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ON THE CONNECTION BETWEEN DIFFERENTIAL POLYNOMIAL RINGS AND POLYNOMIAL RINGS OVER NIL RINGS

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Abstract

In this paper, we study some connections between the polynomial ring $R[y]$ and the differential polynomial ring $R[x; D]$. In particular, we answer a question posed by Smoktunowicz, which asks whether $R[y]$ is nil when $R[x; D]$ is nil, provided that R is an algebra over a field of positive characteristic and D is a locally nilpotent derivation.

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1. Introduction

Let R be a noncommutative, associative ring, not necessarily with unity. We define the ring of polynomials in the indeterminate y with coefficients in R to be $R[y]$. We say that an additive map $D : R \to R$ is a derivation if $D(ab) = D(a)b + aD(b)$ for all $a, b \in R$. We recall that the differential polynomial ring $R[x; D]$ is, as a set, given by all polynomials of the form $a_0 + a_1x + \cdots + a_nx^n$ with $n \geq 0$ and $a_i \in R$; multiplication is given by $xr = rx + D(r)$ for $r \in R$, extended using associativity and linearity. Throughout this paper, we will refer to the standard polynomial ring as $R[y]$ and the differential polynomial ring as $R[x; D]$ to avoid confusion when comparing polynomials in the respective rings.

Recently, there has been considerable interest in comparing differential polynomial rings over nil rings with polynomial rings. For example, Smoktunowicz proved in [3] that in the differential polynomial ring, $J(R[x; D]) = I[x; D]$ for some nil ideal I, when D is a locally nilpotent derivation and R is an algebra over a field of characteristic $p > 0$, thus extending a classical result by Amitsur [1]. On the other hand, in their influential paper, Smoktunowicz and Ziembowski proved that there exist a locally nilpotent ring R and a derivation $D : R \to R$ such that $R[x; D]$ is not a Jacobson radical ring [4].

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It is well known that if R is a locally nilpotent ring, then $R[y]$ is locally nilpotent. Our first result generalises this to the differential polynomial ring $R[x; D]$.

Theorem 1.1. *If R is a locally nilpotent ring with D a locally nilpotent derivation, then $R[x; D]$ is a locally nilpotent ring.*

As mentioned above, in [4], Smoktunowicz and Ziembowski showed that if R is locally nilpotent and D is a derivation on R, then $R[x; D]$ is not necessarily a Jacobson radical. Comparing this result with Theorem 1.1, we can see that the additional condition of D being locally nilpotent is crucial to the result. In the counterexample given in [4], the derivation D is an outer derivation. In particular, if D is an inner derivation, $R[x; D]$ will be locally nilpotent because an inner derivation on a locally nilpotent ring is nilpotent.

We now ask the following question.

Question 1.2. If $R[x; D]$ is a nil ring, does it follow that $R[y]$ is nil?

Let us stress that, in general, a polynomial ring over a nil ring need not be nil, due to a famous result by Smoktunowicz [2]. Our next result, Theorem 1.3, gives an answer to Question 1.2 in the affirmative, under the condition that the derivation D is an inner derivation.

Theorem 1.3. *Let R be a nil ring and D an inner derivation on R. If $R[x; D]$ is nil, then $R[y]$ is nil.*

Our final result answers the following question posed by Smoktunowicz [3, Question 2].

Question 1.4. Let F be a field of characteristic $p > 0$, let R be an F-algebra and let D be a locally nilpotent derivation on R. Suppose that $R[x; D]$ is nil. Does it follow that $R[y]$ is nil?

We answer this question in the affirmative in a similar fashion to Theorem 1.3.

Theorem 1.5. *Let F be a field of characteristic $p > 0$, let R be an F-algebra and let D be a locally nilpotent derivation on R. Suppose that $R[x; D]$ is nil. Then $R[y]$ is nil.*

Comparing Theorems 1.3 and 1.5, Theorem 1.3 gives information in the case of nil algebras over fields of characteristic 0; however, in Theorem 1.5, the conditions on the derivation are more general when considering the case of characteristic p, since an inner derivation on a nil ring is nilpotent.

Although Question 1.4 has been answered for characteristic p, we can ask if the result holds in general. In particular, we pose the following question.

Question 1.6. Let F be a field of characteristic 0, let R be an F-algebra and let D be a locally nilpotent derivation on R. Suppose that $R[x; D]$ is nil. Does it follow that $R[y]$ is nil?

The reader may notice that Question 1.6 is a particular case of Question 1.2. However, these questions may have different answers, so we have chosen to state Question 1.6 explicitly as a modification of Question 1.4.

2. Proofs

Proof of Theorem 1.1. Suppose that we have some finitely generated subring S of $R[x; D]$. It suffices to show that S is nilpotent. To do this, let $\{s_1, \ldots, s_r\}$ be a generating set for S. Since each s_i is a differential polynomial, we can write

$$s_i = a_{i,0} + a_{i,1}x + \cdots + a_{i,n_i}x^{n_i}.$$

We define the subring T of R as

$$T = \langle a_{i,j}, D^k(a_{i,j})\rangle \quad \text{for all } i, j, k.$$

Since T is a finitely generated subring of R, which is locally nilpotent, there exists some m such that $T^m = 0$. By the fact that D is a locally nilpotent derivation, there are only finitely many terms of the form $D^k(a_{i,j})$ in T. Suppose that we have some arbitrary polynomial s of S. Then $s^m = 0$, as every coefficient in s^m will be a sum of l-tuples of elements in T, where l is greater than or equal to m.

Thus, S is nilpotent, so $R[x; D]$ is locally nilpotent. □

Proof of Theorem 1.3. Since D is an inner derivation, for any $r \in R$, we may write $D(r) = [a, r] = ar - ra$, where $a \in R$ is fixed. Additionally, by definition of a differential polynomial ring, we know that $D(r) = xr - rx$ for all $r \in R$.

Suppose for the sake of contradiction that $R[y]$ is not nil. Let

$$f(y) = a_0 + a_1y + a_2y^2 + \cdots + a_ny^n \in R[y]$$

be a polynomial that is not nilpotent. We note that $[x - a, r] = 0$ for all $r \in R$ because

$$\begin{aligned}(x - a)r - r(x - a) &= xr - ar - rx + ra\\ &= (xr - rx) - (ar - ra)\\ &= D(r) - D(r)\\ &= 0.\end{aligned}$$

Let $\alpha : R[y] \to R[x; D]$ be a map defined by the rule

$$\alpha(b_0 + b_1y + \cdots + b_ny^n) = b_0 + b_1(x - a) + \cdots + b_n(x - a)^n,$$

where each $b_i \in R$. Let $g_1(y), g_2(y)$ be two arbitrary elements of $R[y]$ and introduce $g(y) = g_1(y)g_2(y)$. Thus, since $(x - a)$ commutes with all elements of the ring R, we notice that $\alpha(g(y)) = g(x - a) = g_1(x - a)g_2(x - a) = \alpha(g_1(y))\alpha(g_2(y))$ and also $\alpha(g_1(y) + g_2(y)) = \alpha(g_1(y)) + \alpha(g_2(y))$. So, α is a homomorphism and, by our definition, it is injective. Now, for our polynomial $f(y)$, the subring $\langle f(y)\rangle \subset R[y]$ is isomorphic to $\langle f(x - a)\rangle \subset R[x; D]$. Since $f(y)$ is not nilpotent, $f(x - a)$ is not nilpotent, contradicting the assumption that $R[x; D]$ is nil.

Hence, $R[y]$ is nil. □

We will prove Theorem 1.5 in the same vein as Theorem 1.3, by constructing an isomorphism between certain subrings of the differential polynomial ring and the polynomial ring. However, the isomorphisms will be constructed in a different way.

PROOF OF THEOREM 1.5. Let $r \in R$ be any element. Since $\mathrm{char}(F) = p$, we can see that $D^{p^m}(r) = x^{p^m} \cdot r - r \cdot x^{p^m}$ for any $m \in \mathbb{N}$ [3, page 559].

Suppose for the sake of contradiction that $R[y]$ is not nil. We have some polynomial

$$f(y) = a_0 + a_1 y + a_2 y^2 + \cdots + a_n y^n \in R[y]$$

such that f is not nilpotent. Since D is a locally nilpotent derivation, we can find some power k such that

$$D^{p^k}(a_i) = x^{p^k} a_i - a_i x^{p^k} = 0$$

for all i; that is, all coefficients of f commute with x^{p^k}. Consider

$$f(x^{p^k}) = a_0 + a_1 x^{p^k} + a_2 x^{2p^k} + \cdots + a_n x^{np^k} \in R[x; D].$$

Similarly to the proof of Theorem 1.3, the element x^{p^k} commutes with the elements of the ring R, so the subring $\langle f(x^{p^k}) \rangle \subset R[x; D]$ is isomorphic to $\langle f(y) \rangle \subset R[y]$. Then, since $f(y)$ is not nilpotent, $f(x^{p^k})$ is not nilpotent, which contradicts the assumption that $R[x; D]$ is nil.

Hence, $R[y]$ is nil. □

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