventionally, $0^- = 0$ and $I^+ = I$. If $N > N^-$ then $N - N^-$ is an atom of N, and all atoms are of this form. Conversely, if $N = N^- > 0$ then there is a strictly increasing sequence of projections in $\mathcal{N}$ which converge to N. Similar remarks apply for N^+. We shall make continual use of the fact that the rank-one operator $x \mapsto \langle x, f\rangle e$, which we write as ef^*, belongs to $\mathcal{T}(\mathcal{N})$ if and only if there is an $N \in \mathcal{N}$ such that $e \in \operatorname{ran}(N^+)$ and $f \in \operatorname{ran}(N^\perp)$. See [**2**] for further properties of nest algebras.

Example 2.1. Let $\mathcal{H} := \ell^2(\mathbb{N})$ and let $\{e_i\}_{i=1}^\infty$ be the standard basis. For $n \in \mathbb{N}$, let N_n be the projection onto the span of $\{e_1, \dots, e_n\}$ and let $\mathcal{N} := \{N_n : n \in \mathbb{N}\} \cup \{I\}$. This is a nest, and $\mathcal{T}(\mathcal{N})$ is the nest algebra of all infinite upper triangular operators with respect to the standard basis. By slight abuse of notation, we write $\mathcal{T}(\mathbb{N})$ for this algebra.

We now recall Ringrose's description of the Jacobson radical of a nest algebra, in terms of diagonal seminorms and diagonal ideals.

Definition 2.2. Let $\mathcal{N}$ be a nest and fix $N < I$ in $\mathcal{N}$. The *diagonal seminorm function* $i_N^+(X)$ is defined for $X \in \mathcal{T}(\mathcal{N})$ by

$$i_N^+(X) := \inf\{\|(M - N)X(M - N)\| : M > N \text{ in } \mathcal{N}\}.$$

Likewise, for $N > 0$, the diagonal seminorm function $i_N^-(X)$ is

$$i_N^-(X) := \inf\{\|(N - M)X(N - M)\| : M < N \text{ in } \mathcal{N}\}.$$

It is straightforward to see that the functions $i_N^\pm$ are submultiplicative seminorms on $\mathcal{T}(\mathcal{N})$ and dominated by the norm, and so their kernels are norm-closed two-sided ideals of $\mathcal{T}(\mathcal{N})$.

Definition 2.3. Let $\mathcal{N}$ be a nest. The *diagonal ideals* are the ideals

$$\mathcal{I}_N^+ := \{X \in \mathcal{T}(\mathcal{N}) : i_N^+(X) = 0\} \quad (\text{for } N < I)$$

and

$$\mathcal{I}_N^- := \{X \in \mathcal{T}(\mathcal{N}) : i_N^-(X) = 0\} \quad (\text{for } N > 0).$$

The diagonal ideals can be viewed as generalizations of those ideals of upper triangular $n \times n$ matrices consisting of all the matrices which vanish at a particular diagonal entry. Indeed, if $N > N^-$ then

$$\mathcal{I}_N^- = \{X \in \mathcal{T}(\mathcal{N}) : (N - N^-)X(N - N^-) = 0\}. \tag{2.1}$$

However, if $N = N^-$ then I_N^- is the set of operators *asymptotically* vanishing close to N (from below). More precisely, in the case of $\mathcal{T}(\mathbb{N})$, $\mathcal{I}_N^-$ is of the form (2.1) for all $N < I$ and I_I^- is the compact operators of $\mathcal{T}(\mathbb{N})$. See § 6 for a detailed discussion of the primitive ideals in this algebra.

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Ringrose gave the following description of the Jacobson radical in terms of these diagonal ideals.

Theorem 2.4 (see [23, Theorem 5.3]). *The Jacobson radical of $\mathcal{T}(\mathcal{N})$ is the intersection of the diagonal ideals of $\mathcal{T}(\mathcal{N})$.*

A key point to bear in mind is that although the diagonal ideals are related to the primitive ideals, as the next result quoted shows, they were not known to be primitive. Lance [13] asked whether the diagonal ideals were primitive and, in his study of the diagonal ideals and their quotients, proved a number of results which are entailed by primitivity. In Theorem 3.7 we show that the diagonal ideals are in fact primitive ideals.

The following useful result shows that each primitive ideal of a nest algebra is associated with a unique diagonal ideal.

Theorem 2.5 (see [23, Theorem 4.9]). *Every primitive ideal of $\mathcal{T}(\mathcal{N})$ contains exactly one diagonal ideal.*

Based on this result, we adopt the following notation.

Definition 2.6. If $\mathcal{P}$ is a primitive ideal of the nest algebra $\mathcal{T}(\mathcal{N})$, write $\mathcal{I}_{\mathcal{P}}$ for the unique diagonal ideal contained in $\mathcal{P}$.

Finally, we close the section by recalling Larson's ideal [14], $\mathcal{R}^{\infty}_{\mathcal{N}}$.

Definition 2.7. Let $\mathcal{R}^{\infty}_{N}$ be the set of $X \in \mathcal{T}(\mathcal{N})$ such that, given $\epsilon > 0$, we can find a collection $\{N_i - M_i : i \in \mathbb{N}\}$ of pairwise orthogonal intervals of $\mathcal{N}$ which sum to I and such that $\|(N_i - M_i)X(N_i - M_i)\| < \epsilon$.

Ringrose [23, Theorem 5.4] provides an alternative description of the Jacobson radical which is formally very similar to Larson's ideal. The only difference is the requirement that the collections of pairwise orthogonal intervals must be *finite*. However, this makes an enormous difference to the size of the ideal, as the following example shows.

Example 2.8. Let $\mathcal{N}$ be the canonical nest on $\ell^2(\mathbb{N})$. Then $\mathcal{R}_{\mathcal{N}}$ is the set of zero-diagonal compact operators in $\mathcal{T}(\mathbb{N})$ and $\mathcal{R}^{\infty}_{\mathcal{N}}$ is the set of all zero-diagonal operators in $\mathcal{T}(\mathbb{N})$. Note in particular that $\mathcal{T}(\mathbb{N}) = \mathcal{D}(\mathbb{N}) \oplus \mathcal{R}^{\infty}_{\mathcal{N}}$ but that $\mathcal{T}(\mathbb{N}) \neq \mathcal{D}(\mathbb{N}) \oplus \mathcal{R}_{\mathcal{N}}$ (for example, the right-hand side fails to contain the unilateral backward shift).

3. The diagonal ideals are primitive

The main result of this section is Theorem 3.7, in which we prove that the diagonal ideals of a nest algebra are primitive. We start by recalling some basic facts about primitive ideals which can be found in many standard texts on ring theory or Banach algebras; see, e.g. [1, Chapter III].

Remark 3.1. Let A be a unital Banach algebra. The (left) primitive ideals of A are the annihilators of left A-modules, or, equivalently, the kernels of the irreducible representations of A. If P is any primitive ideal of A then there is a maximal left ideal L

of A such that P is the kernel of the left regular representation of A on A/L. Thus, P is the largest two-sided ideal of A contained in L and is equal to

$$\{x \in A : xA \subseteq L\}.$$

From this, together with the maximality of L, it follows easily that $x \notin P$ if and only if there are $a, b \in A$ such that $e - axb \in L$ (where e is the unit of A). Finally, of course, the Jacobson radical is, by definition, the intersection of all the primitive ideals of A. Analogously, the *right* primitive ideals are the kernels of right A-modules, and each right primitive ideal is the kernel of the right module action of A on the quotient A/R of A by some maximal right ideal. The intersection of the maximal right primitive ideals is also the (same) Jacobson radical.

Lemma 3.3 will enable us to convert arbitrary upper triangular operators to block diagonal form. It relies on the following useful technical lemma, which we quote in full.

Lemma 3.2 (see [19, Lemma 2.2]). *Let $X \in B(\mathcal{H})$ and let P_n, Q_n $(n \in \mathbb{N})$ be sequences of projections such that $\operatorname{dist}(P_n X Q_n, \mathcal{F}_{4n-4}) > 1$ for all n, where $\mathcal{F}_k$ denotes the set of operators of rank not greater than k. Then there are orthonormal sequences $x_i \in P_i\mathcal{H}$ and $y_i \in Q_i\mathcal{H}$ such that $\langle x_i, X y_j\rangle = 0$ for all $i \neq j$, and $\langle x_i, X y_i\rangle$ is real and greater than 1 for all $i \in \mathbb{N}$.*

Lemma 3.3. *Suppose $X \in \mathcal{T}(\mathcal{N})$ but $X \notin \mathcal{I}_N^-$ for some $N = N^- > 0$ in $\mathcal{N}$. Then there are $A, B \in \mathcal{T}(\mathcal{N})$ and a sequence N_k of nest projections strictly increasing to N such that*

$$AXB = \sum_{k=1}^{\infty} (N_k - N_{k-1}) AXB (N_k - N_{k-1}),$$

and each of the terms $(N_k - N_{k-1})AXB(N_k - N_{k-1})$ has norm greater than 1.

Proof. Rescaling if necessary, assume $i_N^-(X) > 1$. Choose a sequence $N_k \in \mathcal{N}$ which increases strictly to N. We shall inductively construct a subsequence N_{k_n} such that $\operatorname{dist}((N_{k_n} - N_{k_{n-1}})X(N_{k_n} - N_{k_{n-1}}), \mathcal{F}_{4n-4}) > 1$ for all k, and the result will follow from an easy application of Lemma 3.2. Take $k_1 := 1$ and suppose $k_1 < k_2 < \cdots < k_{n-1}$ to have been chosen with the desired property.

Suppose for a contradiction that $\operatorname{dist}((N_k - N_{k_{n-1}})X(N_k - N_{k_{n-1}}), \mathcal{F}_{4n-4}) \leq 1$ for all $k > k_{n-1}$. Fix an a with $1 < a < i_N^-(X)$ and for each $k \geq k_{n-1}$ find $F_k \in \mathcal{F}_{4n-4}$ such that

$$\|(N_k - N_{k_{n-1}})X(N_k - N_{k_{n-1}}) - F_k\| < a.$$

The sequence F_k is norm-bounded and so has a w^*-convergent subsequence, $F_{m_j} \to F$. But $F \in \mathcal{F}_{4n-4}$ since $\mathcal{F}_{4n-4}$ is w^*-closed and, by the lower semicontinuity of the norm,

$$\begin{aligned} i_N^-(X) &\leq \|(N - N_{k_{n-1}})X(N - N_{k_{n-1}}) - F\| \\ &\leq \liminf_{j\to\infty} \|(N_{m_j} - N_{k_{n-1}})X(N_{m_j} - N_{k_{n-1}}) - F_{m_j}\| \\ &\leq a, \end{aligned}$$

which is a contradiction. Thus, we find $k_n > k_{n-1}$ with which to continue the induction.

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With N_{k_n} chosen, apply Lemma 3.2 to obtain unit vectors x_n, y_n in the range of $N_{k_n} - N_{k_{n-1}}$ such that $\langle x_m, Xy_n\rangle = 0$ for all $m \neq n$, and $\langle x_m, Xy_m\rangle > 1$ for all $m \in \mathbb{N}$. Set

$$A := \sum_{n=1}^{\infty} x_{3n}x_{3n+1}^* \quad \text{and} \quad B := \sum_{n=1}^{\infty} y_{3n+1}y_{3n+2}^*.$$

Then $A, B \in \mathcal{T}(\mathcal{N})$, since the terms of both sums are of the form $N_{k_m}RN_{k_m}^{\perp}$, and $AXB = \sum_{n=1}^{\infty}\langle x_{3n+1}, Xy_{3n+1}\rangle x_{3n}y_{3n+2}^*$, so that $AXB = \sum_{n=1}^{\infty}(N_{k_{3n+2}} - N_{k_{3n-1}})AXB(N_{k_{3n+2}} - N_{k_{3n-1}})$ and each of the terms of the sum has norm greater than 1. □

The following, unfortunately rather technical, definition is central to our analysis in this section.

Definition 3.4. Fix a nest $\mathcal{N}$ and a projection $N \in \mathcal{N}$. Say that a set $\mathcal{S}$ of operators in $B(\mathcal{H})$ are of Type-S if there exists a strictly increasing sequence N_n in $\mathcal{N}$ which converges to N, and a sequence of unit vectors $x_n = (N_n - N_{n-1})x_n$ such that for each $X \in \mathcal{S}$ both $Xx_n \to 0$ and $X^*x_n \to 0$.

Clearly, if $\mathcal{S} \subseteq \mathcal{T}(\mathcal{N})$ is of Type-S, then it lies in both a proper left ideal of $\mathcal{T}(\mathcal{N})$ and a proper right ideal of $\mathcal{T}(\mathcal{N})$. Note, however, that it need not lie in a proper two-sided ideal; for example, consider the singleton $\{I - U\}$ where U is the unilateral backward shift on $\ell^2(\mathbb{N})$. This is Type-S with respect to the sequences N_{2^n} and $x_n := 2^{(1-n)/2}\sum_{i=2^{n-1}}^{2^n-1} e_i$ but does not lie in a proper two-sided ideal of $\mathcal{T}(\mathbb{N})$. In fact, this example is the prototype of the analysis which follows, and an analogous sequence is at the heart of the proof of the next lemma. Note also that, strictly speaking, 'Type-S' is a property which a set has with respect to a particular $\mathcal{N}$ and $N \in \mathcal{N}$. In the following arguments these will always be easily discerned from the context.

Lemma 3.5. *Fix a nest $\mathcal{N}$ and a projection $N = N^- > 0$ in $\mathcal{N}$, and let $\{X_i : i \in \mathbb{N}\}$ be a set of Type-S. Let $X \in \mathcal{T}(\mathcal{N})$ but $X \notin \mathcal{I}_N^-$. Then there are $A, B \in \mathcal{T}(\mathcal{N})$ such that $\{I - AXB\} \cup \{X_i : i \in \mathbb{N}\}$ is also of Type-S.*

Proof. Take a sequence $N_n \in \mathcal{N}$ which increases strictly to N, and unit vectors $x_n = (N_n - N_{n-1})x_n$ such that $X_ix_n, X_i^*x_n \to 0$ for all $i \in \mathbb{N}$.

By Lemma 3.3 there are A, B in $\mathcal{T}(\mathcal{N})$ and a sequence of nest projections strictly increasing to N such that AXB is block diagonal with respect to these projections and each of the blocks has norm greater than 1. Since N_k and x_k demonstrate the Type-S property, so does any subsequence of theirs; so, replacing N_k with a subsequence, we may assume that each interval $N_k - N_{k-1}$ dominates a block of AXB. Multiplying AXB by a diagonal projection to select only those blocks which are dominated by an $N_k - N_{k-1}$ and have norm greater than 1, and replacing X with the resulting operator, we may now assume that X is block diagonal with respect to N_k, and that all the blocks $(N_k - N_{k-1})X(N_k - N_{k-1})$ have norm greater than 1.

We shall inductively construct a new sequence of unit vectors $y_n = (N_{k_n} - N_{k_{n-1}})y_n$ for a subsequence (k_n), together with contractions

$$A_n = (N_{k_n} - N_{k_{n-1}})A_n(N_{k_n} - N_{k_{n-1}})$$

and

$$B_n = (N_{k_n} - N_{k_{n-1}})B_n(N_{k_n} - N_{k_{n-1}})$$

in $\mathcal{T}(\mathcal{N})$ such that

$$\max\{\|(I - A_n X B_n)y_n\|, \|(I - A_n X B_n)^* y_n\|\} < 1/n$$

and $\max\{\|X_i y_n\|, \|X_i^* y_n\|\} < 1/n$ for all $1 \le i \le n$. The result will then follow by taking $A := \sum_{n=1}^{\infty} A_n$ and $B := \sum_{n=1}^{\infty} B_n$.

To perform the induction, fix n and suppose k_m, y_m, A_m and B_m have been chosen for all $m < n$. (To start the induction when $n = 1$, define $k_0 := 0$ and observe that no other features of the preceding steps are used in the induction step which follows.)

Note that for all sufficiently large m,

$$\max\{\|X_i x_m\|, \|X_i^* x_m\|\} < 1/(2n^2)$$

for all $1 \le i \le n$. Thus, taking $N = 4n^2$, we can pick $m_1 < m_2 < \cdots < m_N$ such that $m_1 > k_{n-1} + 1$, each $m_j > m_{j-1} + 1$, and for all $1 \le i \le n$ and $1 \le j \le N$, $\max\{\|X_i x_{m_j}\|, \|X_i^* x_{m_j}\|\} < 1/(2n^2)$.

Set $k_n := m_N$ and $y_n := N^{-1/2} \sum_{j=1}^{N} x_{m_j}$, which is a unit vector since the x_{m_j} are pairwise orthogonal. For each $1 < j \le N$, the interval $N_{m_j - 1} - N_{m_{j-1}}$ dominates a diagonal block of X which has norm greater than 1. Thus, we can choose vectors e_j and f_j in $N_{m_j-1} - N_{m_{j-1}}$ with $\|e_j\| \ge \|f_j\| = 1$ and $e_j = X f_j$, and set

$$A_n := \sum_{j=2}^{N} \|e_j\|^{-1} x_{m_{j-1}} e_j^* \quad \text{and} \quad B_n := \sum_{j=2}^{N} \|e_j\|^{-1} f_j x_{m_j}^*.$$

Since

$$x_{m_{j-1}} e_j^* = N_{m_{j-1}}(x_{m_{j-1}} e_j^*) N_{m_{j-1}}^{\perp} \quad \text{and} \quad f_j x_{m_j}^* = N_{m_j - 1}(f_j x_{m_j}^*) N_{m_j - 1}^{\perp},$$

each of the terms of the sums are in $\mathcal{T}(\mathcal{N})$, and the ranges and cokernels of the terms are pairwise orthogonal, so that both sums converge strongly. Now, clearly, for each $1 \le i \le n$, $\|X_i y_n\| \le N^{-1/2} \sum_{j=1}^{N} \|X_i x_{m_j}\| < N^{1/2}/2n^2 = 1/n$ and, likewise, $\|X_i^* y_n\| < 1/n$. Further, $A_n X B_n = \sum_{j=2}^{N} x_{m_{j-1}} x_{m_j}^*$, so that

$$\|(I - A_n X B_n) y_n\| = N^{-1/2} \left\| \sum_{j=1}^{N} x_{m_j} - \sum_{j=2}^{N} x_{m_{j-1}} \right\| = N^{-1/2} < 1/n,$$

and $(A_n X B_n)^* = \sum_{j=2}^{N} x_{m_j} x_{m_{j-1}}^* = \sum_{j=1}^{N-1} x_{m_{j+1}} x_{m_j}^*$, so that

$$\|(I - A_n X B_n)^* y_n\| = N^{-1/2} \left\| \sum_{j=1}^{N} x_{m_j} - \sum_{j=1}^{N-1} x_{m_{j+1}} \right\| = N^{-1/2} < 1/n.$$

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Note also that each of the e_j, f_j, x_{m_j} for $1 \leq j \leq N$, lie in the range of $N_{m_N} - N_{m_1-1} \leq N_{k_n} - N_{k_{n-1}}$. Thus, $A_n = (N_{k_n} - N_{k_{n-1}})A_n(N_{k_n} - N_{k_{n-1}})$, $B_n = (N_{k_n} - N_{k_{n-1}})B_n(N_{k_n} - N_{k_{n-1}})$ and $y_n = (N_{k_n} - N_{k_{n-1}})y_n$.

Having met all the requirements, the induction proceeds as stated, and we let $A := \sum_{n=1}^{\infty} A_n$ and $B := \sum_{n=1}^{\infty} B_n$. Clearly, for any fixed $i \in \mathbb{N}$,

$$\max\{\|X_i y_n\|, \|X_i^* y_n\|\} < 1/n$$

for all sufficiently large n and so $X_i y_n, X_i^* y_n \to 0$. Moreover, since A and B are block diagonal with respect to N_{n_k}, as is X, it follows that $(I - AXB)y_n = (I - A_n X B_n)y_n \to 0$ and $(I - AXB)^* y_n = (I - A_n X B_n)^* y_n \to 0$, and we are done. □

Lemma 3.6. *Fix a nest $\mathcal{N}$ and a projection $N \in \mathcal{N}$, and let $\mathcal{S}_i$ ($i \in \mathbb{N}$) be a countable collection of countable sets of Type-S which form a chain (i.e. for any i, j, either $S_i \subseteq S_j$ or $S_j \subseteq S_i$). Then $\bigcup_{i \in \mathbb{N}} \mathcal{S}_i$ is also of Type-S.*

Proof. The proof is a routine countability argument. Recall that the strong operator topology on $\mathcal{N}$ is metrizable; let d be a metric for it. Enumerate $\bigcup_{i \in \mathbb{N}} \mathcal{S}_i$ and let the sets $\mathcal{C}_n$ ($n \in \mathbb{N}$) consist of the first n terms of that enumeration. Fix n and suppose N_m and x_m have been chosen for $m < n$ so that $N_1 < N_2 < \cdots < N_{n-1} < N$, $x_m = (N_m - N_{m-1})x_m$ and $\max\{\|Xx_m\|, \|X^* x_m\|\} < 1/m$ for all $X \in \mathcal{C}_m$. Each $X \in \mathcal{C}_n$ belongs to some $\mathcal{S}_i$ and, since $\mathcal{C}_n$ is finite and $\{S_i\}$ is a chain, $\mathcal{C}_n$ is contained in some $\mathcal{S}_i$. Therefore, $\mathcal{C}_n$ is of Type-S. Using this fact, we can find $N_{n-1} < N_n < N$ with $d(N_n, N) < 1/n$ and $x_n = (N_n - N_{n-1})x_n$ such that $\max\{\|Xx_n\|, \|X^* x_n\|\} < 1/n$ for all $X \in \mathcal{C}_n$. Continue this inductively to construct a strictly increasing sequence $N_n \to N$ and $x_n = (N_n - N_{n-1})$ for all $n \in \mathbb{N}$ such that $\max\{\|Xx_n\|, \|X^* x_n\|\} < 1/n$ for all $X \in \mathcal{C}_n$ (taking $k_0 = 0$ to start the induction). Each $X \in \bigcup_{i \in \mathbb{N}} \mathcal{S}_i$ belongs to $\mathcal{C}_n$ for all sufficiently large n, and so the result follows with the vectors so chosen. □

Theorem 3.7. *Assume the continuum hypothesis and let $\mathcal{N}$ be a nest. Then the diagonal ideals of $\mathcal{T}(\mathcal{N})$ are primitive ideals.*

Proof. The result is trivial when the diagonal ideal is of type $\mathcal{I}_N^-$ with $N^- < N$ or $\mathcal{I}_N^+$ with $N^+ > N$; in either case, the diagonal ideal is the kernel of the representation $X \mapsto EX|_{E\mathcal{H}}$, where E is an atom of $\mathcal{N}$, whose range is therefore all of $B(E\mathcal{H})$ and so is irreducible. For the remainder of the proof, consider only diagonal ideals which are not of this type.

Next, let $\mathcal{I}$ be a diagonal ideal of $\mathcal{T}(\mathcal{N})$ and suppose that $\mathcal{I} = \mathcal{I}_N^-$ for some $N = N^- > 0$ in $\mathcal{N}$. It is enough to construct operators $A_X, B_X \in \mathcal{T}(\mathcal{N})$ for each operator $X \in \mathcal{T}(\mathcal{N}) \setminus \mathcal{I}_N^-$, such that the collection $\{I - A_X X B_X : X \in \mathcal{T}(\mathcal{N}) \setminus \mathcal{I}_N^-\}$ generates a proper left ideal of $\mathcal{T}(\mathcal{N})$. Then, there is a maximal left ideal $\mathcal{L}$ which contains this family of operators, and the kernel of the left regular representation of $\mathcal{T}(\mathcal{N})$ on $\mathcal{T}(\mathcal{N})/\mathcal{L}$ is a primitive ideal, $\mathcal{P}$, which by Remark 3.1 must exclude all $X \notin \mathcal{I}_N^-$. Thus, $\mathcal{P} \subseteq \mathcal{I}_N^-$. Since every primitive ideal contains a diagonal ideal [**23**, Theorem 4.9], and the distinct diagonal ideals are incomparable [**23**, Lemma 4.7], it follows that $\mathcal{I}_N^- = \mathcal{P}$ and so is primitive.

Now consider the case when $\mathcal{I} = \mathcal{I}_N^+$ for some $N = N^+ < I$ in $\mathcal{N}$. By the same reasoning, it is enough to find $A_X, B_X \in \mathcal{T}(\mathcal{N})$ such that $\{I - A_X X B_X : P \in \mathcal{T}(\mathcal{N}) \setminus \mathcal{I}_N^+\}$ is contained in a proper left ideal of $\mathcal{T}(\mathcal{N})$. To do this, we take adjoints and seek $A_X, B_X \in \mathcal{T}(\mathcal{N})^* = \mathcal{T}(\mathcal{N}^\perp)$ such that $\{I - A_X X B_X : X \in \mathcal{T}(\mathcal{N}^\perp) \setminus \mathcal{I}_{N^\perp}^-\}$ is contained in a proper right ideal of $\mathcal{T}(\mathcal{N}^\perp)$. Since $\mathcal{N}$ is an arbitrary nest, we can replace $\mathcal{N}^\perp$ with $\mathcal{N}$ to recast this as a second problem about $\mathcal{I}_N^-$ in $\mathcal{T}(\mathcal{N})$, namely, to find $A_X, B_X \in \mathcal{T}(\mathcal{N})$ for each $X \in \mathcal{T}(\mathcal{N}) \setminus \mathcal{I}_N^-$ such that $\{I - A_X X B_X : X \in \mathcal{T}(\mathcal{N}) \setminus \mathcal{I}_N^-\}$ generates a proper *right* ideal of $\mathcal{T}(\mathcal{N})$. We shall show in fact that the choice can be made so that the same set of operators $\{I - A_X X B_X\}$ serves to generate both a proper left ideal and a proper right ideal. We shall construct these operators using transfinite recursion.

The cardinality of $\mathcal{T}(\mathcal{N}) \setminus \mathcal{I}_N^-$ is equal to the cardinality of the continuum, since every operator can be represented as a countable array of complex numbers. Since we are assuming the continuum hypothesis, $\mathcal{T}(\mathcal{N}) \setminus \mathcal{I}_N^-$ has cardinality $\aleph_1$ and so it can be put in bijective correspondence with the set of ordinals $a < \omega_1$ (where ω_1 denotes the first uncountable ordinal). Write this correspondence as X_a $(a < \omega_1)$. To run the transfinite recursion, we suppose that for some $a < \omega_1$ we have operators A_b, B_b in $\mathcal{T}(\mathcal{N})$ for all $b < a$, and describe how to obtain A_a, B_a. First, if the set $\{I - A_b X_b B_b : b < a\}$ is of Type-S, observe that $\{I - A_b X_b B_b : b < a\}$ is a countable collection and use Lemma 3.5 to find $A_a, B_a \in \mathcal{T}(\mathcal{N})$ such that $\{I - A_b X_b B_b : b \le a\}$ is also of Type-S. On the other hand, if it happens that $\{I - A_b P_b B_b : b < a\}$ is not of Type-S, set $A_a = B_a = 0$. (This is a sink terminal state which, as we shall prove momentarily, is never in fact reached.)

Note that, formally, Lemma 3.5 assumes a countably infinite collection of predecessors. However, the case of finite a, or even $a = 1$, can be covered by padding the collection of predecessors with countably many repeated zeros. Note also that the recursion step involves an arbitrary choice of operators, which can easily be resolved using the axiom of choice.

Having described a rule to construct A_a, B_a with $(A_b, B_b)_{b<a}$ given, we apply the principle of transfinite recursion to obtain $(A_a, B_a)_{a<\omega_1}$, where the transition rule from the previous paragraph applies for every $a < \omega_1$. We next note that for every $a < \omega_1$, $\mathcal{S}_a := \{I - A_b X_b B_b : b \le a\}$ is of Type-S; if this were not true, then we could find the least a such that $\mathcal{S}_a$ is not Type-S. Thus, for each of the countably many $b < a$, $\mathcal{S}_b$ is countable and of Type-S and so, by Lemma 3.6, $\bigcup_{b<a} \mathcal{S}_b$ is Type-S. But $\bigcup_{b<a} \mathcal{S}_b = \{I - A_b X_b B_b : b < a\}$ and so, by the recursion step, $\mathcal{S}_a = \{I - A_b X_b B_b : b \le a\}$ is also Type-S. Thus, by contradiction, each $\mathcal{S}_a$ is of Type-S and, in particular, generates a proper left ideal of $\mathcal{T}(\mathcal{N})$ and a proper right ideal of $\mathcal{T}(\mathcal{N})$. Now, in general, the union of any chain of sets, each of which generates a proper left (respectively, right) ideal, will also generate a proper left (respectively, right) ideal. Thus, $\{I - A_a X_a B_a : a < \omega_1\} = \bigcup_{a<\omega_1} \mathcal{S}_a$ generates a proper left ideal and a proper right ideal, and the result follows. □

Corollary 3.8. *Assuming the continuum hypothesis, the diagonal ideals of $\mathcal{T}(\mathcal{N})$ are also right primitive ideals, that is to say, the annihilators of simple right modules.*

Proof. The conjugate-linear anti-isomorphism $X \mapsto X^*$ maps $\mathcal{T}(\mathcal{N})$ to $\mathcal{T}(\mathcal{N}^\perp)$, maps diagonal ideals to diagonal ideals and converts left modules into right modules. □

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We remark in passing that Theorem 3.7 provides a new proof of Ringrose's characterization of the Jacobson radical of a nest algebra; in view of Theorem 2.5,

$$\bigcap_{\mathcal{I}\text{ diagonal}} \mathcal{I} \subseteq \bigcap_{\mathcal{P}\text{ primitive}} \mathcal{P},$$

and the reverse inclusion follows from Theorem 3.7. Insofar as our result assumes the continuum hypothesis and also assumes $\mathcal{H}$ is separable, this is, of course, substantially less general than Ringrose's original proof.

4. The left ideals of a nest algebra

In this section, we study the left ideals of nest algebras. Definition 4.1 gives a method of specifying left ideals, and in Theorem 4.4 we shall see that every left ideal can be specified in this way. We then introduce a stronger property (Definition 4.7) which specifies many closed left ideals, including the maximal left ideals. This leads to insights into the structure of left ideals (Proposition 4.18), which we apply in the following sections.

Definition 4.1. Let $\mathcal{L}$ be a left ideal of $\mathcal{T}(\mathcal{N})$. We say that $\mathcal{L}$ is *constructible* if there is a net indexed by a directed set A consisting of pairs (N_α, x_α) of projections $N_\alpha \in \mathcal{N}$ and vectors $x_\alpha \in \mathcal{H}$ such that

$$\mathcal{L} = \{X \in \mathcal{T}(\mathcal{N}) : \lim_{\alpha\in A} \|(I - N_\alpha)Xx_\alpha\| = 0\}$$

for every $X \in \mathcal{T}(\mathcal{N})$.

Lemma 4.2. *$\mathcal{T}(\mathcal{N})$ is itself a constructible ideal and, in general, the constructible ideal $\mathcal{L}$, specified by the net $(N_\alpha, x_\alpha)_{\alpha\in A}$, is equal to $\mathcal{T}(\mathcal{N})$ if and only if $\lim_\alpha N_\alpha^\perp x_\alpha = 0$.*

Proof. If $N_\alpha^\perp x_\alpha \to 0$ then for any fixed $X \in \mathcal{T}(\mathcal{N})$, $\|N_\alpha^\perp X x_\alpha\| = \|N_\alpha^\perp X N_\alpha^\perp x_\alpha\| \leq \|X\|\|N_\alpha^\perp x_\alpha\|$ and so $X \in \mathcal{L}$. Conversely, if $N_\alpha^\perp x_\alpha \not\to 0$, then $I \notin \mathcal{L}$ and so $\mathcal{L}$ is proper. □

Note that if $(N_\alpha, x_\alpha)_{\alpha\in A}$ is a net in $\mathcal{N}\times\mathcal{H}$ and $X \in \mathcal{T}(\mathcal{N})$, then $N_\alpha^\perp X x_\alpha = N_\alpha^\perp X N_\alpha^\perp x_\alpha$, for all α, and so without loss we can always assume that $x_\alpha = N_\alpha^\perp x_\alpha$.

The following interpolation result of Katsoulis, Moore and Trent enables us to see that all left ideals are constructible. In this context, we remark that the results of [**11**] have a precursor in Lance's [**13**, Theorem 2.3], introduced to study the radical and diagonal ideals.

Theorem 4.3 (see [11, Theorem 4]). *Let $X_1, \dots, X_n$ and Y be in $\mathcal{T}(\mathcal{N})$. Then there are $A_1, \dots, A_n$ in $\mathcal{T}(\mathcal{N})$ such that*

$$Y = \sum_{i=1}^n A_i X_i$$

if and only if

$$\sup\left\{\frac{\|N^\perp Y x\|^2}{\sum_{i=1}^n \|N^\perp X_i x\|^2} : N \in \mathcal{N}, x \in \mathcal{H}\right\} < \infty \tag{4.1}$$

(where 0/0 is interpreted as 0).

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Theorem 4.4. *Every left ideal of a nest algebra is constructible.*

Proof. Let $\mathcal{L}$ be a fixed left ideal of the nest algebra $\mathcal{T}(\mathcal{N})$ and take A to be the set of all 4-tuples (F, ϵ, N, x) where F is a finite subset of $\mathcal{L}$, $\epsilon > 0$, $N \in \mathcal{N}$ and $x \in \mathcal{H}$, subject to the constraint that $\|N^{\perp}Xx\| < \epsilon$ for all $X \in F$. This is a directed set if we say $(F, \epsilon, N, x) \leq (F', \epsilon', N', x')$ when $F \subseteq F'$ and $\epsilon \geq \epsilon'$, as the relation is clearly reflexive and transitive, and any pair of members of A, (F, ϵ, N, x) and (F', ϵ', N', x'), is dominated by $(F \cup F', \min\{\epsilon, \epsilon'\}, 0, 0)$. Define a net on A with values in $\mathcal{N} \times \mathcal{H}$ by the mapping which takes $\alpha := (F, \epsilon, N, x) \in A$ to (N_α, x_α) where $N_\alpha := N$, and $x_\alpha := x$. We shall see that this net specifies $\mathcal{L}$ exactly.

On the one hand, trivially, if $X \in \mathcal{L}$ then for any $\epsilon > 0$, the tuple $\alpha_0 := (\{X\}, \epsilon, 0, 0)$ belongs to A and so for any $\alpha \geq \alpha_0$, $\|N_\alpha^{\perp}Xx_\alpha\| < \epsilon$. Next, suppose on the other hand that $Y \in \mathcal{T}(\mathcal{N}) \setminus \mathcal{L}$.

Let an arbitrary $\alpha_0 := (\{X_1, \ldots, X_n\}, \epsilon, M, x)$ in A be given. Since $Y \notin \mathcal{L}$, there do not exist any $A_1, \ldots, A_n$ in $\mathcal{T}(\mathcal{N})$ such that $\sum_{i=1}^{n} A_i X_i = Y$. Thus, by Theorem 4.3, the supremum (4.1) is infinite, and so we can find $N \in \mathcal{N}$ and $y \in \mathcal{H}$ such that

$$\|N^{\perp}X_i y\| < \epsilon \|N^{\perp}Yy\|$$

for each $i = 1, \ldots, n$. Rescaling y, we obtain N and y such that $\|N^{\perp}X_i y\| < \epsilon$ and $\|N^{\perp}Yy\| = 1$. Thus, $\beta := (\{X_1, \ldots, X_n\}, \epsilon, N, y)$ is in A, and we have $\beta \geq \alpha_0$ and $\|N_\beta^{\perp}Yx_\beta\| = 1$. In other words, the net $\|N_\alpha^{\perp}Yx_\alpha\|$ is frequently equal to 1, and so $\|N_\alpha^{\perp}Yx_\alpha\| \not\to 0$. □

Example 4.5. The set $\mathcal{F}_\mathcal{N}$ of finite-rank operators in $\mathcal{T}(\mathcal{N})$ is a two-sided ideal of $\mathcal{T}(\mathcal{N})$ but is not norm-closed. We can specify this with the following net. Let A consist of the set of pairs (F, x), where F is a finite-dimensional subspace of $\mathcal{H}$ and x is a vector which is orthogonal to F. For $\alpha = (F, x) \in A$, define $x_\alpha := x$ and $N_\alpha = 0$. Say $(F, x) \leq (G, y)$ in A if $F \subseteq G$. Clearly, $T \in \mathcal{T}(\mathcal{N})$ belongs to $\mathcal{F}_\mathcal{N}$ if and only if there is a finite-dimensional space F such that T vanishes on $F^{\perp}$. Since the vectors in the pairs are unbounded, the condition $\|N_\alpha^{\perp}Tx_\alpha\| < 1$ for all $\alpha \geq (F, 0)$ is equivalent to T vanishing on $F^{\perp}$.

Example 4.6. The set $\mathcal{K}_\mathcal{N}$ of compact operators in $\mathcal{T}(\mathcal{N})$ is a norm-closed two-sided ideal of $\mathcal{T}(\mathcal{N})$. We can specify it with the following net, which is similar to the previous example. Let A consist of the set of pairs (F, x), where F is a finite-dimensional subspace of $\mathcal{H}$ and x is a unit vector which is orthogonal to F. Again, for $\alpha = (F, x) \in A$, define $x_\alpha := x$ and $N_\alpha = 0$, and say $(F, x) \leq (G, y)$ in A if $F \subseteq G$. By spectral theory, an operator $T \in \mathcal{T}(\mathcal{N})$ belongs to $\mathcal{K}_\mathcal{N}$ if and only if for any $\epsilon > 0$ there is a finite-dimensional space F such that $\|T|_{F^{\perp}}\| < \epsilon$, which is readily seen to be equivalent to $\|N_\alpha^{\perp}Tx_\alpha\| \to 0$.

The contrast between the last two examples, in which the net was unbounded in one case and bounded in the other, motivates the following definition.

Definition 4.7. Let $\mathcal{L}$ be a left ideal of $\mathcal{T}(\mathcal{N})$. We say that $\mathcal{L}$ is *strongly constructible* if it is constructible and a net $(N_\alpha, x_\alpha)_{\alpha \in A}$ specifying $\mathcal{L}$ can be found in which all the vectors $x_\alpha = N_\alpha^{\perp}x_\alpha$ have norm 1.

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Proposition 4.8. *Strongly constructible ideals are norm-closed.*

Proof. Let $\mathcal{L}$ be strongly constructible and specified by $(N_\alpha, x_\alpha)_{\alpha\in A}$, where $\|x_\alpha\| = 1$ for all $\alpha \in A$. Suppose the sequence of $X_n \in \mathcal{L}$ converges in norm to $X \in \mathcal{T}(\mathcal{N})$. Given $\epsilon > 0$, find a fixed $n \in \mathbb{N}$ such that $\|X - X_n\| < \epsilon/2$, and $\alpha_0 \in A$ such that $\|N_\alpha^\perp X_n x_\alpha\| < \epsilon/2$ for all $\alpha \geq \alpha_0$. Then

$$\|N_\alpha^\perp X x_\alpha\| \leq \|X - X_n\|\|x_\alpha\| + \|N_\alpha^\perp X_n x_\alpha\|, < \epsilon. \qquad \square$$

Proposition 4.9. *The maximal left ideals of $\mathcal{T}(\mathcal{N})$ are strongly constructible.*

Proof. Let $\mathcal{L}$ be a maximal left ideal which we suppose to be specified by the net $(N_\alpha, x_\alpha)_{\alpha\in A}$. Without loss, assume that each $x_\alpha = N_\alpha^\perp x_\alpha$. By Lemma 4.2, $x_\alpha \not\to 0$, and so there is an $\epsilon_0 > 0$ such that $\|x_\alpha\|$ is frequently at least ϵ_0. Let $A' := \{\alpha \in A : \|x_\alpha\| \geq \epsilon_0\}$ and $x'_\alpha = x_\alpha/\|x_\alpha\|$ for $\alpha \in A'$. Now A' is a directed set and (N_α, x'_α) is a net on it. Again, by Lemma 4.2, the net $(N_\alpha, x'_\alpha)_{\alpha\in A'}$ specifies a proper ideal; furthermore, this ideal contains $\mathcal{L}$, as for $X \in \mathcal{L}$,

$$\|N_\alpha^\perp X x'_\alpha\| \leq \frac{1}{\epsilon_0}\, \|N_\alpha^\perp X x_\alpha\|$$

for all $\alpha \in A'$, and the net on the right converges to zero since $(N_\alpha, x_\alpha)_{\alpha\in A'}$ is a subnet of $(N_\alpha, x_\alpha)_{\alpha\in A}$. By maximality, the ideal which $(N_\alpha, x'_\alpha)_{\alpha\in A'}$ specifies must equal $\mathcal{L}$. $\square$

Proposition 4.10. *Arbitrary intersections of strongly constructible ideals are strongly constructible.*

The proof is a consequence of the following simple result about nets.

Lemma 4.11. *Fix a set X and suppose that we have a family of nets in X indexed by a set K, which we denote $(x_\alpha^{(k)})_{a\in A_k}$. Then we can find a net $(x_\alpha)_{\alpha\in A}$ in X with the property that for any $E \subseteq X$, $(x_\alpha)_{\alpha\in A}$ is eventually in E if and only if for each $k \in K$, $(x_\alpha^{(k)})_{\alpha\in A_k}$ is eventually in E.*

Proof. Define A to be set the set of pairs (σ, k), where σ is a section map on the fibre bundle of A_k over K (i.e. for each $k \in K$, $\sigma(k) \in A_k$), and k is an arbitrary member of K. Put a relation on A by declaring $(\sigma, k) \leq (\tau, l)$ if $\sigma(i) \leq_i \tau(i)$ for all $i \in K$ (the relation $\leq_i$ is the directed relation defined on A_i). This is a reflexive and transitive relation. Moreover, if (σ, k) and (τ, l) are in A then for each $i \in K$ we can find an element of A_i which dominates both $\sigma(i)$ and $\tau(i)$. By the axiom of choice, there is therefore a section map ρ such that $\rho(i)$ dominates both $\sigma(i)$ and $\tau(i)$ for all $i \in K$. Taking an arbitrary $i \in K$, then (ρ, i) dominates both (σ, k) and (τ, l) in A. Thus, A is a directed set, and we define the net $(x_{(\sigma,k)})_{(\sigma,k)\in A}$ by $x_{(\sigma,k)} := x_{\sigma(k)}^{(k)}$.

Now, on the one hand, suppose that $(x_{(\sigma,k)})$ is eventually in $E \subseteq X$. Thus, there is a $(\sigma_0, k_0) \in A$ such that $x_{(\sigma,k)} \in E$ for all $(\sigma, k) \geq (\sigma_0, k_0)$. Fix $k \in K$ and consider $\alpha_0 := \sigma_0(k) \in A_k$. If $\alpha \geq_k \alpha_0$ then define $\sigma(i) := \sigma_0(i)$ for all $i \neq k$ and $\sigma(k) := \alpha$. Then $(\sigma, k) \geq (\sigma_0, k_0)$ and so $x_\alpha^{(k)} = x_{(\sigma,k)} \in E$. This shows that for each $k \in K$, $(x_\alpha^{(k)})_{\alpha\in A_k}$ is eventually in E.

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Conversely, let $E \subseteq X$ and suppose that for every $k \in K$, $(x_\alpha^{(k)})_{\alpha \in A_k}$ is eventually in E. That is to say, for each $k \in K$, we can find an $\alpha_0 \in A_k$ such that $x_\alpha^{(k)} \in E$ for all $\alpha \geq_k \alpha_0$ in A_k. Again, by the axiom of choice, we pick one such α_0 for each $k \in K$ and obtain a section σ_0 such that for each $k \in K$ and $\alpha \geq_k \sigma_0(k)$ in A_k, we have $x_\alpha^{(k)} \in E$. Pick an arbitrary $k_0 \in K$ and then suppose $(\sigma, k) \geq (\sigma_0, k_0)$. This means that, in particular, $\sigma(k) \geq_k \sigma_0(k)$, so that $x_{(\sigma,k)} = x_{\sigma(k)}^{(k)} \in E$. We conclude that the net $(x_{(\sigma,k)})_{(\sigma,k)\in A}$ is eventually in E. □

The proof of Proposition 4.10 now follows straightforwardly.

Proof of Proposition 4.10. Let $\mathcal{L}_k$ $(k \in K)$ be a collection of strongly constructible left ideals. Writing $\mathcal{H}_1$ for the set of unit vectors in $\mathcal{H}$, for each $k \in K$ there are directed sets A_k and nets $(N_\alpha^{(k)}, x_\alpha^{(k)}) \in \mathcal{N} \times \mathcal{H}_1$ for $\alpha \in A_k$ such that an $X \in \mathcal{T}(\mathcal{N})$ belongs to $\mathcal{L}_k$ if and only if $\lim_{\alpha \in A_k} \|(I - N_\alpha^{(k)}) X x_\alpha^{(k)}\| = 0$.

By Lemma 4.11, find a new net $(N_\alpha, x_\alpha)_{\alpha \in A}$ in $\mathcal{N} \times \mathcal{H}_1$ which is eventually in a subset of $\mathcal{N} \times \mathcal{H}_1$ if and only if each of the $(N_\alpha^{(k)}, x_\alpha^{(k)})_{\alpha \in A_k}$ are eventually in that set. Fix $X \in \mathcal{T}(\mathcal{N})$ and let $\epsilon > 0$ be given. Let

$$E_\epsilon := \{(N, x) \in \mathcal{N} \times \mathcal{H}_1 : \|(I - N) X x\| < \epsilon\}.$$

Clearly, $X \in \bigcap_{k \in K} \mathcal{L}_k$ if and only if for every $k \in K$ and every $\epsilon > 0$, $(N_\alpha^{(k)}, x_\alpha^{(k)})_{\alpha \in A_k}$ is eventually in E_ϵ. This happens if and only if for every $\epsilon > 0$, $(N_\alpha, x_\alpha)_{\alpha \in A}$ is eventually in E_ϵ, which in turn happens if and only if $\lim_{\alpha \in A} \|(I - N_\alpha) X x_\alpha\| = 0$. Thus, $\bigcap_{k \in K} \mathcal{L}_k$ is strongly constructible. □

Corollary 4.12. *Every proper left ideal $\mathcal{L}$ of $\mathcal{T}(\mathcal{N})$ is contained in a smallest strongly constructible left ideal, which we shall call the* strongly constructible hull *of $\mathcal{L}$.*

Corollary 4.13. *The primitive ideals of $\mathcal{T}(\mathcal{N})$ are strongly constructible.*

Proof. Every primitive ideal is the intersection of the maximal left ideals which contain it [**1**, § 24, Proposition 12 (iv)]. The result follows by Propositions 4.9 and 4.10. □

Example 4.14. In particular, the maximal two-sided ideals of $\mathcal{T}(\mathcal{N})$, being primitive, are strongly constructible. Recall that the *strong radical* of a unital algebra is the intersection of all its maximal two-sided ideals. In [**17**, Theorem 3.2], we saw that if $\mathcal{T}(\mathcal{N})$ is a continuous nest algebra then any norm-closed, two-sided ideal of $\mathcal{T}(\mathcal{N})$ that contains the strong radical is the intersection of the maximal two-sided ideals which contain it. Thus, by Propositions 4.10 and 4.13, all such ideals are strongly constructible.

Corollary 4.15. *All norm-closed, two-sided ideals of a continuous nest algebra which contain the strong radical are strongly constructible.*

Question 4.16. Is every norm-closed left ideal of a nest algebra strongly constructible?

Strongly constructible ideals are also characterized by two ostensibly weaker conditions.

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Proposition 4.17. *Let $\mathcal{L}$ be a proper left ideal of $\mathcal{T}(\mathcal{N})$. The following are equivalent.*

1. *$\mathcal{L}$ is strongly constructible.*
2. *$\mathcal{L}$ can be specified by a net $(N_\alpha, x_\alpha)_{\alpha\in A}$, where $\|x_\alpha\| \leq 1$ for all $\alpha \in A$.*
3. *$\mathcal{L}$ can be specified by a net $(N_\alpha, x_\alpha)_{\alpha\in A}$, where $\|x_\alpha\|$ is bounded.*

Proof. Clearly, $(1) \Rightarrow (2) \Rightarrow (3)$, and so it remains to prove $(3) \Rightarrow (1)$. Suppose $(N_\alpha, x_\alpha)_{\alpha\in A}$ specifies $\mathcal{L}$ and $\|x_\alpha\|$ is bounded. Since $\mathcal{L}$ is proper, by Lemma 4.2, $x_\alpha \not\to 0$, and so there is an ϵ_0 such that $\|x_\alpha\|$ is frequently at least ϵ_0. For each $k \in \mathbb{N}$, set

$$A_k := \{\alpha \in A : \|x_\alpha\| \geq \epsilon_0/k\}.$$

Each A_k is a directed set (with the order relation inherited from A), and the restricted net $(N_\alpha, x_\alpha)_{\alpha\in A_k}$ defines a left ideal $\mathcal{L}_k$. Since the x_α are bounded away from zero on A_k, we can normalize and see that each $\mathcal{L}_k$ is strongly constructible. It remains to check that $\mathcal{L} = \bigcap_{k\in\mathcal{N}} \mathcal{L}_k$, and then the result will follow by Proposition 4.10.

Clearly, since each $A_k \subseteq A$, also $\mathcal{L} \subseteq \mathcal{L}_k$ and so $\mathcal{L} \subseteq \bigcap_{k\in\mathcal{N}} \mathcal{L}_k$. Suppose $X \notin \mathcal{L}$. Then $(I - N_\alpha)Xx_\alpha \not\to 0$, and so there is an $\epsilon_1 > 0$ such that $\|(I - N_\alpha)Xx_\alpha\| \geq \epsilon_1$ frequently. Choose $k > \epsilon_0\|X\|/\epsilon_1$ so that then whenever $\|(I - N_\alpha)Xx_\alpha\| \geq \epsilon_1$ then

$$\|X\|\|x_\alpha\| \geq \|(I - N_\alpha)Xx_\alpha\| \geq \epsilon_1 > \|X\|\frac{\epsilon_0}{k},$$

and thus $\alpha \in A_k$. It follows that $\|(I - N_\alpha)Xx_\alpha\| \geq \epsilon_1$ frequently on A_k, and so $X \notin \mathcal{L}_k$. □

Proposition 4.18. *Let $\mathcal{L}$ be a maximal left ideal in $\mathcal{T}(\mathcal{N})$ and let P_n be a sequence of pairwise orthogonal projections in $\mathcal{L}$. There is a subsequence P_{n_k} such that the projection $\sum_{n=1}^{\infty} P_{n_k}$ belongs to $\mathcal{L}$.*

Proof. By Proposition 4.9, $\mathcal{L}$ is strongly constructible, say by a net (N_α, x_α), where each x_α is a unit vector in the range of N_α. By Kelley's theorem, this net has a universal subnet, which specifies a proper ideal containing $\mathcal{L}$, hence it in fact specifies $\mathcal{L}$ itself. Thus, we may assume that (N_α, x_α) is universal.

The proof now proceeds by means of a fairly routine diagonal argument. For any $S \subseteq \mathbb{N}$ write $P(S) := \sum_{n\in S} P_n$. Take $S_0 := \mathbb{N}$ and split S_0 into two infinite sets, S_0' and S_0''. If $\|P(S_0')x_\alpha\|$ and $\|P(S_0'')x_\alpha\|$ are each eventually greater than $1/\sqrt{2}$ then $\|P(S_0)x_\alpha\|^2 = \|P(S_0')x_\alpha\|^2 + \|P(S_0'')x_\alpha\|^2$ is eventually greater than 1, which is impossible. Since (N_α, x_α) is universal, at least one of $\|P(S_0')x_\alpha\|$, $\|P(S_0'')x_\alpha\|$ is eventually no greater than $1/\sqrt{2}$; without loss suppose that $\|P(S_0')x_\alpha\| \leq 1/\sqrt{2}$ eventually, and set $S_1 := S_0'$.

Now decompose $S_1 = S_1' \cup S_1''$ in the same way as the union of infinite subsets and, as before, we conclude that at least one of $\|P(S_1')x_\alpha\|$, $\|P(S_1'')x_\alpha\|$ is eventually no greater than $(1/\sqrt{2})^2$. Take S_2 to be one of S_1', S_1'' for which this holds. Proceeding in this way, we obtain a sequence $S_0 \supseteq S_1 \supseteq S_2 \supseteq \dots$ of infinite subsets of $\mathbb{N}$ such that for each k, eventually $\|P(S_k)x_\alpha\| \le (1/\sqrt{2})^k$. Now take n_k to be the kth element of S_k in order, which is a strictly increasing sequence, and let $S := \{n_k\}$. Thus, $S \setminus S_k$ is finite for all k.

Finally, write $P := P(S)$ and, given $\epsilon > 0$, take k such that $(1/\sqrt{2})^k < \epsilon$. For all sufficiently large α,

$$\begin{aligned}\|Px_\alpha\| &= \|(PP(S_k) + PP(S_k)^\perp)x_\alpha\| \\ &\le \|P(S_k)x_\alpha\| + \|P(S \setminus S_k)x_\alpha\| \\ &\le \epsilon + \sum_{n \in S \setminus S_k} \|P_n x_\alpha\|.\end{aligned}$$

But the sum in the last line is finite and so is eventually less than ϵ. We can conclude that $\|N_\alpha^\perp P x_\alpha\| = \|Px_\alpha\| \to 0$, so that $P \in \mathcal{L}$. □

Corollary 4.19. *Let $\mathcal{J}$ be a maximal right ideal in $\mathcal{T}(\mathcal{N})$ and let P_n be a sequence of pairwise orthogonal projections in $\mathcal{R}$. There is a subsequence P_{k_n} such that the projection $\sum_{k=1}^{\infty} P_{k_n}$ belongs to $\mathcal{R}$.*

Proof. The result follows by taking adjoints and working in $\mathcal{T}(\mathcal{N}^\perp)$. □

5. Atomic nest algebras

In this section we focus on atomic nest algebras and relate the character of primitive ideals to the family of diagonal operators they contain. Observe that if $\mathcal{P}$ is a primitive ideal of $\mathcal{T}(\mathcal{N})$ then $\mathcal{P} \cap \mathcal{D}(\mathcal{N})$ is a norm-closed two-sided ideal of the C*-star algebra $\mathcal{D}(\mathcal{N})$ and is therefore a $*$-ideal. In many interesting cases, the nest is multiplicity-free so that $\mathcal{D}(\mathcal{N})$ is an abelian C*-algebra.

Proposition 5.1. *Let $\mathcal{N}$ be an atomic nest and $\mathcal{J}$ a two-sided ideal in $\mathcal{T}(\mathcal{N})$. Then, $\mathcal{J}$ is a maximal two-sided ideal if and only if $\mathcal{J} \cap \mathcal{D}(\mathcal{N})$ is a maximal two-sided ideal of $\mathcal{D}(\mathcal{N})$.*

Proof. Suppose $\mathcal{J}$ is maximal. Then, by [**20**, Theorem 3.8], $\mathcal{J}$ contains $\mathcal{R}_\mathcal{N}^\infty$. It follows that $\mathcal{J} = (\mathcal{J} \cap \mathcal{D}(\mathcal{N})) \oplus \mathcal{R}_\mathcal{N}^\infty$. If $\mathcal{J} \cap \mathcal{D}(\mathcal{N})$ is not maximal then there is a larger proper ideal $\mathcal{D}_0$ of $\mathcal{D}(\mathcal{N})$. But then $\mathcal{D}_0 \oplus \mathcal{R}_\mathcal{N}^\infty$ is a proper ideal of $\mathcal{T}(\mathcal{N})$ and strictly larger than $\mathcal{J}$, contrary to fact.

Suppose on the other hand that $\mathcal{J} \cap \mathcal{D}(\mathcal{N})$ is maximal. By [**6**, Theorem 10.2], $\mathcal{R}_\mathcal{N}^\infty$ is generated as a two-sided ideal by a generator which is the sum of three commutators $[G_i, P_i]$ $(i = 1, 2, 3)$, where $G_i \in \mathcal{T}(\mathcal{N})$ and P_i is a projection in the core $\mathcal{C}(\mathcal{N})$ of $\mathcal{T}(\mathcal{N})$. (Recall that the *core* of a nest algebra is the abelian von Neumann algebra generated by $\mathcal{N}$.) Now, since $\mathcal{J} \cap \mathcal{D}(\mathcal{N})$ is a maximal ideal of $\mathcal{D}(\mathcal{N})$, and the P_i are in the centre of $\mathcal{D}(\mathcal{N})$, it follows that one of $P_i, P_i^\perp$ must lie in $\mathcal{J} \cap \mathcal{D}(\mathcal{N})$ for each i. Thus, in any event, the commutators $[G_i, P_i] = [G_i, P_i^\perp]$ belong to $\mathcal{J}$ and so $\mathcal{J}$ contains $\mathcal{R}_\mathcal{N}^\infty$. Thus, again, $\mathcal{J} = (\mathcal{J} \cap \mathcal{D}(\mathcal{N})) \oplus \mathcal{R}_\mathcal{N}^\infty$. If $\mathcal{J}$ is not maximal then there is a larger proper ideal $\mathcal{J}_0$ of $\mathcal{T}(\mathcal{N})$. But then since $\mathcal{J}_0$ also contains $\mathcal{R}_\mathcal{N}^\infty$, $\mathcal{J}_0 = (\mathcal{J}_0 \cap \mathcal{D}(\mathcal{N})) \oplus \mathcal{R}_\mathcal{N}^\infty$ and so $\mathcal{J}_0 \cap \mathcal{D}(\mathcal{N})$ is a proper ideal of $\mathcal{D}(\mathcal{N})$ and larger than $\mathcal{J} \cap \mathcal{D}(\mathcal{N})$, contrary to fact. □

The proof of Proposition 5.1 is deceptively straightforward. In fact, the result cited from [20] depends on Marcus, Spielman and Srivastava's proof [15] of the paving theorem. Recall (Definition 2.6) that we write $\mathcal{I}_\mathcal{P}$ for the unique diagonal ideal contained by the primitive ideal $\mathcal{P}$.

Proposition 5.2. *Let $\mathcal{N}$ be an atomic nest, let $\mathcal{P}$ be a primitive ideal of $\mathcal{T}(\mathcal{N})$ and suppose $\mathcal{P} \neq \mathcal{I}_\mathcal{P}$. Then there are non-zero projections in $\mathcal{P} \setminus \mathcal{I}_\mathcal{P}$.*

Proof. We shall prove the result in the case when $\mathcal{I}_\mathcal{P} = \mathcal{I}_N^-$ for some $N > 0$ in $\mathcal{N}$. If, instead, $\mathcal{I}_\mathcal{P} = \mathcal{I}_N^+$ for some $N < I$ then we take adjoints and apply the result to $\mathcal{I}_{N^\perp}^- \subsetneq \mathcal{P}^* \subseteq \mathcal{T}(\mathcal{N})^* = \mathcal{T}(\mathcal{N}^\perp)$. In this case, $\mathcal{P}$ is a *right* primitive ideal of $\mathcal{T}(\mathcal{N}^\perp)$ and so we shall take care that our proof accommodates the case where $\mathcal{P}$ is either left or right primitive.

If $\mathcal{P}$ is a left primitive ideal, let $\mathcal{J}$ be a maximal left ideal such that $\mathcal{P}$ is the kernel of the left regular module action of $\mathcal{T}(\mathcal{N})$ on $\mathcal{T}(\mathcal{N})/\mathcal{J}$. In the case where $\mathcal{P}$ is right primitive, let $\mathcal{J}$ be a maximal right ideal such that $\mathcal{P}$ is the kernel of the right regular module action of $\mathcal{T}(\mathcal{N})$ on $\mathcal{T}(\mathcal{N})/\mathcal{J}$.

Suppose that $N^- < N$. Note that $\operatorname{rank}(N - N^-)$ cannot be finite; if it were then $\mathcal{I}_\mathcal{P} = \mathcal{I}_N^-$ would be a maximal ideal of $\mathcal{T}(\mathcal{N})$ and so $\mathcal{P} = \mathcal{I}_\mathcal{P}$, contrary to hypothesis. If $\operatorname{rank}(N - N^-) = \infty$ then the only proper ideal strictly containing $\mathcal{I}_N^-$ is $\{X \in \mathcal{T}(\mathcal{N}) : (N - N^-)X(N - N^-) \text{ is compact}\}$, which must therefore equal $\mathcal{P}$. Any finite-rank projection of the form $P = (N - N^-)P(N - N^-)$ will serve to establish the result in this case.

For the remainder of the proof, assume that $N = N^-$ and take $X \in \mathcal{P} \setminus \mathcal{I}_N^-$. By Lemma 3.3, there are $A, B \in \mathcal{T}(\mathcal{N})$ such that AXB is block diagonal with respect to some sequence M_k of nest projections strictly increasing to N, and each of the blocks has norm greater than 1. Replacing X with AXB we can assume $X = \sum_{k=1}^\infty (M_k - M_{k-1})X(M_k - M_{k-1})$, where the norm of each term is greater than 1.

Consider the sequence of intervals $M_{2k+1} - M_{2k}$. These are each in $\mathcal{I}_N^-$ and so in $\mathcal{J}$. By Proposition 4.18 and Corollary 4.19, whether $\mathcal{J}$ is assumed to be maximal right or maximal left, there is a subsequence k_n such that $\mathcal{J}$ contains $\sum_{n=1}^\infty M_{2k_n+1} - M_{2k_n}$. Then, for each n, find an atom $N_n^+ - N_n \leq M_{2k_n+1} - M_{2k_n}$. Choose vectors e_n, f_n, g_n such that $e_n e_n^* \leq N_n^+ - N_n$ and f_n and g_n are in the range of $M_{2k_n+2} - M_{2k_n+1}$ with $\|f_n\| > \|g_n\| = 1$ and $f_n = Xg_n$. Thus,

$$V := \sum_{n=1}^\infty e_n e_{n+1}^* = \left(\sum_{n=1}^\infty \|f_n\|^{-1} e_n f_n^*\right) X \left(\sum_{n=1}^\infty \|f_n\|^{-1} g_n e_{n+1}^*\right),$$

where both of the sums converge strongly and are in $\mathcal{T}(\mathcal{N})$ because

$$e_n f_n^* = M_{2k_n+1}(e_n f_n^*) M_{2k_n+1}^\perp$$

and

$$g_n e_{n+1}^* = M_{2k_n+2}(g_n e_{n+1}^*) M_{2k_{n+1}}^\perp = M_{2k_n+2}(g_n e_{n+1}^*) M_{2k_n+2}^\perp,$$

since $k_{n+1} \geq k_n + 1$.

Thus, $V \in \mathcal{P}$. Let $P := \sum_{k=1}^\infty e_{2k} e_{2k}^* \leq \sum_{k=1}^\infty N_{2k}^+ - N_{2k}$, which is dominated by a projection in $\mathcal{J}$ and so is also in $\mathcal{J}$. We shall show that $P \in \mathcal{P}$.

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Suppose for a contradiction that $P \notin \mathcal{P}$. It follows, as observed in Remark 3.1, that there are $A, B \in \mathcal{T}(\mathcal{N})$ such that $I - APB \in \mathcal{J}$. We can assume that $A = AP$ and $B = PB$. Write $A = A_1 + A_2$, where

$$A_1 := \sum_{k=1}^{\infty} N_{2k-1}^{\perp} A(N_{2k}^{+} - N_{2k}) \quad \text{and} \quad A_2 := A - A_1,$$

so that $A_2(N_{2k}^{+} - N_{2k}) = N_{2k-1}A(N_{2k}^{+} - N_{2k})$. Likewise, write $B = B_1 + B_2$, where

$$B_1 := \sum_{k=1}^{\infty} (N_{2k}^{+} - N_{2k})BN_{2k+1} \quad \text{and} \quad B_2 := B - B_1,$$

so that $(N_{2k}^{+} - N_{2k})B_2 = (N_{2k}^{+} - N_{2k})BN_{2k+1}^{\perp}$. The sums for A_1 and B_1 converge strongly because the sequences of terms are norm-bounded and have pairwise orthogonal ranges and cokernels.

Now set $A_2' := A_2V^*$ and $B_2' := V^*B_2$. From the following computations we see that A_2' and B_2' are in $\mathcal{T}(\mathcal{N})$, since the terms of the sums are in $\mathcal{T}(\mathcal{N})$:

$$A_2' = A_2PV^* = \sum_{k=1}^{\infty} A_2(N_{2k}^{+} - N_{2k})V^* = \sum_{k=1}^{\infty} N_{2k-1}A(N_{2k}^{+} - N_{2k})V^*N_{2k-1}^{\perp}$$

$$B_2' = V^*PB_2 = \sum_{k=1}^{\infty} V^*(N_{2k}^{+} - N_{2k})B_2 = \sum_{k=1}^{\infty} N_{2k+1}^{+}V^*(N_{2k}^{+} - N_{2k})BN_{2k+1}^{\perp}.$$

Furthermore, since $VV^* = \sum_{k=1}^{\infty} e_k e_k^*$ and $V^*V = \sum_{k=1}^{\infty} e_{k+1}e_{k+1}^* = \sum_{k=2}^{\infty} e_k e_k^*$, we have that

$$A_2 = A_2P = A_2PV^*V = A_2'V \in \mathcal{P}$$

and

$$B_2 = PB_2 = VV^*PB_2 = VB_2' \in \mathcal{P}.$$

Since $I - (A_1 + A_2)P(B_1 + B_2) \in \mathcal{J}$, it now follows that also $I - A_1PB_1 \in \mathcal{J}$.

Now note that

$$\begin{aligned} A_1PB_1 &= \sum_{k=1}^{\infty} A_1(N_{2k}^{+} - N_{2k})B_1 \\ &= \sum_{k=1}^{\infty} N_{2k-1}^{\perp}A(N_{2k}^{+} - N_{2k})BN_{2k+1} \\ &= \sum_{k=1}^{\infty} (N_{2k}^{+} - N_{2k-1})A(N_{2k}^{+} - N_{2k})B(N_{2k+1} - N_{2k}) \\ &= \sum_{k=1}^{\infty} (N_{2k}^{+} - N_{2k-1})C_k(N_{2k+1} - N_{2k}), \end{aligned}$$

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where $C_k := A(N_{2k}^+ - N_{2k})B$. We can decompose A_1PB_1 in two ways, either as

$$\sum_{k=1}^{\infty}(N_{2k}^+ - N_{2k})C_k(N_{2k+1} - N_{2k}) + \sum_{k=1}^{\infty}(N_{2k} - N_{2k-1})C_k(N_{2k+1} - N_{2k})$$

or as

$$\sum_{k=1}^{\infty}(N_{2k}^+ - N_{2k-1})C_k(N_{2k}^+ - N_{2k}) + \sum_{k=1}^{\infty}(N_{2k}^+ - N_{2k-1})C_k(N_{2k+1} - N_{2k}^+).$$

These two cases are of the form $PY + Z$ and $YP + Z$, respectively, where in both cases Z is nilpotent. Recall that $P \in \mathcal{J}$ and so, whether $\mathcal{J}$ is a maximal left ideal or a maximal right ideal, we conclude that $I - Z \in \mathcal{J}$, which is impossible since this would be invertible and $\mathcal{J}$ is proper. From this contradiction we conclude that $P \in \mathcal{P}$. □

Theorem 5.3. *Let $\mathcal{N}$ be an atomic nest and let $\mathcal{P}$ be a primitive ideal of $\mathcal{T}(\mathcal{N})$.*

1. *If $\mathcal{P} \cap \mathcal{D}(\mathcal{N})$ is a maximal two-sided ideal of $\mathcal{D}(\mathcal{N})$ then $\mathcal{P}$ is a maximal two-sided ideal of $\mathcal{T}(\mathcal{N})$.*
2. *If $\mathcal{P} \cap \mathcal{D}(\mathcal{N})$ is equal to $\mathcal{I} \cap \mathcal{D}(\mathcal{N})$ for some diagonal ideal $\mathcal{I}$ then $\mathcal{P}$ is a diagonal ideal and, in fact, $\mathcal{P} = \mathcal{I}$.*

Proof. Case (1) is just Proposition 5.1. To prove Case (2), suppose that $\mathcal{P} \cap \mathcal{D}(\mathcal{N}) = \mathcal{I} \cap \mathcal{D}(\mathcal{N})$ for some diagonal ideal $\mathcal{I}$. First, observe that $\mathcal{I}_{\mathcal{P}} \cap \mathcal{D}(\mathcal{N}) \subseteq \mathcal{P} \cap \mathcal{D}(\mathcal{N}) = \mathcal{I} \cap \mathcal{D}(\mathcal{N})$. Now, distinct diagonal ideals contain complementary projections (see the proof of [**23**, Lemma 4.8] for this fact) and so $\mathcal{I}$ must equal $\mathcal{I}_{\mathcal{P}}$. But now, if $\mathcal{P} \neq \mathcal{I}_{\mathcal{P}}$ then by Proposition 5.2, $\mathcal{P}$ contains projections which are not in $\mathcal{I}_{\mathcal{P}}$, contrary to hypothesis. □

We can now distinguish three classes of primitive ideals based on the diagonal operators they contain. The first class ($\Pi_{\max}$) consists of primitive ideals for which $\mathcal{P} \cap \mathcal{D}(\mathcal{N})$ is a maximal ideal of $\mathcal{D}(\mathcal{N})$, and this consists of the maximal two-sided ideals of $\mathcal{T}(\mathcal{N})$. The second class ($\Pi_{\min}$) consists of primitive ideals for which $\mathcal{P} \cap \mathcal{D}(\mathcal{N}) = \mathcal{I} \cap \mathcal{D}(\mathcal{N})$ for some diagonal ideal $\mathcal{I}$, and this class consists of diagonal ideals. The third class (Π_{int}) consists of the remaining primitive ideals for which $\mathcal{P} \cap \mathcal{D}(\mathcal{N})$ takes neither its minimal nor its maximal values.

The maximal ideals of a general nest algebra were completely described in [**20**, Corollary 3.10]. In particular, when $\mathcal{N}$ is atomic the ideals in $\Pi_{\max}$ are precisely the ideals of the form $\mathcal{D}_0 \oplus \mathcal{R}_{\mathcal{N}}^{\infty}$, where $\mathcal{D}_0$ is a maximal two-sided ideal of $\mathcal{D}(\mathcal{N})$. The ideals in $\Pi_{\min}$ are the primitive ideals which are also diagonal ideals. Trivially, all ideals of the form $\mathcal{I}_N^-$ where $N > N^-$ (or, equivalently, I_N^+ where $N < N^+$) are included in this class. (See the first paragraph of the proof of Theorem 3.7 for details.) By Theorem 3.7, if we assume the continuum hypothesis then $\Pi_{\min}$ consists of *all* the diagonal ideals. Without the assumption of the continuum hypothesis we cannot say which diagonal ideals belong to $\Pi_{\min}$. The structure of Π_{int} is more delicate. In the following section we will see examples of representatives of all three classes.

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6. The infinite upper triangular operators

Throughout this section, let $\mathcal{H} = \ell^2(\mathbb{N})$ and consider the algebra $\mathcal{T}(\mathbb{N})$ of all upper triangular operators with respect to the standard basis of $\ell^2(\mathbb{N})$. Recall that we write $\{e_i\}_{i=1}^{\infty}$ for the standard basis and let N_n be the projection onto the span of $\{e_1, \dots, e_n\}$, and $\mathcal{N} := \{N_n : n \in \mathbb{N}\} \cup \{0, I\}$. Then $\mathcal{T}(\mathbb{N}) := \mathcal{T}(\mathcal{N})$ is the algebra of infinite upper triangular operators with respect to the e_i, and $\mathcal{R}_{\mathcal{N}}^{\infty}$ is simply the ideal of infinite *strictly* upper triangular operators. Moreover, the diagonal ideals of $\mathcal{T}(\mathbb{N})$ are precisely the ideals $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \dots; \mathcal{I}_\infty$, where $\mathcal{I}_n := \mathcal{I}_{N_n}^{-}$, for $1 \le n < \infty$ and $\mathcal{I}_\infty := \mathcal{I}_I^{-}$. Note that $\mathcal{I}_\infty$ coincides with the compact operators of $\mathcal{T}(\mathbb{N})$, a fact which we develop below.

6.1. The quasitriangular algebra

Let $\mathcal{K}(\mathcal{H})$ be the set of all compact operators in $B(\mathcal{H})$ and write $\mathcal{QT}(\mathbb{N})$ for the quasitriangular algebra $\mathcal{T}(\mathbb{N}) + \mathcal{K}(\mathcal{H})$. By [**10**] and, in more generality, [**7**], $\mathcal{QT}(\mathbb{N})$ is a norm-closed algebra in $B(\mathcal{H})$ and the canonical isomorphism between $\mathcal{QT}(\mathbb{N})/\mathcal{K}(\mathcal{H})$ and $\mathcal{T}(\mathbb{N})/(\mathcal{T}(\mathbb{N}) \cap \mathcal{K}(\mathcal{H}))$ is isometric.

Corollary 6.1. *Assuming the continuum hypothesis,* $\mathcal{T}(\mathbb{N})/(\mathcal{T}(\mathbb{N}) \cap \mathcal{K}(\mathcal{H}))$ *is a left (respectively, right) primitive algebra.*

Proof. $\mathcal{K}(\mathcal{H}) \cap \mathcal{T}(\mathbb{N}) = \mathcal{I}_\infty$, which is a left primitive ideal by Theorem 3.7 and a right primitive ideal by Corollary 3.8. □

Corollary 6.2. *Assuming the continuum hypothesis,* $\mathcal{QT}(\mathbb{N})/\mathcal{K}(\mathcal{H})$ *is a left (respectively, right) primitive algebra, and* $\mathcal{K}(\mathcal{H})$ *is a left (respectively, right) primitive ideal in* $\mathcal{QT}(\mathbb{N})$.

6.2. A catalogue of primitive ideals

Clearly, $\Pi_{\min}$ contains $\{\mathcal{I}_1, \mathcal{I}_2, \dots\}$. Assuming the continuum hypothesis, by Theorem 3.7,

$$\Pi_{\min} = \{\mathcal{I}_1, \mathcal{I}_2, \dots\} \cup \{\mathcal{I}_\infty\}.$$

By [**20**, Corollary 3.10], the ideals of $\Pi_{\max}$ are precisely the ideals of the form $\mathcal{D}_0 \oplus \mathcal{R}_{\mathcal{N}}^{\infty}$, where $\mathcal{D}_0$ is a maximal ideal of $\mathcal{D}(\mathcal{N})$. In this case, $\mathcal{D}(\mathcal{N})$ is naturally identified with $\ell^\infty(\mathbb{N})$ and its maximal ideal space, with the sequences vanishing at points of $C(\beta\mathbb{N})$. The maximal ideals of $\mathcal{T}(\mathbb{N})$ corresponding to points of $\mathbb{N}$ are precisely the $\mathcal{I}_n$, and so we can write

$$\Pi_{\max} = \{\mathcal{I}_1, \mathcal{I}_2, \dots\} \cup \{\mathcal{D}_x \oplus \mathcal{R}_{\mathcal{N}}^{\infty} : x \in \beta\mathbb{N} \setminus \mathbb{N}\},$$

where $\mathcal{D}_x$ is the maximal ideal of $\mathcal{D}(\mathcal{N})$ corresponding to sequences in $\ell^\infty(\mathbb{N})$ vanishing at $x \in \beta\mathbb{N}$.

There remains the set Π_{int} of primitive ideals which are neither diagonal ideals nor maximal ideals. These are the primitive ideals $\mathcal{P}$ where $\mathcal{P} \cap \mathcal{D}(\mathcal{N})$ is a closed ideal of $\mathcal{D}(\mathcal{N})$ corresponding to an ideal of $\ell^\infty(\mathbb{N})$ which strictly contains $c_0(\mathbb{N})$ and is not maximal. We cannot give a complete catalogue of these ideals, but we can provide a rich set of examples.

Consider the following special case of a general construction of epimorphisms between nest algebras, taken from [**3**, Corollary 5.3]. Let $0 \le m_k < n_k < +\infty$ be integers such that

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the intervals $(m_k, n_k]$ are pairwise disjoint, and let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. Suppose that $\lim_{k\in\mathcal{U}} n_k - m_k = +\infty$. Let $U_k : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be the partial isometry mapping e_i to e_{i-m_k} when $m_k < i \le n_k$ and zero otherwise. For $X \in \mathcal{T}(\mathbb{N})$, define

$$\phi(X) := \lim_{k\in\mathcal{U}} U_k X U_k^*,$$

where convergence is in the weak operator topology and the limit always exists by weak operator topology compactness of the unit ball. Then, by [3, Corollary 5.3], this map is an epimorphism of $\mathcal{T}(\mathbb{N})$ onto $\mathcal{T}(\mathbb{N})$. Note also that ϕ is a *-homomorphism of the diagonal of $\mathcal{T}(\mathbb{N})$ onto itself.

If ϕ is such an epimorphism of $\mathcal{T}(\mathbb{N})$ onto $\mathcal{T}(\mathbb{N})$ and π is an irreducible representation of $\mathcal{T}(\mathbb{N})$ then clearly $\pi \circ \phi$ is also an irreducible representation of $\mathcal{T}(\mathbb{N})$. If $\ker\pi$ is in $\Pi_{\max}$ then so is $\ker\pi\circ\phi$. However, as we shall see, if $\ker\pi \in \Pi_{\min}\setminus\Pi_{\max}$ then $\ker\pi\circ\phi$ will be in Π_{int}, and this provides a rich supply of examples of primitive ideals in Π_{int}.

Assuming the continuum hypothesis, $\mathcal{I}_\infty \in \Pi_{\min}\setminus\Pi_{\max}$, so consider the primitive ideal $\mathcal{P} = \phi^{-1}(\mathcal{I}_\infty)$. Note that ϕ annihilates $\mathcal{I}_\infty$ and so $\mathcal{I}_\infty$ is the unique diagonal ideal in $\mathcal{P}$. Writing $\Delta(X)$ for the diagonal expectation $\sum_{k=1}^{\infty}(N_{n_k} - N_{m_k})X(N_{n_k} - N_{m_k})$, observe that $\ker\Delta \subseteq \ker\phi \subseteq \mathcal{P}$ and so $\mathcal{P} \ne \mathcal{I}_\infty$. Thus, $\mathcal{P} \notin \Pi_{\min}$. On the other hand, $\mathcal{P} \notin \Pi_{\max}$ since, by [20, Theorem 3.8], every maximal ideal of $\mathcal{T}(\mathbb{N})$ contains $\mathcal{R}_{\mathcal{N}}^{\infty}$, but $\mathcal{P}$ does not contain the unilateral backward shift U since $\phi(U) = U \notin \mathcal{I}_\infty$. Thus, $\mathcal{P} \notin \Pi_{\min}$ and $\mathcal{P} \notin \Pi_{\max}$, and so $\mathcal{P} \in \Pi_{\mathrm{int}}$.

In fact, this construction readily yields uncountably many incomparable ideals in Π_{int}. Fix projections $P_k := N_{n_k} - N_{m_k}$, where $\lim_{k\to+\infty} n_k - m_k = +\infty$, and let $\mathcal{U}$ be a fixed free ultrafilter. As is well known, we can find an uncountable collection Σ of infinite subsets of $\mathbb{N}$ with the property that distinct members of Σ intersect only in finite sets. For $\sigma \in \Sigma$, list the elements of σ in order as s_k and build an ultrafilter epimorphism $\phi_\sigma : \mathcal{T}(\mathbb{N}) \to \mathcal{T}(\mathbb{N})$ as above, this time employing the intervals P_{s_k} and the ultrafilter $\mathcal{U}$. Write $\Delta_\sigma(X)$ for the diagonal expectation $\sum_{k\in\sigma} P_k X P_k$. As before, $\ker\Delta_\sigma \subseteq \ker\phi_\sigma$. Now, for any $\sigma \ne \sigma'$, $\phi_\sigma^{-1}(\mathcal{I}_\infty) \ne \phi_{\sigma'}^{-1}(\mathcal{I}_\infty)$, as otherwise

$$\phi_\sigma^{-1}(\mathcal{I}_\infty) = \phi_{\sigma'}^{-1}(\mathcal{I}_\infty) \supseteq \ker\Delta_\sigma + \ker\Delta_{\sigma'} + \mathcal{I}_\infty = \mathcal{T}(\mathbb{N}).$$

We can also exhibit infinite chains of ideals in Π_{int}; since $\phi^{-1}(\mathcal{I}_\infty) \supsetneq \mathcal{I}_\infty$, the ideals $\mathcal{P}_k := \phi^{-1}(\phi^{-1}(\cdots\phi^{-1}(\mathcal{I}_\infty)\cdots))$ form a chain of distinct ideals in Π_{int} for any fixed epimorphism $\phi : \mathcal{T}(\mathcal{N}) \to \mathcal{T}(\mathcal{N})$.

6.3. Some properties of ideals in Π_{int}

Although the ultrafilter epimorphism construction of ideals in Π_{int} is not representative, we can prove some properties which all ideals in Π_{int} share with the ultrafilter construction. These results are, however, tightly bound to the case of $\mathcal{T}(\mathbb{N})$ (especially Proposition 6.3) and it is unclear how they might be extended.

Proposition 6.3. *Let $\mathcal{P}$ be a primitive ideal of $\mathcal{T}(\mathbb{N})$ and suppose $\mathcal{P} \supsetneq \mathcal{I}_\infty$. Then there is an increasing sequence of integers n_k such that $\mathcal{P}$ contains*

$$\{X \in \mathcal{T}(\mathcal{N}) : (N_{n_k} - N_{n_{k-1}})X(N_{n_k} - N_{n_{k-1}}) = 0 \ \ \textit{for all } k\}.$$

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Proof. Let $\mathcal{L}$ be a maximal left ideal such that $\mathcal{P}$ is the kernel of the left-regular representation on $\mathcal{T}(\mathcal{N})/\mathcal{L}$. By Proposition 5.2, $\mathcal{P}$ contains a projection $P \notin \mathcal{I}_I^-$. Choose a subsequence of nest projections N_{n_k} such that

$$\operatorname{rank}(N_{n_{k+1}} - N_{n_k})P \geq \operatorname{rank} N_{n_k}$$

for all k. We shall show that if

$$\mathcal{S} := \{X \in \mathcal{T}(\mathcal{N}) : (N_{n_{2k+2}} - N_{n_{2k}})X(N_{n_{2k+2}} - N_{n_{2k}}) = 0 \text{ for all } k\}$$

then $\mathcal{S} \subseteq \mathcal{P}$. By Remark 3.1, since $\mathcal{S}$ is a two-sided ideal of $\mathcal{T}(\mathcal{N})$, if $\mathcal{S} \subseteq \mathcal{L}$ then $\mathcal{S} \subseteq \mathcal{P}$, so suppose for a contradiction that $\mathcal{S} \nsubseteq \mathcal{L}$. By maximality of $\mathcal{L}$, $\mathcal{S} + \mathcal{L} = \mathcal{T}(\mathcal{N})$, and so there is an $X \in \mathcal{S}$ such that $I - X \in \mathcal{L}$. Decompose X as $Y_0 + Y_1$ where

$$Y_0 := \sum_{k=1}^{\infty}(N_{n_{2k+1}} - N_{n_{2k-1}})X(N_{n_{2k+1}} - N_{n_{2k-1}})$$

and $Y_1 := X - Y_0$. Observe that therefore

$$(N_{n_{k+2}} - N_{n_k})Y_1(N_{n_{k+2}} - N_{n_k}) = 0 \tag{6.1}$$

for all k.

Now take fixed arbitrary $M < N < I$ in $\mathcal{N}$ and consider two cases. First, if $N - M$ does not dominate any $N_{n_{k+1}} - N_{n_k}$ then there must be a k such that $N - M \leq N_{n_{k+2}} - N_{n_k}$, and so $(N - M)Y_1(N - M) = 0$. On the other hand, if $N - M$ does dominate some $N_{n_{k+1}} - N_{n_k}$, take k to be the largest possible (which exists since $N < I$) and observe that, by (6.1),

$$\begin{aligned}\operatorname{rank}(N - M)Y_1(N - M) &= \operatorname{rank} N_{n_k}(N - M)Y_1(N - M)\\ &\leq \operatorname{rank} N_{n_k}\\ &\leq \operatorname{rank}(N_{n_{k+1}} - N_{n_k})P\\ &\leq \operatorname{rank}(N - M)P.\end{aligned}$$

It follows that in either case

$$\operatorname{rank}(N - M)Y_1(N - M) \leq \operatorname{rank}(N - M)P.$$

Since the right-hand side is infinite if $N = I$, the inequality is valid for all $M < N$ in $\mathcal{N}$. It follows immediately from [**21**, Theorem 2.6] that Y_1 factors through P as $Y_1 = APB$ for some $A, B \in \mathcal{T}(\mathcal{N})$, and so $Y_1 \in \mathcal{P} \subseteq \mathcal{L}$, whence $I - Y_0 \in \mathcal{L}$.

However, since $X \in \mathcal{S}$, the terms of the sum for Y_0 are

$$\begin{aligned}&(N_{n_{2k+1}} - N_{n_{2k-1}})X(N_{n_{2k+1}} - N_{n_{2k-1}})\\ &\quad = (N_{n_{2k}} - N_{n_{2k-1}})X(N_{n_{2k+1}} - N_{n_{2k}})\end{aligned}$$

so that Y_0 is nilpotent of order 2. Thus, $I - Y_0$ cannot belong to the proper left ideal $\mathcal{L}$, which is a contradiction. □

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Let E_i $(i \in \mathbb{N})$ be a set of pairwise orthogonal intervals of $\mathcal{N}$. For $\sigma \subseteq \mathbb{N}$, let $P_\sigma := \sum_{i\in\sigma} E_i$ and $\Delta_\sigma(X) := \sum_{i\in\sigma} E_i X E_i$. For convenience, write Δ for $\Delta_{\mathbb{N}}$. The last result shows that, at least in $\mathcal{T}(\mathbb{N})$, primitive ideals which are not in $\Pi_{\min}$ must contain $\ker\Delta$ for suitable $\{E_i\}$. The next two lemmas explore the consequences of a primitive ideal containing $\ker\Delta$ and hold for general nest algebras.

Lemma 6.4. *Let $\mathcal{P}$ be a primitive ideal of $\mathcal{T}(\mathcal{N})$ and suppose $\ker\Delta \subseteq \mathcal{P}$. Then $\Sigma := \{\sigma \subseteq \mathbb{N} : \ker\Delta_\sigma \subseteq \mathcal{P}\}$ is an ultrafilter.*

Proof. Σ itself is non-empty since $\mathbb{N} \in \Sigma$, and the sets in Σ are non-empty since $\ker\Delta_\emptyset = \mathcal{T}(\mathcal{N})$. If $\tau \supseteq \sigma$ and $\sigma \in \Sigma$ then $\ker\Delta_\tau \subseteq \ker\Delta_\sigma \subseteq \mathcal{P}$, and so $\tau \in \Sigma$. If $\sigma, \tau \in \Sigma$ then $\ker\Delta_{\sigma\cap\tau} = \ker\Delta_\sigma + \ker\Delta_\tau \subseteq \mathcal{P}$, and so $\sigma \cap \tau \in \Sigma$. Thus, Σ is a filter.

Let $\pi : \mathcal{T}(\mathcal{N}) \to \mathcal{L}(V)$ be an irreducible representation with $\mathcal{P} = \ker\pi$. For any $\sigma \subseteq \mathbb{N}$ and $X \in \mathcal{T}(\mathcal{N})$, $P_\sigma X - X P_\sigma \in \ker\Delta$ and so $\pi(P_\sigma)$ commutes with $\pi(\mathcal{T}(\mathcal{N}))$. Thus, $\operatorname{ran}(\pi(P_\sigma))$ is an invariant subspace of $\pi(\mathcal{T}(\mathcal{N}))$, and so $\pi(P_\sigma) = 0, I$. Suppose that $\pi(P_\sigma) = I$ and so $\pi(P_{\sigma^c}) = 0$. Then, for any $X \in \mathcal{T}(\mathcal{N})$,

$$X - \Delta_\sigma(X) - \Delta_{\sigma^c}(X) \in \ker\Delta \subseteq \mathcal{P},$$

and so

$$\pi(X) = \pi(\Delta_\sigma(X) + \Delta_{\sigma^c}(X)) = \pi(\Delta_\sigma(X)P_\sigma + \Delta_{\sigma^c}(X)P_{\sigma^c}) = \pi(\Delta_\sigma(X)),$$

whence $\ker\Delta_\sigma \subseteq \mathcal{P}$ and $\sigma \in \Sigma$. Likewise, if $\pi(P_\sigma) = 0$, then $\sigma^c \in \Sigma$. Thus, Σ is an ultrafilter. □

Lemma 6.5. *Let $\mathcal{P}$ be a primitive ideal of $\mathcal{T}(\mathcal{N})$ and suppose $\ker\Delta \subseteq \mathcal{P}$. Suppose that for each i we can decompose E_i as the sum $E_i^0 + E_i^1$ of intervals of $\mathcal{N}$. Then $\mathcal{P}$ contains one of $\ker\Delta^j$, where $\Delta^j(X) = \sum_{i=1}^\infty E_i^j X E_i^j$.*

Proof. Each E_i is decomposed into the sum of two intervals which share a common endpoint. Let σ be the set of i for which the shared endpoint is the upper endpoint of E_i^0 and the lower endpoint of E_i^1. Clearly, σ^c is then the set of i for which the upper endpoint of E_i^1 equals the lower endpoint of E_i^0. By Lemma 6.4, $\mathcal{P}$ contains one of $\ker\Delta_\sigma, \ker\Delta_{\sigma^c}$. Without loss of generality, assume $\ker\Delta_\sigma \subseteq \mathcal{P}$. Let $P := \sum_{i\in\sigma} E_i^0$ and observe that for each $i \in \sigma$ there is an $N_i \in \mathcal{N}$ such that $E_i^0 = N_i E_i$ and $E_i^1 = N_i^\perp E_i$, and thus

$$\Delta_\sigma(P^\perp X P) = \sum_{i\in\sigma} E_i^1 X E_i^0 = \sum_{i\in\sigma} E_i N_i^\perp X N_i E_i = 0.$$

If π is an irreducible representation with $\ker\pi = \mathcal{P}$ then $\pi(P^\perp X P) = 0$, and so the range of $\pi(P)$ is an invariant subspace of $\pi(\mathcal{T}(\mathcal{N}))$, whence one of $P, P^\perp \in \mathcal{P}$. If $P \in \mathcal{P}$,

$$\ker\Delta^1 \subseteq \ker\Delta_\sigma + P\mathcal{T}(\mathcal{N}) \subseteq \mathcal{P},$$

while if $P^\perp \in \mathcal{P}$,

$$\ker\Delta^0 \subseteq \ker\Delta_\sigma + \mathcal{T}(\mathcal{N})P^\perp \subseteq \mathcal{P}.$$ □

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Theorem 6.6. *Let $\mathcal{P} \in \Pi_{int}$ in $\mathcal{T}(\mathbb{N})$. Then there is a free ultrafilter $\mathcal{U}$ and a sequence of pairwise orthogonal finite-rank intervals E_i such that $\lim_{i\in\mathcal{U}} \operatorname{rank} E_i = +\infty$ and $\mathcal{P}$ contains*

$$\{X \in \mathcal{T}(\mathbb{N}) : \lim_{i\in\mathcal{U}} \|E_i X E_i\| = 0\}.$$

Moreover, given any decomposition of the E_i as the sums of intervals $E_i^0 + E_i^1$, we can replace $\{E_i\}$ with one of $\{E_i^0\}$ or $\{E_i^1\}$.

Proof. The existence of the intervals follows from Proposition 6.3. Let $\mathcal{U}$ be the ultrafilter obtained in Lemma 6.4. If $\lim_{i\in\mathcal{U}} \|E_i X E_i\| = 0$ then, given $\epsilon > 0$, there is a $\sigma \in \mathcal{U}$ such that $\|E_i X E_i\| < \epsilon$ for all $i \in \sigma$. Thus, taking $X' := X - \Delta_\sigma(X)$, we see that $\|X - X'\| = \|\Delta_\sigma(X)\| \leq \epsilon$ and that $\Delta_\sigma(X') = 0$, whence $X' \in \mathcal{P}$. Thus, X is a limit point of $\mathcal{P}$ and, since $\mathcal{P}$ is norm closed, $X \in \mathcal{P}$.

Given a decomposition $E_i = E_i^0 + E_i^1$, we know from Proposition 6.5 that one of $\ker \Delta^j$ $(j = 0, 1)$ is in $\mathcal{P}$. Without loss, suppose $\ker \Delta^0 \subseteq \mathcal{P}$. Again, by Lemma 6.4, $\mathcal{U}^0 := \{\sigma : \ker \Delta_\sigma^0 \subseteq \mathcal{P}\}$ is an ultrafilter. Now let $\sigma \in \mathcal{U}^0$. Since $\mathcal{U}$ is an ultrafilter, one of $\sigma, \sigma^c \in \mathcal{U}$. But if $\sigma^c \in \mathcal{U}$ then

$$\mathcal{T}(\mathbb{N}) = \ker \Delta_\sigma^0 + \ker \Delta_{\sigma^c} \subseteq \mathcal{P},$$

which is impossible. Thus, $\sigma \in \mathcal{U}$ and so, since σ was arbitrary, $\mathcal{U}^0 \subseteq \mathcal{U}$. But $\mathcal{U}_0$ is also an ultrafilter, so in fact $\mathcal{U}_0 = \mathcal{U}$. Thus, we may replace $\{E_i\}$ with $\{E_i^0\}$.

Now it follows that the limit of the ranks of the intervals must be $+\infty$; otherwise, after finitely many decompositions, we could conclude that $\mathcal{P} \supseteq \mathcal{R}_{\mathcal{N}}^\infty$ and so $\mathcal{P} \in \Pi_{\max}$. Similarly, if $\mathcal{U}$ were not free then $\mathcal{P}$ would contain $\{X : E_{i_0} X E_{i_0} = 0\}$ for some $i_0 \in \mathbb{N}$ and, after finitely many decompositions if necessary, we would see that $\mathcal{P} \supseteq \mathcal{I}_n$ for some n, again contrary to hypothesis. □

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