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Entropies and volume growth of unstable manifolds

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Abstract. Let f be a [-10.5pc]C^2 diffeomorphism on a compact manifold. Ledrappier and Young introduced entropies along unstable foliations for an ergodic measure μ. We relate those entropies to covering numbers in order to give a new upper bound on the metric entropy of μ in terms of Lyapunov exponents and topological entropy or volume growth of sub-manifolds. We also discuss extensions to the $C^{1+\alpha}$, $\alpha > 0$, case.

Key words: entropy, Lyapunov exponents, volume growth
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1. *Introduction*

Entropy is a fundamental invariant in dynamics. It can be defined in the topological, ergodic or differentiable categories and quantifies the dynamical complexity. The classical result on the connection between entropy and Lyapunov exponents is the Margulis–Ruelle inequality [17]. It states that for a C^1 map f on a compact manifold M and an ergodic invariant Borel probability measure μ,

$$h(f, \mu) \leq \sum_{i=1}^{u} \lambda_i(f, \mu) \cdot \dim E^i,$$

where $\lambda_i(f, \mu)$, $1 \leq i \leq u$, are the positive Lyapunov exponents and E^i, $1 \leq i \leq u$, are the corresponding Oseledec's vector bundles. As perhaps first observed by Katok, this inequality implies that measures with positive entropy of surface diffeomorphisms are hyperbolic, i.e., without zero Lyapunov exponents. Katok [6] was then able to analyze such dynamics using Pesin theory in the $C^{1+\alpha}$ setting.

Also using Pesin theory, Newhouse [12] proved another bound for the entropy of an ergodic measure by the volume growth of sub-manifolds which are transverse to its stable manifolds. In the C^1 setting with dominated splitting, without using Pesin theory, Saghin [18] and Guo *et al* [5] bounded above the metric entropy by a mixture between the positive Lyapunov exponents and the volume growth of some sub-manifold. By using Ledrappier and Young's result [11], Cogswell [3] proved that the volume growth of local unstable manifolds is larger than the metric entropy. Cogswell's proof assumes C^2 smoothness since this is required in Ledrappier and Young's work. On the topological side, for $C^{1+\alpha}$ diffeomorphisms, Przytycki [14] proved that the topological entropy is bounded above by the growth rate of some differential forms. Later Kozlovski [8] showed that it is an equality if the system is C^∞.

In this paper, we generalize Cogswell's idea from [3] to establish a more general upper bound for C^2 systems without assuming dominated splitting. We bound the entropy of a measure by a combination of Lyapunov exponents (as in Ruelle's inequality) and various growths of unstable manifolds such as volume growth (as in Newhouse's inequality [12]). In a forthcoming work, we will use this new bound to extend the previously mentioned Katok's hyperbolicity argument beyond dimension two.

Our proof is a combination of Ledrappier and Young's entropy formula [11] and Pesin theory. We also discuss some extensions for hyperbolic measures in the $C^{1+\alpha}$ case.

MAIN THEOREM. *Let f be a C^2 diffeomorphism on a compact manifold M and let μ be any ergodic, invariant probability measure. Consider its positive Lyapunov exponents $\lambda_1 > \cdots > \lambda_u$ and the corresponding ith local unstable manifolds $W^i_{\mathrm{loc}}(x)$ for almost every $x \in M$ and $i = 1, \ldots, u$.*

Then the entropy $h(f, \mu)$ is bounded, for any index $1 \le i \le u$, by the sum of the almost everywhere volume growth of $W^i_{\mathrm{loc}}(x)$ and the transverse Lyapunov exponents $\lambda_{i+1}, \ldots, \lambda_u$, repeated according to multiplicity.

In this inequality, the volume growth can be replaced by fibered entropy or topological entropy of $W^i_{\mathrm{loc}}(x)$.

We give complete and precise statements in the next section after introducing the required notions. See in particular Theorem B.

1.1. *Definitions.* Let f be a $C^{1+\alpha}$ ($\alpha > 0$) diffeomorphism on a compact manifold M, that is, f is differentiable and its differential is Hölder-continuous with some positive exponent α. Let μ be an ergodic probability measure. Oseledec's theorem [13] states that there are an invariant measurable subset R_μ with full measure, an invariant measurable decomposition $T_{R_\mu}M = E^1 \oplus E^2 \oplus \cdots \oplus E^l$ and finitely many numbers $\lambda_1 > \lambda_2 > \cdots > \lambda_l$ such that for any $x \in R_\mu$ and any non-zero vector $v \in E^j_x$, we have

$$\lim_{n\to\pm\infty} \frac{1}{n} \log\|Df^n_x(v)\| = \lambda_j.$$

We list the positive Lyapunov exponents as $\lambda_1 > \lambda_2 > \cdots > \lambda_u$. By Pesin theory, for $1 \le i \le u$ and for any $x \in R_\mu$, the i*th global unstable manifold*

$$W^i(x) \triangleq \left\{ y \in M \,\middle|\, \limsup_{n\to+\infty} \frac{1}{n} \log d(f^{-n}(x), f^{-n}(y)) \le -\lambda_i \right\}$$

is a $C^{1+\alpha}$ immersed sub-manifold.

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We define the *ith local unstable manifold*,

$$W^i_\rho(x) \triangleq \text{ connected part of } W^i(x) \cap B(x, \rho) \text{ containing } x,$$

where $B(x, \rho)$ is the ball centered at x with radius ρ. At each $x \in R_\mu$, we fix a positive number $r(x)$ such that $W^i_{r(x)}(x)$ is an embedded sub-manifold. We remark that by Pesin theory (see [4, Theorem 16, p. 195]), the function $r : R_\mu \to (0, +\infty)$ can be chosen in a measurable way.

Definition 1.1. Let f be a $C^{1+\alpha}$ diffeomorphism on a compact manifold M and let μ be an invariant measure. For $1 \le i \le u$, we say that a measurable partition ξ^i is *subordinate to* W^i if for μ-almost every x:

- $\xi^i(x) \subset W^i(x)$;
- $\xi^i(x)$ contains an open neighborhood of x with respect to the intrinsic topology on $W^i(x)$.

Remark 1.2.

- We refer to [15] for background on measurable partitions and associated systems of conditional measures.
- Lemma 9.1.1 in [11] shows the existence of increasing subordinate measurable partitions. Here a partition η is called *increasing* if $\eta(x) \subset f(\eta(f^{-1}(x)))$ for μ-almost every x.

From now on, we fix a family of measurable partitions $\{\xi^i\}_{1\le i\le u}$ subordinate to $\{W^i\}_{1\le i\le u}$. For $1 \le i \le u$, let $\{\mu^i_x\}$ be the family of conditional measures with respect to the measurable partition ξ^i. Ledrappier and Young [11] have defined the *entropy along the ith unstable foliation* $h_i(f, \mu)$ (for more detail, see Proposition 2.6) by a fibered version of Brin and Katok's formula, namely:

$$h_i(f, \mu) \triangleq \lim_{\tau\to 0} \liminf_{n\to+\infty} -\frac{1}{n} \log \mu^i_x(V^i(x, n, \tau)),$$

where

$$V^i(x, n, \tau) \triangleq \{y \in W^i_{r(x)}(x) |\ d(f^j(x), f^j(y)) \le \tau,\ 0 \le j \le n-1\}.$$

We remark that here in the definition of the dynamical ball $V^i(x, n, \tau)$, we use the global metric d on M, unlike the definition in [11] that uses the intrinsic metric on the sub-manifold $W^i(x)$. But since we only consider the case when $\tau \to 0$, our definition of $h_i(f, \mu)$ coincides with theirs.

The volume of a sub-manifold $\gamma \subset M$ of constant dimension is denoted by $\mathrm{Vol}(\gamma)$. The lower volume growth of such a sub-manifold $\gamma \subset M$ with $\mathrm{Vol}(\gamma) < \infty$ is

$$\underline{v}(f, \gamma) \triangleq \liminf_{n\to\infty} \frac{1}{n} \log^+ \mathrm{Vol}(f^n(\gamma)),$$

where $\log^+ a = \max\{0, \log a\}$.

We now introduce the key concepts of our results. They are well defined by Lemma 2.1 in §2.

Definition 1.3. Given $1 \le i \le u$, the *μ-almost everywhere lower volume growth rate of W^i* is the μ-almost everywhere value of

$$\underline{\mathrm{v}}_i(f, \mu) \triangleq \inf_{\rho} \underline{\mathrm{v}}(f, W^i_\rho(x)).$$

Let $E(n, \varepsilon, \gamma)$ denote a maximal (n, ε) separated subset of a C^1 sub-manifold γ. For the definitions of separated subset and some other basic concepts in ergodic theory, see the book [19].

Definition 1.4. Given $1 \le i \le u$, the *μ-almost everywhere lower topological entropy of W^i* is the μ-almost everywhere value of

$$\underline{h}^i_{\mathrm{top}}(f, \mu) \triangleq \inf_{\rho} \lim_{\varepsilon \to 0} \liminf_{n \to +\infty} \frac{1}{n} \log \# E(n, \varepsilon, W^i_\rho(x)).$$

Remark 1.5. Recall that the topological entropy of $W^i_\rho(x)$ is

$$h_{\mathrm{top}}(f, W^i_\rho(x)) \triangleq \lim_{\varepsilon \to 0} \limsup_{n \to +\infty} \frac{1}{n} \log \# E(n, \varepsilon, W^i_\rho(x)).$$

Hence we have $\underline{h}^i_{\mathrm{top}}(f, \mu) \le \inf_\rho h_{\mathrm{top}}(f, W^i_\rho(x))$ for any $x \in R_\mu$. Note that $\inf_\rho h_{\mathrm{top}}(f, W^i_\rho(x))$ is also μ-almost everywhere constant.

For $1 \le i \le u,\ x \in R_\mu$ and $\lambda > 0$, define

$$N_\lambda(\mu^i_x, n, \varepsilon) \triangleq \min \left\{ \# C \subset R_\mu : \mu^i_x \left(\bigcup_{y \in C} V^i(y, n, \varepsilon) \right) \ge \lambda \right\}.$$

Definition 1.6. Given $1 \le i \le u$, the *upper fibered Katok entropy of W^i* is the μ-almost everywhere value of

$$\overline{h}^K_i(f, \mu) \triangleq \inf_{\lambda} \lim_{\varepsilon \to 0} \limsup_{n \to +\infty} \frac{1}{n} \log N_\lambda(\mu^i_x, n, \varepsilon).$$

Similarly, the *lower fibered Katok entropy of W^i* is the μ-almost everywhere value of

$$\underline{h}^K_i(f, \mu) \triangleq \inf_{\lambda} \lim_{\varepsilon \to 0} \liminf_{n \to +\infty} \frac{1}{n} \log N_\lambda(\mu^i_x, n, \varepsilon).$$

Remark 1.7. The above definition is analogous to the formula of Katok in [6], expressing the metric entropy as the growth rate of the cardinality of maximal separated sets.

1.2. *Main results.*

From now on, when we mention a $C^{1+\alpha}$ diffeomorphism, we always assume that $\alpha > 0$.

THEOREM A. *Let f be a $C^{1+\alpha}$ diffeomorphism on a compact manifold M. Let μ be an ergodic measure. List the positive Lyapunov exponents of μ as $\lambda_1 > \lambda_2 > \cdots > \lambda_u > 0$. Then, for $1 \le i \le u$, the entropy along the ith unstable foliation satisfies:*

(1) $h_i(f, \mu) = \underline{h}^K_i(f, \mu) = \overline{h}^K_i(f, \mu)$;
(2) $h_i(f, \mu) \le \underline{h}^i_{\mathrm{top}}(f, \mu)$;
(3) $h_i(f, \mu) \le \underline{\mathrm{v}}_i(f, \mu)$.

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Let $h(f, \mu)$ be the entropy of μ. When f is C^2, Ledrappier and Young have shown the following entropy formula [11, Theorem C]: for any $1 \leq i \leq u$,

$$h(f, \mu) = h_i(f, \mu) + \sum_{j=i+1}^{u} \lambda_j \cdot \gamma_j,$$

where $\gamma_1, \ldots, \gamma_u$ are some transverse dimensions satisfying $\gamma_j \leq \dim E^j$. Therefore, Theorem A immediately implies the following result.

THEOREM B. *Let f be a C^2 diffeomorphism on a compact manifold M. Let μ be an ergodic measure. List the positive Lyapunov exponents of μ as $\lambda_1 > \lambda_2 > \cdots > \lambda_u$. Then, for $1 \leq i \leq u$,*

$$h(f, \mu) = \underline{h}_i^K(f, \mu) + \sum_{j=i+1}^{u} \lambda_j \cdot \gamma_j,$$

$$h(f, \mu) \leq \underline{h}_{\mathrm{top}}^i(f, \mu) + \sum_{j=i+1}^{u} \lambda_j \cdot \gamma_j,$$

$$h(f, \mu) \leq \underline{\mathrm{v}}_i(f, \mu) + \sum_{j=i+1}^{u} \lambda_j \cdot \gamma_j.$$

The above contains the Main Theorem from the introduction.

When the measure μ is hyperbolic, i.e., when μ has no zero Lyapunov exponents, the result in Theorem B is true for $i = u$ without the C^2 assumption.

THEOREM C. *Let f be a $C^{1+\alpha}$ diffeomorphism on a compact manifold M. Let μ be an ergodic measure. If μ is hyperbolic, then*

$$h_u(f, \mu) = h(f, \mu).$$

Remark 1.8. As a consequence of Theorems A and C,

$$h(f, \mu) = \underline{h}_u^K(f, \mu) = \overline{h}_u^K(f, \mu).$$

Moreover, this quantity is bounded above both by $\underline{h}_{\mathrm{top}}^u(f, \mu)$ and by $\underline{\mathrm{v}}_u(f, \mu)$.

1.3. *Remarks.* Let us explain our motivation beyond the desire to prove natural inequalities.

Theorem B will be used in a forthcoming work to study some entropy-hyperbolic diffeomorphisms (as suggested by Buzzi [2]). More precisely, we will find a non-empty C^∞ open set of diffeomorphisms which are not uniformly hyperbolic but whose ergodic measures of entropy close to the topological entropy are nevertheless hyperbolic and of given index.

Theorem C extends by a simple argument the Ledrappier–Young entropy formula in the $C^{1+\alpha}$ setting assuming hyperbolicity. This is used in some ongoing work by other authors (J. Buzzi, S. Crovisier and O. Sarig).

Note that Brown [1] gave this $C^{1+\alpha}$ generalization without the hyperbolicity assumption. More precisely, he gave a proof of a uniform bi-Lipschitz property of the stable

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holonomies inside center-unstable manifolds. However, his argument is technical and only a preprint at the time we are writing this. Hence we believe that our simple, half-page argument has some interest.

2. *Basic properties*

In this section, we list some basic results that will be used later.

LEMMA 2.1. *Let f be a $C^{1+\alpha}$ diffeomorphism on a compact manifold M. Let μ be an ergodic measure. Then the following four functions are constant almost everywhere.*

$$\inf_{\rho} \underline{v}(f, W^i_\rho(x)),$$

$$\inf_{\rho} \lim_{\varepsilon\to 0} \liminf_{n\to+\infty} \frac{1}{n} \log \#E(n, \varepsilon, W^i_\rho(x)),$$

$$\inf_{\lambda} \lim_{\varepsilon\to 0} \limsup_{n\to+\infty} \frac{1}{n} \log N_\lambda(\mu^i_x, n, \varepsilon),$$

$$\inf_{\lambda} \lim_{\varepsilon\to 0} \liminf_{n\to+\infty} \frac{1}{n} \log N_\lambda(\mu^i_x, n, \varepsilon).$$

Proof. We first explain the measurability of these functions.

Since the infimum or the limit of a sequence of measurable functions is also measurable, it is enough to check that $\mathrm{Vol}(f^n(W^i_\rho(x)))$, $E(n, \varepsilon, W^i_\rho(x))$, $N_\lambda(\mu^i_x, n, \varepsilon)$ are measurable with respect to x.

Recall that a result of Pesin theory gives that the ith local unstable manifold $W^i_\rho(x)$ varies measurably with respect to x (see [4, Theorem 16, p. 195]). This gives the measurability of $\mathrm{Vol}(f^n(W^i_\rho(x)))$ and $E(n, \varepsilon, W^i_\rho(x))$ by noting that the composition of measurable functions is still measurable. Since the family of conditional measures μ^i_x of μ (with respect to measurable subordinate partitions; see Definition 1.1) varies measurably with respect to x, one can get that $N_\lambda(\mu^i_x, n, \varepsilon)$ is also measurable.

Once we get the measurability, one can check that these functions are f-invariant. Hence, by ergodicity, they are constant almost everywhere. □

Recall that $r(x) > 0$, $x \in R_\mu$ is such that $W^i_{r(x)}(x)$ is an embedded sub-manifold. Indeed, by Pesin theory, we can assume for any $x \in R_\mu$, $r(x)$ is such that

$$\lim_{n\to+\infty} \frac{1}{n} \log r(f^n(x)) = 0.$$

In light of this, we introduce in the following a collection of results in classical Pesin theory. For more detail, see [11, §8] and [9, Proposition 3.3] (which originates from Part I in [7]).

LEMMA 2.2. *Let f be a $C^{1+\alpha}$ diffeomorphism on a compact manifold M and let μ be an ergodic measure. List the positive Lyapunov exponents of μ as $\lambda_1 > \lambda_2 > \cdots > \lambda_u$. For any $\varepsilon > 0$, we can find an increasing sequence of measurable sets $\Lambda^\varepsilon_1 \subset \Lambda^\varepsilon_2 \subset \cdots \subset \Lambda^\varepsilon_k \cdots \subset R_\mu$ and a sequence of numbers $\{r_k\}_{k\geq 1}$ with $0 < r_k < 1$ and $r_k \to 0$ such that:*

- $\bigcup_k \Lambda^\varepsilon_k = R_\mu$;
- $f^n(\Lambda^\varepsilon_k) \subset \Lambda^\varepsilon_{k+n}$ *for all* $k, n \geq 1$;

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- *for any* $x \in \Lambda_k^\varepsilon$, $r_k \le r(x)$ *and any* $y \in W^i_{r_k}(x)$, $1 \le i \le u$,

$$d(f^{-n}(x), f^{-n}(y)) \le r_k^{-1} e^{-n(\lambda_i - \varepsilon)} d(x, y) \quad \text{for all } n \ge 0;$$

- *there is a constant* K *such that for* $k \ge 1$, $x \in \Lambda_k^\varepsilon$, $\rho \le r_k$ *and* $1 \le i \le u$,

$$\mathrm{Vol}(W^i_\rho(x)) \le K \cdot \rho^{\sum_{j=1}^i \dim E^j};$$

- $e^{-\varepsilon} \le r_{k+1}/r_k \le e^{\varepsilon}$ *for all* $k \ge 1$.

Remark 2.3.

- Here, for example, one can choose $r_k = e^{-\varepsilon k}$.
- Note that $W^i(x)$ is tangent to $\sum_{l=1}^i \dim E_x^l$ at x. These small numbers $\{r_k\}_{k \ge 1}$ indicate the size of Pesin charts. When $W^i_\rho(x)$ is in the Pesin chart of x, we can assume that it is contained in a small cone around x and therefore its volume is determined by its radius up to a uniform constant K.

A result of standard Pesin theory (e.g., remarks below Lemma 2.2.3 in [10]) shows the following lemma.

LEMMA 2.4. *Let* f *be a* $C^{1+\alpha}$ *diffeomorphism on a compact manifold* M *and let* μ *be a hyperbolic ergodic measure. Given* $\varepsilon > 0$ *and* $x \in R_\mu$, *assume that* $x \in \Lambda_k^\varepsilon$ *for some* k. *Then*

$$S^{cu}(x) \subset W^u(x) \quad \text{where } S^{cu}(x) \triangleq \{y \in M \mid d(f^{-n}(x), f^{-n}(y)) \le r_k e^{-n\varepsilon} \text{ for all } n \ge 0\}.$$

Remark 2.5. Roughly speaking, $S^{cu}(x)$ above is just the set of points whose backward trajectory always stays in the same Pesin chart of the backward trajectory of x. Hence, in general, $S^{cu}(x)$ is the local center unstable manifold of x. But, when the measure is hyperbolic, the above lemma says that $S^{cu}(x)$ reduces to the local unstable manifold.

For two measurable partitions ξ and η, $\xi \vee \eta$ denotes the partition $\{\xi(x) \cap \eta(x)\}_{x \in R_\mu}$ and $\xi^+ = \vee_{n=0}^{+\infty} f^n \xi$. Let $H_\mu(\xi|\eta)$ denote the mean conditional entropy and let $h_\mu(f, \xi)$ denote the entropy of ξ with respect to f (i.e., $h_\mu(f, \xi) \triangleq H_\mu(\xi|f(\xi^+))$). We note that if ξ is an increasing partition (i.e., $\xi(x) \subset f(\xi(f^{-1}(x)))$ for μ-almost every x), we have $h_\mu(f, \xi) = H_\mu(\xi|f\xi)$.

The following result of Ledrappier and Young justifies the definition of the entropies along unstable foliations.

PROPOSITION 2.6. [11, Proposition 7.2.1] *Let* f *be a* $C^{1+\alpha}$ *diffeomorphism on a compact manifold* M *and let* μ *be an ergodic measure. For* $1 \le i \le u$ *and for any increasing partition* ξ^i *subordinate to* W^i *and for* μ*-almost every point* x,

$$\begin{aligned} h_i(f, \mu) &\triangleq \lim_{\tau \to 0} \liminf_{n \to +\infty} -\frac{1}{n} \log \mu_x^i(V^i(x, n, \tau)) \\ &= \lim_{\tau \to 0} \limsup_{n \to +\infty} -\frac{1}{n} \log \mu_x^i(V^i(x, n, \tau)) = H_\mu(\xi^i | f\xi^i). \end{aligned}$$

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Remark 2.7.

- Note that the functions of x that appear in Proposition 2.6 are f-invariant and therefore constant μ-almost everywhere. They do not depend on the choice of the subordinate partition ξ^i. So it is proper to denote them by $h_i(f, \mu)$. See Lemma 3.12 in [10] for more detail.
- Ledrappier and Young [11] assumed C^2 smoothness. But their proof of Proposition 2.6 in their §9 only uses Pesin theory and $C^{1+\alpha}$ smoothness.

We say that η is finer than ξ, denoted by $\xi \leq \eta$, if $\eta(x) \subset \xi(x)$ for μ-almost every x. For partitions with finite mean entropy, the finer partition has larger entropy. The following is an extension of this property to non-finite partitions.

LEMMA 2.8. [16, Property 8.7] *Let f be a homeomorphism on a compact metric space X. Assume that μ is an f-invariant probability measure. Let ξ, η be two measurable partitions (possibly with infinite mean entropy) with η being finer than ξ. If the mean conditional entropy $H_\mu(\eta \mid f(\xi^+))$ is finite, then $h_\mu(f, \xi) \leq h_\mu(f, \eta)$.*

Remark 2.9. Rohlin's article [16] mainly discusses entropy theory for endomorphisms where most definitions and properties are stated by using f^{-1}. Since here we assume that f is a homeomorphism, our statement is parallel to the original statement of Property 8.7 in [16].

LEMMA 2.10. *Let f be a homeomorphism on a compact metric space X. Assume that μ is an f-invariant probability measure. Let ξ, η be two increasing measurable partitions with $h_\mu(f, \xi) < +\infty$, $h_\mu(f, \eta) < +\infty$. Then, for any integer $n \geq 1$,*

$$h_\mu(f, \xi \vee \eta) = h_\mu(f, \xi \vee f^n \eta).$$

Proof. Since ξ, η are increasing, for any $n \geq 1$, we have

$$f^n \xi \vee f^n \eta \leq \xi \vee f^n \eta \leq \xi \vee \eta.$$

In order to apply Lemma 2.8, we note that

$$\begin{aligned} H_\mu(\xi \vee \eta \mid f((\xi \vee f^n \eta)^+)) &= H_\mu(\xi \vee \eta \mid f\xi \vee f^{n+1}\eta) \\ &= H_\mu(\xi \mid f\xi \vee f^{n+1}\eta) + H_\mu(\eta \mid \xi \vee f^{n+1}\eta) \\ &\leq H_\mu(\xi \mid f\xi) + H_\mu(\eta \mid f^{n+1}\eta) \\ &= h_\mu(f, \xi) + n h_\mu(f, \eta) \\ &< +\infty. \end{aligned}$$

By Lemma 2.8, we have

$$h_\mu(f, \xi \vee f^n \eta) \leq h_\mu(f, \xi \vee \eta).$$

To conclude, we prove the converse inequality by applying the previous one to $\xi_1 = f^n \eta$ and $\eta_1 = \xi$, obtaining

$$h_\mu(f, \xi \vee \eta) = h_\mu(f, f^n \xi \vee f^n \eta) \leq h_\mu(f, \xi \vee f^n \eta).$$ □

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The following is an extension of [10, Lemma 3.1.2]. The main difference is that here we only assume that one of the two partitions is subordinate. The proof is essentially identical to [10, Lemma 3.1.2]. For completeness, we present it.

LEMMA 2.11. *Let f be a $C^{1+\alpha}$ diffeomorphism on a compact manifold M and let μ be an ergodic measure. Let ξ^u be an increasing partition subordinate to W^u. Assume that β is a measurable partition satisfying:*

(1) *β is increasing;*
(2) *for μ-almost every x, $\beta(x) \subset W^u(x)$;*
(3) *for μ-almost every x, $\mathrm{diam}((f^{-n}(\beta))(x)) \to 0$.*

Then

$$h_\mu(f, \xi^u \vee \beta) = h_\mu(f, \beta).$$

Proof. Since both ξ^u and β are increasing and their entropies with respect to f and μ are finite, by Lemma 2.10, for any $n \geq 1$, we have

$$\begin{aligned} h_\mu(f, \xi^u \vee \beta) &= h_\mu(f, (f^n\xi^u) \vee \beta) \\ &= H_\mu((f^n\xi^u) \vee \beta \mid (f^{n+1}\xi^u) \vee f\beta) \\ &= H_\mu(\beta \mid (f^{n+1}\xi^u) \vee f\beta) + H_\mu(f^n\xi^u \mid (f^{n+1}\xi^u) \vee \beta) \\ &= H_\mu(\beta \mid (f^{n+1}\xi^u) \vee f\beta) + H_\mu(\xi^u \mid (f\xi^u) \vee f^{-n}\beta). \end{aligned}$$

By the third assumption on β, $f^{-n}\beta$ tends increasingly to the partition ε into points. Note that $H_\mu(\xi^u \mid (f\xi^u) \vee f^{-n}\beta) \leq H_\mu(\xi^u \mid f\xi^u) < +\infty$, by [16, Property 5.11]; the second term $H_\mu(\xi^u \mid (f\xi^u) \vee f^{-n}\beta)$ above goes to $H_\mu(\xi^u \mid \varepsilon) = 0$ as $n \to +\infty$. So it is sufficient to prove that $H_\mu(\beta \mid (f^{n+1}\xi^u) \vee f\beta) \to H(\beta \mid f\beta) = h_\mu(f, \beta)$. First note that

$$H_\mu(\beta \mid (f^{n+1}\xi^u) \vee f\beta) \leq H(\beta \mid f\beta).$$

Write the conditional measures of μ with respect to $(f^{n+1}\xi^u) \vee f\beta$ as $\{\mu_x^n\}_{x \in M}$ and the conditional measures with respect to $f\beta$ as $\{\mu_x\}_{x \in M}$. By definition,

$$\begin{aligned} H_\mu(\beta \mid (f^{n+1}\xi^u) \vee f\beta) &= \int -\log \mu_x^n(\beta(x))\, d\mu(x), \\ H(\beta \mid f\beta) &= \int -\log \mu_x(\beta(x))\, d\mu(x). \end{aligned}$$

Let

$$\Omega_n = \{x \mid f\beta(x) \subset f^{n+1}\xi^u(x)\}.$$

Since $\xi^u(x)$ contains an open neighborhood of x in $W^u(x)$ with respect to the sub-manifold topology, by assumptions 2 and 3 on β, $\{\Omega_n\}$ is a non-decreasing sequence and $\mu(\Omega_n) \to 1$ as $n \to +\infty$. For $x \in \Omega_n$, by definition, $(f^{n+1}\xi^u)(x) \cap (f\beta)(x) = (f\beta)(x)$. Then one can show that this implies that

$$-\log \mu_x^n(\beta(x)) = -\log \mu_x(\beta(x)) \quad \text{for } \mu\text{-almost every } x \in \Omega_n.$$

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Hence the non-negative functions $\{-\log \mu^n_{(\cdot)}(\beta(\cdot))\}$ tend pointwise to $-\log \mu_{(\cdot)}(\beta(\cdot))$. By Fatou's lemma,

$$\lim_{n\to+\infty} H_\mu(\beta \mid (f^{n+1}\xi^u) \vee f\beta) \geq H(\beta \mid f\beta). \qquad \square$$

3. *Proof of Theorem A*

We prove the assertions in Theorem A one by one.

3.1. $h_i(f,\mu) = \underline{h}_i^K(f,\mu) = \overline{h}_i^K(f,\mu)$.

We first prove that $h_i(f,\mu) \leq \underline{h}_i^K(f,\mu)$.

By Proposition 2.6 and by removing a set of zero measure from R_μ if necessary, we can assume that

$$\begin{aligned}\lim_{\varepsilon\to 0}\limsup_{n\to+\infty} -\frac{1}{n}\log \mu_x^i(V^i(x,n,\varepsilon)) &= \lim_{\varepsilon\to 0}\liminf_{n\to+\infty} -\frac{1}{n}\log \mu_x^i(V^i(x,n,\varepsilon)) \\ &= h_i(f,\mu) \quad \text{for all } x \in R_\mu.\end{aligned}$$

We write $h_i(f,\mu)$ as h_i for short.

For any $\eta, \varepsilon > 0$, define

$$\Delta_\eta^\varepsilon \triangleq \bigg\{x \in R_\mu \,\bigg|\, \liminf_{n\to+\infty} -\frac{1}{n}\log \mu_x^i(V^i(x,n,2\varepsilon)) > h_i - \eta\bigg\}.$$

Then $\bigcup_{\varepsilon>0}\Delta_\eta^\varepsilon = R_\mu$.

For $j \in \mathbb{N}$ and $p \in \Delta_\eta^\varepsilon$, define

$$\Delta_\eta^\varepsilon(p,j) \triangleq \{x \in \Delta_\eta^\varepsilon \mid \mu_p^i(V^i(x,n,2\varepsilon)) \leq e^{-n(h_i-\eta)} \text{ for all } n \geq j\}.$$

By definition,

$$\Delta_\eta^\varepsilon(p,j) \subset \Delta_\eta^\varepsilon(p,j+1), \quad \mu_p^i\bigg(\bigcup_j \Delta_\eta^\varepsilon(p,j)\bigg) = \mu_p^i(\Delta_\eta^\varepsilon).$$

Fix any $\lambda > 0$ and $p \in R_\mu$. Choose ε small enough and N large enough such that

$$\mu_p^i(\Delta_\eta^\varepsilon(p,j)) \geq 1 - \frac{\lambda}{2} \quad \text{for all } j \geq N.$$

For $n \in \mathbb{N}$, let $C_n \subset R_\mu$ be a subset such that $\#C_n = N_\lambda(\mu_p^i, n, \varepsilon)$ and $\mu_p^i(\bigcup_{y\in C_n} V^i(y,n,\varepsilon)) \geq \lambda$. Hence we have

$$\mu_p^i\bigg(\Delta_\eta^\varepsilon(p,n) \cap \bigg(\bigcup_{y\in C_n} V^i(y,n,\varepsilon)\bigg)\bigg) \geq \frac{\lambda}{2} \quad \text{for all } n \geq N.$$

Let $A_n \subset C_n$ be such that for each $y \in A_n$, we have $V^i(y,n,\varepsilon) \cap \Delta_\eta^\varepsilon(p,n) \neq \emptyset$. For $y \in A_n$, we fix any $\widetilde{y} \in V^i(y,n,\varepsilon) \cap \Delta_\eta^\varepsilon(p,n)$. Then we have

$$V^i(y,n,\varepsilon) \subset V^i(\widetilde{y},n,2\varepsilon).$$

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Hence, for $n \geq N$,

$$\frac{\lambda}{2} \leq \mu_p^i\bigg(\bigcup_{y\in A_n} V^i(\widetilde{y}, n, 2\varepsilon)\bigg) \leq N_\lambda(\mu_p^i, n, \varepsilon) \times \sup_{x\in\Delta_\eta^\varepsilon(p,n)} \mu_p^i(V^i(x, n, 2\varepsilon))$$
$$\leq N_\lambda(\mu_p^i, n, \varepsilon) \times e^{-n(h_i-\eta)}.$$

Therefore, for any ε small enough (depending on η) and n large enough,

$$N_\lambda(\mu_p^i, n, \varepsilon) \geq \frac{\lambda}{2} \cdot e^{n(h_i-\eta)}.$$

Then, by the arbitrariness of η and λ, we get

$$h_i(f, \mu) \leq \inf_\lambda \lim_{\varepsilon\to 0} \liminf_{n\to+\infty} \frac{1}{n} \log N_\lambda(\mu_p^i, n, \varepsilon) = \underline{h}_i^K(f, \mu).$$

Next we prove that $h_i(f, \mu) \geq \overline{h}_i^K(f, \mu)$. The arguments are similar to above.

For any $\eta, \varepsilon > 0$, define

$$\Omega_\eta^\varepsilon \triangleq \bigg\{x \in R_\mu \,\bigg|\, \limsup_{n\to+\infty} -\frac{1}{n} \log \mu_x^i\bigg(V^i\bigg(x, n, \frac{\varepsilon}{2}\bigg)\bigg) < h_i + \eta\bigg\}.$$

Then $\bigcup_{\varepsilon>0} \Delta_\eta^\varepsilon = R_\mu$.

For $j \in \mathbb{N}$ and $p \in \Delta_\eta^\varepsilon$, define

$$\Omega_\eta^\varepsilon(p, j) \triangleq \bigg\{x \in \Delta_\eta^\varepsilon \,\bigg|\, \mu_p^i\bigg(V^i\bigg(x, n, \frac{\varepsilon}{2}\bigg)\bigg) \geq e^{-n(h_i+\eta)} \text{ for all } n \geq j\bigg\}.$$

By definition,

$$\Omega_\eta^\varepsilon(p, j) \subset \Omega_\eta^\varepsilon(p, j+1), \quad \mu_p^i\bigg(\bigcup_j \Omega_\eta^\varepsilon(p, j)\bigg) = \mu_p^i(\Omega_\eta^\varepsilon).$$

Fix any $\lambda > 0$ and $p \in R_\mu$. Choose ε small enough and N large enough such that

$$\mu_p^i(\Omega_\eta^\varepsilon(p, j)) \geq \lambda \quad \text{for all } j \geq N.$$

For $n \in \mathbb{N}$, let $F_n \subset \Omega_\eta^\varepsilon(p, n)$ be a maximal (n, ε) separated set of $\Omega_\eta^\varepsilon(p, n) \cap \xi^i(p)$. Then $\{V^i(y, n, \varepsilon)\}_{y\in F_n}$ covers $\Omega_\eta^\varepsilon(p, n) \cap \xi^i(p)$. Hence $\#F_n \geq N_\lambda(\mu_p^i, n, \varepsilon)$. And we also have

$$y_1, y_2 \in F_n,\ y_1 \neq y_2 \Longrightarrow V^i\bigg(y_1, n, \frac{\varepsilon}{2}\bigg) \bigcap V^i\bigg(y_2, n, \frac{\varepsilon}{2}\bigg) = \emptyset.$$

Hence, for $n \geq N$,

$$N_\lambda(\mu_p^i, n, \varepsilon) \leq \#F_n \leq \frac{1}{\sup_{x\in\Omega_\eta^\varepsilon(p,n)} \mu_p^i(V^i(x, n, (\varepsilon/2)))} \leq e^{n(h_i+\eta)}.$$

Then, by the arbitrariness of η and λ, we get

$$h_i(f, \mu) \geq \inf_\lambda \lim_{\varepsilon\to 0} \limsup_{n\to+\infty} \frac{1}{n} \log N_\lambda(\mu_p^i, n, \varepsilon) = \overline{h}_i^K(f, \mu).$$

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3.2. $h_i(f,\mu) \le \underline{h}^i_{\mathrm{top}}(f,\mu)$.

Since ξ^i is a partition subordinate to W^i, for any $\rho > 0$, we assume that for any $x \in R_\mu$, $\mu^i_x(W^i_\rho(x)) > 0$.

For $\rho, \varepsilon, \eta > 0$, $n, j \in \mathbb{N}$ and $p \in R_\mu$, let $F^\varepsilon_\eta(p, j, n)$ be an (n, ε)-separated subset of $\Delta^\varepsilon_\eta(p, j) \cap W^i_\rho(p)$ with maximum cardinality. It is a cover and hence we have

$$\begin{aligned}\mu^i_p(\Delta^\varepsilon_\eta(p, j) \cap W^i_\rho(p)) &\le \mu^i_p\bigg(\bigcup_{x\in F^\varepsilon_\eta(p,j,n)} V^i(x, n, \varepsilon)\bigg)\\ &\le \#F^\varepsilon_\eta(p, j, n) \times \sup_{x\in\Delta^\varepsilon_\eta(p,j)} \mu^i_p(V^i(x, n, \varepsilon))\\ &\le \#F^\varepsilon_\eta(p, j, n) \times e^{-n(h_i-\eta)} \quad \text{for all } j,\ \text{for all } n \ge j.\end{aligned}$$

Hence

$$\#F^\varepsilon_\eta(p, j, n) \ge \frac{\mu^i_p(\Delta^\varepsilon_\eta(p, j) \cap W^i_\rho(p))}{e^{n(h_i-\eta)}} \quad \text{for all } j,\ \text{for all } n \ge j.$$

Choose ε small enough (depending on η) and j large enough such that $\mu^i_p(\Delta^\varepsilon_\eta(p, j) \cap W^i_\rho(x)) > 0$. Then, taking $\liminf_{n\to+\infty}(1/n) \log$ on both sides, we have

$$\liminf_{n\to+\infty} \frac{1}{n} \log \#E(n, \varepsilon, W^i_\rho(x)) \ge \liminf_{n\to+\infty} \frac{1}{n} \log \#F^\varepsilon_\eta(p, j, n) \ge h_i - \eta.$$

Since η and ρ are arbitrary, we get $h_i(f,\mu) \le \underline{h}^i_{\mathrm{top}}(f,\mu)$.

3.3. $h_i(f,\mu) \le \underline{v}_i(f,\mu)$.

Applying Lemma 2.2 for any $\varepsilon > 0$, we obtain an increasing sequence of measurable sets $\{\Lambda^\varepsilon_k \subset R_\mu\}$.

Let us first note that, for any $k, n \in \mathbb{N}$, any $x \in \Lambda^\varepsilon_k$, any $\tau \le r_k e^{-n\varepsilon}$ and any y with $f^n(y) \in W^i_\tau(f^n(x))$,

$$d(f^{n-j}(x), f^{n-j}(y)) \le r^{-1}_{k+n} e^{-j(\lambda_i-\varepsilon)} d(f^n(x), f^n(y)) \quad \text{for all } 0 \le j \le n.$$

Hence, for any $k, n \in \mathbb{N}$, any $x \in \Lambda^\varepsilon_k$ and any $\tau \le r_k e^{-n\varepsilon}$, $f^n(V^i(x, n, \tau))$ contains an ith local sub-manifold $W^i_{r_{k+n}\tau}(f^n(x))$.

Since the function $\mu^i_p(V^i(p, n, \tau))$ is non-decreasing with respect to τ, for any sequence $\{\tau_n\}$ with $\tau_n \to 0$, we have, for $p \in R_\mu$,

$$\liminf_{n\to+\infty} -\frac{1}{n} \log \mu^i_p(V^i(p, n, \tau_n)) \ge \liminf_{n\to+\infty} -\frac{1}{n} \log \mu^i_p(V^i(p, n, \tau)) \quad \text{for all } \tau > 0.$$

Hence, in particular, for $k \in \mathbb{N}$, $p \in \Lambda^\varepsilon_k$ and $x \in \Lambda^\varepsilon_k \cap \xi^i(p)$,

$$\liminf_{n\to+\infty} -\frac{1}{n} \log \mu^i_p(V^i(x, n, r_k e^{-n\varepsilon})) \ge h_i.$$

For any $j \in \mathbb{N}$, $p \in \Lambda^\varepsilon_k$ and $\rho > 0$ with $W^i_\rho(p) \subset \xi^i(p)$, define

$$\Lambda^{\varepsilon,p}_{k,j} = \{x \in \Lambda^\varepsilon_k \cap W^i_\rho(p) \mid \mu^i_p(V^i(x, n, r_k e^{-n\varepsilon})) \le e^{-n(h_i-\varepsilon)} \text{ for all } n \ge j\}.$$

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By definition,

$$\Lambda_{k,j}^{\varepsilon,p} \subset \Lambda_{k,j+1}^{\varepsilon,p}, \quad \mu_p^i\bigg(\bigcup_j \Lambda_{k,j}^{\varepsilon,p}\bigg) = \mu_p^i(\Lambda_k^\varepsilon \cap W_\rho^i(p)).$$

Let $F_{n,j,k}^{\varepsilon,p}$ be an $(n, r_k e^{-n\varepsilon})$-separated subset of $\Lambda_{k,j}^{\varepsilon,p}$ with maximum cardinality. Then we have

$$\begin{aligned}\mu_p^i(\Lambda_{k,j}^{\varepsilon,p}) &\le \mu_p^i\bigg(\bigcup_{x\in F_{n,j,k}^{\varepsilon,p}} V^i(x,n,r_k e^{-n\varepsilon})\bigg)\\ &\le \#F_{n,j,k}^{\varepsilon,p} \times \sup_{x\in\Lambda_{k,j}^{\varepsilon,p}} \mu_p^i(V^i(x,n,r_k e^{-n\varepsilon})).\end{aligned} \tag{$*$}$$

Note that for any $x \in F_{n,j,k}^{\varepsilon,p} \subset W_\rho^i(p)$, $V^i(x,n,r_k e^{-n\varepsilon}) \subset W_{2\rho}^i(p)$ for all n such that $r_k e^{-n\varepsilon} < \rho$. Since the sets $\{f^n(V^i(x,n,\frac{1}{2}r_k e^{-n\varepsilon}))\}_{x\in F_{n,j,k}^{\varepsilon,p}}$ are mutually disjoint, for all large n, we have

$$\mathrm{Vol}(f^n(W_{2\rho}^i(x))) \ge \sum_{x\in F_{n,j,k}^{\varepsilon,p}} \mathrm{Vol}\bigg(f^n\bigg(V^i\bigg(x,n,\frac{1}{2}r_k e^{-n\varepsilon}\bigg)\bigg)\bigg).$$

Recall that each $f^n(V^i(x,n,\frac{1}{2}r_k e^{-n\varepsilon}))$ contains an ith local unstable manifold $W^i_{(1/2)r_{k+n}r_k e^{-n\varepsilon}}(f^n(x))$. Thus

$$\mathrm{Vol}(f^n(W_{2\rho}^i(x))) \ge \#F_{n,j,k}^{\varepsilon,p} \times K \times \big(\tfrac{1}{2}r_{k+n}r_k e^{-n\varepsilon}\big)^{\sum_{l=1}^i \dim E^l}, \tag{$**$}$$

where K is the constant from Lemma 2.2.

Combining $(*)$ and $(**)$, for all large n,

$$\begin{aligned}\mathrm{Vol}(f^n(W_{2\rho}^i(x))) &\ge \frac{\mu_p^i(\Lambda_{k,j}^{\varepsilon,p})}{\sup_{x\in\Lambda_{k,j}^{\varepsilon,p}} \mu_p^i(V^i(x,n,r_k e^{-n\varepsilon}))} \times K \times \bigg(\frac{1}{2}r_{k+n}r_k e^{-n\varepsilon}\bigg)^{\sum_{l=1}^i \dim E^l}\\ &\ge \frac{\mu_p^i(\Lambda_{k,j}^{\varepsilon,p})}{e^{-n(h_i-\varepsilon)}} \times K \times \bigg(\frac{1}{2}r_{k+n}r_k e^{-n\varepsilon}\bigg)^{\sum_{l=1}^i \dim E^l}\\ &\qquad \text{for all } k\in\mathbb{N},\ \text{for all } p\in\Lambda_k^\varepsilon,\ \text{for all } j\in\mathbb{N}.\end{aligned}$$

Since $\mu_p^i(W_\rho^i(p)) > 0$, we choose k, j large enough such that $\mu_p^i(\Lambda_{k,j}^{\varepsilon,p}) > 0$. Taking $\liminf_{n\to+\infty}(1/n)\log$ on both sides, we have

$$\liminf_{n\to+\infty} \frac{1}{n}\log \mathrm{Vol}(f^n(W_{2\rho}^i(p))) \ge h_i - \varepsilon - 2\varepsilon\sum_{l=1}^i \dim E^l \quad \text{for all } k\in\mathbb{N},\ \text{for all } p\in\Lambda_k^\varepsilon.$$

Hence we have

$$\liminf_{n\to+\infty} \frac{1}{n}\log \mathrm{Vol}(f^n((W_{2\rho}^i(p))) \ge h_i - \varepsilon - 2\varepsilon\sum_{l=1}^i \dim E^l \quad \text{for all } p\in R_\mu.$$

By the arbitrariness of ε and ρ, we get the conclusion.

4. *Proof of Theorem C*

We now explain how to deduce our Theorem C based on the arguments of Ledrappier and Young in [10].

Proof. Let ξ^u be any increasing measurable partition subordinate to W^u. By Proposition 2.6, $h_u(f,\mu) = H_\mu(\xi^u | f\xi^u) = h_\mu(f,\xi^u)$. Hence $h_u(f,\mu) \le h(f,\mu)$. So it is sufficient to prove that $h_u(f,\mu) \ge h(f,\mu)$.

In the following argument, some properties only hold for μ-almost every x. But, without loss of generality, we assume that these properties hold for any $x \in R_\mu$.

For $\varepsilon > 0$ and $x \in \Lambda_k^\varepsilon$, let $S^{cu}(x)$ be the set in Lemma 2.4. By Lemma 2.4.2 in [10], there is a measurable partition ξ with $H_\mu(\xi) < +\infty$ such that $\xi^+(x) \subset S^{cu}(x),\ x \in R_\mu$, where $\xi^+ = \vee_{n=0}^{+\infty} f^n\xi$. Since $H_\mu(\xi) < +\infty$, we can assume that $h_\mu(f,\xi) \ge h(f,\mu) - \varepsilon$.

By Lemma 2.4, $\xi^+(x) \subset W^u(x),\ x \in R_\mu$.

We note the following facts.

- By Lemma 3.2.1 in [10], we have
$$h_u(f,\mu) = H_\mu(\xi^u | f\xi^u) = h_\mu(f, \xi^u \vee \xi^+).$$
- Since ξ^+ is increasing, $\xi^+(x) \subset W^u(x)$ and $\operatorname{diam}(f^{-n}(\xi^+(x))) \to 0$ for μ-almost every x, Lemma 2.11 yields
$$h_\mu(f, \xi^u \vee \xi^+) = h_\mu(f, \xi^+).$$
- Since ξ^+ is increasing,
$$h_\mu(f,\xi^+) = H_\mu(\xi^+ | f(\xi^+)) = H_\mu(\xi \vee f(\xi^+) | f(\xi^+)) = H_\mu(\xi | f(\xi^+)) = h_\mu(f,\xi).$$

Hence we have
$$h_u(f,\mu) = h(f,\xi) \ge h(f,\mu) - \varepsilon.$$

Since ε is arbitrarily small, we get the conclusion. □

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