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ARTICLE

Definability in the local structure of the ω-Turing degrees

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Abstract
This article continues the study of the definability in the local substructure $\mathcal{G}_{T,\omega}$ of the ω-Turing degrees, initiated in (Sariev and Ganchev 2014). We show that the class $\mathbf{I}$ of the intermediate degrees is definable in $\mathcal{G}_{T,\omega}$.

Keywords: Turing reducibility; ω-Turing degrees; degree structures; definability; jump classes

1. Introduction

One of the major research themes in computability theory is the exploration of degree structures. Each degree structure arises as a formal way to model the comparison of the informational content of given objects. Usually the comparison is described as a reflexive and transitive reducibility relation $\leq$ on the set Ω of the given objects. The equivalence classes of the objects reducible to each other are called *degrees*, and the preorder $\leq$ on Ω induces a partial order on the degrees.

Among the most explored examples of such structures is that of the Turing degrees. This structure classifies the informational content of the sets of natural numbers via the Turing reducibility $\leq_T$. Namely, the set A is *Turing reducible to* the set B if and only if there is an algorithm that, on any input $x \in \omega$, determines the membership of x in A in finitely many steps and making finitely many membership queries to B.

There are different points of view from which a degree structure can be investigated. For example, one can explore the algebraic complexity of the structure. Another one is the exploration of the definable sets in the structure. The investigation of the complexity of the structure involves also an analysis of the group of the structure automorphisms, in particular whether there are nontrivial automorphisms.

What are the typical definability problems, which one can investigate? Suppose that $\mathcal{D} = (\mathbf{D}, \leq, \mathbf{0})$ is a degree structure possessing an ordering relation $\leq$ and a least element $\mathbf{0}$. Moreover, suppose that the structure is augmented with a jump operation $'$, which is monotone (i.e., $\mathbf{a} \leq \mathbf{b} \Rightarrow \mathbf{a}' \leq \mathbf{b}'$) and strictly expansive (i.e., $\mathbf{a} < \mathbf{a}'$). Naturally one can ask for the first-order definability of the jump operation in the language of the structure order. Problems also arise about the first-order definability of degree classes, determined originally in the terms of the jump operation. The same definability problems can be transferred to its local structure (i.e., the substructure consisting of the degrees bounded by the jump of the least element) as well. As an interesting special case one can ask about the definability of the classes from the jump hierarchy.

The jump hierarchy was first introduced for the local substructure of the Turing degrees in (Cooper 1972) and (Soare 1974). In this hierarchy, the elements of the local substructure are partitioned in classes depending on their "closeness" to the least element or to its first jump in the terms of the jump operation. Namely, a degree in the local substructure is *low*$_n$ iff its n-th jump is as low as possible – the same as the n-th jump of the least element. Similarly, a degree is *high*$_n$ iff its n-th jump is as high as possible – the same as the n-th jump of the jump of the least element. The corresponding degree classes are denoted by $\mathbf{L}_n$ and $\mathbf{H}_n$ respectively:

$$\mathbf{L}_n = \{\mathbf{x} \mid \mathbf{x}^{(n)} = \mathbf{0}^{(n)}\} \text{ and } \mathbf{H}_n = \{\mathbf{x} \mid \mathbf{x}^{(n)} = \mathbf{0}^{(n+1)}\}.$$

We set $\mathbf{L}$ to be the class of the degrees which are low$_n$ for some $n < \omega$, and $\mathbf{H}$ – the class of the degrees which are high$_n$ for some $n < \omega$:

$$\mathbf{L} = \bigcup \mathbf{L}_n \text{ and } \mathbf{H} = \bigcup \mathbf{H}_n.$$

Finally, we shall refer to the degrees in the local substructure, which are neither in $\mathbf{H}$ nor in $\mathbf{L}$, as *intermediate* ($\mathbf{I}$).

This article concerns the problem of the first-order definability of the classes $\mathbf{L}$, $\mathbf{H}$, and $\mathbf{I}$ in the local substructure of the ω-Turing degrees $\mathcal{D}_{T,\omega}$. Unlike the well-known structures $\mathcal{D}_T$ and $\mathcal{D}_e$ of the Turing and the enumeration degrees, $\mathcal{D}_{T,\omega}$ is induced by a reducibility on the set $\mathcal{S}_\omega$ of the sequences of sets of natural numbers. The study of the degree structures induced by such reducibilities has been initiated by Soskov. In the paper (Soskov 2007), he introduces the ω-enumeration reducibility $\leq_\omega$, generalizing the enumeration reducibility over sequences.

The structure that is object of the current work, $\mathcal{D}_{T,\omega}$, is introduced in (Sariev and Ganchev 2014) as a "Turing" analogue of $\mathcal{D}_\omega$ in the following way. First, to each sequence $\mathcal{A} = \{A_k\}_{k<\omega}$ in $\mathcal{S}_\omega$, a jump-class $J_\mathcal{A}$ is assigned:

$$J_\mathcal{A} = \{\deg_T(X) \mid A_k \leq_T X^{(k)} \text{ uniformly in } k\}.$$

Then the ω-Turing reducibility $\leq_{T,\omega}$, which induces $\mathcal{D}_{T,\omega}$, is set in such a way that for each sequences $\mathcal{A}$ and $\mathcal{B}$ in $\mathcal{S}_\omega$, $\mathcal{A} \leq_{T,\omega} \mathcal{B}$ if and only if $J_\mathcal{B} \subseteq J_\mathcal{A}$. The relation $\leq_{T,\omega}$ is a preorder on $\mathcal{S}_\omega$ and induces the structure $\mathcal{D}_{T,\omega}$ of the ω-Turing degrees in the standard way. The least element of $\mathcal{D}_{T,\omega}$ is the degree of the sequence $\{\emptyset\}_{k<\omega}$.

The jump $\mathcal{A}'$ of a sequence $\mathcal{A}$ is defined so that the class $J_{\mathcal{A}'}$ consists exactly of the jumps of the Turing degrees in $J_\mathcal{A}$, that is, so that $J_{\mathcal{A}'} = J'_\mathcal{A}$. The jump operator on sequences is monotone and thus induces a jump operation $'$ in $\mathcal{D}_{T,\omega}$. Just like the jump operation in $\mathcal{D}_T$, the range of the jump operation in $\mathcal{D}_{T,\omega}$ is exactly the cone above the first jump of the least element $\mathbf{0}_{T,\omega}$. In other words, a general jump inversion theorem is valid for $\mathcal{D}_{T,\omega}$. Moreover, even a stronger statement turns out to be true, namely for every ω-Turing degree $\mathbf{a}$ above $\mathbf{0}_{T,\omega}{}'$ there is a least degree with jump equal to $\mathbf{a}$. This property is true neither for $\mathcal{D}_T$ nor for $\mathcal{D}_e$.

Using this jump inversion theorem, one may consider a natural copy of the structure $\mathcal{D}_T$, which is definable in $\mathcal{D}_{T,\omega}$ augmented by the jump operation. Even more, the automorphism groups of the structures $\mathcal{D}_T$ and $\mathcal{D}_{T,\omega}{}'$ are isomorphic. This closeness between $\mathcal{D}_{T,\omega}$ and $\mathcal{D}_T$ makes the ω-Turing degrees interesting for investigation.

Just like the Turing and the enumeration degree structures, the jump operation in each one of $\mathcal{D}_\omega$ and $\mathcal{D}_{T,\omega}$ gives rise to the corresponding local substructure: $\mathcal{G}_\omega$ and $\mathcal{G}_{T,\omega}$. What do we know about the definability of the jump classes in these two local structures? First, in both $\mathcal{G}_\omega$ and $\mathcal{G}_{T,\omega}$, for each n the classes $\mathbf{H}_n$ and $\mathbf{L}_n$ are first-order definable, as it is shown in (Ganchev and Soskova 2012) and (Sariev and Ganchev 2014) respectively. The first-order definability of the classes $\mathbf{L}$ and $\mathbf{H}$, as well as of the class $\mathbf{I}$ of the intermediate degrees in $\mathcal{G}_\omega$, is shown in (Ganchev and Sariev 2015). Note that the latter does not hold in the local substructures neither of the Turing degrees nor of the enumeration degrees.

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All of these definability results rely on the existence of a class of remarkable degrees having no analogue in either $\mathcal{R}$ (the structure of the computably enumerable degrees), $\mathcal{G}_T$, or $\mathcal{G}_e$. These degrees are denoted by $\mathbf{o}_n$, $n < \omega$, and are defined so that $\mathbf{o}_n$ is the least degree whose n-th jump is equal to the $(n+1)$-th jump of $\mathbf{0}_\omega$. In other words, $\mathbf{o}_n$ is the least high$_n$ degree. The degrees $\mathbf{o}_n$ are also connected to low$_n$ degrees. Indeed, a degree in $\mathcal{G}_\omega$ is low$_n$ *iff* it forms a minimal pair with $\mathbf{o}_n$. The same connections hold and in $\mathcal{G}_{T,\omega}$.

Each one of the degrees $\mathbf{o}_n$ turns out to be definable both in $\mathcal{G}_\omega$ and $\mathcal{G}_{T,\omega}$ (Ganchev and Soskova 2012), (Sariev and Ganchev 2014) and hence such are the classes $\mathbf{H}_n$ and $\mathbf{L}_n$, for each $n < \omega$. Nevertheless, the corresponding first-order definitions of the degrees $\mathbf{o}_n$ are quite different. The definition in $\mathcal{G}_\omega$ of $\mathbf{o}_n$, given in (Ganchev and Soskova 2012), is based on the notion of Kalimullin pairs – a notion first introduced and studied by Kalimullin in the context of the enumeration degrees. Just like in the Turing degrees, Kalimullin pairs do not exist in the structure of the ω-Turing degrees. The definition here is based on the notion of *noncuppable* degrees. Namely, in (Sariev and Ganchev 2014) it is shown that for each $n < \omega$,

$$\mathbf{o}_{n+1} \text{ is the greatest degree below } \mathbf{o}_n \text{which is noncuppable to } \mathbf{o}_n.$$

Further, to obtain a first-order definition of the class $\mathbf{I}$ (no matter in $\mathcal{G}_\omega$ or $\mathcal{G}_{T,\omega}$) it is sufficient to define by a first-order formula the class $\mathfrak{O} = \{\mathbf{o}_n \mid n < \omega\}$ (of course, in the corresponding local structure). Indeed, for each degree $\mathbf{x}$ in the corresponding local structure,

$$\mathbf{x} \in \mathbf{L} \iff (\exists n)[\mathbf{x} \in \mathbf{L}_n] \iff (\exists n)[\mathbf{x} \wedge \mathbf{o}_n = \mathbf{0}] \iff (\exists \mathbf{o} \in \mathfrak{O})[\mathbf{x} \wedge \mathbf{o} = \mathbf{0}],$$

$$\mathbf{x} \in \mathbf{H} \iff (\exists n)[\mathbf{x} \in \mathbf{H}_n] \iff (\exists n)[\mathbf{o}_n \leq \mathbf{x}] \iff (\exists \mathbf{o} \in \mathfrak{O})[\mathbf{o} \leq \mathbf{x}],$$

and

$$\mathbf{x} \in \mathbf{I} \iff (\forall n)[\mathbf{x} \notin \mathbf{H}_n \text{ and } \mathbf{x} \notin \mathbf{L}_n] \iff$$

$$\iff (\forall n)[\mathbf{o}_n \not\leq \mathbf{x} \text{ and } \mathbf{x} \wedge \mathbf{o}_n \neq \mathbf{0}] \iff$$

$$\iff (\forall \mathbf{o} \in \mathfrak{O})[\mathbf{o} \not\leq \mathbf{x} \text{ and } \mathbf{x} \wedge \mathbf{o} \neq \mathbf{0}].$$

In the case of $\mathcal{G}_\omega$, (Ganchev and Sariev 2015) exploit additional connections between the $\mathbf{o}_n$ degrees and the elements of the Kalimullin pairs in order to find a definition for the class $\mathfrak{O}$ and hence, definitions for $\mathbf{L}$, $\mathbf{H}$, and $\mathbf{I}$.

The existing strong parallel between the ω-Turing and the ω-enumeration degree structures yields the hypothesis that the classes of the low, high, and intermediate degrees are first-order definable also in $\mathcal{G}_{T,\omega}$. In this article we find an interesting cupping property of the $\mathbf{o}_n$ degrees – for each degree $\mathbf{b}$ above $\mathbf{o}_n$, the equation $\mathbf{o}_n \vee \mathbf{x} = \mathbf{b}$ always has a least solution. Further we use the above property as our main component in the first-order definition of the classes $\mathfrak{O}$, $\mathbf{L}$, $\mathbf{H}$, and $\mathbf{I}$.

Finally as a consequence of the first-order definability of $\mathfrak{O}$, we shall conclude the definability of an interesting class of degrees. More precisely, we shall call a degree *almost zero* iff it contains such a sequence whose members, when considered in isolation, are Turing equivalent to the corresponding member of the least ω-sequence, $\{\emptyset\}_{k<\omega}$, but not necessary uniformly. A characterization of the almost zero degrees in the local structure by Sariev and Ganchev reveals that these degrees are exactly those that are below each $\mathbf{o}_n$. Having this, we shall be able to define the class of the almost zero degrees in $\mathcal{G}_{T,\omega}$.

2. Preliminaries

2.1 Basic notations

We shall denote the set of all natural numbers by ω. If not stated otherwise, a, b, c, ...shall stand for natural numbers, A, B, C, ...for sets of natural numbers, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, ...for degrees, and

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$\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, ...for sequences of sets of natural numbers. We shall further follow the convention: whenever a sequence is denoted by a calligraphic Latin letter, then we shall use the roman style of the same Latin letter, indexed with a natural number, say k, to denote the k-th element of the sequence (we always start counting from 0). Thus $\mathcal{A} = \{A_k\}_{k<\omega}$, $\mathcal{B} = \{B_k\}_{k<\omega}$, $\mathcal{C} = \{C_k\}_{k<\omega}$, etc. We shall denote the set of all sequences (*of length* ω) of sets of natural numbers by $\mathcal{S}_\omega$.

As usual $A \oplus B$ shall stand for the set $\{2x \mid x \in A\} \cup \{2x+1 \mid x \in B\}$.

2.2 The Turing degrees

We assume that the reader is familiar with the notion of Turing reducibility, $\leq_T$, and with the structure of the Turing degrees.

For every natural number e and every function $f \in \omega^\omega$, we denote by $\{e\}^f$ the partial function computed by the oracle Turing machine with index e and using f as an oracle. Thus for arbitrary sets $A, B \subseteq \omega$, $A \leq_T B \iff \chi_A = \{e\}^{\chi_B}$ for some natural number e. Given a natural number x, we shall indicate by $\{e\}^f(x)\downarrow$ that the function $\{e\}^f$ is defined in x. In case that $\{e\}^f$ is not defined in x, we shall write $\{e\}^f(x)\uparrow$.

Recall, that A is computably enumerable (*c.e.*) in B *iff* there is an enumeration f of A (i.e., a surjection from ω onto A), such that $f = \{e\}^{\chi_B}$ for some natural number e.

By A' we shall denote the Turing jump of the set A, that is,

$$A' = \{x \mid \{x\}^{\chi_A}(x)\downarrow\}.$$

Recall that A' is c.e. in A and every set c.e. in A is Turing reducible to A'. Further, the jump operator preserves uniformly the Turing reducibility, that is, there is a computable function f, such that for arbitrary sets A and B if A is Turing reducible to B via the oracle Turing machine with index e, then A' is Turing reducible to B' via the oracle Turing machine with index $f(e)$.

The relation $\leq_T$ is a preorder on the powerset $\mathcal{P}(\omega)$ of the natural numbers and induces a nontrivial equivalence relation $\equiv_T$. The equivalence classes under $\equiv_T$ are called Turing degrees. The Turing degree which contains the set A is denoted by $\deg_T(A)$. The set of all Turing degrees is denoted by $\mathbf{D}_T$. The Turing reducibility between sets induces a partial order $\leq$ on $\mathbf{D}_T$ by

$$\deg_T(A) \leq \deg_T(B) \iff A \leq_T B.$$

We denote by $\mathcal{D}_T$ the partially ordered set $(\mathbf{D}_T, \leq)$. The least element of $\mathcal{D}_T$ is the Turing degree $\mathbf{0}_T$ of $\emptyset$. Also, the Turing degree of $A \oplus B$ is the least upper bound of the degrees of A and B. Therefore, $\mathcal{D}_T$ is an upper semilattice with least element.

The jump operation gives rise to the local substructure $\mathcal{G}_T$, consisting of all degrees below $\mathbf{0}'_T$ – the jump of the least Turing degree. In fact, $\mathcal{G}_T$ is exactly the collection of all Δ^0_2 Turing degrees.

Finally we need the following definition, which we shall use in order to characterize ω-Turing reducibility.

Definition 1. *Let $\mathcal{A} = \{A_k\}_{k<\omega} \in \mathcal{S}_\omega$. The jump-sequence $\mathcal{P}(\mathcal{A})$ of $\mathcal{A}$ is the sequence $\{P_k(\mathcal{A})\}_{k<\omega}$ defined inductively as:*

1. $P_0(\mathcal{A}) = A_0$;
2. $P_{k+1}(\mathcal{A}) = P_k(\mathcal{A})' \oplus A_{k+1}$.

Note that $\mathcal{P}(\mathcal{A})$ is a monotone sequence with respect to Turing reducibility. Moreover, every member of the jump-sequence can solve the halting problem of the previous member.

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3. The ω-Turing Degrees

In this section we shall give the basic definitions and list the main properties of the ω-Turing degrees which we shall need in the last section. All of these definitions and proofs can be found in (Sariev and Ganchev 2014).

3.1 The structure $\mathcal{D}_{T,\omega}$

The ω-Turing reducibility $\leq_{T,\omega}$ is a relation between sequences of sets of natural numbers. In the introduction, the original definition is given in terms of the so-called *jump class* of a sequence. Here we use an equivalent one based on the uniform Turing reducibility.

Definition 2. *Let $\mathcal{A}, \mathcal{B} \in \mathcal{S}_\omega$. We shall say that $\mathcal{A}$ is ω-Turing reducible to $\mathcal{B}$, denoted by $\mathcal{A} \leq_{T,\omega} \mathcal{B}$, if for every n,*

$$A_n \leq_T P_n(\mathcal{B}) \text{ uniformly in } n.$$

In other words, $\mathcal{A} \leq_{T,\omega} \mathcal{B}$ *iff* there is a computable function f, such that for every natural number k, $\chi_{A_k} = \{f(k)\}^{P_k(\mathcal{B})}$.

As one may expect, the relation $\leq_{T,\omega}$ is reflexive and transitive, and the relation $\equiv_{T,\omega}$ defined by

$$\mathcal{A} \equiv_{T,\omega} \mathcal{B} \iff \mathcal{A} \leq_{T,\omega} \mathcal{B} \;\&\; \mathcal{B} \leq_{T,\omega} \mathcal{A}$$

is an equivalence relation.

The equivalence classes under $\equiv_{T,\omega}$ are called ω-Turing degrees. In particular the equivalence class $\deg_{T,\omega}(\mathcal{A}) = \{\mathcal{B} \mid \mathcal{A} \equiv_{T,\omega} \mathcal{B}\}$ is called the ω-Turing degree of $\mathcal{A}$. We shall denote the set of all ω-Turing degrees by $\mathbf{D}_{T,\omega}$.

The relation $\leq$ defined by

$$\mathbf{a} \leq \mathbf{b} \iff (\exists \mathcal{A} \in \mathbf{a})(\exists \mathcal{B} \in \mathbf{b})[\mathcal{A} \leq_{T,\omega} \mathcal{B}] \tag{1}$$

is a partial order on $\mathbf{D}_{T,\omega}$. By $\mathcal{D}_{T,\omega}$ we shall denote the structure $(\mathbf{D}_{T,\omega}, \leq)$.

The ω-Turing degree $\mathbf{0}_{T,\omega}$ of the sequence $\emptyset_\omega = \{\emptyset\}_{k<\omega}$, whose every member is the empty set, is the least element in $\mathcal{D}_{T,\omega}$.

Further, the ω-Turing degree of the sequence $\mathcal{A} \oplus \mathcal{B} = \{A_k \oplus B_k\}_{k<\omega}$ is the least upper bound $\mathbf{a} \vee \mathbf{b}$ of the pair of degrees $\mathbf{a} = \deg_{T,\omega}(\mathcal{A})$ and $\mathbf{b} = \deg_{T,\omega}(\mathcal{B})$.

Thus $\mathcal{D}_{T,\omega}$ is an upper semi-lattice with least element.

From the definitions of the jump-sequence and of the ω-Turing reducibility follows that each sequence is ω-Turing equivalent with its jump-sequence, that is, for all $\mathcal{A} \in \mathcal{S}_\omega$,

$$\mathcal{A} \equiv_{T,\omega} \mathcal{P}(\mathcal{A}). \tag{2}$$

Note that there exist only countably many computable functions, which implies that there could be only countably many sequences ω-Turing reducible to a given sequence. In particular every ω-Turing degree cannot contain more than countably many sequences and hence there are continuum many ω-Turing degrees.

3.2 Embedding of the Turing degrees

Given a set $A \subseteq \omega$, denote by $A \uparrow \omega$ the sequence $(A, \emptyset, \emptyset, \ldots, \emptyset, \ldots)$. From the definition of $\leq_{T,\omega}$ and the uniformity of the jump operation, we have that for all sets A and B,

$$A \uparrow \omega \leq_{T,\omega} B \uparrow \omega \iff A \leq_T B. \tag{3}$$

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The last equivalence means that the mapping $\kappa : \mathbf{D}_T \to \mathbf{D}_{T,\omega}$, defined by

$$\kappa(\mathbf{x}) = \deg_{T,\omega}(X \uparrow \omega),$$

where X is an arbitrary set in $\mathbf{x}$, is an embedding of $\mathcal{D}_T$ into $\mathcal{D}_{T,\omega}$. Also, the so defined embedding κ preserves the least element and the binary least upper bound operation.

The above-mentioned properties of the embedding κ are summarized in the following proposition.

Proposition 1. *For all Turing degrees $\mathbf{a}, \mathbf{b} \in \mathbf{D}_T$,*

1. $\mathbf{a} \leq \mathbf{b} \iff \kappa(\mathbf{a}) \leq \kappa(\mathbf{b})$;
2. $\kappa(\mathbf{a} \vee \mathbf{b}) = \kappa(\mathbf{a}) \vee \kappa(\mathbf{b})$;
3. $\kappa(\mathbf{0}_T) = \mathbf{0}_{T,\omega}$.

3.3 Jump operator

Before we define the jump operator on degrees, we need to define a jump of a sequence of sets of natural numbers.

Definition 3. *Let $\mathcal{A} \in \mathcal{S}_\omega$. The ω-Turing jump of the sequence $\mathcal{A}$ is the sequence*

$$\mathcal{A}' = (P_1(\mathcal{A}), A_2, A_3, \ldots, A_k, \ldots).$$

Note that for each k, $P_k(\mathcal{A}') = P_{1+k}(\mathcal{A})$. Hence:

$$\mathcal{A}' \equiv_{T,\omega} \{P_{k+1}(\mathcal{A})\}. \tag{4}$$

The jump is strictly increasing and monotone operation, that is,

$$\mathcal{A} \lneq_{T,\omega} \mathcal{A}' \text{ and } \mathcal{A} \leq_{T,\omega} \mathcal{B} \Rightarrow \mathcal{A}' \leq_{T,\omega} \mathcal{B}'.$$

This allows to define a jump operation on the ω-Turing degrees by setting

$$\mathbf{a}' = \deg_{T,\omega}(\mathcal{A}'),$$

where $\mathcal{A}$ is an arbitrary sequence in $\mathbf{a}$. Clearly $\mathbf{a} < \mathbf{a}'$ and $\mathbf{a} \leq \mathbf{b} \Rightarrow \mathbf{a}' \leq \mathbf{b}'$.

Also the jump operation on ω-Turing degrees agrees with the jump operation on the Turing degrees, that is, we have

$$\kappa(\mathbf{x}') = \kappa(\mathbf{x})', \text{ for all } \mathbf{x} \in \mathbf{D}_T.$$

We shall denote by $\mathcal{A}^{(n)}$ the n-th iteration of the jump operator on $\mathcal{A}$. Let us note that

$$\mathcal{A}^{(n)} = (P_n(\mathcal{A}), A_{n+1}, A_{n+2}, \ldots) \equiv_{T,\omega} \{P_{n+k}(\mathcal{A})\}_{k<\omega}. \tag{5}$$

It is clear that if $\mathcal{A} \in \mathbf{a}$, then $\mathcal{A}^{(n)} \in \mathbf{a}^{(n)}$, where $\mathbf{a}^{(n)}$ denotes the n-th iteration of the jump operation on the ω-Turing degree $\mathbf{a}$.

3.4 Jump inversion

Just like the Turing degrees, each ω-Turing degree $\mathbf{b}$ above ${\mathbf{0}_{T,\omega}}^{(n)}$ is a n-th jump of some degree $\mathbf{x}$. But in $\mathcal{D}_{T,\omega}$ it holds a much stronger jump inversion property.

Proposition 2. *For each $n < \omega$, if ${\mathbf{0}_{T,\omega}}^{(n)} \leq \mathbf{b}$, then there is a least degree $\mathbf{x} \in \mathbf{D}_{T,\omega}$ such that $\mathbf{x}^{(n)} = \mathbf{b}$.*

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We shall denote this degree by $\mathbf{I}^n(\mathbf{b})$. An explicit representative of $\mathbf{I}^n(\mathbf{b})$ can be given by setting

$$I^n(\mathcal{B}) = (\underbrace{\emptyset, \emptyset, \ldots, \emptyset}_{n}, B_0, B_1, \ldots, B_k, \ldots), \tag{6}$$

where $\mathcal{B} \in \mathbf{b}$ is arbitrary.

From here it follows that for every given $n < \omega$, the operation $\mathbf{I}^n$ is monotone. Also its range is downwards closed.

The following proposition summarizes the properties we shall use later.

Proposition 3. *Let* $\mathbf{a}, \mathbf{b} \in \mathbf{D}_{T,\omega}$ *be such that* ${\mathbf{0}_{T,\omega}}^{(n)} \leq \mathbf{a}, \mathbf{b}$. *Then:*

1. $\mathbf{a} \leq \mathbf{b} \iff \mathbf{I}^n(\mathbf{a}) \leq \mathbf{I}^n(\mathbf{b})$;
2. $(\forall \mathbf{x} \in \mathbf{D}_{T,\omega})[\mathbf{x} \leq \mathbf{I}^n(\mathbf{b}) \Rightarrow \mathbf{x} = \mathbf{I}^n(\mathbf{x}^{(n)})]$.

3.5 The local structure, jump hierarchy, and the $\mathbf{o}_n$ degrees

The degree substructure lying beneath the first jump of the least element is usually referred to as the *local structure* of the degree structure. In the case of the ω-Turing degrees, we shall denote this structure by $\mathcal{G}_{T,\omega}$,

$$\mathcal{G}_{T,\omega} = (\{\mathbf{x} \in \mathbf{D}_{T,\omega} \mid \mathbf{x} \leq {\mathbf{0}_{T,\omega}}'\}, \leq),$$

where $\leq$ is the inherited from $\mathcal{D}_{T,\omega}$ order.

When considering a local structure, one is usually concerned with questions about the definability of some classes of degrees, which have a natural definition either in the context of the global structure (e.g., the classes of the high and the low degrees) or in the context of the basic objects from which the degrees are built (e.g., the class of the Turing degrees containing a c.e. set).

Recall that a degree in the local structure is said to be *high*$_n$ for some n *iff* its n-th jump is as high as possible. Similarly a degree in the local structure is said to be *low*$_n$ for some n *iff* its n-th jump is as low as possible. The degrees that are neither high$_n$ nor low$_n$ for any n shall be referred to as *intermediate* degrees. The corresponding degree classes are described in the following definition.

Definition 4. *Let* $\mathbf{a} \leq {\mathbf{0}_{T,\omega}}'$. *Then:*

1. $\mathbf{a} \in \mathbf{L}_n \iff \mathbf{a}^{(n)} = {\mathbf{0}_{T,\omega}}^{(n)}$;
2. $\mathbf{a} \in \mathbf{H}_n \iff \mathbf{a}^{(n)} = {\mathbf{0}_{T,\omega}}^{(n+1)}$;
3. $\mathbf{L} = \bigcup \mathbf{L}_n$ *and* $\mathbf{H} = \bigcup \mathbf{H}_n$;
4. $\mathbf{a} \in \mathbf{I} \iff \mathbf{a} \notin \mathbf{L} \cup \mathbf{H}$.

Using the corresponding results for the structure of the Turing degrees, it is easy to see that the jump hierarchy is non-degenerative, that is, that $\mathbf{I} \neq \emptyset$ and that for all $n < \omega$, $\mathbf{H}_{n+1} \setminus \mathbf{H}_n \neq \emptyset$ and $\mathbf{L}_{n+1} \setminus \mathbf{L}_n \neq \emptyset$.

Our definition of the classes $\mathbf{L}$ and $\mathbf{H}$ relies heavily on the following notion.

Definition 5. *For each* $n < \omega$, *set* $\mathbf{o}_n$ *to be the least* n*-th jump invert of* ${\mathbf{0}_{T,\omega}}^{(n+1)}$, *that is,*

$$\mathbf{o}_n = \mathbf{I}^n({\mathbf{0}_{T,\omega}}^{(n+1)}).$$

In other words, for each n, $\mathbf{o}_n$ is the least degree, satisfying the equality $\mathbf{x}^{(n)} = {\mathbf{0}_{T,\omega}}^{(n+1)}$. Note also that $\mathbf{o}_0 = {\mathbf{0}_{T,\omega}}'$. The following proposition summarizes the basic properties of the degrees $\mathbf{o}_n$.

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Proposition 4. *For all $n < \omega$ and $\mathbf{x} \leq \mathbf{0}_{T,\omega}{}'$,*

1. *the sequence $\mathcal{O}_n = (\emptyset, \emptyset', \dots, \emptyset^{(n-1)}, \emptyset^{(n+1)}, \emptyset^{(n+2)}, \dots) \in \mathbf{o}_n$;*
2. $\mathbf{0}'_\omega = \mathbf{o}_0 > \mathbf{o}_1 > \mathbf{o}_2 > \dots > \mathbf{o}_n > \dots$;
3. $[\mathbf{0}_{T,\omega}, \mathbf{o}_n] \cong [\mathbf{0}_{T,\omega}{}^{(n)}, \mathbf{0}_{T,\omega}{}^{n+1}]$;
4. $\mathbf{I}^n(\mathbf{x}^{(n)}) = \mathbf{x} \wedge \mathbf{o}_n$.

The next proposition presents a characterization of the classes $\mathbf{H}_n$ and $\mathbf{L}_n$ that does not involve directly the jump operation.

Proposition 5. *Let $\mathbf{x} \leq \mathbf{0}_{T,\omega}{}'$. Then:*

1. $\mathbf{x} \in \mathbf{H}_n \iff \mathbf{o}_n \leq \mathbf{x}$;
2. $\mathbf{x} \in \mathbf{L}_n \iff \mathbf{x} \wedge \mathbf{o}_n = \mathbf{0}_{T,\omega}$.

Note that from the characterizations from Proposition 5, in order to show that the classes $\mathbf{H}_n$ and $\mathbf{L}_n$ are first-order definable in $\mathcal{G}_{T,\omega}$, it is sufficient to show this for the degree $\mathbf{o}_n$. For the definition of the classes $\mathbf{H} = \bigcup \mathbf{H}_n$, $\mathbf{L} = \bigcup \mathbf{L}_n$ and $\mathbf{I} = \mathcal{G}_{T,\omega} \setminus (\mathbf{H} \cup \mathbf{L})$ it is sufficient to show the definability of the set $\mathfrak{O} = \{\mathbf{o}_n \mid n < \omega\}$.

In (Sariev and Ganchev 2014) it is shown that each one of the degrees $\mathbf{o}_n$ is first-order definable in $\mathcal{G}_{T,\omega}$. The definitions rely on the notion of a *noncuppable*[1] degree. Namely, the following proposition holds.

Proposition 6. *For each $n < \omega$,*

$$\mathbf{o}_{n+1} \textit{ is the greatest degree beneath } \mathbf{o}_n\textit{, that is noncuppable to } \mathbf{o}_n.$$

Now note that the above definition is equivalent to a first-order formula in the language of the structure order $\leq$ and also that $\mathbf{o}_0 = \mathbf{0}_{T,\omega}{}'$ is first-order definable as the greatest element in $\mathcal{G}_{T,\omega}$. So using an induction on n, one can easily show the first-order definability of each one of the degrees $\mathbf{o}_n$. Thus we have the following theorem.

Theorem 1. *For every $n < \omega$, the classes $\mathbf{H}_n$ and $\mathbf{L}_n$ are first-order definable in the local structure of the ω-Turing degrees.*

3.6 Almost zero and minimal degrees

Further we shall need the notion of *almost zero* (*a.z.*) degrees.

Definition 6. *We shall say that the ω-Turing degree $\mathbf{x}$ is a.z. iff there is a representative $\mathcal{X} \in \mathbf{x}$ such that:*

$$(\forall k)[P_k(\mathcal{X}) \equiv_T \emptyset^{(k)}]. \tag{7}$$

It is clear that the class of the *a.z.* degrees is downward closed. Note also that there are continuum many *a.z.* degrees and hence not all *a.z.* degrees are in $\mathcal{G}_{T,\omega}$.

The *a.z.* degrees in the local structure $\mathcal{G}_{T,\omega}$ are exactly the degrees bounded by every degree $\mathbf{o}_n$, that is,

$$\mathbf{x} \in \mathcal{G}_{T,\omega} \text{ is } a.z. \iff (\forall n < \omega)[\mathbf{x} \leq \mathbf{o}_n]. \tag{8}$$

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We finish this section with some observations concerning the minimal[2] ω-Turing degrees. As has been shown in (Sariev and Ganchev 2014) there are exactly countably many minimal ω-Turing degrees and all of them are bounded by $\mathbf{0}_{T,\omega}{}'$. This follows from the characterization of the minimal degrees in $\mathcal{D}_{T,\omega}$:

Proposition 7. *An ω-Turing degree is minimal, if and only if it contains a sequence of the form*

$$(\underbrace{\emptyset, \emptyset, \dots, \emptyset}_{n}, A, \emptyset, \dots, \emptyset, \dots),$$

where the Turing degree of A is a minimal cover of $\mathbf{0}_T^{(n)}$ and $A' \equiv_T \emptyset^{(n+1)}$.

Note that no *a.z.* degree is minimal. Since each *a.z.* degree bounds only *a.z.* degrees, then no *a.z.* degree bounds a minimal degree. In converse, one can easily show that each of the degrees $\mathbf{o}_n$ bounds (countably many) minimal degrees.

4. Definability in $\mathcal{G}_{T,\omega}$

In this last section we shall show how to first order define the set $\mathfrak{O} = \{\mathbf{o}_n \mid n < \omega\}$. This definition is based on observations for the local theory of the Turing degrees. Finally, from the fact that the set $\mathfrak{O} = \{\mathbf{o}_n \mid n < \omega\}$ is first-order definable in $\mathcal{G}_{T,\omega}$, by (5), we conclude the definability of the classes $\mathbf{H}$, $\mathbf{L}$ and $\mathbf{I}$.

Further, if $\mathcal{D} = (\mathbf{D}, \leq, \vee)$ is an upper semi-lattice, and $\mathbf{a}, \mathbf{l}, \mathbf{r} \in \mathbf{D}$ are such that $\mathbf{l} \leq \mathbf{r}$, then by $\mathrm{Cup}(\mathbf{a}, \mathbf{l}, \mathbf{r})$ we shall denote the set of all solutions $\mathbf{x}$ of the equation $\mathbf{a} \vee \mathbf{x} = \mathbf{r}$ such that $\mathbf{l} \leq \mathbf{x} \leq \mathbf{r}$,

$$\mathrm{Cup}_{\mathcal{D}}(\mathbf{a}, \mathbf{l}, \mathbf{r}) = \{\mathbf{x} \mid \mathbf{l} \leq \mathbf{x} \leq \mathbf{r} \text{ and } \mathbf{a} \vee \mathbf{x} = \mathbf{r}\}.$$

Let us now consider a degree $\mathbf{b}$ in the local substructure $\mathcal{G}_{T,\omega}$ of the ω-Turing degrees, which is above $\mathbf{o}_1$. Since $(\emptyset, \emptyset^{(2)}, \emptyset^{(3)}, \dots) \in \mathbf{o}_1$, then $\mathbf{b}$ contains a sequence of the form $(B, \emptyset^{(2)}, \emptyset^{(3)}, \dots)$. Then it is easy to notice that the degree containing the sequence $B \uparrow \omega$ is the least degree, which cups $\mathbf{o}_1$ to $\mathbf{b}$. Using an analogous reasoning, in the following lemma, we show that each of the degrees $\mathbf{o}_n$ satisfies the formula:

$$\Phi(\mathbf{a}) \rightleftharpoons (\forall \mathbf{b} \geq \mathbf{a})[\mathrm{Cup}_{\mathcal{G}_{T,\omega}}(\mathbf{a}, \mathbf{0}_{T,\omega}, \mathbf{b}) \text{ has a least element}].$$

Lemma 1. *For each $n < \omega$, $\Phi(\mathbf{o}_n)$.*

Proof. Indeed, fix a natural number n. Let $\mathbf{b} \in \mathcal{G}_{T,\omega}$ be a degree above $\mathbf{o}_n$. Recall that $\mathbf{o}_n$ contains the sequence $\mathcal{O}_n = (\emptyset, \emptyset', \dots, \emptyset^{(n-1)}, \emptyset^{(n+1)}, \emptyset^{(n+2)}, \dots)$. Hence there are sets $B_0, B_1, \dots, B_{n-1}$ such that the sequence $(B_0, B_1, \dots, B_{n-1}, \emptyset^{(n+1)}, \emptyset^{(n+2)}, \dots)$ belongs to $\mathbf{b}$. Note that the degree $\widetilde{\mathbf{b}} = \deg_{T,\omega}(B_0, B_1, \dots, B_{n-1}, \emptyset, \emptyset, \dots)$ is in the set $\mathrm{Cup}_{\mathcal{G}_{T,\omega}}(\mathbf{o}_n, \mathbf{0}_{T,\omega}, \mathbf{b})$. Now, let $\mathbf{x}$ be an arbitrary degree, which cups $\mathbf{o}_n$ to $\mathbf{b}$ and let $\mathcal{X} \in \mathbf{x}$. Hence, for all $i < n$, we have that $B_i \leq_T P_i(\mathcal{X} \oplus \mathcal{O}_n) \equiv_T P_i(\mathcal{X})$. Thus, $\widetilde{\mathbf{b}} \leq \mathbf{x}$ and therefore, $\widetilde{\mathbf{b}}$ is the least element of $\mathrm{Cup}_{\mathcal{G}_{T,\omega}}(\mathbf{o}_n, \mathbf{0}_{T,\omega}, \mathbf{b})$. □

Further, recall that no *a.z.* degree bounds a minimal degree. On the other hand, for each n, $\mathbf{o}_n$ bounds a minimal degree. Thus the property

$$\Psi(\mathbf{a}) \rightleftharpoons (\exists \mathbf{m})[\mathbf{m} \text{ is a minimal degree } \& \ \mathbf{m} < \mathbf{a}]$$

separates $\mathfrak{O}$ from the *a.z.* degrees.

In fact, the formula $\Phi \& \Psi$ defines exactly the degrees $\mathbf{o}_n$. In order to prove it, first we need some additional observations concerning the local theory of the Turing degrees. By (Posner

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and Robinson 1981), for all Turing degrees $\mathbf{a}, \mathbf{b} \in \mathbf{D}_T(\mathbf{0}_T, \mathbf{0}'_T)$, there is a (low) Turing degree $\mathbf{c} \in \mathbf{D}_T(\mathbf{0}_T, \mathbf{0}'_T)$, such that $\mathbf{a} \vee \mathbf{c} = \mathbf{0}'_T$ and $\mathbf{b} \not\leq \mathbf{c}$. Hence, if $\mathbf{a}$ is a Turing degree strictly between the least element $\mathbf{0}_T$ and its first jump $\mathbf{0}'_T$, then the set

$$\mathrm{Cup}_{\mathcal{D}_T}(\mathbf{a}, \mathbf{0}_T, \mathbf{0}'_T) = \{\mathbf{x} \mid \mathbf{0}_T \leq \mathbf{x} \leq \mathbf{0}'_T \text{ and } \mathbf{a} \vee \mathbf{x} = \mathbf{0}'_T\}$$

is not empty, and does not have a least element. The result in (Posner and Robinson 1981) can be relativized straightforward.

Thus for each Turing degree $\mathbf{d}$, if $\mathbf{a}, \mathbf{b} \in \mathbf{D}_T(\mathbf{d}, \mathbf{d}')$, then there is a (low over $\mathbf{d}$) degree $\mathbf{c}$ in $\mathbf{D}_T(\mathbf{d}, \mathbf{d}')$, which cups $\mathbf{a}$ to $\mathbf{d}'$ and avoids $\mathbf{b}$. Using this relativization, we conclude that if $\mathbf{a}$ is in the interval $\mathbf{D}_T(\mathbf{d}, \mathbf{d}')$, then the set $\mathrm{Cup}_{\mathcal{D}_T}(\mathbf{a}, \mathbf{d}, \mathbf{d}')$ is not empty, but does not have a least element.

Now we are ready to prove that $\Phi\&\Psi$ defines only degrees in $\mathfrak{O}$.

Lemma 2. *For each* $\mathbf{a} \leq \mathbf{0}_{T,\omega}{}'$, *if* $\mathcal{G}_{T,\omega} \models \Phi(\mathbf{a})\&\Psi(\mathbf{a})$, *then* $\mathbf{a} \in \mathfrak{O}$.

Proof. Suppose that $\mathbf{a}$ is a degree in $\mathcal{G}_{T,\omega}$, such that $\Phi(\mathbf{a})$ and $\Psi(\mathbf{a})$ hold in $\mathcal{G}_{T,\omega}$, but $\mathbf{a}$ is not in $\mathfrak{O}$. Since $\mathbf{a}$ bounds a minimal degree, then $\mathbf{a}$ is not *a.z.* degree. By (8), there exists a greatest natural number n, such that $\mathbf{a} < \mathbf{o}_n$. Since $(\emptyset, \emptyset', \dots, \emptyset^{(n-1)}, \emptyset^{(n+1)}, \emptyset^{(n+2)}, \dots)$ is an element of $\mathbf{o}_n$, then there are sets $A_n, A_{n+1}, A_{n+2}, \dots$ such that the sequence $\mathcal{A} = (\emptyset, \emptyset', \dots, \emptyset^{(n-1)}, A_n, A_{n+1}, A_{n+2}, \dots)$ has a degree $\mathbf{a}$. By the choice of n, $\mathbf{a}$ is not below $\mathbf{o}_{n+1}$, so $\emptyset^{(n)} <_T P_n(\mathcal{A})$. On the other hand, $\mathbf{a} \leq \mathbf{0}_{T,\omega}{}'$ and $\mathbf{a} \neq \mathbf{o}_n$, so $P_n(\mathcal{A}) <_T \emptyset^{(n+1)}$.

Since $\mathbf{a} < \mathbf{o}_n$ and $\Phi(\mathbf{a})$, then there is a least degree $\mathbf{x}$ such that $\mathbf{a} \vee \mathbf{x} = \mathbf{o}_n$. Note that $\mathbf{x} \leq \mathbf{o}_n$, so it contains a sequence $\mathcal{X}$ of the form $(\emptyset, \emptyset', \dots, \emptyset^{(n-1)}, X_n, X_{n+1}, X_{n+2}, \dots)$, for some sets $X_n, X_{n+1}, X_{n+2}, \dots$. From $\mathbf{a} \vee \mathbf{x} = \mathbf{o}_n$ we have that $P_n(\mathcal{A}) \oplus P_n(\mathcal{X}) \equiv_T \emptyset^{(n+1)}$ and since $P_n(\mathcal{A}) <_T \emptyset^{(n+1)}$, then $\emptyset^{(n)} <_T P_n(\mathcal{X})$.

Further, note that if the set of natural numbers Y has a Turing degree in the set $\mathrm{Cup}_{\mathcal{D}_T}(\deg_T(P_n(\mathcal{A})), \mathbf{0}_T^{(n)}, \mathbf{0}_T^{(n+1)})$, then $\mathrm{Cup}_{\mathcal{G}_{T,\omega}}(\mathbf{a}, \mathbf{0}_{T,\omega}, \mathbf{o}_n)$ contains the ω-Turing degree of the sequence $(\emptyset, \emptyset', \dots, \emptyset^{(n-1)}, Y, \emptyset, \emptyset, \dots)$. By the relativization of the result of Posner and Robinson for $\mathbf{d} = \mathbf{0}_T^{(n)}$, $\mathrm{Cup}_{\mathcal{D}_T}(\deg_T(P_n(\mathcal{A})), \mathbf{0}_T^{(n)}, \mathbf{0}_T^{(n+1)})$ has an element not equal to $\mathbf{0}_T^{(n+1)}$. Therefore, $\mathrm{Cup}_{\mathcal{G}_{T,\omega}}(\mathbf{a}, \mathbf{0}_{T,\omega}, \mathbf{o}_n)$ has an element strictly below $\mathbf{o}_n$, and so $P_n(\mathcal{X}) <_T \emptyset^{(n+1)}$.

Finally, since $\emptyset^{(n)} <_T P_n(\mathcal{X}) <_T \emptyset^{(n+1)}$, again by the relativization of the Posner and Robinson result, there is a set Y, such that:

$$\emptyset^{(n)} <_T Y <_T \emptyset^{(n+1)},\ P_n(\mathcal{A}) \oplus Y \equiv_T \emptyset^{(n+1)} \text{ and } P_n(\mathcal{X}) \not\leq_T Y.$$

Then the ω-Turing degree $\mathbf{y}$ of the sequence $(\emptyset, \emptyset', \dots, \emptyset^{(n-1)}, Y, \emptyset, \emptyset, \dots)$ is an element of $\mathrm{Cup}_{\mathcal{G}_{T,\omega}}(\mathbf{a}, \mathbf{0}_{T,\omega}, \mathbf{o}_n)$, which is not above $\mathbf{x}$. A contradiction. □

Combining Lemmas 1 and 2 together, we have that for each degree $\mathbf{a}$ below $\mathbf{0}_{T,\omega}{}'$, $\mathbf{a} \in \mathfrak{O} \iff \mathcal{G}_{T,\omega} \models \Phi(\mathbf{a})\&\Psi(\mathbf{a})$. In order to conclude our main result, we only have to notice that $\Phi\&\Psi$ is equivalent to a first-order formula in the language of the partial orders.

Theorem 2. *The classes* $\mathbf{H}, \mathbf{L}$, *and* $\mathbf{I}$ *are first-order definable in* $\mathcal{G}_{T,\omega}$

A direct consequence of the latter and (8) is the following corollary.

Corollary 1. *The class of the a.z. degrees is first-order definable in* $\mathcal{G}_{T,\omega}$.

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Notes

1 Let $\mathcal{D} = (\mathbf{D}, \leq, \vee)$ is an upper semi-lattice. Let $\mathbf{a}, \mathbf{b} \in \mathbf{D}$ and $\mathbf{a} < \mathbf{b}$. We shall say that $\mathbf{a}$ is *noncuppable* to $\mathbf{b}$ iff the least upper bound of $\mathbf{a}$ and any $\mathbf{x}$ strictly less than $\mathbf{b}$ is not equal to $\mathbf{b}$:

$$(\forall \mathbf{x} < \mathbf{b})[\mathbf{a} \vee \mathbf{x} < \mathbf{b}].$$

2 A degree $\mathbf{m}$ is said to be minimal in a degree structure $\mathcal{D} = (\mathbf{D}, \mathbf{0}, \leq)$, if the only degree strictly less than $\mathbf{m}$ is the least element $\mathbf{0}$ of $\mathcal{D}$. Also, $\mathbf{m}$ is a minimal cover of $\mathbf{a}$ iff $\mathbf{a} < \mathbf{m}$ and the interval $\mathbf{D}(\mathbf{a}, \mathbf{m})$ is empty.

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