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VOLUME–VOLATILITY DYNAMICS IN AN INTERTEMPORAL ASSET PRICING MODEL

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This paper embeds time-varying volatility into a dynamic equilibrium model of returns and trading. The model allows us to ask how time-varying volatility might affect the relation among return autocorrelation, volatility, and trading volume, as opposed to the pairwise relations that have been studied previously. It is shown analytically that, with time-varying volatility, the relationship between volume and stock return autocorrelation is ambiguous even if agents have symmetric information, which may explain the contradictory findings in the empirical literature. In the numerical exercise, the model is simulated in a way that mimics the persistent volatility of high-frequency stock data documented in numerous empirical studies. Specially, the time-varying volatility of stock returns is approximated with a highly persistent chaotic tent map, which is known to have the same autocorrelation coefficients as an AR(1) process. The simulated data can approximate GARCH-type behavior very well. Whereas in the simulated data, no significant relation between volume and return autocorrelation can be found, there is a significantly positive relation between volume and one-step-ahead stock return volatility. The ambiguous volume–persistence and positive volume–volatility relations are confirmed empirically by using four heavily traded individual stocks. Therefore, the data simulated from the highly stylized asset pricing model with deterministic time-varying volatility can mimic well the volume–return dynamics revealed in the observed data in these two respects.

1. INTRODUCTION

There are relatively few theoretical studies of the joint relationship among stock return autocorrelation, volatility, and volume. This paper embeds time-varying volatility into a dynamic equilibrium model of returns and trading. Hence, this model allows us to ask how time-varying volatility might affect the relations among return autocorrelation, volatility, and trading volume, as opposed to the pairwise relations that have been studied previously. Most previous theoretical studies of the joint behavior of returns and trading volume do not involve dynamic optimizing behavior by agents. Wang (1994) proposed a dynamic equilibrium model of stock trading to study the relation between volume and the nature of heterogeneity among investors. His model assumes that agents are heterogeneous in their investment

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opportunities and/or in their information about the stock's future payoff. Without information asymmetry, volume is always negatively related to the return autocorrelation. With information asymmetry, however, the relationship between volume and the serial correlation of returns is ambiguous. This information asymmetry is one potential explanation for the contradictory empirical findings concerning the relation between stock return autocorrelation and volume: Campbell et al. (1993) and Conrad et al. (1994) found a negative relation, whereas Morse (1978) and Antoniewicz (1992) found a positive relation; LeBaron (1992) found that the sign depends on the detrending procedure applied to the volume series; Tauchen et al. (1996) found no influence of volume on the autocorrelation of stock returns.

In this paper, I show that if stock return volatility is time varying, the relationship between volume and stock return autocorrelation is ambiguous even if agents have symmetric information. Moreover, under the assumption of time-varying volatility, one can study the correlatedness of volume with the one-period-ahead return volatility, which has been the central issue of numerous empirical studies [Lamoureux and Lastrapes (1990), Bollerslev et al. (1992), Gallant et al. (1992), Brock and LeBaron (1996), and others].

I extend a special case of Wang's model—the case of symmetric information—by assuming that some of the shocks in the economy have autocorrelated time-varying but deterministic volatility. Ideally, one would use a time-varying stochastic volatility. However, if the volatility is stochastic, there is no closed-form solution for the intertemporal optimization problem because the form of the value function is unknown. Therefore, the analysis of the dynamics between volume and returns becomes intractable. With deterministic time-varying volatility, it is shown analytically that the predicted relation between stock return autocorrelation and volume is not monotonic: It can be positive or negative, depending on the probability structure of the shocks.

The relation between volume and volatility is more complicated and cannot be derived analytically. Thus, we rely on numerical exercises. In the simulations reported in the following section, we model the deterministic volatility process as a chaotic tent map, which is known to be similar to an AR(1) process [Sakai and Tokumaru (1980)]. The simulated stock returns mimic GARCH behavior quite well. Thus, modeling the volatility as an autocorrelated time-varying deterministic process can be viewed as an approximation to the ARCH or GARCH process [Engle (1982), Bollerslev (1986)], which is one of the most prominent tools for characterizing changing variance of financial time series. The ambiguous relation between the serial correlation of stock returns and volume as well as the positive relation between volume and the volatility of stock returns are confirmed in a series of empirical and numerical examples. Therefore, the data generated from the simple dynamic asset pricing model with time-varying volatility can mimic well the volume–persistence and volume–volatility relations revealed in the observed data.

The rest of the paper is organized as follows: Section 2 describes the economic model, Section 3 reports the empirical and simulation results, and Section 4 presents concluding remarks.

2. THEORETICAL FRAMEWORK

In this section we modify Wang's model by assuming that a subset of the shocks in the economy have time-varying (as opposed to constant) volatility and that agents maximize their finite (as opposed to infinite) lifetime utility. It is shown that there is a rational expectations equilibrium in which the market-clearing price is linear in the payoffs of risky assets in the economy.

In the economy, all agents can trade both a bond (riskless asset) and a stock (risky asset). Bonds guarantee a gross rate of return $R \equiv 1+r$. The supply of bonds is infinitely elastic. The dividend payoff of each share of stock is D_t in period t. The total supply of stock shares per capita is fixed and is normalized to one. There are two types of traders in the economy who differ from each other in their investment opportunities: type A traders have the opportunity to invest in a risky asset that is not available to type B traders. The nontradable asset has an excess rate of return of q_t for period t, the mean of which, conditional upon information in period $t-1$, is denoted Z_{t-1}. The percentage of type A traders is ω and, consequently, the percentage of type B traders is $1-\omega$.

The dividend payoff of each share of stock in period t is assumed to follow an AR(1) process:

$$D_t = a_D D_{t-1} + \epsilon_{D,t}, \tag{1}$$

with $0 \leq a_D < 1$ and $\epsilon_{D,t} \sim$ i.i.d. $N(0, \sigma_D^2)$. The excess rate of return on the nontradable asset q_{t+1} for period $t+1$ is assumed to follow the process

$$\begin{aligned} q_{t+1} &= Z_t + \epsilon_{q,t+1}, \qquad \epsilon_{q,t} \sim \text{i.i.d. } N\left(0, \sigma_q^2\right), \\ Z_t &= a_Z Z_{t-1} + \epsilon_{Z,t}, \qquad \epsilon_{Z,t} \sim N\left(0, \sigma_{Z,t}^2\right), \end{aligned} \tag{2}$$

with $0 \leq a_Z < 1$ and $\sigma_{Z,t}^2 = f(\sigma_{Z,t-1}^2; t)$. It is assumed that the innovations $\epsilon_{D,t}$, $\epsilon_{q,t}$, and $\epsilon_{Z,t}$ are jointly normal and uncorrelated except for $\epsilon_{D,t}$ and $\epsilon_{q,t}$, the covariance of which, denoted $\sigma_{D,q}$, is strictly positive.

Note that the particular source of time-varying volatility is attached to the returns of the nontradable asset, not the dividends. It would be a more direct approach to assume that time-varying volatility of stock returns comes from the payoffs on traded stock. However, it is well known that firms smooth dividends over time [Lintner (1956), Merchant (1989)]. Therefore, it is unlikely that the dividend process would exhibit highly autocorrelated time-varying volatility. Moreover, if the daily dividend process has GARCH-like errors, low-frequency dividends should exhibit conditional heteroskedasticity of the GARCH form as well [see Drost and Nijman (1993)]. However, there is little empirical literature documenting GARCH-like behavior of dividend movements.[1]

For all traders, preferences are assumed to be additively time separable with exponential per-period utility that has constant absolute risk aversion (CARA).

All traders maximize their expected lifetime utility:

$$\max E_t \left[-\sum_{s=t}^{T} \beta^{s-t} \exp(-\gamma c_s) - \beta^{T-t+1} \exp(-\alpha W_{T+1}) \right] \tag{3}$$
$$\text{subject to } W_{t+1} = (W_t - c_t)R + \boldsymbol{X}_t' \mathcal{Q}_{t+1},$$

where E_t is the expectation operator conditioned upon traders' information in period t, c_t is an agent's consumption in period t, W_t is his wealth at t, X_t is his portfolio, $\mathcal{Q}_{t+1}$ is the excess returns on risky assets at $t+1$, β is the discount factor, γ is the risk aversion coefficient, and $\alpha = (r\gamma)/R$.[2] The salvage term $\beta^{T-t+1} \exp(-\alpha W_{T+1})$ can be interpreted as a bequest motive, and this particular form allows for analytical solution of the Bellman equation.

Denote the stock price in period t as P_t. The information structure in this economy is that both types of traders observe P_t, D_t, and Z_t at time t. Therefore, all traders have the information set $\mathcal{I}_t = \{D_s, P_s, Z_s \mid s \leq t\}$.

Given the assumed process for D_t and Z_t, the maximization problem in equation (3) can be solved recursively, starting by solving the agents' maximization problem at time T:

$$\max E_T[-\exp(-\gamma c_T) - \beta \exp(-\alpha W_{T+1})]$$
$$\text{subject to } W_{T+1} = (W_T - c_T)R + \boldsymbol{X}_T' \mathcal{Q}_{T+1}.$$

The terminal price P_{T+1} is assumed to be the discounted sum of the future dividend payoffs:

$$P_{T+1} = E_{T+1}\left(\sum_{j=1}^{\infty} R^{-j} D_{T+1+j} \right).$$

In doing so, we implicitly assume that the stock lives infinitely. The excess return on the stock at $T+1$ is

$$\begin{aligned} Q_{T+1} &\equiv P_{T+1} + D_{T+1} - RP_T \\ &= aRD_T - RP_T \end{aligned}$$

with $a = a_D/(R - a_D)$. The excess return on the nontradable asset at $T+1$ is

$$q_{T+1} = Z_T + \epsilon_{q,T+1}.$$

We have the following result.

PROPOSITION 1. *The economy described above has a rational expectations equilibrium in which the equilibrium stock price is*

$$P_t = p_{0,t} + aD_t - p_{Z,t} Z_t, \tag{4}$$

where $a = a_D/(R - a_D)$ and where $p_{0,t}$ and $p_{Z,t}$ are deterministic sequences.

Proof. See Appendix A. ∎

Given the equilibrium price in equation (4), the excess return per share of the stock, $Q_{t+1} \equiv P_{t+1} + D_{t+1} - RP_t$, is

$$Q_{t+1} = p_{0,t+1} - Rp_{0,t} + (Rp_{Z,t} - a_Z p_{Z,t+1})Z_t + (1+a)\epsilon_{D,t+1} - p_{Z,t+1}\epsilon_{Z,t+1}. \tag{5}$$

The conditional variance of returns, denoted $\mathrm{Var}_t(Q_{t+1})$, is

$$\begin{aligned}\underset{t}{\mathrm{Var}}(Q_{t+1}) &= \underset{t}{\mathrm{Var}}[(1+a)\epsilon_{D,t+1} - p_{Z,t+1}\epsilon_{Z,t+1}] \\ &= (1+a)^2\sigma_D^2 + p_{Z,t+1}^2\sigma_{Z,t+1}^2.\end{aligned}$$

Therefore, the stock return volatility is time varying.

The optimal stock demand of type B traders, denoted X_t^B, is[3]

$$X_t^B = f_{0,t}^B + f_{Z,t}^B Z_t, \tag{6}$$

where $f_{0,t}^B$ and $f_{Z,t}^B$ depend on $\sigma_{Z,t}^2$. Trading volume at time t can be written as $1-\omega$ times the absolute change in the optimal holding of stock by type B traders. Therefore, volume equals

$$\begin{aligned}V_t &= (1-\omega)\left|X_t^B - X_{t-1}^B\right| \\ &= (1-\omega)\left|\, f_{0,t}^B - f_{0,t-1}^B + f_{Z,t}^B Z_t - f_{Z,t-1}^B Z_{t-1}\right|.\end{aligned} \tag{7}$$

Note that a change in $\sigma_{Z,t+1}^2$ or in Z_t will trigger an increase in volume, whereas a change in D_t will not have any impact on volume. If $\sigma_{Z,t+1}^2$ is high or Z_t is low, type A traders demand more stock because the risky private investment opportunity is less attractive. While they rebalance their portfolio, the equilibrium price changes to accommodate this excess demand, and volume increases. A change in D_t, however, will not have any influence on the expected excess return on the stock [see equation (5)] and thus will have no influence on the optimal demand of agents. Therefore, we should observe a positive relation between volume and volatility in this economy.

The ambiguous relation between volume and expected return is given in the following proposition.

PROPOSITION 2. *Given the current excess return and volume, next period's expected excess return is*

$$\begin{aligned}E(\tilde{Q}_{t+1} \mid \tilde{Q}_t, V_t) &= \lambda_{0,t+1} + \lambda_{1,t+1}V_t^2 + \lambda_{2,t+1}\tilde{Q}_t \\ &\quad + \lambda_{3,t+1}V_t^2\tilde{Q}_t + \textit{high-order terms},\end{aligned} \tag{8}$$

where $\tilde{Q}_{t+1} \equiv Q_{t+1} - p_{0,t+1} + Rp_{0,t}$, *that is, the random part of the excess return, and* $\lambda_{0,t+1}$, $\lambda_{1,t+1}$, $\lambda_{2,t+1}$, *and* $\lambda_{3,t+1}$ *are functions of* $\sigma_{Z,t+1}^2$. *The coefficient* $\lambda_{3,t+1}$ *can be positive or negative, depending on the parameters in the economy.*

Proof. See Appendix B. ■

From Proposition 2, the sign of $\lambda_{3,t+1}$ is ambiguous. Therefore, the model described earlier does not imply any monotonic relation between volume and the

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first-order autocorrelation of stock returns. One can further write V_t as

$$V_t = \tilde{V}_t + \bar{V},$$

where $\bar{V} = E(V_t)$ is the mean volume. To the same order approximation, equation (8) becomes

$$E(\tilde{Q}_{t+1} \mid \tilde{Q}_t, V_t) = \phi_{0,t+1} + \phi_{1,t+1}\tilde{V}_t + (\phi_{2,t+1} + \phi_{3,t+1}\tilde{V}_t)\tilde{Q}_t + \text{high-order terms}, \tag{9}$$

with $\phi_{0,t+1} = \lambda_{0,t+1} + \lambda_{1,t+1}\bar{V}^2$, $\phi_{1,t+1} = 2\lambda_{1,t+1}\bar{V}$, $\phi_{2,t+1} = \lambda_{2,t+1} + \lambda_{3,t+1}\bar{V}^2$, $\phi_{3,t+1} = 2\lambda_{3,t+1}\bar{V}$. Again, since $\lambda_{3,t+1}$ can be positive or negative and $\bar{V} > 0$, the sign of $\phi_{3,t+1}$ is ambiguous.

The reason for the ambiguous relation between volume and return autocorrelation is that the excess returns on the stock depend not only on the expected return on the nontradable asset Z_t but also on the time-varying volatility $\sigma^2_{Z,t}$. Consider one-period changes in Z_t and $\sigma^2_{Z,t}$. If a high Z_t is followed by a low volatility $\sigma^2_{Z,t+1}$, Q_t is low because the higher expected return on the nontradable asset decreases type A traders' demand for the stock. The lower risk in the nontradable asset leads to a decrease in Q_{t+1}. In this case, the correlation between Q_t and Q_{t+1} is more likely to be positive and we should observe a positive relation between volume and return autocorrelation. If a high Z_t is followed by a high volatility, Q_t decreases and Q_{t+1} increases because of low P_t and high P_{t+1}. In this case, the relation between volume and return autocorrelation is negative. Persistent movements in $\sigma^2_{Z,t}$, along the lines of the GARCH-type behavior that seems to characterize financial markets, can lead to even more complicated time-series patterns of returns. Therefore, the relation between volume and return autocorrelation is nonmonotonic.

3. VOLUME, VOLATILITY, AND SERIAL CORRELATION OF STOCK RETURNS: EMPIRICAL AND NUMERICAL RESULTS

Using the model of the preceding section, it was possible to show analytically that time-varying volatility of the payoffs on the nontradable asset results in an ambiguous relation between volume and the serial correlation of stock returns. The positive relation between return volatility and volume cannot be derived analytically, however, because both the coefficients in the volume equation (7) and conditional second moments of returns are functions of the time-varying volatility of nontradable asset $\sigma^2_{Z,t}$. Thus, other methods must be used to address the more complicated question of the relation among volume, volatility, and serial correlation of stock returns implied by the economic model. We try to answer this question by examining a number of numerical simulations and comparing the results with observed data. Both the simulations and the data analysis support the prediction of Proposition 2 that the relation between volume and serial correlation can be of either sign. Furthermore, the data generated from the artificial economic world described in Section 2 exhibit a highly significant positive correlation between volume and the one-step-ahead volatility of stock returns. Intuitively, one would

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expect a positive relation, as explained in the preceding section. This significant relation is further confirmed empirically by using observed data.

3.1. Data

The observed data set comprises daily log closing-price differences and the number of shares traded on the New York Stock Exchange for four heavily traded stocks: Boeing (BA), International Business Machine (IBM), Coca Cola (KO), and Minnesota Mining & Manufacturing (MMM). The sample period is from April 5, 1987, to October 14, 1996, for 3,000 observations on each series. We choose individual stock data instead of index data for two reasons: First, in the artificial world described in Section 2 there is only one risky asset traded in the economy and volume is the net trade of this single stock. With aggregate volume data, if traders buy or sell one stock in exchange for others, the net trade of the risky asset might be overestimated. Second, the problem of nonsynchronous trading may lead to spurious positive autocorrelation in an index return [Lo and MacKinlay (1990)].

Many previous empirical studies have found that long time spans of volume data tend to be nonstationary. Alternatively, the ratio of the number of shares traded to the number of shares outstanding, known as turnover, is used in some studies. However, it does not remove the low-frequency variations in the volume data completely and an additional detrending procedure has to be done [see, e.g., Campbell et al. (1993) and Brock and LeBaron (1996)]. For the volume series used here, we conduct the Phillips and Perron (1988) test with and without a linear trend and find no evidence supporting the unit root hypothesis. There is, however, a significant linear trend in our four-volume series. Therefore, we use the linearly detrended volume series for the following empirical exercise. The return data and the transformed volume data are plotted in Figure 1.

3.2. Simulation Algorithm

The simulated data are obtained by calibrating the theoretical model with the risk aversion coefficient $\gamma = 1.00$ and the discount factor $\beta = 0.9995$, which implies approximately a consistent interest rate of $r = 0.05\%$ (=0.0005). The proportion of type A traders (ω) in the economy is set equal to 0.01. The autocorrelation coefficient of dividends (a_D) as well as of the expected return on the nontradable asset (a_Z) are set equal to 0.99. The variances of dividends (σ_D^2) and the nontradable asset (σ_q^2) are set to 1 and the covariance ($\sigma_{D,q}$) equals 0.25.

The time-varying volatility of Z_t is modeled by the chaotic asymmetric tent-map dynamics. The asymmetric tent map is the continuous piecewise linear map $f_{h,[0,b]}\colon [0, b] \mapsto [0, b]$ defined as

$$\begin{aligned} f_{h,[0,b]}(x) &= \frac{2x}{1+h} \quad \text{if} \quad 0 \le x \le \frac{(1+h)b}{2} \\ &= \frac{2(b-x)}{1-h} \quad \text{if} \quad \frac{(1+h)b}{2} \le x \le b, \end{aligned}$$

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where $-1 < h < 1$. The (asymmetric) tent maps have the property that the autocorrelation coefficients coincide exactly with the autocorrelation coefficients of an AR(1) process [Sakai and Tokumaru (1980)]. Modeling volatility in this way can be considered as a first-order approximation for a GARCH process [Bollerslev (1986)]. The main limitation of using a tent map is that the generated volatility would look uniformly distributed, which is different from a GARCH(1,1) process or a stochastic volatility with Gaussian innovations. Thus, it is unlikely that data simulated from this simple tent-map process would match well the higher moments in observed data. One could use a more complicated deterministic volatility process to mimic the higher moments of observed returns. For example, Gallant et al. (1997) show that, by fitting the variance with a chaotic Mackey–Glass sequence, the conditional density fits the daily standard and Poor's 500 Composite Index better than a standard stochastic volatility model. Because higher moments are not central to the issues examined in this paper, this additional complication does not seem worthwhile.

In our simulation, we set $h = 0.9$, that is, the first-order autocorrelation of the time-varying volatility $\sigma^2_{Z,t}$ equals 0.9, so as to mimic the persistence of the volatility on stock returns found in high-frequency financial time series. The volatility varies between 0 and b. We set $b = 1$, 10, 50, and 100 in various settings, denoted setting A, B, C, and D, respectively. This implies that, on average, the volatility of Z_t contributes 3%, 19%, 56%, and 74% of stock return volatility in simulation setting A, B, C, and D, respectively.[4]

Given the entire path of volatility $\{\sigma^2_{Z,t}\}_{t=0}^{T+1}$, we first compute the solution of the model described in Section 2 for the terminal period T and then iteratively compute the solution for $t = 1, \ldots, T-1$ backward, as described in Appendix A. For each setting of the upper bound of volatility, we simulate 1,000 data sets, each with a length of 15,000 periods. We discard the last 12,000 periods to ensure that the effect of terminal conditions on the maximization problem does not distort the results. To be more precise, for $y_t = (r_t, V_t)'$ with $r_t \equiv Q_t/P_{t-1}$ denoting stock return per dollar, the simulated data set $\{y^i_t\}_{t=1}^{T}$ is generated by the structural model described in Section 2 with $T = 15{,}000$ and $i = 1, 2, \ldots, 1{,}000$.

3.3. Simulation and Empirical Results

Figures 2 and 3 show the scatterplots of stock return and detrended volume series from the observed data and simulated data, respectively. The scatterplots of the observed as well as the simulated data reveal a positive contemporaneous relationship between volume and the magnitude of returns: days with small absolute returns tend to be days with lower-than-average volume, whereas days with large absolute returns are high-volume days. Table 1 reports this significant relation by regressing the absolute value of returns on volume data for observed as well as simulated data. As surveyed by Karpoff (1987), a V-shaped relation between volume and stock returns has been found by virtually all empirical investigators of the return-volume relation in equity markets. This consistent empirical

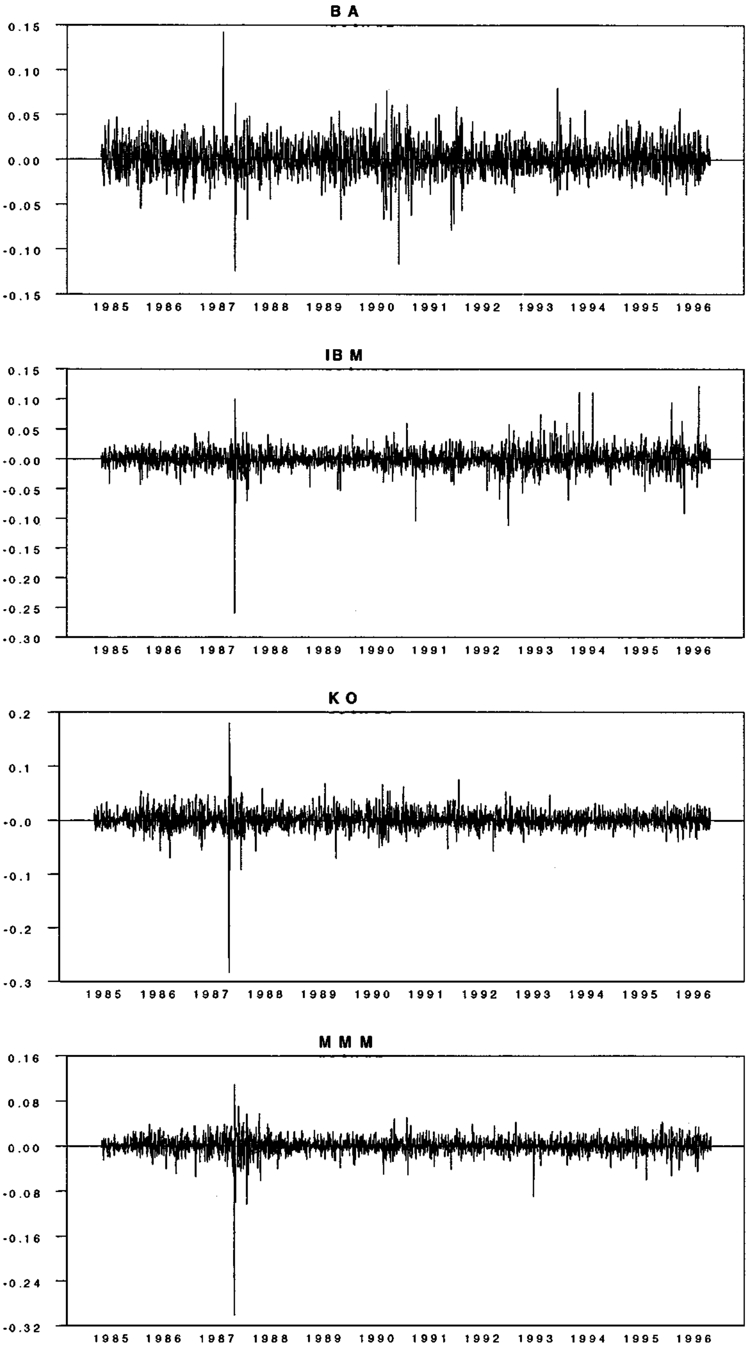

FIGURE 1. Return (left) and volume series (right).

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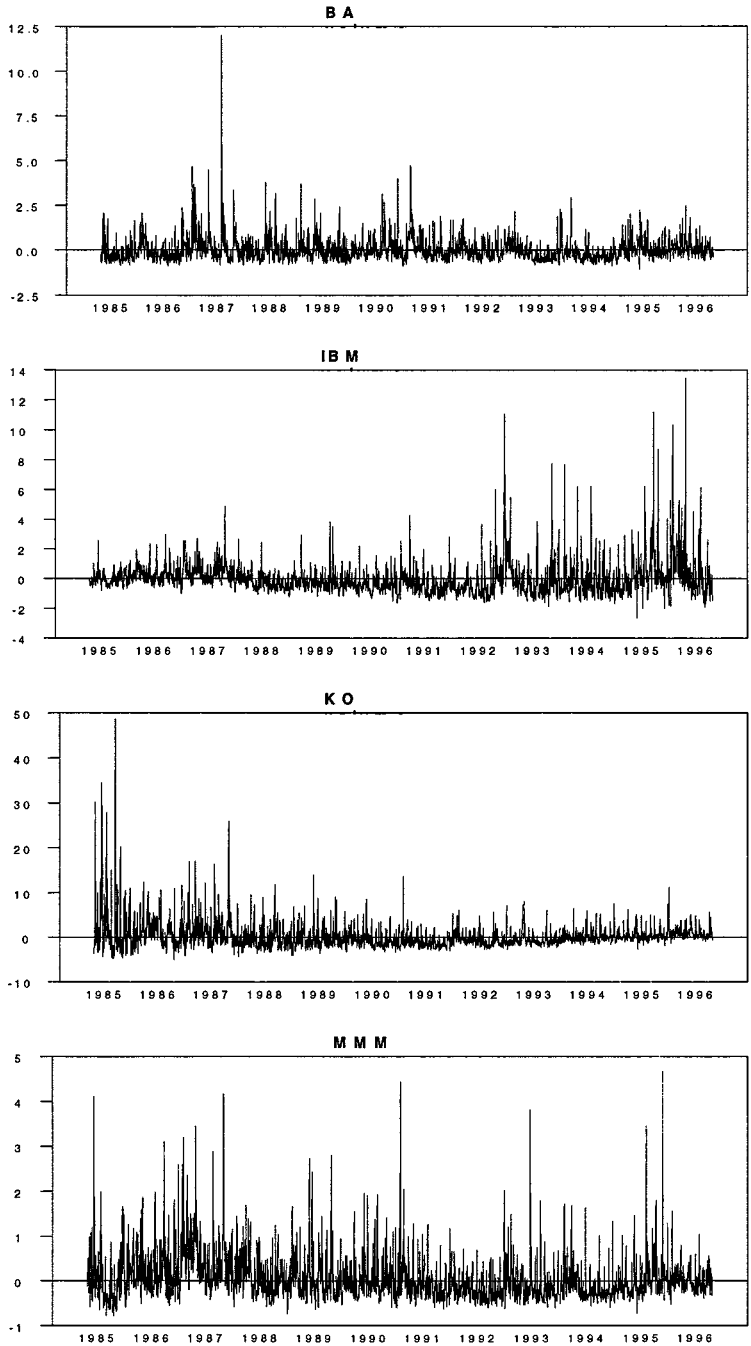

Figure 1. (Continued.)

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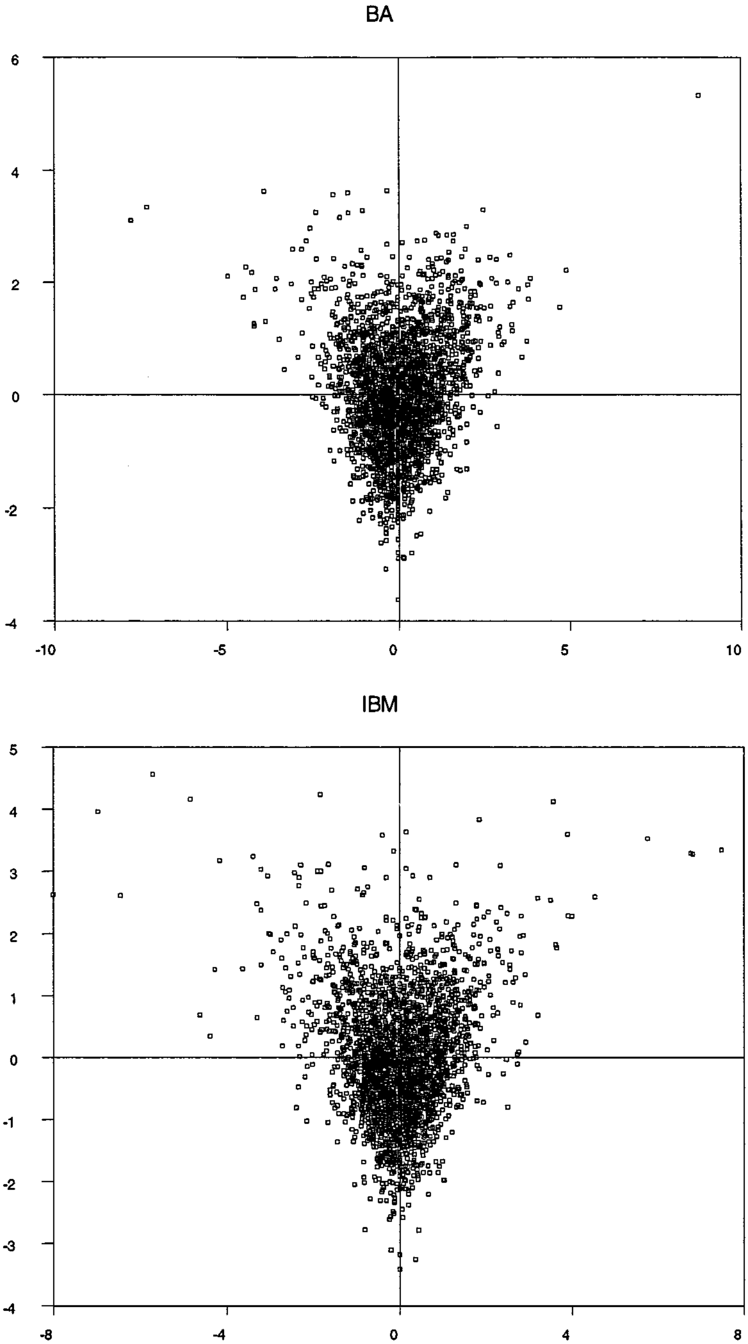

FIGURE 2. Scatterplots of return vs. detrended volume, observed data.

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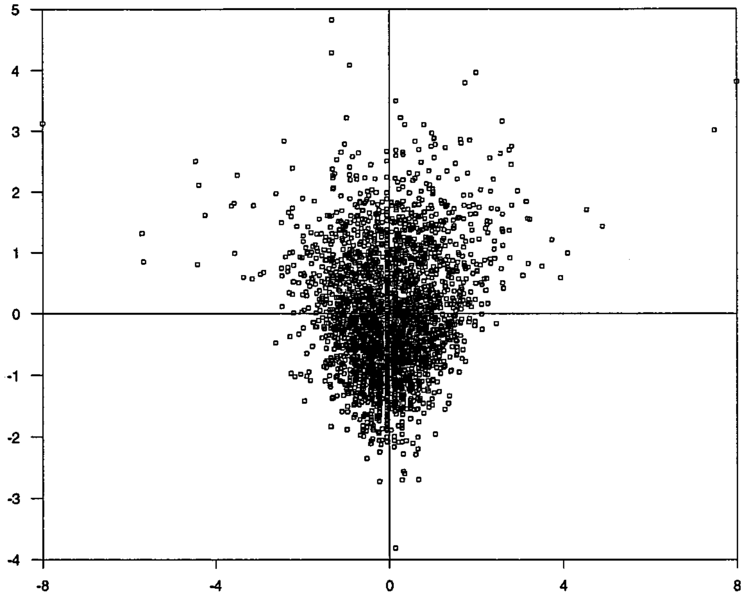
KO



MMM

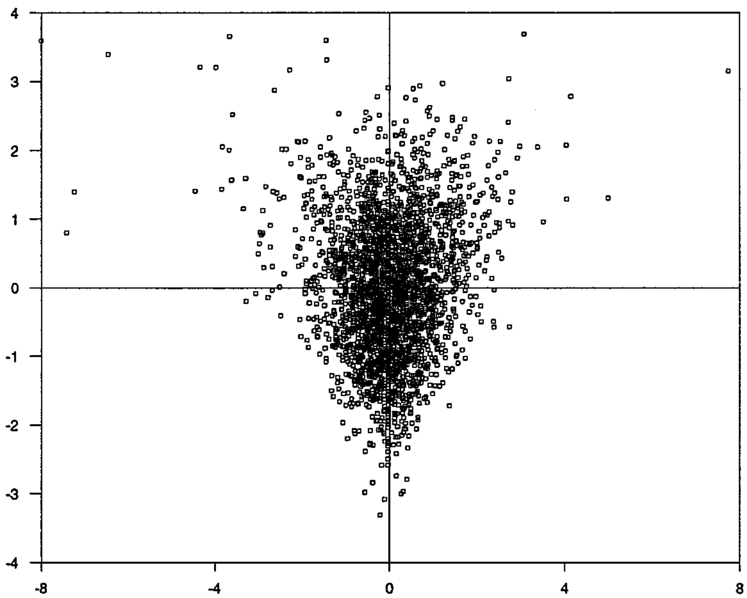


Figure 2. (Continued.)

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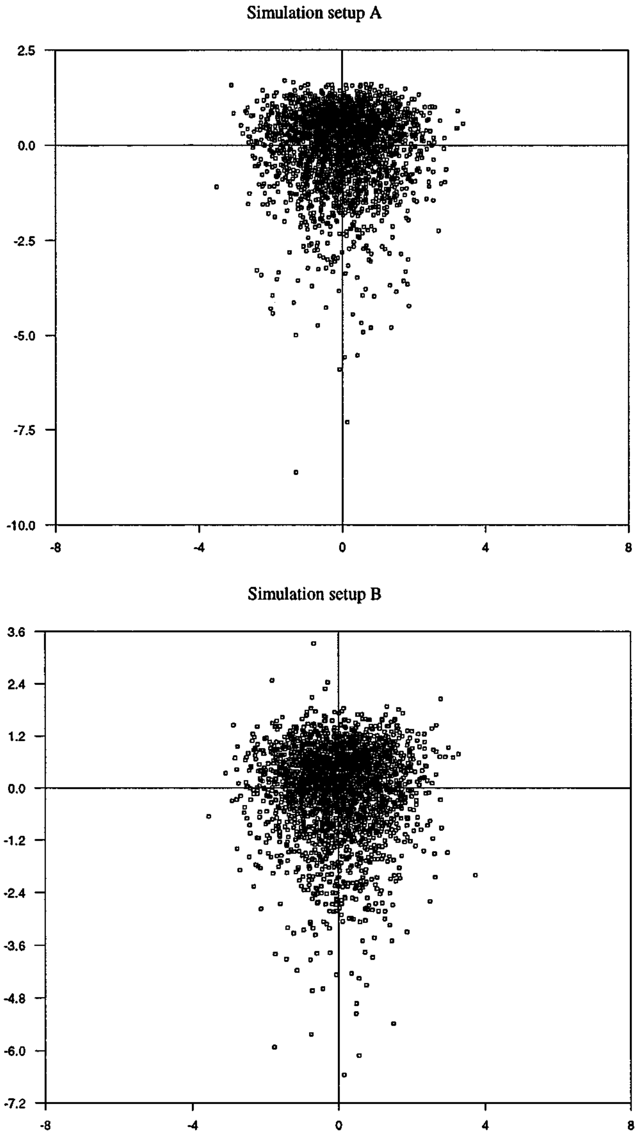

FIGURE 3. Scatterplots of return vs. volume, simulated data.

result can be reproduced by the data generated from the model introduced in Section 2.

Table 2 and Table 3 report the relationship between volume and the first autocorrelation of stock returns in the observed and simulated data, respectively. In Table 3, we report the proportion of individual t-statistics that are less than -1.64,[5] the averaged standard errors, and the centered R^2 averaged over 1,000 regressions

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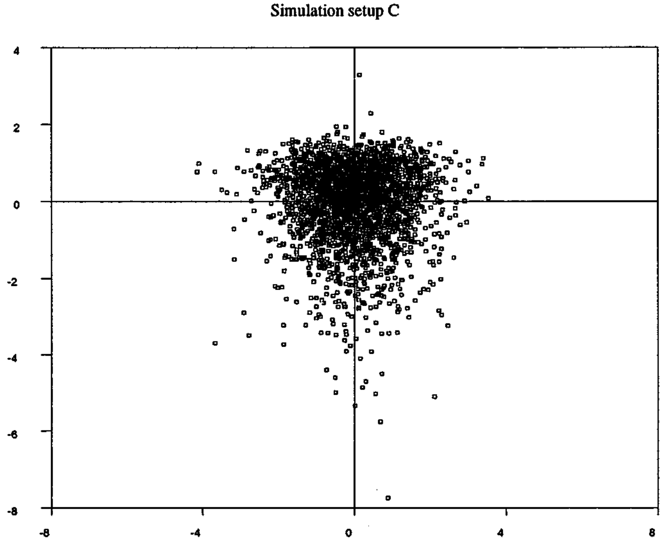

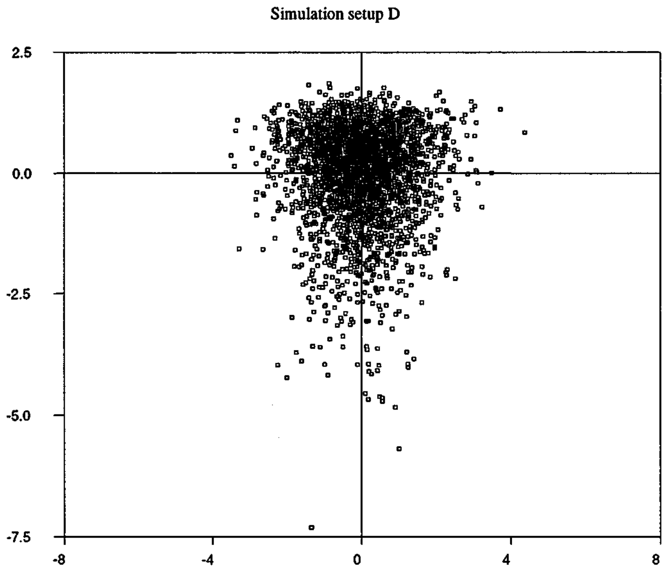

Figure 3. (Continued.)

of the simulated time series. From the second column of Table 2, we can see that the first-order autocorrelations of individual stock returns are not all positive, as found for the returns on a value-weighted stock index by, for example, Campbell et al. (1993). This characteristic is exhibited in the simulated data. From the second column of Table 3, only 0.5%, 5%, 6%, and 7% of the individual t-statistics are less than -1.64 for simulation settings A, B, C, and D, respectively. The adjusted

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Table 1. Volume and absolute value of return

$\lvert r_t \rvert = a + bV_t$		
Sample	b^a	R^{2b}
BA	0.485(0.016)***	0.230
IBM	0.345(0.010)***	0.264
KO	0.083(0.004)***	0.102
MMM	0.618(0.026)***	0.164
Setting A	0.016(0.012)*	0.001
Setting B	0.021(0.008)**	0.011
Setting C	0.025(0.005)***	0.017
Setting D	0.046(0.011)***	0.053

[a] Numbers in parentheses are standard errors. For simulated data the standard errors and R^2 are averaged over 1,000 simulations. Significance levels are $* =$ 10%, $** = 5\%$, and $*** = 1\%$: $* = 0.1$, $** = 0.5$, and $*** = 0.01$.
[b] R^2's are the adjusted R^2's.

Table 2. Volume and first autocorrelation of return, individual stock data[a]

$r_{t+1} = \gamma_0 + \left(\gamma_1 + \gamma_2 V_t + \gamma_3 V_t^2\right) r_t$				
Sample	γ_1	γ_2	γ_3	R^2
BA	0.033(0.018)[b]			0.001
	0.053(0.025)[c]	−0.022(0.009)[c]		0.003
	0.066(0.025)[c]	−0.055(0.037)	0.003(0.003)	0.003
IBM	−0.014(0.018)			0.000
	−0.032(0.028)	0.004(0.016)		0.000
	−0.019(0.024)	−0.019(0.038)	0.002(0.003)	0.002
KO	−0.041(0.018)[c]			0.002
	−0.047(0.035)	0.001(0.018)		0.002
	−0.021(0.026)	−0.028(0.032)	0.001(0.001)	0.010
MMM	−0.054(0.018)[c]			0.003
	−0.005(0.029)	−0.044(0.036)		0.008
	−0.054(0.028)[b]	0.224(0.120)[b]	−0.067(0.027)[c]	0.019

[a] In parentheses are the heteroskedasticity-consistent standard errors. R^2's are the adjusted R^2's.
[b] $p = 0.1$.
[c] $p = 0.5$.

R^2's averaged over the 1,000 simulated return series are less than 0.01% for all simulation settings.

To investigate the relation between volume and the first autocorrelation of returns, we regress the one-day-ahead return on the current return and the return interacted with volume. We also regress the one-day-ahead return on the current

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Table 3. Volume and first autocorrelation of return, simulated data[a]

$$r_{t+1} = \gamma_0 + \left(\gamma_1 + \gamma_2 V_t + \gamma_3 V_t^2\right) r_t,$$
$$\bar{\gamma}_j = \frac{1}{1000}\sum_{i=1}^{1000} \gamma_{j,i}, \quad j = 1, 2, 3, 4.$$

Sample		$\bar{\gamma}_1$	$\bar{\gamma}_2$	$\bar{\gamma}_3$	R^2
Setting A		0.000(0.018)			0.000
	$t < -1.64$	**0.005**			
		0.000(0.003)	0.010(0.810)		0.000
	$t < -1.64$	**0.005**	**0.062**		
		−0.002(0.041)	0.119(2.319)	−1.339(2.604)	0.001
	$t < -1.64$	**0.056**	**0.061**	**0.071**	
Setting B		0.000(0.018)			0.000
	$t < -1.64$	**0.059**			
		0.000(0.026)	−0.005(0.332)		0.000
	$t < -1.64$	**0.051**	**0.070**		
		0.000(0.031)	−0.001(0.557)	−0.037(1.527)	0.001
	$t < -1.64$	**0.043**	**0.070**	**0.045**	
Setting C		−0.001(0.018)			0.000
	$t < -1.64$	**0.063**			
		0.000(0.028)	−0.013(0.179)		0.001
	$t < -1.64$	**0.046**	**0.131**		
		0.000(0.033)	−0.013(0.355)	−0.013(0.722)	0.001
	$t < -1.64$	**0.030**	**0.090**	**0.069**	
Setting D		−0.003(0.018)			0.000
	$t < -1.64$	**0.078**			
		0.001(0.028)	−0.027(0.145)		0.001
	$t < -1.64$	**0.052**	**0.178**		
		0.001(0.034)	−0.029(0.309)	0.000(0.544)	0.002
	$t < -1.64$	**0.031**	**0.103**	**0.100**	

[a] In parentheses are the standard errors averaged over 1,000 simulations. The bold numbers represent the percentage of the numbers of t-statistics less than −1.64. R^2's are the adjusted R^2's.

return, the return interacted with volume and squared volume, which is motivated by equation (9). This regression exercise is similar to that conducted by Campbell et al. (1993). The analytical result predicted in Proposition 2 is confirmed by the observed individual stocks as well as the simulated realization: With time-varying volatility, the correlation between volume and return persistence is not always negative, as found by Campbell et al. (1993), or is not always positive, as found by Morse (1978) and Antoniewicz (1992). Table 2 shows that, when regressing the one-day-ahead return on the current return and the return interacted with volume, the relation between volume and the autocorrelation of return (the coefficient γ_2) is significantly negative for the BA series but insignificant for IBM, KO, and MMM. By including the additional term involving squared volume interacted with return,

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the estimate of γ_2 becomes insignificant for the BA series. For IBM and KO, the estimate of γ_2 remains insignificant. For MMM, the estimate of γ_2 is significantly positive whereas the coefficient of squared volume interacted with return (γ_3) is significantly negative. This suggests that the relation between the first autocorrelation of returns and volume is nonlinear and nonmonotonic for the MMM series.

Table 3 reports the results from the simulated data. From Table 3, only 6%, 7%, 13%, and 17% of the individual t-statistics of the coefficient on volume interacted with stock return are less than -1.64 for simulation settings A, B, C, and D, respectively. Furthermore, the average adjusted R^2's are less than 0.002 for all simulation samples, which implies that the effect of trading volume on the autocorrelation of returns is negligible. The results are not improved by including the additional terms involving volume and squared volume interacted with return: only 6%, 7%, 9%, and 10% of the individual t-statistics of the coefficient on volume interacted with return are less than -1.64 for simulation settings A, B, C, and D, respectively. In addition, for all four simulation settings, none out of 1,000 simulated series has estimated coefficients on volume and volume squared interacted with return (i.e., γ_2 and γ_3) that are jointly significantly different from zero at the 5% level. In short, our result supports the finding by Tauchen et al. (1996), who used a nonparametric impulse response analysis and found no relation between volume and first-order autocorrelation of stock returns.

We now turn to the analysis of the relation between volume and return volatility. First, we regress the simulated theoretical volatility of stock returns on simulated volume data. The volatility of (per dollar) returns can be directly expressed as

$$\sigma^2_{r,t+1} \equiv \operatorname*{Var}_t(r_{t+1}) = \operatorname*{Var}_t\left(\frac{Q_{t+1}}{P_t}\right) = \frac{(1+a)^2\sigma^2_D + p^2_{z,t+1}\sigma^2_{Z,t+1}}{P^2_t}. \tag{10}$$

Thus, the regression equation is

$$\sigma^2_{r,t+1} = c_0 + c_1 V_t + e_t, \qquad e_t \sim \text{i.i.d.}\left(0, \sigma^2_e\right). \tag{11}$$

If the volatility of return per share is constant over time, one should find a negative relation between volume and the volatility of return per dollar.[6]

Next, we empirically examine the relation between volume and volatility for the four individual stock return series. Because volatility is not directly observable in the real world, we use the fitted values from a GARCH(1,1) model.[7] Such a model is given by

$$\begin{aligned} r_{t+1} &= \beta_0 + \beta_1 r_t + \varepsilon_{t+1}, \\ \varepsilon_{t+1} \mid I_t &\sim N(0, h_{t+1}), \\ h_{t+1} &= a_0 + a_1\varepsilon^2_t + b_1 h_t. \end{aligned} \tag{12}$$

The estimated volatility, denoted $\hat{h}_{t+1}$, is then regressed on volume data. The reason for doing this two-step procedure instead of putting volume as an explanatory variable directly into the volatility equation, as in Lamoureux and Lastrapes (1990),

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is that the high correlation between contemporaneous volatility and volume causes severe problems of convergence. Because the estimated (as well as simulated) volatilities exhibit high persistence, we regress the one-step-ahead volatility on current volatility and volume

$$\hat{h}_{t+1} = \phi_0 + \phi_1 \hat{h}_t + \phi_2 V_t. \tag{13}$$

For the sake of comparison, we perform the same exercise for the simulated data as well.

Table 4 and Table 5 summarize our findings of a positive correlation between volume and one-step-ahead return volatility. From Table 4, we can see that the regression coefficient c_1 obtained by regressing the theoretical volatilities on volume data, both generated from our artificial economy, is highly significant and positive. Furthermore, from the lower panel of Table 5, if one approximates the simulated volatility of stock returns by estimating a GARCH(1,1) model, the estimated

TABLE 4. Volume and the theoretical volatility

$\sigma^2_{r,t+1} = c_0 + c_1 V_t$

	c_1	s.e.[a]	$t > 1.64$[b]	R^{2c}
Setting A	0.000	0.004	0.044	0.000
Setting B	0.199	0.023	1.000	0.025
Setting C	0.375	0.027	1.000	0.054
Setting D	0.333	0.023	1.000	0.066

[a] s.e. is the standard error of c_1 estimates averaged over 1,000 simulations. [b] The fourth column ($t > 1.64$) reports the percentage of t statistics larger than 1.64. [c] R^2's are the averaged R^2 over 1,000 simulations.

TABLE 5. Volume and return volatility[a]

$$r_{t+1} = \beta_0 + \beta_1 r_t + \varepsilon_{t+1}, \quad \epsilon_{t+1}|I_t \sim N(0, h_{t+1})$$
$$h_{t+1} = a_0 + a_1 \varepsilon^2_{t+1} + b_1 h_t$$
$$h_{t+1} = \phi_0 + \phi a_1 h_t + \phi_2 V_t$$

Sample	a_1	b_1	ϕ_2	R^{2c}
BA	0.215 (0.020)***[b]	0.328 (0.040)***	0.012 (0.000)***	0.401
IBM	0.117 (0.005)***	0.894 (0.010)***	0.042 (0.003)***	0.812
KO	0.096 (0.004)***	0.851 (0.001)***	0.084 (0.009)***	0.892
MMM	0.093 (0.003)***	0.861 (0.007)***	0.058 (0.005)***	0.837
Setting A	0.006 (0.770)	0.267 (0.252)	0.000 (0.939)	0.331
Setting B	0.012 (0.885)	0.813 (0.523)	0.002 (1.000)	0.582
Setting C	0.025 (0.979)	0.845 (0.737)	0.017 (1.000)	0.737
Setting D	0.042 (1.000)	0.886 (0.984)	0.018 (1.000)	0.841

[a] For the individual stocks, it is the standard error in the parenthesis and for the simulated data, it is the percentage of t statistics larger than 1.64. [b] *** represents significant level of one percent. [c] R^2's are the adjusted R^2's of the regression in equation (12).

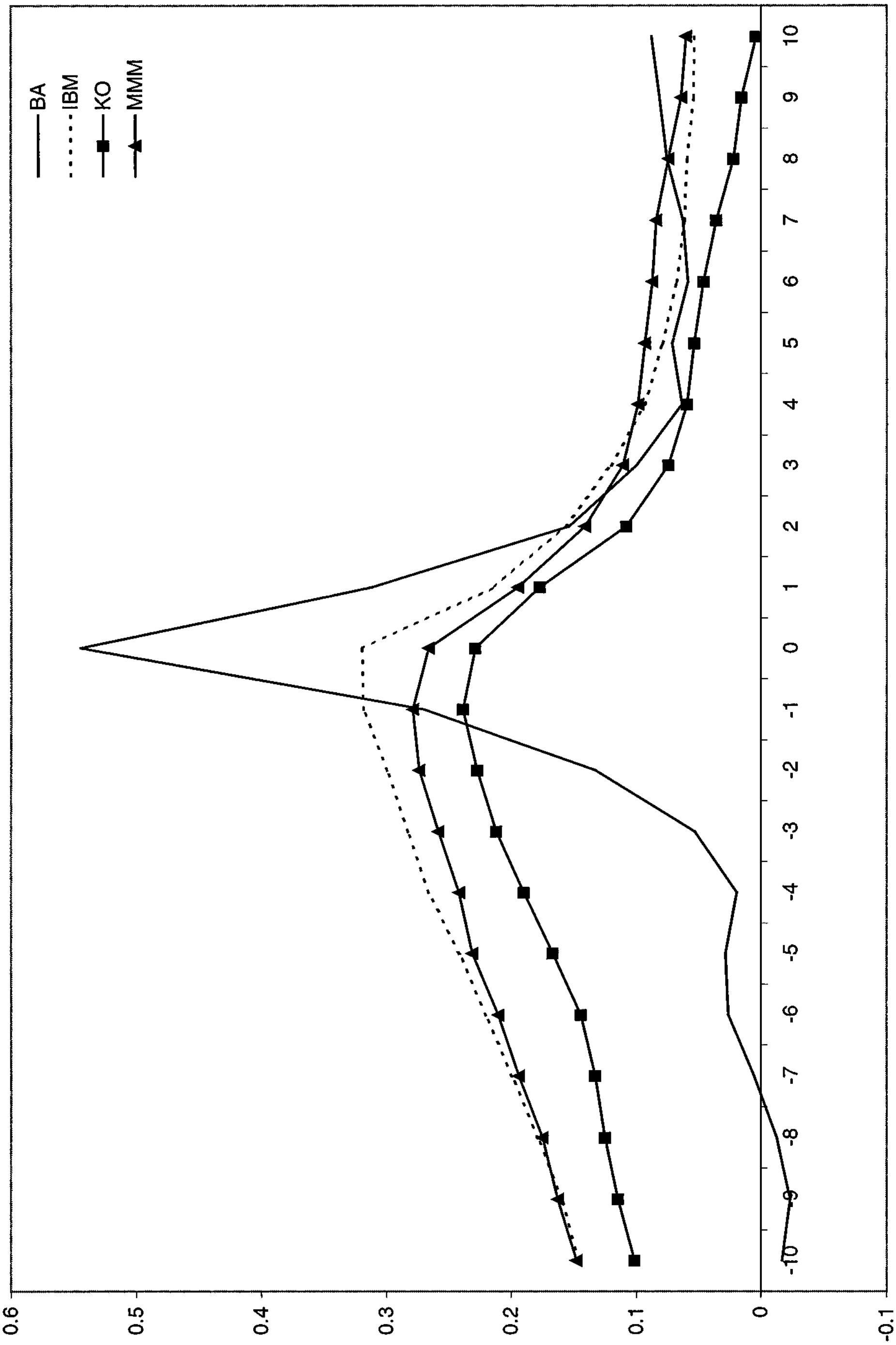

FIGURE 4. Cross correlations of volume$_{t+j}$ and volatility$_t$, observed data.

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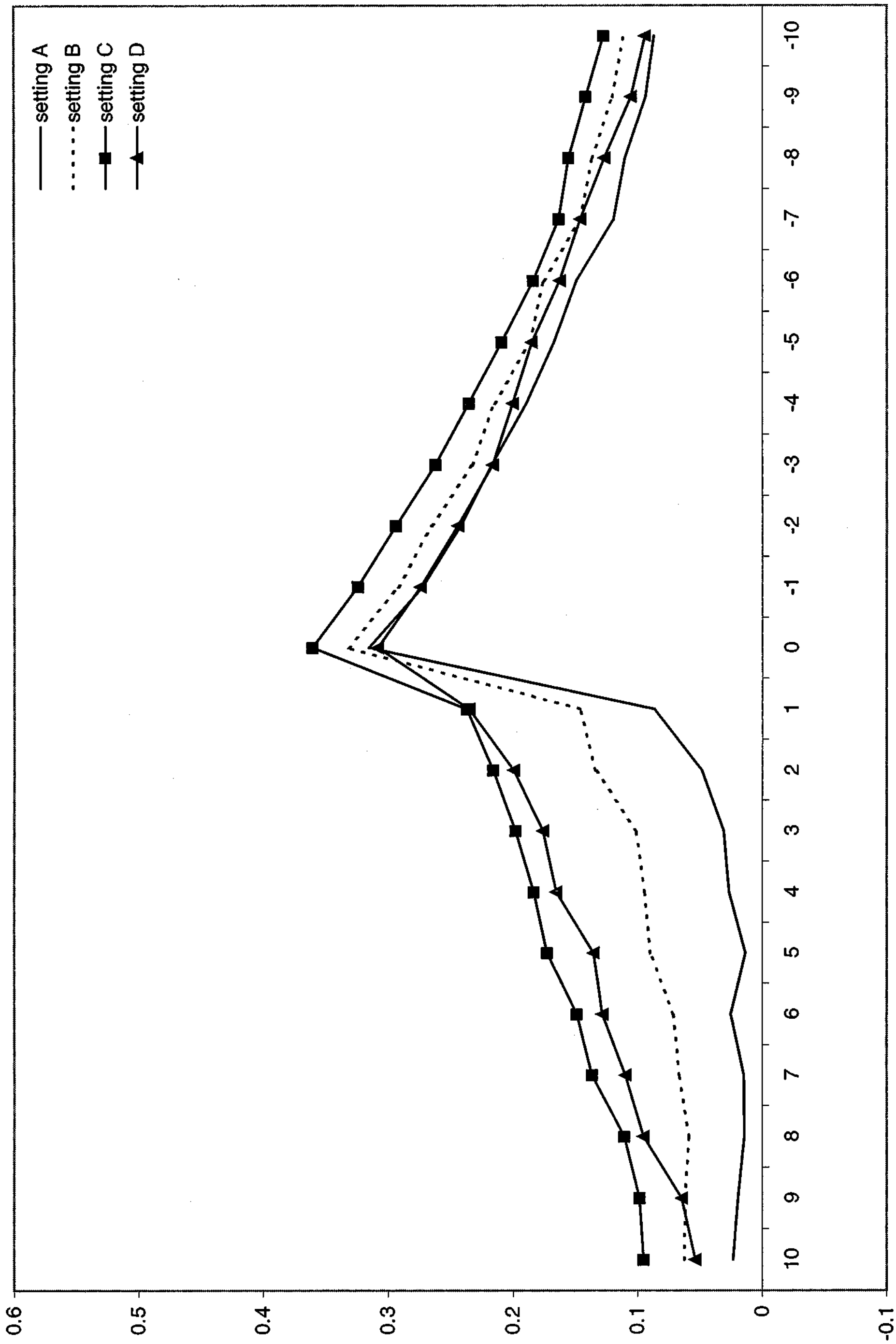

FIGURE 5. Cross correlations of volume$_{t+1}$ and volatility$_t$, simulated data.

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persistence in the (estimated) volatility, measured by $a_1 + b_1$ in equation (12), is very close to the calibrated value of 0.9 for the autocorrelation in the time-varying volatility except for setting A. This result can be considered as a justification for approximating a GARCH(1,1) process with the deterministic chaotic process employed in the simulation. For the volatility of all individual stock returns, the estimated ϕ_2''s are positive and highly significant. This finding is consistent with results obtained in other empirical work cited in Section 1. The above reported results confirm that much of the movement in the volatility of daily stock returns is related to volume. Furthermore, to compare the correlation pattern between volume and volatility, Figures 4 and 5 plot the cross correlations of volume (from lag 10 to lead 10) and volatility of observed and simulated data, respectively. For the observed data, the volatility is the GARCH(1,1) estimate whereas for the simulated data it is the theoretical volatility defined in equation (10). Figures 4 and 5 show that the correlation pattern between volume and volatility is qualitatively matched by the data simulated from the theoretical model.

4. CONCLUSION

This paper investigates the relation between volume, stock return autocorrelation, and volatility. I extend a special case of Wang's (1994) model, the case of symmetric information, by assuming that a subset of the shocks in the economy has autocorrelated time-varying volatility. It is shown that if the volatility is time varying, the relationship between volume and stock return autocorrelation is ambiguous, even if agents have symmetric information. This may explain the contradictory findings in the empirical literature studying the relation between volume and the persistence of stock returns. Moreover, under the assumption of time-varying volatility, one can study the correlatedness of volume with the one-period-ahead return volatility.

It is shown analytically that time-varying volatility of the payoffs on the non-tradable asset results in an ambiguous relation between volume and the serial correlation of stock returns. The relation between return volatility and volume cannot be derived analytically. We use numerical simulations as well as empirical exercises to address this more complicated question. In the numerical exercise, we simulate the model in a way that mimics the persistent volatility of high-frequency stock data documented in numerous empirical studies. Specifically, we approximate time-varying volatility of stock returns with a highly persistent chaotic tent map, which is known to have the same autocorrelation coefficients as an AR(1) process. The simulated data can approximate GARCH-type behavior very well. Whereas in the simulated data, no significant relation between volume and return autocorrelation can be found, there is a significant positive relation between volume and one-step-ahead stock return volatility. The ambiguous volume–persistence and positive volume–volatility relations are confirmed empirically by using four heavily traded individual stocks. Therefore, the data simulated from the highly stylized asset pricing model with deterministic volatility can mimic well the volume–return dynamics revealed in the observed data in these two respects.

NOTES

1. Using the annual growth rate of dividends for Boeing, IBM, Coca Cola, and Minnesota Mining & Manufacturing from 1973 to 1998, we conduct a likehood ratio test of constant volatility vs. the GARCH(1,1) process. For all four series, one cannot reject the hypothesis that volatility is constant.

2. Thus, for a type A trader, $\boldsymbol{X}_t = (X_t, y_t)'$, where X_t is his stock shares, y_t is his investment in the nontradable asset, and $\mathcal{Q}_{t+1} = (Q_{t+1}, q_{t+1})'$. For type B traders, $\boldsymbol{X}_t = X_t$ and $\mathcal{Q}_{t+1} = Q_{t+1}$ because they cannot invest in the nontradable asset.

3. See Appendix A.

4. These pecentages are obtained from $\mathrm{Var}_t(Q_{t+1}) = (1+a)^2\sigma_D^2 + (p_{Z,t+1})^2\sigma_{Z,t+1}^2$. In the numerical simulations, the maximum values of $p_{Z,t}$ for $b = 1, 10, 50$, and 100 are 16.5, 14.5, 15.3, and 16.1, respectively.

5. This is the 5% level for a one-tailed test, or the 10% level for a two-tailed test.

6. If the volatility of return per share is constant over time, i.e., $\sigma_{Z,t}^2 = c$ for all t, the one-step-ahead conditional volatility of return per dollar is $\sigma_{r,t+1}^2 = c/P_t^2$. Since volume V_t and squared price P_t^2 are positively related [see equations (4) and (7)], $\sigma_{r,t+1}^2$ will be positively related to $1/V_t$.

7. Nelson (1992) shows that the volatility can be estimated very precisely from high-frequency data, even when the true model for volatility is unknown.

8. There seems to be a typo in Wang's equations (A17) and (A18) (1994, p. 161).

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APPENDIX A: PROOF OF PROPOSITION 1

As in Appendix B of Wang (1994), we take three steps to prove Proposition 1: First, we conjecture that the equilibrium price function has the form as in equation (4). Second, we solve the optimization problem of both types of traders, given the conjectured price function. Finally, we verify the conjectured price function by imposing the market-clearing condition. The proof here differs from Wang's in two crucial aspects: (i) The coefficients in the conjected price function are time varying, whereas in Wang's case they are constant. (ii) Since we assume that the investor has a finite life horizon, the optimization problem is solved recursively.

Define

$$\epsilon_{D,t} = \delta\varepsilon_{D,t},$$

$$\epsilon_{Z,t} = \sigma_t\varepsilon_{z,t},$$

$$\epsilon_{q,t} = \kappa\varepsilon_{q,t},$$

where $\sigma_t^2 = f(\sigma_{t-1}^2; t)$. If the price takes the conjectured form as in equation (4),

$$P_t = p_{0,t} + aD_t - p_{Z,t}Z_t,$$

the excess return per share of the stock, Q_{t+1}, is

$$Q_{t+1} = p_{0,t+1} - Rp_{0,t} + [Rp_{Z,t} - a_Z p_{Z,t+1}]Z_t + (1+a)\epsilon_{D,t+1} - p_{Z,t+1}\epsilon_{Z,t+1}. \quad \textbf{(A.1)}$$

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Define $\Psi_t \equiv (1, Z_t)'$, which can be written as

$$\Psi_t = a_\Psi \Psi_{t-1} + b_{\Psi,t}\boldsymbol{\epsilon}_t,$$

with $a_\Psi = \begin{pmatrix} 1 & 0 \\ 0 & a_Z \end{pmatrix}$, $b_{\Psi,t} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and

$$\epsilon_t \equiv (\epsilon_{D,t} \quad \epsilon_{Z,t} \quad \epsilon_{q,t})' \sim N(0, \Sigma_t), \qquad \Sigma_t = \begin{pmatrix} \delta^2 & 0 & \rho \\ 0 & \sigma_t^2 & 0 \\ \rho & 0 & \kappa^2 \end{pmatrix}.$$

The excess returns of the type A traders $\mathcal{Q}^A_{t+1} = (Q_{t+1}, q_{t+1})'$ can be written as

$$\mathcal{Q}^A_{t+1} = \boldsymbol{e}^A_{\mathcal{Q}} \Psi_t + \boldsymbol{b}^A_{\mathcal{Q}} \epsilon_{t+1}, \tag{A.2}$$

with

$$\boldsymbol{e}^A_{\mathcal{Q}} = \begin{pmatrix} 0 & Rp_{Z,t} - a_Z p_{Z,t+1} \\ 0 & 1 \end{pmatrix}; \qquad \boldsymbol{b}^A_{\mathcal{Q}} = \begin{pmatrix} 1+a & -p_{Z,t+1} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Analogously, the excess returns of the type B traders $\mathcal{Q}^B_{t+1} = Q_{t+1}$ can be written as

$$\mathcal{Q}^B_{t+1} = \boldsymbol{e}^B_{\mathcal{Q}} \Psi_t + \boldsymbol{b}^B_{\mathcal{Q}} \epsilon_{t+1}, \tag{A.3}$$

with

$$\boldsymbol{e}^B_{\mathcal{Q}} = (0 \quad Rp_{Z,t} - a_Z p_{Z,t+1}); \qquad \boldsymbol{b}^B_{\mathcal{Q}} = (1+a \quad -p_{Z,t+1} \quad 0).$$

With this notation, both the type A and type B traders' optimization problem can be expressed in the form of the Bellman equation:

$$J(W_t; \Psi_t; t) = \max\left\{-\beta^t e^{-\gamma c_t} + E[J(W_{t+1}; \Psi_{t+1}; t+1) \mid \mathcal{I}_t]\right\}$$
$$\text{subject to } W_{t+1} = (W_t - c_t)R + \boldsymbol{X}_t' \mathcal{Q}_{t+1}.$$

This equation must hold for all $0 < t \le T$. Because of the salvage term in equation (3), we have

$$J(W_{T+1}, \Psi_{T+1}; T+1) = -\beta^{T+1} e^{-\alpha W_{T+1}}. \tag{A.4}$$

The trial solution for the value function considered here is

$$J(W_t; \Psi_t; t) = -\beta^t e^{-\frac{r\gamma}{R} W_t - \frac{1}{2}\Psi_t' \boldsymbol{v}_t \Psi_t}, \tag{A.5}$$

where $\boldsymbol{v}_t$ is a 2×2 matrix of following form:

$$\boldsymbol{v}^i_t = \begin{pmatrix} v^i_{1,t} & v^i_{3,t} \\ v^i_{3,t} & v^i_{2,t} \end{pmatrix},$$

for $i = A, B$. Define

$$\boldsymbol{v}^i_{aa,t+1} \equiv a'_\Psi \mathbf{v}^i_{t+1} a_\Psi,$$
$$\boldsymbol{v}^i_{ab,t+1} \equiv a'_\Psi \mathbf{v}^i_{t+1} b_\Psi,$$
$$\boldsymbol{v}^i_{bb,t+1} \equiv b'_\Psi \mathbf{v}^i_{t+1} b_\Psi,$$

for $i = A, B$.

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We first compute the solution for the period T. To do so, we compute the agents' value functions and policy functions, taking as given the previous wealth W_T and the vector of the state variables $\mathbf{\Psi}_T$. Agents' optimization problem at time T is

$$\max E_T(e^{-\gamma c_T} - \beta e^{-\alpha W_{T+1}})$$
$$\text{subject to } W_{T+1} = (W_T - c_T)R + \boldsymbol{X}_T' \mathcal{Q}_{T+1}. \tag{A.6}$$

If the price takes the conjectured form as in equation (4), the equilibrium price at T is

$$P_T = p_{0,T} + aD_T - p_{Z,T} Z_t.$$

The terminal price P_{T+1} is assumed to be the discounted sum of the future dividend payoffs:

$$P_{T+1} = E_{T+1}\left(\sum_{j=1}^{\infty} R^{-j} D_{T+1+j}\right).$$

In doing so, we implicitly assume that the stock lives infinitely. The excess return on stock and on the nontradable asset at $T+1$ is

$$\begin{aligned} Q_{T+1} &= P_{T+1} + D_{T+1} - RP_T, \\ &= -Rp_{0,T} + Rp_{Z,T} Z_T + (1+a)\epsilon_{D,T+1}, \\ q_{T+1} &= Z_T + \epsilon_{q,T+1}. \end{aligned}$$

Given the market-clearing condition

$$\omega X_T^A + (1-\omega)\, X_T^B = 1$$

and the terminal condition $\boldsymbol{v}_{T+1}^A = \boldsymbol{v}_{T+1}^B = 0$ from equation (A.6), we can fully determine the value function and the optimal investment-consumption policies. The type A traders' optimal demand of stock at T is given by

$$X_T^A = \frac{-R\kappa^2 p_{0,T}}{\alpha(1+a)^2(\delta^2\kappa^2 - \rho^2)} + \frac{R\kappa^2 p_{Z,T} - (1+a)\rho}{\alpha(1+a)^2(\delta^2\kappa^2 - \rho^2)} Z_T$$

and the type B traders' optimal demand of stock at T is

$$X_T^B = \frac{-Rp_{0,T}}{\alpha(1+a)^2\delta^2} + \frac{Rp_{Z,T}}{\alpha(1+a)^2\delta^2} Z_T$$

with

$$\begin{aligned} p_{0,T} &= \frac{-\alpha(1+a)^2\delta^2(\delta^2 - \rho^2)}{R(\delta^2\kappa^2 - \rho^2 + \omega\rho^2)}, \\ p_{Z,T} &= \frac{(1+a)\omega\delta^2\kappa\rho}{R\kappa^2(\delta^2\kappa^2 - \rho^2 + \omega\rho^2)}. \end{aligned} \tag{A.7}$$

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For the period $t \leq T$, one can solve the model in exactly the same way as in Wang [1994, Eq. (A18), p. 161]. The following recursive equations[8] have to hold for $t \leq T$:

$$v_{1,t}^A = \frac{1}{R}\left[v_{1,t+1}^A - \left(v_{3,t+1}^A\right)^2 s_{t+1}^A + \frac{\kappa^2}{\Delta_{t+1}^A}\left(\pi_{t+1} - p_{Z,t+1}v_{3,t+1}s_{t+1}^A\right)^2\right]$$
$$+2\left(\frac{r}{R}\ln r - \frac{1}{R}\ln\beta\sqrt{\frac{s_{t+1}^A}{\sigma_{t+1}^2}} - \ln R\right),$$

$$v_{2,t}^A = \frac{a_Z^2 v_{2,t+1}^A}{R}\left(1 - v_{2,t+1}^A s_{t+1}^A\right)$$
$$+\frac{\kappa^2\left[Rp_{Z,t} - a_Z p_{Z,t+1}\left(1 - v_{2,t+1}^A s_{t+1}^A\right)\right]^2}{R\Delta_{t+1}^A}$$
$$-\frac{2(1+a)\rho\left[Rp_{Z,t} - a_Z p_{Z,t+1}\left(1 - v_{2,t+1}^A s_{t+1}^A\right)\right]}{R\Delta_{t+1}^A} + \frac{(1+a)^2\delta^2 + p_{Z,t+1}^2 s_{t+1}^A}{R\Delta_{t+1}^A},$$

$$v_{3,t}^A = \frac{a_Z v_{3,t+1}^A}{R}\left(1 - v_{2,t+1}^A s_{t+1}^A\right)$$
$$+\frac{\kappa^2\left(\pi_{t+1} - p_{Z,t+1}v_{3,t+1}^A s_{t+1}^A\right)\left[Rp_{Z,t} - a_Z p_{Z,t+1}\left(1 - v_{2,t+1}^A s_{t+1}^A\right)\right]^2}{R\Delta_{t+1}^A}$$
$$-\frac{(1+a)\rho\left(\pi_{t+1} - p_{Z,t+1}v_{3,t+1}^A s_{t+1}^A\right)}{R\Delta_{t+1}^A},$$

$$v_{1,t}^B = \frac{1}{R}\left[v_{1,t+1}^B - \left(v_{3,t+1}^B\right)^2 s_{t+1}^B + \frac{1}{\Delta_{t+1}^B}\left(\pi_{t+1} - p_{Z,t+1}v_{3,t+1}s_{t+1}^B\right)^2\right]$$
$$+2\left(\frac{r}{R}\ln r - \frac{1}{R}\ln\beta\sqrt{\frac{s_{t+1}^A}{\sigma_{t+1}^2}} - \ln R\right),$$

$$v_{2,t}^B = \frac{a_Z^2 v_{2,t+1}^B}{R}\left(1 - v_{2,t+1}^B s_{t+1}^B\right) + \frac{\left[Rp_{Z,t} - a_Z p_{Z,t+1}\left(1 - v_{2,t+1}^B s_{t+1}^B\right)\right]^2}{R\Delta_{t+1}^B},$$

$$v_{3,t}^B = \frac{a_Z v_{3,t+1}^B}{R}\left(1 - v_{2,t+1}^B s_{t+1}^B\right)$$
$$+\frac{\left(\pi_{t+1} - p_{Z,t+1}v_{3,t+1}^B s_{t+1}^B\right)\left[Rp_{Z,t} - a_Z p_{Z,t+1}\left(1 - v_{2,t+1}^B s_{t+1}^B\right)\right]^2}{R\Delta_{t+1}^B},$$

where

$$s_{t+1}^A = \frac{\sigma_{t+1}^2}{1 + v_{2,t+1}^A\sigma_{t+1}^2},$$

$$s_{t+1}^B = \frac{\sigma_{t+1}^2}{1 + v_{2,t+1}^B\sigma_{t+1}^2},$$

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$$\Delta^A_{t+1} = (1+a)^2(\delta^2\kappa^2 - \rho^2) + \kappa^2 p^2_{Z,t+1} s^A_{t+1},$$
$$\Delta^B_{t+1} = (1+a)^2\delta^2 + p^2_{Z,t+1} s^B_{t+1},$$
$$\pi_{t+1} = p_{0,t+1} - Rp_{0,t}.$$

Moreover, from the market-clearing condition

$$\omega X^A_t + (1-\omega)X^B_t = 1$$

one can derive the following two additional equations:

$$p_{0,t} = \frac{1}{R}\left[p_{0,t+1} - \frac{\omega\kappa^2\Delta^B_{t+1}s^A_{t+1}v^A_{3,t+1} + (1-\omega)\Delta^A_{t+1}s^B_{t+1}v^B_{3,t+1} + \alpha\Delta^A_{t+1}\Delta^B_{t+1}}{\omega\kappa^2\Delta^B_{t+1} + (1-\omega)\Delta^A_{t+1}}\right],$$

$$p_{Z,t} = \frac{a_Z p_{Z,t+1}\left[\omega\kappa^2\Delta^B_{t+1}\left(1 - s^A_{t+1}v^A_{2,t+1}\right) + (1-\omega)\Delta^A_{t+1}\left(1 - s^B_{t+1}v^B_{2,t+1}\right)\right]}{R\left[\omega\kappa^2\Delta^B_{t+1} + (1-\omega)\Delta^A_{t+1}\right]}$$
$$+ \frac{\omega(1+a)\rho\Delta^B_{t+1}}{R\left[\omega\kappa^2\Delta^B_{t+1} + (1-\omega)\Delta^A_{t+1}\right]}.$$

This system of first-order difference equations can be solved backward, given the end condition $v^A_{T+1} = v^B_{T+1} = 0$. ■

APPENDIX B: PROOF OF PROPOSITION 2

Since trading volume at time t can be written as

$$V_t = (1-\omega)\left| f^B_{0,t} - f^B_{0,t-1} + f^B_{Z,t}Z_t - f^B_{Z,t-1}Z_{t-1}\right|,$$

define $Y_t \equiv f^B_{0,t} - f^B_{0,t-1} + f^B_{Z,t}Z_t - f^B_{Z,t-1}Z_{t-1}$. Then Y_t is normally distributed as

$$Y_t \sim N\left(\mu_t, \left[f^B_{Z,t}\right]^2\chi^2_t + \left[f^B_{Z,t-1}\right]^2\chi^2_{t-1}\right)$$

with

$$\mu_t \equiv f^B_{0,t} - f^B_{0,t-1}$$

and

$$\chi^2_t = \sum_{j=0}^{\infty} a^j_Z \sigma^2_{t-j}.$$

Denote the covariance matrix of $[\tilde{Q}_{t+1}, (\tilde{Q}_t, Y_t)']$ as

$$\sum_t \equiv \begin{pmatrix} \sum_{aa,t} & \sum_{ab,t} \\ \sum_{ba,t} & \sum_{bb,t} \end{pmatrix}.$$

Using a similar argument as in Appendix B of Wang (1994), we can derive the following result, which is the counterpart of equation (B10) in Wang (1994, p. 165), with time-varying volatility:

$$E\left(\tilde{Q}_{t+1} \mid \tilde{Q}_t, V_t\right) = \sum\nolimits_{ab,t}\sum\nolimits_{bb,t}^{-1}\left[\begin{pmatrix}\tilde{Q}_t \\ -\mu_t\end{pmatrix} + \begin{pmatrix}0 \\ V_t\end{pmatrix} G(\tilde{Q}_t, V_t, \mu_t)\right], \qquad \textbf{(B.1)}$$

where

$$G(\tilde{Q}_t, V_t, \mu_t) = \frac{f_1 - f_2}{f_1 + f_2},$$

and

$$f_1 = \exp\left[-\frac{1}{2}\begin{pmatrix}\tilde{Q}_t \\ V_t - \mu_t\end{pmatrix}' \sum_{bb,t}^{-1} \begin{pmatrix}\tilde{Q}_t \\ V_t - \mu_t\end{pmatrix}\right];$$

$$f_2 = \exp\left[-\frac{1}{2}\begin{pmatrix}\tilde{Q}_t \\ -V_t - \mu_t\end{pmatrix}' \sum_{bb,t}^{-1} \begin{pmatrix}\tilde{Q}_t \\ -V_t - \mu_t\end{pmatrix}\right].$$

The Taylor expansion with respect to $\tilde{Q}_t$ and V_t of the right-hand side of (B.1) is

$$E(\tilde{Q}_{t+1} \mid \tilde{Q}_t, V_t) = \lambda_{0,t+1} + \lambda_{1,t+1}V_t^2 + \lambda_{2,t+1}\tilde{Q}_t + \lambda_{3,t+1}V_t^2\tilde{Q}_t + \text{high-order terms},$$

where

$$\begin{aligned}
\lambda_{0,t+1} &= -\left(\kappa_{Q_{t+1},Q_t} h_{Q_t,Y_t} + \kappa_{Q_{t+1},Y_t} h_{Y_t,Y_t}\right)\mu_t; \\
\lambda_{1,t+1} &= -\left(\kappa_{Q_{t+1},Q_t} h_{Q_t,Y_t} + \kappa_{Q_{t+1},Y_t} h_{Y_t,Y_t}\right) h_{Y_t,Y_t}\mu_t; \\
\lambda_{2,t+1} &= \kappa_{Q_{t+1},Q_t} h_{Q_t,Q_t} + \kappa_{Q_{t+1},Y_t} h_{Q_t,Y_t}; \\
\lambda_{3,t+1} &= \left(\kappa_{Q_{t+1},Q_t} h_{Q_t,Y_t} + \kappa_{Q_{t+1},Y_t} h_{Y_t,Y_t}\right) h_{Q_t,Y_t}; \\
\kappa_{Q_{t+1},Q_t} &= (Rp_{Z,t-1} - a_Z p_{Z,t})(Rp_{Z,t} - a_Z p_{Z,t+1}) a_Z \chi_{t-1}^2 + p_{Z,t}(a_Z p_{Z,t+1} - r p_{Z,t})\sigma_t^2; \\
\kappa_{Q_{t+1},Y_t} &= a_z (Rp_{Z,t} - a_Z p_{Z,t+1})\left(a_z f_{Z,t}^B - f_{Z,t-1}^B\right)\chi_{t-1}^2 - (a_Z p_{Z,t+1} - r p_{Z,t}) f_{Z,t}^B \sigma_t^2; \\
h_{Q_t,Q_t} &= |\Omega_t|^{-1}\left[\left(a_z f_{Z,t}^B - f_{Z,t-1}^B\right)^2 \chi_{t-1}^2 + \left(f_{Z,t}^B\right)^2 \sigma_t^2\right]; \\
h_{Q_t,Y_t} &= |\Omega_t|^{-1}\left[p_{Z,t} f_{Z,t}^B \sigma_t^2 - \left(Rp_{Z,t-1} - a_Z p_{Z,t}\right)\left(a_z f_{Z,t}^B - f_{Z,t-1}^B\right)\chi_{t-1}^2\right]; \\
\boldsymbol{h}_t &\equiv \begin{pmatrix} h_{Q_t,Q_t} & h_{Q_t,Y_t} \\ h_{Y_t,Q_t} & h_{Y_t,Y_t}\end{pmatrix} = [\mathrm{VC}(Q_t, Y_t)]^{-1} = \begin{pmatrix} \kappa_{Q_t,Q_t} & \kappa_{Q_t,Y_t} \\ \kappa_{Y_t,Q_t} & \kappa_{Y_t,Y_t}\end{pmatrix}^{-1}; \\
|\Omega_t| &= \det(\boldsymbol{h}_t);
\end{aligned}$$

with VC denoting the variance–covariance matrix. Since the signs of κ_{Q_{t+1},Q_t}, κ_{Q_{t+1},Y_t}, and h_{Q_t,Y_t} depend on the signs of $Rp_{Z,t-1} - a_Z p_{Z,t}$, $Rp_{Z,t} - a_Z p_{Z,t+1}$, $a_Z p_{Z,t+1} - r p_{Z,t}$, and $a_z f_{Z,t}^B - f_{Z,t-1}^B$, which are not unambiguous in the case of time-varying volatility, the sign of $\lambda_{3,t+1}$ is not unambiguous. In our numerical simulations, we could indeed observe both positive and negative values for $\lambda_{3,t+1}$. ■