(The Bride's Chair)^2

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Nick's problem

This story begins in Nick Lord's workshop session at the Joined Up Mathematics conference at Keele in April 2008. He had decided to talk about Figure 1 and the results concerning it featured in [1]. He was also having an interesting e-mail exchange with Douglas Rogers, who pointed out that iterating such constructions twice produced a triangle homothetic to the original. The exchange focused on the analogous iterated Vecten configuration which John Mason recently discussed in [2]; but Nick could not see a similarly quick argument for the results in [1], which he thus set as a 'homework problem' for the workshop participants.

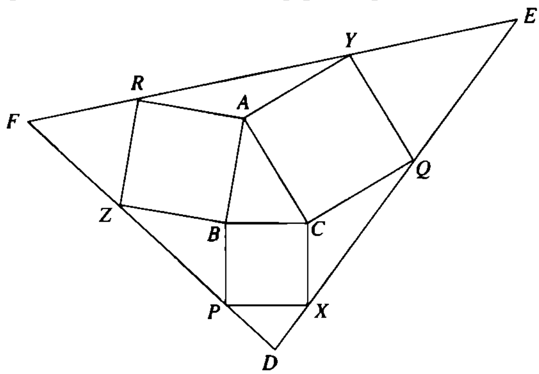

FIGURE 1

Figure 1 shows a triangle *ABC* and squares *BCXP*, *CAYQ*, *ABZR* drawn outwards on the sides, as in Euclid's proof of Pythagoras's theorem; but the triangle is not necessarily right-angled. This is the figure sometimes called the Bride's Chair. (Roger Webster, in the Note which first set this hare running [3], lists several other aliases: the Franciscan's cowl, the peacock's tail and the windmill.) The corners of adjacent squares *YR*, *ZP*, *XQ* are then joined, and these joins are produced to form a triangle *DEF*. We have called this the *square tangent triangle* of ΔABC, and its sides are the square tangent lines.

The focus of discussion was how ΔDEF is related to ΔABC. In particular, Nick's homework problem was to show that, if the process is repeated by constructing the square tangent triangle ΔIJK of ΔDEF, then the triangles ΔABC and ΔIJK are homothetic; that is, they are similar with their corresponding sides parallel.

A property of homothetic figures is that the lines joining corresponding points are concurrent in a centre of perspective. This is illustrated for homothetic triangles in Figure 2, for the cases where the triangles are the same way up and opposite ways up. Nick asked the supplementary question: can the centre of perspective of ΔABC and ΔIJK be identified as one of the known 'centres' of ΔABC?

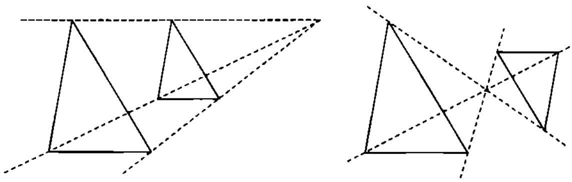

FIGURE 2

After the conference Brian Trustrum, John and Douglas sent proofs to Nick. Brian's proof was trigonometrical, Douglas used a method involving scalar and vector triple products, whilst John devised a proof using the methods of pure geometry which depended on the following lemma.

Lemma: The square tangent lines of ΔABC are perpendicular to the medians.

We offer two proofs of this: John's proof using transformations and an algebraic proof using an Argand diagram.

Proof using transformations: Figure 3 shows ΔABC and the squares $ACQY$ and $BARZ$ drawn outwards on the sides AC and BA. Complete the parallelograms $AYSR$ and $CABT$.

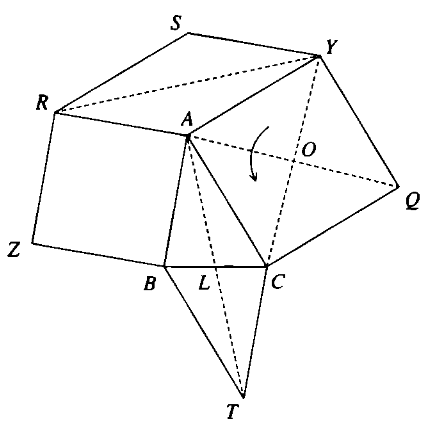

FIGURE 3

The idea of the proof is to show that (with ΔABC lettered in the positive sense) a positive quarter-turn about O, the centre of square $ACQY$, takes the parallelogram $AYSR$ to $CABT$. This follows by combining the following observations.

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- A positive quarter-turn about O takes Y to A and A to C.
- A positive quarter-turn about A takes the directed line segment $\overrightarrow{AR}$ to $\overrightarrow{AB}$.
- $\overrightarrow{YS}$ is equal and parallel to $\overrightarrow{AR}$, and $\overrightarrow{CT}$ is equal and parallel to $\overrightarrow{AB}$.

It follows that a positive quarter-turn about O takes $\overrightarrow{YS}$ to $\overrightarrow{AB}$ and $\overrightarrow{AR}$ to $\overrightarrow{CT}$. It therefore takes the parallelogram $AYSR$ to $CABT$.

So the diagonal YR is perpendicular to AT; and AT passes through L, the midpoint of BC. Therefore YR, which lies along the square tangent line EF, is perpendicular to the median AL.

Proof using an Argand diagram: Let $A, B, \ldots$ represent the complex numbers $a, b, \ldots$. Then, in Figure 1, the displacement $\overrightarrow{AC}$ corresponds to $c - a$, $\overrightarrow{AY}$ corresponds to $i(c - a)$, $\overrightarrow{BA}$ corresponds to $a - b$ and $\overrightarrow{AR}$ corresponds to $i(a - b)$. Therefore

$$y = a + i(c - a), \qquad r = a + i(a - b),$$

so $r - y = i(2a - b - c)$, which corresponds to $\overrightarrow{YR}$.

But $l = \frac{1}{2}(b + c)$, so $\overrightarrow{AL}$ corresponds to $l - a = \frac{1}{2}(b + c - 2a)$. Therefore

$$l - a = \tfrac{1}{2}i(r - y).$$

That is, $\overrightarrow{AL}$ is perpendicular to $\overrightarrow{YR}$, and half its length.

The problem generalised

This lemma casts a new light on Nick's problem. For the main result (though not the supplementary question) is only about the shape and orientation of the triangles ABC and IJK, not their size or location. So Nick's problem is a particular case of a more general conjecture.

General conjecture: If the sides of ΔDEF are perpendicular to the medians of ΔABC, and the sides of ΔIJK are perpendicular to the medians of ΔDEF, then ABC and IJK are homothetic.

(A neat way of stating this is:

If the sides of ΔDEF are perpendicular to the medians of ΔABC,
then the sides of ΔABC are perpendicular to the medians of ΔDEF.)

We are not yet home, but John drew our attention to a comparable known result in which the word 'perpendicular' is replaced by 'parallel' (see [4, section 473]).

Known result: If the sides of ΔPQR are parallel to the medians of ΔABC, and the sides of ΔUVW are parallel to the medians of ΔPQR, then ΔABC and ΔUVW are homothetic.

Again we offer two proofs.

Proof using the intercept theorem: We begin by finding two particular triangles with sides parallel to the medians of ΔABC and ΔPQR.

In Figure 4, G is the centroid of ΔABC; AG is produced to X so that $GL = LX$. Then $AG = 2GL = GX$ and $AM = MC$, so XC is parallel to GM. The sides of ΔCGX are therefore parallel to the medians of ΔABC. It follows that ΔCGX and ΔPQR are homothetic.

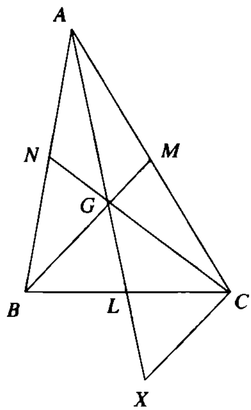

FIGURE 4

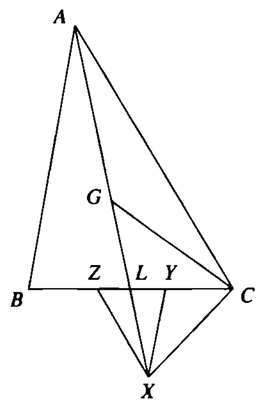

FIGURE 5

Now let Y and Z be the points of trisection of CB (see Figure 5). Then $CY = \frac{2}{3}CL$, so that Y is the centroid of ΔCGX, and $YL = LZ$. Figure 5 therefore contains a diagram like Figure 4, but with ΔCGX and ΔXYZ in place of ΔABC and ΔCGX. It follows that the sides of ΔXYZ are parallel to the medians of ΔCGX, and therefore of ΔPQR; so ΔXYZ and ΔUVW are homothetic.

But, in Figure 5, $AL = 3LX$, $BL = 3LY$ and $CL = 3LZ$. So ΔABC and ΔXYZ are homothetic, with L as centre of perspective. Therefore ΔABC and ΔUVW are homothetic.

Proof using vectors: Since we are concerned only with shape and orientation, not with location, it is best to use free vectors. In Figure 6, $\overrightarrow{BC} = \mathbf{x}$, $\overrightarrow{CA} = \mathbf{y}$ and $\overrightarrow{AB} = \mathbf{z}$, where $\mathbf{x} + \mathbf{y} + \mathbf{z} = \mathbf{0}$. Then $\overrightarrow{AL} = \overrightarrow{AB} + \overrightarrow{BL} = \mathbf{z} + \frac{1}{2}\mathbf{x}$, and similarly $\overrightarrow{BM} = \mathbf{x} + \frac{1}{2}\mathbf{y}$, $\overrightarrow{CN} = \mathbf{y} + \frac{1}{2}\mathbf{z}$. Therefore, if QR, RP, PQ are parallel to AL, BM, CN respectively,

$$\overrightarrow{QR} = \mu\left(\mathbf{z} + \tfrac{1}{2}\mathbf{x}\right), \quad \overrightarrow{RP} = \mu\left(\mathbf{x} + \tfrac{1}{2}\mathbf{y}\right), \quad \overrightarrow{PQ} = \mu\left(\mathbf{y} + \tfrac{1}{2}\mathbf{z}\right)$$

for some scalar μ.

Repeating this process,

$$\begin{aligned}\overrightarrow{VW} &= \nu\left(\overrightarrow{PQ} + \tfrac{1}{2}\overrightarrow{QR}\right) = \nu\mu\left(\mathbf{y} + \tfrac{1}{2}\mathbf{z} + \tfrac{1}{2}\left(\mathbf{z} + \tfrac{1}{2}\mathbf{x}\right)\right) \\ &= \nu\mu\left(\mathbf{y} + \mathbf{z} + \tfrac{1}{4}\mathbf{x}\right) = -\tfrac{3}{4}\nu\mu\mathbf{x} = -\tfrac{3}{4}\nu\mu\overrightarrow{BC}\end{aligned}$$

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for some scalar ν, since $\mathbf{x} + \mathbf{y} + \mathbf{z} = \mathbf{0}$. Similarly $\overrightarrow{WU} = -\frac{3}{4}\nu\mu\overrightarrow{CA}$ and $\overrightarrow{UV} = -\frac{3}{4}\nu\mu\overrightarrow{AB}$. Therefore ΔABC and ΔUVW are homothetic. (Note the minus signs, which show that, if the directions of the medians are chosen in a consistent manner, then the two triangles will be opposite ways up.)

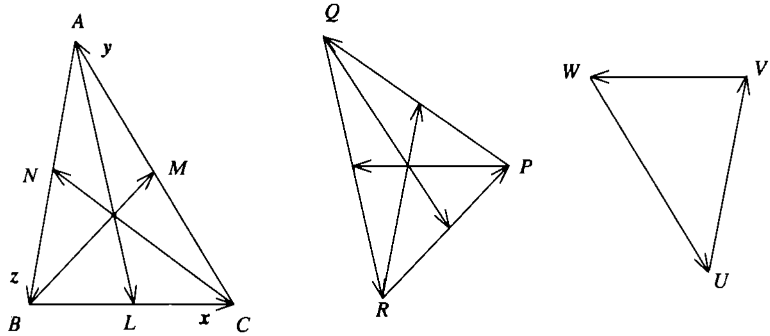

FIGURE 6

This result is about shape and orientation, but not size and location, so it is helpful to use the notation $\{\Delta ABC\}$ to describe the set of all triangles in the plane which are homothetic with ΔABC. Then if **M** denotes the operation which transforms a triangle into another triangle whose sides are parallel to the medians of the first, we can write

$$\{\Delta ABC\} \overset{\mathbf{M}}{\rightarrow} \{\Delta PQR\} \overset{\mathbf{M}}{\rightarrow} \{\Delta UVW\}.$$

What has been proved is that the sets $\{\Delta ABC\}$ and $\{\Delta UVW\}$ are the same, so that $\mathbf{M}^2 = \mathbf{I}$, the identity transformation.

The 'general conjecture' can be considered in the same way, but the sides of ΔDEF are perpendicular (rather than parallel) to the medians of ΔABC, and the sides of ΔIJK are perpendicular to the medians of ΔDEF. So to get from ΔABC to ΔDEF, and from ΔDEF to ΔIJK, the transformation **M** must be combined with a quarter-turn **Q**, so that

$$\{\Delta ABC\} \overset{\mathbf{QM}}{\rightarrow} \{\Delta DEF\} \overset{\mathbf{QM}}{\rightarrow} \{\Delta IJK\}.$$

The general conjecture states that the sets $\{\Delta ABC\}$ and $\{\Delta IJK\}$ are the same, so $(\mathbf{QM})^2 = \mathbf{I}$.

Two observations are sufficient to complete the proof.

- The transformations **Q** and **M** commute. It makes no difference to the shape or orientation whether we first construct a triangle parallel to the medians and then rotate it through a right angle, or first rotate the triangle and then construct a new triangle parallel to the medians. That is, $\mathbf{QM} = \mathbf{MQ}$, so that $(\mathbf{QM})^2 = \mathbf{Q}^2\mathbf{M}^2$.
- After two quarter-turns each line is transformed into a parallel line, so that any triangle is transformed into a homothetic

triangle (the opposite way up). This means that, as a transformation between sets of homothetic triangles, $\mathbf{Q}^2 = \mathbf{I}$.

Therefore $(\mathbf{QM})^2 = \mathbf{Q}^2\mathbf{M}^2 = \mathbf{II} = \mathbf{I}$. This proves the general conjecture, and in particular solves Nick's original problem.

There remains Nick's supplementary question, to identify the centre of perspective for the double square tangent triangle transformation. Since Nick's construction is just one of any number of possibilities covered by the general conjecture, it is unlikely that Nick's centre coincides with any of the known triangle centres. Indeed, Tony Robin kindly used computer algebra on the standard reference triangle to confirm that it is not in Kimberling's list of triangle centres [5].

Scale factors

Tony also alerted us to the fact that, in some cases, the scale factors of the homotheties involved may be of interest. This will be illustrated with two special cases of the general conjecture.

Before describing these we prove the existence of triangles whose sides are parallel *and equal* to the medians of a triangle ABC, and we establish some results relating their side-lengths and area to those of ΔABC. In Figure 4, $\overrightarrow{GX} = 2\overrightarrow{GL} = \frac{2}{3}\overrightarrow{AL}$, $\overrightarrow{XC} = 2\overrightarrow{GM} = \frac{2}{3}\overrightarrow{BM}$, $\overrightarrow{CG} = \frac{2}{3}\overrightarrow{CN}$. Hence any triangle homothetic to ΔCGX and $1\frac{1}{2}$ times its linear size has its sides parallel and equal to the medians of ΔABC. Such a triangle (e.g. ΔPQR in Figure 6 if the scale factor $\mu = 1$) is called a *triangle of medians* of ΔABC.

Denote the side lengths of ΔABC by a, b, c, its area by Δ, the length of the median through A by m_A, and the area of the triangle of medians by Δ_m.

The triangle of medians theorem: (i) $m_A^2 + m_B^2 + m_C^2 = \frac{3}{4}(a^2 + b^2 + c^2)$; (ii) $\Delta_m = \frac{3}{4}\Delta$; (iii) $\dfrac{m_A^2 + m_B^2 + m_C^2}{\Delta_m} = \dfrac{a^2 + b^2 + c^2}{\Delta}$.

Proofs:

(i) This follows immediately from Apollonius' theorem, $b^2 + c^2 = 2m_A^2 + 2\left(\frac{1}{2}a\right)^2$, so $m_A^2 = -\frac{1}{4}a^2 + \frac{1}{2}b^2 + \frac{1}{2}c^2$, and similarly $m_B^2 = \frac{1}{2}a^2 - \frac{1}{4}b^2 + \frac{1}{2}c^2$, $m_C^2 = \frac{1}{2}a^2 + \frac{1}{2}b^2 - \frac{1}{4}c^2$.

(ii) In Figure 4 $\text{area}(\Delta LXC) = \text{area}(\Delta LGB)$, hence $\text{area}(\Delta CGX) = \text{area}(\Delta CGB) = \frac{1}{3}\,\text{area}(\Delta ABC)$. So the area of the triangle of medians is $\left(\frac{3}{2}\right)^2 \text{area}(\Delta CGX) = \frac{9}{4} \times \left(\frac{1}{3}\Delta\right) = \frac{3}{4}\Delta$.

(iii) is a direct consequence of (i) and (ii).

Case 1: An anti-pedal triangle

In the general conjecture, take the triangle ΔDEF to be the lines perpendicular to the medians through the vertices of ΔABC. Kimberling [5] calls this the *anti-pedal triangle* of the centroid G of ΔABC. It is illustrated in Figure 7.

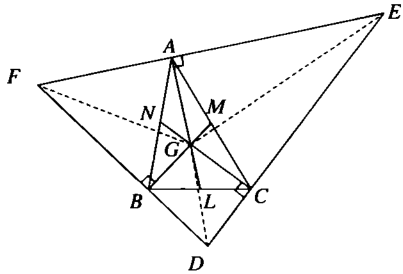

FIGURE 7

The sides of ΔDEF are perpendicular to the sides of the median triangle. Hence it is similar to the triangle of medians, so its side lengths EF, FD, DE can be written as λm_A, λm_B, λm_C for some value of λ. The area of ΔDEF is then $\lambda^2 \Delta_m$, which by the triangle of medians theorem is equal to $\frac{3}{4}\lambda^2\Delta$.

The triangle ΔGEF has base $EF = \lambda m_A$ and height $GA = \frac{2}{3}m_A$, so its area is $\frac{1}{3}\lambda m_A^2$. Similarly the areas of ΔGFD and ΔGDE are $\frac{1}{3}\lambda m_B^2$ and $\frac{1}{3}\lambda m_C^2$. So, using the triangle of medians theorem again,

$$\text{area}\left(\Delta DEF\right) = \tfrac{1}{3}\lambda\left(m_A^2 + m_B^2 + m_C^2\right) = \tfrac{1}{4}\lambda\left(a^2 + b^2 + c^2\right).$$

Therefore $\frac{3}{4}\lambda^2\Delta = \frac{1}{4}\lambda\left(a^2 + b^2 + c^2\right)$, which gives $\lambda = \dfrac{a^2 + b^2 + c^2}{3\Delta}$, and the area ratio of the transformation is

$$\frac{\text{area}\left(\Delta DEF\right)}{\text{area}\left(\Delta ABC\right)} = \frac{\frac{3}{4}\lambda^2\Delta}{\Delta} = \tfrac{3}{4}\lambda^2 = \tfrac{1}{12}k^2, \text{ where } \quad k = \frac{a^2 + b^2 + c^2}{\Delta}.$$

(It can be proved that k is related to the Brocard angle ω by the equation $\cot\omega = \frac{1}{4}k$. See [4, section 474] and [6].)

If you now iterate this construction to obtain the anti-pedal triangle of the centroid of ΔDEF, the ratio $\dfrac{\text{area}\left(\Delta IJK\right)}{\text{area}\left(\Delta DEF\right)}$ is $\frac{1}{12}k'^2$, where

$$k' = \frac{(\lambda m_A)^2 + (\lambda m_B)^2 + (\lambda m_C)^2}{\lambda^2\Delta_m} = \frac{m_A^2 + m_B^2 + m_C^2}{\Delta_m} = \frac{a^2 + b^2 + c^2}{\Delta} = k,$$

by the triangle of medians theorem. So the area ratio of the second transformation is also $\frac{1}{12}k^2$, and $\dfrac{\text{area}\left(\Delta IJK\right)}{\text{area}\left(\Delta ABC\right)} = \left(\frac{1}{12}k^2\right)^2$.

But we know from the general conjecture that ΔIJK is homothetic to ΔABC. It follows that the linear scale factor of this homothety is $\frac{1}{12}k^2$.

Case 2: The square tangent triangle

We can now calculate the area of ΔDEF in Figure 1, by adding together the areas of its component regions.

The central region ΔABC and the triangles in the corners D, E, F fit together to form a triangle congruent to the anti-pedal triangle in Figure 7, whose area is $\frac{1}{12}k^2\Delta$.

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The triangles ΔAYR and ΔABC are each one-half of the congruent parallelograms $AYSR$ and $CABT$ in Figure 3. Hence the area of ΔAYR is Δ, and the same is true of the areas of ΔBZP and ΔCXQ. (This property of the Bride's Chair was the subject of Roger Webster's note [3].)

The three squares have area a^2, b^2 and c^2. Hence the area of ΔDEF is

$$\tfrac{1}{12}k^2\Delta + 3\Delta + \left(a^2 + b^2 + c^2\right) = \tfrac{1}{12}k^2\Delta + 3\Delta + k\Delta = \tfrac{1}{12}(k + 6)^2\Delta.$$

The area ratio of the transformation is therefore

$$\frac{\text{area}\,(\Delta DEF)}{\text{area}\,(\Delta ABC)} = \tfrac{1}{12}(k + 6)^2\,\Delta.$$

By the same argument as in Case 1 above, this is also the area ratio for the second iteration of the square tangent triangle construction, so that

$$\frac{\text{area}\,(\Delta IJK)}{\text{area}\,(\Delta ABC)} = \left(\tfrac{1}{12}(k + 6)^2\right)^2.$$

And since ΔIJK is homothetic to ΔABC, the linear scale factor of this homothety is $\tfrac{1}{12}(k + 6)^2$.

The quantity $k = \left(a^2 + b^2 + c^2\right)/\Delta$ takes its least value $4\sqrt{3}$ when ΔABC is equilateral (exercise!); there is no upper bound. So for the repeated anti-pedal triangle construction the smallest possible linear scale factor is 4, and for the repeated square tangent triangle construction it is $(2 + \sqrt{3})^2$, or nearly 14. So if you want to verify Nick's problem for yourself, you will need a large sheet of paper.

References

1. Nick Lord, Similarity properties of the Bride's Chair, *Math. Gaz.* **91** (July 2007) pp. 328-331.
2. John Mason, Flanking figures, *Math. in School* **37**(1) (January 2008) pp. 28-29.
3. Roger Webster, Bride's Chair revisited, *Math. Gaz.* **78** (November 1994) pp. 345-346.
4. R. A. Johnson, *Advanced Euclidean geometry*, Dover (1960, reprinted 2007).
5. C.Kimberling, Encyclopedia of triangle centres, http://faculty.evansville.edu/ck6/encyclopedia/ETC.html
6. Ron Shail, *Some properties of Brocard points*, *Math. Gaz.* **80** (November 1996) pp. 485-491.

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