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TEMPERED SPECTRAL TRANSFER IN THE TWISTED ENDOSCOPY OF REAL GROUPS

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Abstract Suppose that G is a connected reductive algebraic group defined over $\mathbf{R}$, $G(\mathbf{R})$ is its group of real points, θ is an automorphism of G, and ω is a quasicharacter of $G(\mathbf{R})$. Kottwitz and Shelstad defined endoscopic data associated to (G, θ, ω), and conjectured a matching of orbital integrals between functions on $G(\mathbf{R})$ and its endoscopic groups. This matching has been proved by Shelstad, and it yields a dual map on stable distributions. We express the values of this dual map on stable tempered characters as a linear combination of twisted characters, under some additional hypotheses on G and θ.

Keywords: twisted endoscopy; spectral transfer

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1. Introduction

The theory of endoscopy expresses the harmonic analysis of an algebraic group in terms of the harmonic analysis of smaller so-called endoscopic groups. The group in the present work is a connected reductive algebraic group G defined over $\mathbf{R}$, and the endoscopic groups are denoted by H. Actually, an endoscopic group is only a part of an endoscopic datum upon which the analytic comparison depends, but we shall overlook this briefly.

In standard endoscopy there are several established identities connecting the harmonic analysis of G to H. These identities rely on correspondences between the conjugacy classes of the groups G and H. Perhaps the most basic identities are of the form

$$\sum_{\gamma} \mathcal{O}_{\gamma}(f_H) = \sum_{\delta} \Delta(\gamma, \delta)\mathcal{O}_{\delta}(f), \quad f \in C_c^{\infty}(G(\mathbf{R})) \tag{1.1}$$

[34, Theorem 14.3]. Here, $\mathcal{O}.$ denotes an orbital integral, γ a conjugacy class of $H(\mathbf{R})$ which corresponds to a conjugacy class δ in $G(\mathbf{R})$, $\Delta(\gamma, \delta)$ are scalars, and $f_H \in C_c^{\infty}(H(\mathbf{R}))$. The sum on the left runs over the representatives of a *stable* conjugacy

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class. An alternative way of expressing (1.1) is that, for any Φ in the space of orbital integrals on $G(\mathbf{R})$,

$$\mathrm{Trans}(\Phi) := \sum_{\delta} \Delta(\cdot, \delta)\Phi(\delta)$$

lies in the space of stable orbital integrals on $H(\mathbf{R})$. The map Trans is typically called *geometric transfer*.

The transpose of geometric transfer is a map from stable distributions on $H(\mathbf{R})$ to invariant distributions on $G(\mathbf{R})$. This is what is meant by *spectral transfer*. Tempered spectral transfer identities are dual to the geometric identities above, and have the basic form

$$\sum_{\pi_H \in \Pi_{\varphi_H}} \Theta_{\pi_H}(f_H) = \sum_{\pi \in \Pi_{\varphi}} \Delta(\boldsymbol{\varphi_H}, \pi)\Theta_{\pi}(f), \quad f \in C_c^{\infty}(G(\mathbf{R})). \tag{1.2}$$

Here, the orbital integrals are replaced by Harish-Chandra's characters $\Theta_{\cdot}$ of tempered representations in L-packets. The terms $\Delta(\boldsymbol{\varphi_H}, \pi)$ are scalars called *spectral transfer factors*. They are defined relative to the *geometric transfer factors* $\Delta(\gamma, \delta)$ given in (1.1). The definition of these factors and the proofs of these identities were given by Shelstad originally in [33], and revised more recently in [34, 36].

In *twisted* endoscopy, the group G is supplied with an algebraic automorphism θ and a continuous quasicharacter ω of $G(\mathbf{R})$. The endoscopic groups H are influenced by the twisting data (θ, ω). So too are the underlying conjugacy classes and representations. The foundations of twisted endoscopy were set down by Kottwitz and Shelstad in [20]. They included a conjectural twisted geometric transfer identity extending (1.1). This geometric transfer identity was proved recently by Shelstad in the real case [37].

Extensions of (1.2) to base change were proved in [7, 10, 11]. More generally, the real twisted version of (1.2) was proved in [28] under the assumptions that ω was trivial, θ was of finite order, and the L-packets consisted of (essential limits of) discrete series. There was one further technical assumption in [28] which we will give later. The purpose of this paper is to extend (1.2) to the twisted context only under this technical assumption and the assumption that θ acts semisimply on the centre of G. Our extension of (1.2) is given in Theorem 6.7. We will give just enough background for a precise statement of this theorem, and then discuss its proof.

Suppose that θ is an algebraic automorphism of G which is defined over $\mathbf{R}$. The technical assumption alluded to above requires us to specify a quasisplit group G^* in the inner class of G. By definition, the group G^* has an $\mathbf{R}$-splitting $\mathbf{spl}_{G^*}$, and there is an inner twisting $\psi : G \to G^*$. On may choose $g_\theta \in G^*$ such that $\theta^* = \mathrm{Int}(g_\theta)\psi\theta\psi^{-1}$ preserves $\mathbf{spl}_{G^*}$. One may also choose $u_\sigma \in G^*$ such that $\mathrm{Int}(u_\sigma) = \psi\sigma\psi^{-1}\sigma^{-1}$ for the non-trivial automorphism $\sigma \in \mathrm{Gal}(\mathbf{C}/\mathbf{R})$. In fact, one chooses g_θ and u_σ in G^*_{sc}, the simply connected covering group of the derived group of G^*, with the interior automorphisms interpreted correctly [20, 1.2 and 3.1]. It turns out that

$$\sigma \mapsto g_\theta u_\sigma \sigma(g_\theta)^{-1}\theta^*(u_\sigma)^{-1} \tag{1.3}$$

determines a one-cocycle in the θ^*-coinvariants of the centre of G^*_{sc} [20, Lemma 3.1.A]. Our technical assumption is that this cocycle is a coboundary. The assumption is satisfied for quasisplit G when θ fixes an $\mathbf{R}$-splitting. In particular, it is satisfied for the automorphisms $\tilde{\theta}(N)$ and $\tilde{\theta}$ appearing in [1].

The remaining background for the main theorem pertains to twisted endoscopy, and we expect that the reader has some familiarity with this. Let $(H, \mathcal{H}, \mathsf{s}, \xi)$ be an endoscopic datum for (G, θ, ω) [20, 2.1], and (H_1, ξ_{H_1}) be a corresponding z-pair [20, 2.2]. The group H_1 is a z-extension of H with centre Z_1. For every $f \in C_c^\infty(G(\mathbf{R}))$, there exists a smooth function f_{H_1} on $H_1(\mathbf{R})$ with matching orbital integrals [37], (5.5.1) [20]. Let $\boldsymbol{\varphi_{H_1}}$ be a tempered L-parameter for H_1, and $\varphi_{H_1} \in \boldsymbol{\varphi_{H_1}}$ be an admissible homomorphism. One may define a tempered admissible homomorphism φ of G by composing φ_{H_1} with $\xi \circ \xi_{H_1}^{-1}$ [28, 6]. By the local Langlands correspondence, there are L-packets of (equivalence classes of) irreducible tempered representations $\Pi_{\varphi_{H_1}}$ and Π_φ attached to (the L-parameters of) φ_{H_1} and φ, respectively [26]. For each representation $\pi_{H_1} \in \Pi_{\varphi_{H_1}}$, we have a distribution character $\Theta_{\pi_{H_1}}$. If $\pi \in \Pi_\varphi$ is equivalent to $\omega \otimes \pi \circ \theta$ via an intertwining operator T_π, then one may define a distribution character $\Theta_{\pi, \mathsf{T}_\pi}$ which is twisted by T_π [28, 5.2]. Otherwise, we set $\Theta_{\pi, \mathsf{T}_\pi} = 0$.

Theorem 1.1 (Main Theorem). *Suppose that θ acts semisimply on the centre of G, and that the cocycle defined by (1.3) is a coboundary. Then there exist spectral transfer factors*

$$\Delta(\boldsymbol{\varphi_{H_1}}, \pi) \in \mathbf{C}, \quad \pi \in \Pi_\varphi$$

such that

$$\int_{H_1(\mathbf{R})/Z_1(\mathbf{R})} f_{H_1}(h) \sum_{\pi_{H_1} \in \Pi_{\boldsymbol{\varphi_{H_1}}}} \Theta_{\pi_{H_1}}(h)\, dh = \sum_{\pi \in \Pi_\varphi} \Delta(\boldsymbol{\varphi_{H_1}}, \pi) \Theta_{\pi, \mathsf{T}_\pi}(f)$$

for all $f \in C_c^\infty(G(\mathbf{R}))$.

The crux of this theorem is to produce explicit spectral transfer factors so that the desired identity holds. Although we have done this, we must stress that the transfer factors are defined without showing that they are *canonical*. Indeed, there are certain choices made in the definition of these transfer factors [28, 6.3], and one wishes to show that the transfer factors are independent of these choices. This type of canonicity holds for geometric transfer factors [20, §4.6], and the analogous canonicity for spectral transfer factors is to be expected (see [31] and [36, §12]). One would also expect a form of Whittaker normalization for the twisted spectral transfer factors [20, 5.3], [36, 11]. Rather than trying to cull canonicity and normalization from the complicated spectral transfer factors used here, it would be better to develop a theory that parallels the one given in [36].

Before turning to a discussion of the proof of the main theorem, let us comment on the aforementioned technical assumption on (1.3). It does not seem to be relevant to expected applications, and it simplifies the setting for it avoids any twisting of the endoscopic groups (see [20, 5.4]). However, the cocycle of (1.3) appears to be an essential feature of twisted endoscopy, and future work in this area should accommodate it.

Let us turn to the proof of the main theorem. The proof relies on [28], where ω was taken to be trivial and θ was taken to be of finite order. These assumptions were made on ω and θ in order to satisfy the hypotheses of a twisted version of Harish-Chandra's uniqueness theorem [29, Theorem 15.1], [28, Theorem 1]. This twisted uniqueness theorem has been extended here in appendix A to allow for arbitrary ω and θ. The assumption of θ being semisimple on the centre remains due to changes of variable [28, Corollary 1].

The passage from (essential limits of) discrete series to tempered representations in the proof of the main theorem in *standard* endoscopy employs a simple application of Harish-Chandra's method of parabolic descent [36, 14]. To see that something more is required in the twisted case, consider the example of $G(\mathbf{R}) = \mathrm{SL}(3, \mathbf{R})$. Let ω be trivial, and set the automorphism θ equal to the inverse-transpose map composed with conjugation by a representative $w_0 \in \mathrm{SL}(3, \mathbf{R})$ for the long element in the Weyl group. The dual group $\hat{G}$ is equal to $\mathrm{PGL}(3, \mathbf{C})$. There is a tempered L-packet Π_φ attached to the homomorphism $\varphi : W_{\mathbf{R}} \to \hat{G}$ defined by

$$\varphi(re^{it}) = \begin{bmatrix} e^{it} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-it} \end{bmatrix} Z, re^{it} \in \mathbf{C}^\times, \quad \varphi(\sigma) = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} Z.$$

Here, Z is the centre of $\mathrm{GL}(3, \mathbf{C})$, and $W_{\mathbf{R}} = \mathbf{C}^\times \cup \mathbf{C}^\times \sigma$ is the real Weil group. The image of φ is contained in the proper Levi subgroup

$$\hat{M} = \left\{ \begin{bmatrix} * & 0 & * \\ 0 & 1 & 0 \\ * & 0 & * \end{bmatrix} Z \right\} \cong \mathrm{GL}(2, \mathbf{C}),$$

and this Levi subgroup is dual to the following Levi subgroup of $\mathrm{SL}(3, \mathbf{R})$:

$$M(\mathbf{R}) = \left\{ \begin{bmatrix} * & 0 & * \\ 0 & * & 0 \\ * & 0 & * \end{bmatrix} \right\} \cong \mathrm{GL}(2, \mathbf{R}).$$

Regarding φ as a homomorphism into $\hat{M}$, one obtains an L-packet $\Pi_{\varphi,M}$ of discrete series representations of $M(\mathbf{R})$. In addition, the representations in Π_φ are the irreducible subrepresentations of the representations induced from $\Pi_{\varphi,M}$ [4, 11.3]. The element

$$\mathsf{s} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} w_0 Z \in \mathrm{PGL}(3, \mathbf{C})$$

corresponds to an endoscopic group $H(\mathbf{R}) = \mathrm{PGL}(2, \mathbf{R})$, and the tempered L-packet Π_φ corresponds to an L-packet Π_{φ_H} of discrete series representations [20, p. 24].

In standard endoscopy [34, 7], one may take the centralizer of the split component of an elliptic maximal torus in $H(\mathbf{R})$ to produce a Levi subgroup $M^*(\mathbf{R}) \subseteq G(\mathbf{R})$ which corresponds endoscopically to $H(\mathbf{R})$ and yields an L-packet Π_{φ,M^*} of discrete series. One then uses parabolic descent to M^*. Unfortunately, in our case the elliptic tori of $H(\mathbf{R})$ have trivial split component so $M^* = \mathrm{SL}(3, \mathbf{R})$ and $\Pi_{\varphi,M^*} = \Pi_\varphi$ does *not* consist of discrete series representations.

One might nevertheless associate H with the Levi subgroup M above, for M is preserved by θ and $\Pi_{\varphi,M}$ consists of discrete series. One might then attempt to derive a spectral character identity by parabolic descent to M. The problem here is that in the twisted case the characters on the right-hand side of (1.2) are twisted by an intertwining operator. Although M is preserved by θ, none of its associated parabolic subgroups are. This necessitates the introduction of a Knapp–Stein intertwining operator [17, 6 XIV] into the twisted character, and this operator hinders the usual process of parabolic descent [17, 3 X]. In fact, there is no proper maximal parabolic subgroup of $\mathrm{SL}(3, \mathbf{R})$ which is preserved by θ.

One way out of this bind is to recognize that the representations in Π_φ are *fundamental series* representations (III.3 [13]). Work of Duflo and Bouaziz affords us with characterizations of, and twisted character expansions for, the fundamental series. Using this work, one may prove a twisted spectral transfer identity between $H(\mathbf{R})$ and $\mathrm{SL}(3, \mathbf{R})$ without appealing to parabolic descent. This is the approach we use in general. We have learnt from Clozel that this approach was already present in [9, §2.5].

For the remainder of this introduction, assume that H is any endoscopic group for a general twisting datum (G, θ, ω). Our proof of spectral transfer follows in three stages. In the first stage, the L-packet Π_{φ_H} consists of (essential) discrete series representations, and Π_φ consists of fundamental series representations (Theorem 4.11). In the second stage, the method of coherent continuation is applied to extend spectral transfer to the case that Π_{φ_H} consists of (essential) discrete series representations and Π_φ consists of *limits* of fundamental series representations (Theorem 5.3). In the final stage, it is shown that there is a parabolic subgroup of $G(\mathbf{R})$ which allows us to imitate the parabolic descent argument of standard endoscopy [36, 14]. These three stages are analogous to the three stages of the spectral transfer theorem in standard endoscopy [36, 13–14], [31, 11].

We close with some anticipated consequences of tempered twisted spectral transfer. As in standard endoscopy, one expects to invert the spectral transfer identities for a fixed L-packet Π_φ [33, 5.4], [35]. In so doing, one expects to pair Π_φ with a group-theoretic structure fine enough to isolate individual representations [3, 6]. Such a pairing is of fundamental importance to twisted trace formula comparisons. This is evidenced by Arthur's recent work in classifying automorphic representations of symplectic and orthogonal groups [1, Theorem 2.2.1 and Remark 5]. Indeed, when G is a general linear group the packet Π_φ is a singleton and Theorem 1.1 has important consequences for [1, (2.2.6)].

2. Notation

In this section only, G is a real Lie group which acts upon a non-empty set J. We set

$$N_G(J) = \{g \in G : g \cdot J \subseteq J\},$$
$$Z_G(J) = \{g \in G : g \cdot j = j \text{ for all } j \in J\}.$$

In what follows, the set $N_G(J_1)$ always forms a group. We set $\Omega(G, J)$ equal to the resulting factor group $N_G(J)/Z_G(J)$.

For an automorphism θ of G, we set $\langle\theta\rangle$ equal to the group of automorphisms generated by θ. There is a corresponding semidirect product $G \rtimes \langle\theta\rangle$. When elements of G are written side by side with elements in $\langle\theta\rangle$, we consider them to belong to this semidirect product.

The inner automorphism of an element $\delta \in G$ is defined by

$$\mathrm{Int}(\delta)(x) = \delta x \delta^{-1}, \quad x \in G.$$

It shall be convenient to denote the fixed-point set of $\mathrm{Int}(\delta) \circ \theta$ in G by $G^{\delta\theta}$. We shall abbreviate the notation $\mathrm{Int}(\delta) \circ \theta$ to $\mathrm{Int}(\delta)\theta$ or $\delta\theta$ habitually.

Unless otherwise mentioned, we denote the real Lie algebra of a Lie group using Gothic script. For example, the real Lie group of G is denoted by $\mathfrak{g}$. Suppose that J is Cartan subgroup of a reductive group G. Then the pair of complex Lie algebras $(\mathfrak{g} \otimes \mathbf{C}, \mathfrak{j} \otimes \mathbf{C})$ determines a root system which we denote by $R(\mathfrak{g} \otimes \mathbf{C}, \mathfrak{j} \otimes \mathbf{C})$. We denote the Lie algebra dual to $\mathfrak{g}$ by $\mathfrak{g}^*$. The differential of the inner automorphism $\mathrm{Int}(\delta)$ is the adjoint automorphism $\mathrm{Ad}(\delta)$ on $\mathfrak{g}$. The adjoint automorphism induces an automorphism on $\mathfrak{g}^*$ in the usual way. Often, it shall be convenient to write $\delta \cdot X$ in place of $\mathrm{Ad}(\delta)(X)$ for $X \in \mathfrak{g}$. Similarly, we write $\theta \cdot X$ to mean the differential of θ acting on $X \in \mathfrak{g}$. We extend this slightly abusive notation to the dual spaces, writing $\delta \cdot \lambda$ or even simply $\delta\lambda$ in place of the coadjoint action of δ on $\lambda \in \mathfrak{g}^*$.

Finally, if we take H to be an algebraic group defined over $\mathbf{R}$, we denote its identity component by H^0 and derived subgroup by H_{der}. The group of real points of H is denoted by $H(\mathbf{R})$. This is a real Lie group, and we denote the identity component of $H(\mathbf{R})$ in the real manifold topology by $H(\mathbf{R})^0$.

3. The foundations of real twisted endoscopy

This section is a digest of some early material in [20], in the special case that the field of definition is equal to $\mathbf{R}$. It is essentially a reproduction of [28, Chapter 3], and is included for convenience and completeness.

3.1. Groups and automorphisms

Let G be a connected reductive algebraic group defined over $\mathbf{R}$. We take θ to be an algebraic automorphism of G defined over $\mathbf{R}$, and assume additionally that it acts semisimply on the centre Z_G of G. Set $G(\mathbf{R})$ to be the group of real points of G. Let Γ be the Galois group of $\mathbf{C}/\mathbf{R}$, and let σ be its non-trivial element.

Let us fix a triple

$$(B_0, T_0, \{X\}) \tag{3.1}$$

in which B_0 is a Borel subgroup of G, $T_0 \subseteq B_0$ is a maximal torus of G, and $\{X\}$ is a collection of root vectors corresponding to the simple roots determined by B_0 and T_0. Such triples are called *splittings* of G. If $(B_0, T_0, \{X\})$ is preserved by Γ, then it is called an $\mathbf{R}$-*splitting*.

There is a unique quasisplit group G^* of which G is an inner form [39, Lemma 16.4.8]. This is to say that there is an isomorphism $\psi : G \to G^*$ and $\psi\sigma\psi^{-1}\sigma^{-1} = \mathrm{Int}(u')$ for some $u' \in G^*$. We shall choose u_σ in the simply connected covering group G^*_{sc} of the

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derived group G^*_{der} of G^* so its image under the covering map is u'. We shall then abuse notation slightly by identifying u_σ with u' in equations such as

$$\psi\sigma\psi^{-1}\sigma^{-1} = \text{Int}(u_\sigma). \tag{3.2}$$

As G^* is quasisplit, there is a Borel subgroup B^* defined over $\mathbf{R}$. Applying [40, Theorem 7.5] to B^* and σ, we obtain an $\mathbf{R}$-splitting $(B^*, T^*, \{X^*\})$. Following the convention made for $u_\sigma \in G^*_{\text{sc}}$, we may choose $g_\theta \in G^*_{\text{sc}}$ so the automorphism

$$\theta^* = \text{Int}(g_\theta)\psi\theta\psi^{-1} \tag{3.3}$$

preserves $(B^*, T^*, \{X^*\})$ [39, Theorems 6.2.7 and 6.4.1], [16, 16.5]. Since

$$\sigma(\theta^*) = \sigma\theta^*\sigma^{-1} = \text{Int}(\sigma(g_\theta u_\sigma)g_\theta^{-1}\theta^*(u_\sigma))\theta^*$$

preserves $(B^*, T^*, \{X^*\})$, and the only inner automorphisms which do so are trivial, it follows in turn that $\text{Int}(\sigma(g_\theta u_\sigma)g_\theta^{-1}\theta^*(u_\sigma))$ is trivial and $\sigma(\theta^*) = \theta^*$. This means that the automorphism θ^* is defined over $\mathbf{R}$.

We wish to describe the action of θ induced on the L-group of G. The splitting (3.1) determines a based root datum [39, Proposition 7.4.6] and an action of Γ on the Dynkin diagram of G [4, 1.3]. To the dual based root datum there is attached a dual group $\hat{G}$ defined over $\mathbf{C}$, a Borel subgroup $\mathcal{B} \subseteq \hat{G}$, and a maximal torus $\mathcal{T} \subseteq \mathcal{B}$ [38, 2.12]. Let us fix a splitting $(\mathcal{B}, \mathcal{T}, \{\mathcal{X}\})$ of $\hat{G}$. This allows us to transfer the action of Γ from the Dynkin diagram of $\hat{G}$ to an algebraic action of $\hat{G}$ [38, Proposition 2.13]. This action may be extended trivially to the Weil group $W_{\mathbf{R}}$, which as a set we write as $\mathbf{C}^\times \cup \sigma\mathbf{C}^\times$ [4, 9.4]. The L-group ${}^LG$ is defined by the resulting semidirect product ${}^LG = \hat{G} \rtimes W_{\mathbf{R}}$.

In a parallel fashion, θ induces an automorphism of the Dynkin diagram of G, which then transfers to an automorphism $\hat{\theta}$ on $\hat{G}$. We define ${}^L\theta$ to be the automorphism of ${}^LG$ equal to $\hat{\theta} \times 1_{W_{\mathbf{R}}}$. By definition, the automorphism $\hat{\theta}$ preserves $(\mathcal{B}, \mathcal{T}, \{\mathcal{X}\})$.

We close this section with some remarks concerning Weyl groups. Let us assume for the moment that B_0 and T_0 are preserved by θ, and that T^1 is the identity component of $T^\theta \subseteq T_0$. The torus T^1 contains strongly regular elements [2, pp. 227–228], so its centralizer in G is the maximal torus T_0. Setting the identity component of G^θ equal to G^1 and the Weyl group of G^1 relative to T^1 equal to $\Omega(G^1, T^1)$, we see that we have an embedding

$$\Omega(G^1, T^1) \to \Omega(G, T)^\theta$$

into the θ-fixed elements of the Weyl group $\Omega(G, T)$. In fact, this embedding is an isomorphism [23, Lemma II.1.2].

3.2. Endoscopic data and z-pairs

Endoscopic data are defined in terms of the group G, the automorphism θ, and a cohomology class $\mathbf{a} \in H^1(W_{\mathbf{R}}, Z_{\hat{G}})$, where $Z_{\hat{G}}$ denotes the centre of $\hat{G}$. Let ω be the quasicharacter of $G(\mathbf{R})$ determined by $\mathbf{a}$ [26, pp. 122–123], and let us fix a one-cocycle a in the class $\mathbf{a}$. By definition [20, pp. 17–18], *endoscopic data* for $(G, \theta, \mathbf{a})$ consist of the following:

(1) a quasisplit group H defined over $\mathbf{R}$;

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(2) a split topological group extension

$$1 \to \hat{H} \to \mathcal{H} \overset{c}{\leftrightarrows} W_{\mathbf{R}} \to 1,$$

whose corresponding action of $W_{\mathbf{R}}$ on $\hat{H}$ coincides with the action given by the L-group ${}^{L}H = \hat{H} \rtimes W_{\mathbf{R}}$;

(3) an element $\mathsf{s} \in \hat{G}$ such that $\mathrm{Int}(\mathsf{s})\hat{\theta}$ is a semisimple automorphism [40, 7];

(4) an L-homomorphism [20, p. 18] $\xi : \mathcal{H} \to {}^{L}G$ satisfying

(a) $\mathrm{Int}(\mathsf{s})\, {}^{L}\theta \circ \xi = a' \cdot \xi$ [4, 8.5] for some one-cocycle a' in the class $\mathbf{a}$,

(b) ξ maps $\hat{H}$ isomorphically onto the identity component of $\hat{G}^{\mathsf{s}\hat{\theta}}$, the group of fixed points of $\hat{G}$ under the automorphism $\mathrm{Int}(\mathsf{s})\hat{\theta}$.

Despite requirement 3.2 of this definition, it might not be possible to define an isomorphism between $\mathcal{H}$ and ${}^{L}H$ which extends the identity map on $\hat{H}$. One therefore introduces a z-extension [20, 2.2], [25]

$$1 \to Z_1 \to H_1 \overset{p_H}{\to} H \to 1 \tag{3.4}$$

in which H_1 is a connected reductive group containing a central torus Z_1. The surjection p_H restricts to a surjection $H_1(\mathbf{R}) \to H(\mathbf{R})$.

Dual to (3.4) is the extension

$$1 \to \hat{H} \to \hat{H}_1 \to \hat{Z}_1 \to 1. \tag{3.5}$$

Regarding $\hat{H}$ as a subgroup of $\hat{H}_1$, we may assume that ${}^{L}H$ embeds into ${}^{L}H_1$ and that $\hat{H}_1 \to \hat{Z}_1$ extends to an L-homomorphism

$$p : {}^{L}H_1 \to {}^{L}Z_1.$$

According to [20, Lemma 2.2.A], there is an L-homomorphism $\xi_{H_1} : \mathcal{H} \to {}^{L}H_1$ which extends the inclusion of $\hat{H} \to \hat{H}_1$ and defines a topological isomorphism between $\mathcal{H}$ and $\xi_{H_1}(\mathcal{H})$. Kottwitz and Shelstad call (H_1, ξ_{H_1}) a *z-pair* for $\mathcal{H}$.

Observe that the composition

$$W_{\mathbf{R}} \overset{c}{\to} \mathcal{H} \overset{\xi_{H_1}}{\to} {}^{L}H_1 \overset{p}{\to} {}^{L}Z_1 \tag{3.6}$$

determines a quasicharacter λ_{Z_1} of $Z_1(\mathbf{R})$ via the local Langlands correspondence [4, 9].

3.3. Norm mappings

Our goal here is to fix endoscopic data $(H, \mathcal{H}, \mathsf{s}, \xi)$ as defined in the previous section and to describe a map from the semisimple conjugacy classes of the endoscopic group H to the semisimple θ-conjugacy classes of G. The map uses the quasisplit form G^* as an intermediary. The reference for this section is [20, Chapter 3].

Since we are interested in *semisimple* conjugacy classes, and semisimple elements lie in tori, we shall begin by defining maps between the tori of H and G^*. Suppose that B_H is a Borel subgroup of H containing a maximal torus T_H and that $(\mathcal{B}_H, \mathcal{T}_H, \{\mathcal{X}_H\})$ is the splitting of $\hat{H}$ used in the definition of ${}^{L}H$ (§3.1). Suppose further that B' is a Borel

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subgroup of G^* containing a maximal torus T', and that both are preserved by θ^*.[1] We may assume that $\mathsf{s} \in \mathcal{T}$, $\xi(\mathcal{T}_H) = (\mathcal{T}^{\hat{\theta}})^0$ and $\xi(\mathcal{B}_H) \subseteq \mathcal{B}$. The pairs $(\hat{B}_H, \hat{T}_H)$ and $(\mathcal{B}_H, \mathcal{T}_H)$ determine an isomorphism $\hat{T}_H \cong \mathcal{T}_H$. Similarly, through the pairs $(\hat{B}', \hat{T}')$ and $(\mathcal{B}, \mathcal{T})$, we conclude that $\hat{T}' \cong \mathcal{T}$. We may combine the former isomorphism with requirement 3.2 of §3.2 for the endoscopic map ξ to obtain isomorphisms

$$\hat{T}_H \cong \mathcal{T}_H \overset{\xi}{\cong} (\mathcal{T}^{\hat{\theta}})^0.$$

To connect $(\mathcal{T}^{\hat{\theta}})^0$ with T', we define $T'_{\theta^*} = T'/(1-\theta^*)T'$, and leave it as an exercise to prove that $((\hat{T}')^{\hat{\theta}})^0 \cong \widehat{T'_{\theta^*}}$. Combining this isomorphism with the earlier ones, we obtain in turn that

$$\hat{T}_H \cong \mathcal{T}_H \overset{\xi}{\cong} (\mathcal{T}^{\hat{\theta}})^0 \cong ((\hat{T}')^{\hat{\theta}})^0 \cong \widehat{T'_{\theta^*}}, \tag{3.7}$$

and $T_H \cong T'_{\theta^*}$.

The isomorphic groups T_H and T'_{θ^*} are related to the conjugacy classes, which we now define. The θ^*-*conjugacy* class of an element $\delta \in G^*$ is defined as $\{g^{-1}\delta\theta^*(g) : g \in G^*\}$. The element δ is called θ^*-*semisimple* if the automorphism $\mathrm{Int}(\delta)\theta^*$ preserves a Borel subgroup of G^* and maximal torus thereof. A θ^*-*semisimple* θ^*-*conjugacy class* is a θ^*-conjugacy class of a θ^*-semisimple element. Let $Cl(G^*, \theta^*)$ be the set of all θ^*-conjugacy classes, and let $Cl_{\mathrm{ss}}(G^*, \theta^*)$ be the subset of θ^*-semisimple θ^*-conjugacy classes. With this notation in hand, we look to [20, Lemma 3.2.A], which tells us that there is a bijection

$$Cl_{\mathrm{ss}}(G^*, \theta^*) \to T'_{\theta^*}/\Omega(G^*, T')^{\theta^*},$$

given by taking the coset of the intersection of a θ^*-conjugacy class with T'.

The aforementioned map specializes to give the bijections on either end of

$$Cl_{\mathrm{ss}}(H) \leftrightarrow T_H/\Omega(H, T_H) \to T'_{\theta^*}/\Omega(G^*, T')^{\theta^*} \leftrightarrow Cl_{\mathrm{ss}}(G^*, \theta^*). \tag{3.8}$$

To describe the remaining map in the middle of (3.8), recall from (3.7) that the isomorphism between T_H and T'_{θ^*} is obtained by way of ξ. Using these ingredients and the closing remarks of §3.1, we obtain maps

$$\Omega(H, T_H) \cong \Omega(\hat{H}, \hat{T}_H) \cong \Omega(\hat{H}, \mathcal{T}_H) \to \Omega(\hat{G}^*, \mathcal{T})^{\hat{\theta}} \cong \Omega(G^*, T')^{\theta^*}.$$

This completes the description of the map from $Cl_{\mathrm{ss}}(H)$ to $Cl_{\mathrm{ss}}(G^*, \theta^*)$.

We proceed by describing the map from $Cl_{\mathrm{ss}}(G^*, \theta^*)$ to $Cl_{\mathrm{ss}}(G, \theta)$. The function $m : G \to G^*$ defined by

$$m(\delta) = \psi(\delta) g_\theta^{-1}, \quad \delta \in G \tag{3.9}$$

passes to a bijection from $Cl(G, \theta)$ to $Cl(G^*, \theta^*)$, since

$$m(g^{-1}\delta\theta(g)) = \psi(g)^{-1} m(\delta)\theta^*(\psi(g)).$$

We abusively denote this map on θ^*-conjugacy classes by m as well. It is pointed out in [20, 3.1] that this bijection need not be equivariant under the action of Γ. One of our key assumptions is that the element g_θ of (3.3) may be chosen so that

$$g_\theta u_\sigma \sigma(g_\theta^{-1}) \theta^*(u_\sigma)^{-1} \in (1-\theta^*) Z_{G^*_{\mathrm{sc}}}. \tag{3.10}$$

[1] Readers of [20] should note that we write T' for the torus T occurring there.

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Under this assumption, m is Γ-equivariant [20, (3) Lemma 3.1.A]. Finally, we may combine this bijection with (3.8) to obtain a map

$$\mathcal{A}_{H\backslash G} : Cl_{\rm ss}(H) \rightarrow Cl_{\rm ss}(G,\theta).$$

In keeping with [20, 3.3], we define an element $\delta \in G$ to be *θ-regular* if the identity component of $G^{\delta\theta}$ is a torus. It is said to be *strongly θ-regular* if $G^{\delta\theta}$ itself is abelian. An element $\gamma \in H$ is said to be (*strongly*) *G-regular* if the elements in the image of its conjugacy class under $\mathcal{A}_{H\backslash G}$ are (strongly) regular. An element $\gamma \in H(\mathbf{R})$ is called a *norm* of an element $\delta \in G(\mathbf{R})$ if the θ-conjugacy class of δ equals the image of the conjugacy class of γ under $\mathcal{A}_{H\backslash G}$. It is possible for $\mathcal{A}_{H\backslash G}(\gamma)$ to be a θ-conjugacy class which contains no points in $G(\mathbf{R})$ even though $\gamma \in H(\mathbf{R})$. In this case, one says that γ is *not a norm*. These definitions are carried to the z-extension H_1 in an obvious manner. For example, we say that $\gamma_1 \in H_1(\mathbf{R})$ is a norm of $\delta \in G(\mathbf{R})$ if the image of γ_1 in $H(\mathbf{R})$ under (3.4) is a norm of δ.

As in [20, 3.3], we conclude with a portrayal of the situation when a strongly regular element $\gamma \in H(\mathbf{R})$ is the norm of a strongly θ-regular element $\delta \in G(\mathbf{R})$. We may let $T_H = H^{\gamma}$, as γ is strongly regular. The maximal torus T_H is defined over $\mathbf{R}$, since γ lies in $H(\mathbf{R})$. [20, Lemma 3.3.B] allows us to choose B_H, B', and T' as above so that $\theta^*(B') = B'$, and both T' and the isomorphism $T_H \cong T'_{\theta^*}$ are defined over $\mathbf{R}$. The resulting isomorphism

$$T_H(\mathbf{R}) \cong T'_{\theta^*}(\mathbf{R}) \tag{3.11}$$

is called an *admissible embedding* in [20, 3.3]. The image of γ under this admissible embedding defines a coset in $T'/\Omega(G^*,T')^{\theta^*}$. This coset corresponds to the θ^*-conjugacy class of $m(\delta)$. In fact, by [20, Lemma 3.2.A], there exists some $g_{T'} \in G^*_{\rm sc}$ such that (after $g_{T'}$ has been identified with its image in G^*) this coset equals $g_{T'}m(\delta)\theta^*(g_{T'})^{-1}\Omega(G^*,T')^{\theta^*}$. The element

$$\delta^* = g_{T'}m(\delta)\theta^*(g_{T'})^{-1} \tag{3.12}$$

belongs to T', and it is an exercise to show that $\mathrm{Int}(g_{T'}) \circ \psi$ furnishes an isomorphism between $G^{\delta\theta}$ and $(G^*)^{\delta^*\theta^*}$. Since $\mathrm{Int}(\delta^*) \circ \theta^*$ preserves (B',T'), the torus $(G^*)^{\delta^*\theta^*}$ contains strongly G-regular elements of T' [2, pp. 227–228], so we see in turn that the centralizer of $(G^*)^{\delta^*\theta^*}$ in G^* is T', and $(G^*)^{\delta^*\theta^*} = (T')^{\theta^*}$. By [20, (3.3.6)], the resulting isomorphism

$$G^{\delta\theta} \stackrel{\mathrm{Int}(g_{T'})\psi}{\longrightarrow} (T')^{\theta^*} \tag{3.13}$$

is defined over $\mathbf{R}$.

3.4. Twisted geometric transfer

Twisted geometric transfer is laid out generally in [20, 5.5]. For real groups, it has been proved in [37]. It shall be convenient for us to state twisted geometric transfer in the framework of orbital integrals on the component $G(\mathbf{R})\theta$ of the group $G(\mathbf{R}) \rtimes \langle\theta\rangle$. Let $\delta \in G(\mathbf{R})$ be θ-semisimple and strongly θ-regular, and assume that the quasicharacter ω is trivial on $G^{\delta\theta}(\mathbf{R})$. Let $C_c^{\infty}(G(\mathbf{R})\theta)$ be the space of smooth compactly supported

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functions on the component $G(\mathbf{R})\theta$. Define the *twisted orbital integral* of $f \in C_c^\infty(G(\mathbf{R})\theta)$ at $\delta\theta \in G(\mathbf{R})\theta$ to be

$$\mathcal{O}_{\delta\theta}(f) = \int_{G^{\delta\theta}(\mathbf{R})\backslash G(\mathbf{R})} \omega(g) f(g^{-1}\delta\theta g)\, dg.$$

We wish to match functions in $C_c^\infty(G(\mathbf{R})\theta)$ with functions on the z-extension H_1. Specifically, let $C_c^\infty(H_1(\mathbf{R}), \lambda_{Z_1})$ be the space of smooth functions f_{H_1} on $H_1(\mathbf{R})$ whose support is compact modulo $Z_1(\mathbf{R})$ and which satisfy

$$f_{H_1}(zh) = \lambda_{Z_1}(z)^{-1} f_{H_1}(h), \quad z \in Z_1(\mathbf{R}), h \in H_1(\mathbf{R})$$

(see the end of §3.2). The definition of orbital integrals easily carries over to functions of this type at semisimple regular elements.

Suppose that $\gamma_1 \in H_1(\mathbf{R})$ is a norm of a θ-semisimple strongly θ-regular element $\delta \in G(\mathbf{R})$. According to [37, Corollary 2.2], for every $f \in C_c^\infty(G(\mathbf{R})\theta)$ there exists a function $f_{H_1} \in C^\infty(H_1(\mathbf{R}), \lambda_{Z_1})$ as above such that

$$\sum_{\gamma_1'} \mathcal{O}_{\gamma_1'}(f_{H_1}) = \sum_{\delta'} \Delta(\gamma_1, \delta') \mathcal{O}_{\delta'\theta}(f). \tag{3.14}$$

The sum on the left is taken over representatives in $H_1(\mathbf{R})$ of $H_1(\mathbf{R})$-conjugacy classes contained in the H_1-conjugacy class of γ_1. The sum on the right is taken over representatives in $G(\mathbf{R})$ of θ-conjugacy classes under $G(\mathbf{R})$ contained in the θ-conjugacy class of δ. The terms $\Delta(\gamma_1, \delta')$ are geometric transfer factors, and they are defined in [20, Chapter 4]. Normalization is required for the measures in the orbital integrals to be compatible [20, p. 71]. We assume that the correspondence $f \leftrightarrow f_{H_1}$ induces a continuous map on spaces of orbital integrals, and thereby also a map from stably invariant distributions on $H_1(\mathbf{R})$ to distributions on $G(\mathbf{R})$ as in standard endoscopy [8, Remark 2, 6].

4. Spectral transfer for the fundamental series

The first step in proving spectral transfer in standard endoscopy for real groups is the case of essentially square-integrable representations [36, 13], [33, 4.4]. This was done for twisted endoscopy for θ of finite order and trivial quasicharacter ω [28, Theorem 1]. It amounts to an identity of the form

$$\int_{H_1(\mathbf{R})/Z_1(\mathbf{R})} f_{H_1}(h) \sum_{\pi_{H_1} \in \Pi_{\varphi_{H_1}}} \Theta_{\pi_{H_1}}(h)\, dh = \sum_{\pi \in \Pi_\varphi} \Delta(\boldsymbol{\varphi}_{\boldsymbol{H_1}}, \pi) \Theta_{\pi, \mathsf{U}_\pi}(f) \tag{4.1}$$

for all $f \in C_c^\infty(G(\mathbf{R})\theta)$. The restrictions on θ and ω were due to the lack of a sufficiently general version of Harish-Chandra's uniqueness theorem in the twisted case. The required version of this theorem is proved in the appendix (Proposition A.4). The spectral transfer theorem (4.1) therefore now holds for arbitrary quasicharacter ω and θ acting semisimply on Z_G when Π_φ consists of essentially square-integrable representations.

The purpose of this section is to prove (4.1) when Π_φ consists more generally of fundamental series representations. By 'fundamental series' we have in mind the

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representations presented in [13, III.3]. For convenience, let us call the essentially square-integrable representations the *discrete series*, and highlight some differences with the fundamental series. The discrete series is determined by regular forms of elliptic tori. By contrast, the fundamental series is determined by regular forms of fundamental tori. The main difference here is that elliptic tori are compact modulo the centre, whereas fundamental tori are merely maximally compact modulo the centre. The compact portion of a fundamental torus takes on the role of an elliptic torus in the discrete case, and a fundamental series representation is obtained by inducing the resulting discrete series representation.

Luckily, the character expansions of Bouaziz [6, Proposition 6.1.2], which lie at the core of the spectral transfer theorem for the discrete series, remain the same for the fundamental series. Thus, the foremost tasks in proving spectral transfer for the fundamental series are to convert the language of Duflo into endoscopic parameters, and then to show that this conversion retains the hypotheses necessary for Bouaziz' character expansions. We begin by providing the said endoscopic parameters. We shall do so in the form of six assumptions.

The quadruple $(H, \mathcal{H}, \mathsf{s}, \xi)$ is a fixed set of endoscopic data together with a z-pair (H_1, ξ_{H_1}), as in §3.2. We take an L-parameter $\boldsymbol{\varphi_{H_1}}$ which is the $\hat{H}_1$-conjugacy class of an admissible homomorphism $\varphi_{H_1} : W_{\mathbf{R}} \to {}^L H_1$ [4, 8.2]. We suppose that the composition of φ_{H_1} with ${}^L H_1 \to {}^L Z_1$ corresponds to the quasicharacter $\lambda_{Z_1} : Z_1(\mathbf{R}) \to \mathbf{C}^\times$ of (3.6) under the local Langlands correspondence. The endoscopic Langlands parameter $\boldsymbol{\varphi_{H_1}}$ corresponds to a Langlands parameter $\boldsymbol{\varphi^*}$ of the quasisplit form G^* [28, 6]. Our first assumption is that φ_{H_1} is not contained in a proper parabolic subgroup of ${}^L H_1$. This is equivalent to the assertion that the L-packet $\Pi_{\varphi_{H_1}}$ consists of essentially square-integrable representations [4, (3) 10.3].

Our second assumption is that there exists a strongly θ-regular element $\delta \in G(\mathbf{R})$ which has a *norm* $\gamma \in H(\mathbf{R})$ (§3.3), and $(G^{\delta\theta}/Z_G^{\theta})(\mathbf{R})$ is compact. A strongly θ-regular element in $G(\mathbf{R})$ satisfying the latter compactness condition is called *θ-elliptic* [20, p. 5]. This compactness condition may be translated to a maximal torus. We say that a maximal torus S in G, which is defined over $\mathbf{R}$, is *fundamental* if $R(G, S)$ has no real roots. This is equivalent to $S(\mathbf{R})$ being a maximally compact Cartan subgroup in $G(\mathbf{R})$ [44, Lemma 2.3.5]. Similarly, on the level of Lie algebras, one says that $\mathfrak{s}$ is *fundamental* if $R(\mathfrak{g} \otimes \mathbf{C}, \mathfrak{s}, \otimes \mathbf{C})$ has no real roots.

Lemma 4.1. *The element $\delta \in G(\mathbf{R})$ determines a unique maximal torus S of G which contains $G^{\delta\theta}$. Moreover, the torus S is defined over $\mathbf{R}$, and is fundamental.*

Proof. By definition of strongly θ-regular, $G^{\delta\theta}$ is an abelian group. It contains strongly G-regular elements [2, pp. 227–228], so the identity component of $Z_G(G^{\delta\theta})$ is a maximal torus of G, which is uniquely determined by δ. Suppose first that the centre Z_G is trivial. Then $G^{\delta\theta}(\mathbf{R})$ is compact, since δ is θ-elliptic. The Lie algebra of $G^{\delta\theta}(\mathbf{R})$ is therefore contained in a Cartan subalgebra of the Lie algebra of a maximally compact subgroup of $G(\mathbf{R})$. The centralizer of this Cartan subalgebra in $\mathfrak{g}$ is a fundamental Cartan subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ [18, Proposition 6.60]. The exponential of $\mathfrak{s} \otimes \mathbf{C}$ is a maximal torus S in G [16, Corollary 15.3]. By construction, the torus S is defined over $\mathbf{R}$, and $S(\mathbf{R})$ is maximally compact. Furthermore, S contains $G^{\delta\theta}$ so it is equal to the uniquely determined torus mentioned above.

Now, we remove the assumption that Z_G is trivial, and observe that there is a canonical bijection between the set of maximal tori of G and the set of maximal tori of the semisimple algebraic group G/Z_G, which is induced by the quotient map. The quotient map is defined over $\mathbf{R}$ [39, Theorem 12.2.1]. This bijection therefore passes to a bijection of maximal $\mathbf{R}$-tori. In addition, the quotient map sends δ to an element of $(G/Z_G)(\mathbf{R})$, and it is immediate that this element retains the analogues of the properties of strong θ-regularity and θ-ellipticity. By our earlier argument, we obtain a maximal torus of G/Z_G. The pre-image of this torus under the quotient map is a maximal torus in G with the desired properties. □

By construction, the torus S of Lemma 4.1 is stable under $\mathrm{Int}(\delta)\theta$, and isomorphism (3.13) passes to an isomorphism

$$S^{\delta\theta}(\mathbf{R})^0 \overset{\mathrm{Int}(g_{T'})\psi}{\longrightarrow} (T')^{\theta^*}(\mathbf{R})^0.$$

In fact, this map extends to an isomorphism of the respective centralizers

$$S(\mathbf{R}) \overset{\mathrm{Int}(g_{T'})\psi}{\cong} T'(\mathbf{R}), \tag{4.2}$$

as the commutator of σ and $\mathrm{Int}(g_{T'})\psi$ lies in $\mathrm{Int}(T')$ [20, (3.3.6)] and acts trivially on T'.

One may decompose S into a product of a maximally split subtorus S_d and a maximally anisotropic subtorus S_a [5, Proposition 8.15]. The centralizer $M = Z_G(S_d)$ is a Levi subgroup of G which is defined over $\mathbf{R}$ [5, Proposition 20.4]. By construction, $Z_M \supseteq S_d$, and it therefore follows that S is elliptic in M. The torus S_d is also the split component of the centre of M [5, Proposition 20.6]. The usual notation for the latter is A_M. Observe that, since $\mathrm{Int}(\delta)\theta$ is defined over $\mathbf{R}$ and preserves S, it also preserves $S_d = A_M$ and M.

Our third assumption is that $\boldsymbol{\varphi^*}$ has a representative homomorphism φ^* whose image is minimally contained in a parabolic subgroup of ${}^LG$, and that this parabolic subgroup is dual, in the sense of [4, 3.3 (2)], to an $\mathbf{R}$-parabolic subgroup P of G with Levi component M. In the language of [4, 8.2], this translates as the parabolic subgroup of ${}^LG$ being *relevant*, and φ^* being *admissible* with respect to G. Under this assumption, we set $\boldsymbol{\varphi} = \boldsymbol{\varphi^*}$ with the intention that $\boldsymbol{\varphi}$ be regarded as a Langlands parameter of G.

We choose a Levi subgroup $\mathcal{M}$ of $\hat{G}$ and an admissible homomorphism $\varphi \in \boldsymbol{\varphi}$ such that $\mathcal{M} \cong \hat{M}$ and $\mathcal{M} \rtimes W_{\mathbf{R}}$ is a standard Levi subgroup of ${}^LG$ which contains $\varphi(W_{\mathbf{R}})$ minimally [4, 3.4], [28, 4.1]. We may thus regard φ as an admissible homomorphism into $\mathcal{M} \rtimes W_{\mathbf{R}}$, and derive from it an L-packet $\Pi_{\varphi,M}$ of essentially square-integrable representations of $M(\mathbf{R})$ [4, 10.3 (3) and 11.3].

Our fourth assumption is that the representations in $\Pi_{\varphi,M}$ have unitary central character. From this, the local Langlands correspondence prescribes that the representations in Π_{φ} are the irreducible subrepresentations of the representations induced from those in $\Pi_{\varphi,M}$ [4, 11.3].

Before making our fifth assumption, we must recall some facts about the homomorphism φ and the L-packet $\Pi_{\varphi,M}$. The homomorphism φ is determined by a pair

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$\mu, \lambda \in X_*(\hat{S}) \otimes \mathbf{C}$ [26, 3], [28, 4]. One may regard the elements in this pair as elements in the dual of the complex Lie algebra of S via the isomorphisms $X_*(\hat{S}) \cong X^*(S)$ and

$$X^*(S) \otimes \mathbf{C} \cong \mathfrak{s}^* \otimes \mathbf{C}. \tag{4.3}$$

To be more precise, isomorphism (4.3) is an isomorphism of $\mathbf{R}[\Gamma]$-modules, given that Γ acts on both $X^*(S)$ and $\mathbf{C}$ in the usual way [4, see 9.4]. In other words, isomorphism (4.3) rests upon an isomorphism

$$(X^*(S_a) \otimes i\mathbf{R}) \oplus (X^*(S_d) \otimes \mathbf{R}) \cong \mathfrak{s}^* \tag{4.4}$$

of $\mathbf{R}$-vector spaces. The pair may be lifted to a quasicharacter of $S(\mathbf{R})$ in the following manner. The element μ is $\hat{M}$-regular, and so determines a positive system on $R(M, S)$ [26, Lemma 3.3]. Let $\iota_M \in X_*(\hat{S}) \otimes \mathbf{C}$ be the half-sum of the positive roots of $R(M, S)$. The pair $(\mu - \iota_M, \lambda)$ corresponds to a linear form on $\mathfrak{s}$, and satisfies a condition which allows one to lift to a quasicharacter $\Lambda = \Lambda(\mu - \iota_M, \lambda)$ of $S(\mathbf{R})$ [26, p. 132], [28, 4.1].

By the work of Harish-Chandra, the quasicharacter Λ corresponds to an essentially square-integrable representation of $Z_M(\mathbf{R}) M_{\mathrm{der}}(\mathbf{R})^0$ [14]. Inducing this representation to $M(\mathbf{R})$ produces an irreducible representation $\varpi_\Lambda \in \Pi_{\varphi, M}$ [26, p. 134]. The remaining representations of $\Pi_{\varphi, M}$ are obtained by replacing Λ by $w^{-1}\Lambda = \Lambda(w^{-1} \cdot (\mu - \iota_M), \lambda)$, where $w \in \Omega(M, S)/\Omega_{\mathbf{R}}(M, S)$ (see [28, 4.1]).

Let us consider the differential of the quasicharacter Λ. The differential only records the behaviour of Λ on the identity component $S(\mathbf{R})^0$, and this behaviour is given precisely by $\mu - \iota_M$ [32, 4.1]. The infinitesimal character of ϖ_Λ corresponds to μ, and the restriction of this infinitesimal character to $\mathfrak{s} \cap [\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{s}_a$ is equal to the Harish-Chandra parameter of the underlying representation of $M_{\mathrm{der}}(\mathbf{R})^0$ [17, p. 310].

Our fifth assumption is really two separate regularity assumptions. The first regularity assumption is that μ is $\hat{G}$-regular, that is,

$$\langle \mu, \alpha \rangle \neq 0, \quad \alpha \in R(\hat{G}, \hat{S}).$$

The second regularity assumption pertains to Duflo's characterization of fundamental series representations, and this depends on the behaviour of μ on the anisotropic part $S_a(\mathbf{R})$ of $S(\mathbf{R})^0$ [13, (ii) III.1]. By identifying μ with a linear form in $\mathfrak{s}^* \otimes \mathbf{C}$ under (4.3), the second regularity assumption reads as

$$\langle \mu_{|\mathfrak{s}_a}, \alpha \rangle \neq 0, \quad \alpha \in R(\hat{G}, \hat{S}).$$

Holding this view, the second regularity assumption is equivalent to the $\mathfrak{g} \otimes \mathbf{C}$-regularity of the $\mathfrak{s}_a^* \otimes \mathbf{C}$-component of μ. Alternatively, the $\mu_{|\mathfrak{s}_a}$ may be regarded as the restriction to S_a of $\mu \in X^*(S) \otimes \mathbf{C}$.

We come to our sixth and final assumption. In order for twisted spectral transfer to have any content, we assume that Π_φ is stable under twisting, that is,

$$\Pi_\varphi = \omega \otimes (\Pi_\varphi \circ \theta)$$

(see [28, 4.3]).

We list the six assumptions of this section again for convenience.

Assumption 1. φ_{H_1} is not contained in a proper parabolic subgroup of ${}^L H_1$.

Assumption 2. There exists a strongly θ-regular and θ-elliptic element $\delta \in G(\mathbf{R})$ which has a norm $\gamma \in H(\mathbf{R})$.

Assumption 3. $\boldsymbol{\varphi^*}$ has a representative φ^* whose image is minimally contained in a parabolic subgroup of ${}^L G$ which is dual to an $\mathbf{R}$-parabolic subgroup P with Levi component M.

Assumption 4. The representations in $\Pi_{\varphi,M}$ have unitary central character.

Assumption 5. The elements μ and $\mu_{|\mathfrak{s}_a}$ in $X_*(\hat{S}) \otimes \mathbf{C}$ are $\hat{G}$-regular.

Assumption 6. $\Pi_\varphi = \omega \otimes (\Pi_\varphi \circ \theta)$.

4.1. Fundamental series representations

Our goal here is to show that the L-packet Π_φ, given under the previous assumptions, consists of fundamental series representations as defined by Duflo in [13, III]. By definition, the representations in Π_φ are (equivalence classes of) irreducible subrepresentations of $\mathrm{ind}_{P(\mathbf{R})}^{G(\mathbf{R})} \varpi$, where $\varpi \in \Pi_{\varphi,M}$ [4, 11.3]; parabolic induction throughout is normalized. Recall that $\varpi_\Lambda \in \Pi_{\varphi,M}$ is induced from an irreducible representation of $Z_M(\mathbf{R}) M_{\mathrm{der}}(\mathbf{R})^0$. More precisely, there exists a square-integrable (i.e., discrete series) representation ϖ_0 of $M_{\mathrm{der}}(\mathbf{R})^0$ such that

$$\varpi_\Lambda \cong \mathrm{ind}_{Z_M(\mathbf{R}) M_{\mathrm{der}}(\mathbf{R})^0}^{M(\mathbf{R})} \left(\chi_\varphi \otimes \varpi_0\right), \tag{4.5}$$

where χ_φ is the central character of ϖ_Λ (or any other representation in $\Pi_{\varphi,M}$). Using isomorphism (4.3), one may identify the infinitesimal character of ϖ_Λ with μ. In addition, since $\langle \mu, \alpha^\vee \rangle \in \mathbf{R}$ for all $\alpha \in R(M, S)$ [26, Proof of Lemma 3.3] and χ_φ is unitary on $Z_M(\mathbf{R})$, it follows from Corollary 6.49 [18] that $i\mu \in \mathfrak{s}^*$ (see (4.4)). In particular,

$$\sigma(\mu) = -\mu. \tag{4.6}$$

This infinitesimal character must satisfy three criteria in order for $\mathrm{ind}_{P(\mathbf{R})}^{G(\mathbf{R})} \varpi_\Lambda$ to be in the fundamental series. Two of the three criteria are covered by Assumption 5. The $\mathfrak{g} \otimes \mathbf{C}$-regularity of μ fulfils the criterion that $i\mu$ be *bien polarizable* [13, Lemma 7 II and III.1]. The $\mathfrak{g} \otimes \mathbf{C}$-regularity of the $\mathfrak{s}_a^* \otimes \mathbf{C}$-component of μ fulfils the criterion of $i\mu$ being *standard* [13, (ii) III.1].

To state the third criterion, we define ρ to be the half-sum of positive roots in $R(\mathfrak{g} \otimes \mathbf{C}, \mathfrak{s} \otimes \mathbf{C})$ determined by the regular element $\mu_{|\mathfrak{s}_a}$. The third criterion is that $\mu - \rho$ lifts to a quasicharacter of $S(\mathbf{R})^0$ [13, Remark 2 II.2]. This is equivalent to $i\mu$ being *admissible* in the parlance of Duflo.

Lemma 4.2. *The linear form $\mu - \rho \in \mathfrak{s}^* \otimes \mathbf{C}$ lifts to a quasicharacter of $S(\mathbf{R})^0$.*

Proof. Since $S_d(\mathbf{R})^0 S_a(\mathbf{R})$ is a closed connected subgroup of the same dimension as $S(\mathbf{R})^0$, we see that $S(\mathbf{R})^0 = S_d(\mathbf{R})^0 S_a(\mathbf{R})$. It is clear from the isomorphism $\mathfrak{s}_d \cong S_d(\mathbf{R})^0$ that

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$(\mu-\rho)_{|\mathfrak{s}_d}$ lifts to a quasicharacter of $S_d(\mathbf{R})^0$. To lift $(\mu-\rho)_{|\mathfrak{s}_a}$, we observe that $-\sigma(\mu_{|\mathfrak{s}_a}) = \mu_{|\mathfrak{s}_a}$ (see (4.6)). It follows that $-\sigma(\rho) = \rho$ and

$$\rho = \left(\frac{1}{2}\sum_{\text{imaginary}} \alpha\right) + \left(\frac{1}{2}\sum_{\text{complex}} \alpha + (-\sigma(\alpha))\right),$$

where the rightmost sum is over $-\sigma$-orbits of positive roots. The sum on the left corresponds to ι_M [39, Lemma 15.3.2]. By [32, 4.1], the form $(\mu-\iota_M)_{|\mathfrak{s}_a}$ lifts to $\Lambda = \Lambda(\mu-\iota_M, \lambda)$ on $S_a(\mathbf{R})$. The lemma will therefore be complete once we show that the second sum lifts to $S_a(\mathbf{R})$. For this, we compute that

$$\frac{1}{2}\sum_{\text{complex}} (\alpha - \sigma(\alpha))_{|\mathfrak{s}_a} = \frac{1}{2}\sum_{\text{complex}} \alpha_{|\mathfrak{s}_a} + \alpha_{|\mathfrak{s}_a} = \sum_{\text{complex}} \alpha_{|\mathfrak{s}_a},$$

and note that all integer combinations of roots lift to $S(\mathbf{R})$ [17, (4.15)]. □

We have now verified the criteria μ must satisfy for us to describe Π_φ in Duflo's framework.

Lemma 4.3. *The representation* $\mathrm{ind}_{P(\mathbf{R})}^{G(\mathbf{R})} \varpi_\Lambda$ *is irreducible and equivalent to*

$$\mathrm{ind}_{Z_G(\mathbf{R})G(\mathbf{R})^0}^{G(\mathbf{R})} \left((\chi_\varphi)_{|Z_G(\mathbf{R})} \otimes \mathrm{ind}_{P(\mathbf{R})\cap G(\mathbf{R})^0}^{G(\mathbf{R})^0} \varpi_1\right), \tag{4.7}$$

where ϖ_1 *is defined in terms of* (4.5) *as*

$$\varpi_1 = \mathrm{ind}_{Z_M(\mathbf{R})^0 M_{\mathrm{der}}(\mathbf{R})^0}^{M(\mathbf{R})\cap G(\mathbf{R})^0}((\chi_\varphi)_{|Z_M(\mathbf{R})^0} \otimes \varpi_0).$$

Proof. The reader may verify that the distribution characters of $\mathrm{ind}_{P(\mathbf{R})}^{G(\mathbf{R})} \varpi_\Lambda$ and (4.7) agree on the regular subset of $S(\mathbf{R})$. The equivalence of the two representations then follows from Harish-Chandra's uniqueness theorem [17, Theorem 12.6]. The irreducibility of (4.7) follows in two steps. First, we show that it is equal to representation [13, (8), on p. 172]. The latter representation is written as

$$\mathrm{Ind}_{Z_G(\mathbf{R})G(\mathbf{R})^0}^{G(\mathbf{R})} (\tau \otimes T_g^{G(\mathbf{R})^0}). \tag{4.8}$$

Here, $g \in \mathfrak{g}^*$ is an element which is *admissible*, *bien polarizable*, and *standard*. As discussed above, we may take $g = i\mu$. The expression $T_g^{G(\mathbf{R})^0}$ is defined as $\mathrm{ind}_{P(\mathbf{R})\cap G(\mathbf{R})^0}^{G(\mathbf{R})^0} \varpi_1$ [13, p. 164]. The term τ is an irreducible representation of a metaplectic group, but only its restriction to $Z_G(\mathbf{R})$ is relevant in (4.8). We may take $\tau_{|Z_G(\mathbf{R})} = (\chi_\varphi)_{|Z_G(\mathbf{R})}$. With these substitutions, one sees that (4.7) is equal to (4.8).

The irreducibility of (4.8) follows from [13, Lemma 8 (i) III.6] once we show that $S(\mathbf{R}) = Z_G(\mathbf{R})\, S(\mathbf{R})^0$ [13, Remark 2 III.5]. When G is semisimple, this identity follows from [19, Lemma 10.4]. When G is reductive, the semisimple case reduces the exact sequence

$$H^2(\Gamma, X_*(Z_G^0)) \to H^2(\Gamma, X_*(S)) \to H^2(\Gamma, X_*(S/Z_G^0))$$

to a surjection

$$Z_G^0(\mathbf{R})/Z_G(\mathbf{R})^0 \to S(\mathbf{R})/S(\mathbf{R})^0 \to 1$$

(cf. [32, 4.1]). This surjection implies that $S(\mathbf{R}) = Z_G(\mathbf{R})S(\mathbf{R})^0$. □

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Corollary 4.4. *Every representation of $G(\mathbf{R})$, parabolically induced from an irreducible representation in $\Pi_{\varphi,M}$, is irreducible.*

Proof. The representations of $\Pi_{\varphi,M}$ are obtained by replacing ϖ_Λ by $\varpi_{w^{-1}\Lambda}$, where $w \in \Omega(M, S)/\Omega_{\mathbf{R}}(M, S)$ (see [28, 4.1]). The arguments of the proof are unaffected by replacing Λ by $\dot{w}^{-1}\Lambda$ and μ by $\dot{w}^{-1} \cdot \mu$ for any $\dot{w} \in \Omega(M, S)$. □

Corollary 4.4 tells us that parabolic induction furnishes a bijection between $\Pi_{\varphi,M}$ and Π_φ. Moreover, every representation in Π_φ has an expansion as in (4.7). From the perspective of [13, III], this is equivalent to saying that every representation in Π_φ belongs to the fundamental series of $G(\mathbf{R})$.

4.2. Twisted characters

As claimed earlier, the character expansions of Bouaziz [6, Proposition 6.1.2] apply equally well to discrete series and fundamental series representations. We shall address this claim in the context of the twisted characters. In this section, we suppose that the representation

$$\pi = \mathrm{ind}_{P(\mathbf{R})}^{G(\mathbf{R})} \varpi_\Lambda \in \Pi_\varphi$$

is stable under twisting. More precisely, we suppose that there exists a unitary linear operator $\mathsf{U} = \mathsf{U}_\pi$ on the space V_π of π such that

$$\mathsf{U} \circ \pi(x) = \omega(x)\, \pi^\theta(x) \circ \mathsf{U}, \quad x \in G(\mathbf{R}). \tag{4.9}$$

We define the *twisted character* $\Theta_{\pi,\mathsf{U}}$ as the distribution on $G(\mathbf{R})$ given by

$$f \mapsto \mathrm{tr} \int_{G(\mathbf{R})} f(x\theta)\pi(x)\mathsf{U}\, dx, \quad f \in C_c^\infty(G(\mathbf{R})\theta)$$

(see [28, (34)]). This is the kind of distribution which appears on the right of (4.1).

There is a technical point we must verify in order to use Bouaziz' character formula later on. By the reduction of [28, 5.1], this point needs only to be verified in the case that $G(\mathbf{R})$ is semisimple and connected in the manifold topology. Let us assume for the rest of §4.2 that this is so. In this way, we temporarily remove the quasicharacter ω from the picture, and associate the representation π to a representation of the larger disconnected Lie group

$$L = G(\mathbf{R}) \rtimes \langle \delta\theta \rangle.$$

In the context of this group, $\Theta_{\pi,\mathsf{U}}$ may be identified with the restriction of the distribution character to the connected component $G(\mathbf{R}) \rtimes \delta\theta \subseteq L$. Under this identification, Bouaziz' character formula on L delivers an explicit formula for $\Theta_{\pi,\mathsf{U}}$ on θ-regular and θ-elliptic elements of $G(\mathbf{R})$ [28, Lemma 6]. This was explained in great detail in 5 [28] for discrete series representations. The only noteworthy difference in the case of the fundamental series is in proving that the parameter μ occurring in $\Lambda = \Lambda(\mu - \iota_M, \lambda)$ is *elliptic*, i.e., that its restriction to S_d is trivial (5.2 [6]). Indeed, in the case of discrete series the torus S is elliptic so S_d is itself trivial and there is nothing to prove. This is the technical point alluded to above.

Lemma 4.5. *The parameter μ is elliptic.*

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Proof. The element $\mu \in X^*(S) \otimes \mathbf{C}$ is obtained from $\mu_{H_1} \in X^*(T_{H_1}) \otimes \mathbf{C}$. To be more precise, μ_{H_1} lies in the image of $X^*(T_H) \otimes \mathbf{C} \hookrightarrow X^*(T_{H_1}) \otimes \mathbf{C}$ [28, 6], and μ is obtained through the sequence of Γ-module homomorphism

$$X^*(T_H) = X_*(\mathcal{T}_H) \overset{\xi}{\cong} X_*((\mathcal{T}^{\hat{\theta}})^0) \cong X^*(T'_{\theta *}) \hookrightarrow X^*(T') \overset{(4.2)}{\cong} X^*(S).$$

There is a surjection $S^{\delta\theta}(\mathbf{R}) \to T_H(\mathbf{R})$ [28, Proof of Lemma 12], and under our assumptions $S^{\delta\theta}(\mathbf{R})$ is compact. It follows in turn that $T_H(\mathbf{R})$ is compact, $X^*(T_H)^{1-\sigma} = X^*(T_H)$ [4, 9.4], $\sigma(\mu_{H_1}) = -\mu_{H_1}$ $\sigma(\mu) = -\mu$, and $\mu_{|S_d} = 0$. □

4.3. A parameterization of stable data

There are two sorts of stable data underlying the spectral transfer identity (4.1). The first sort is geometric and is related to the pair of elements $\delta \in G(\mathbf{R})$ and $\gamma_1 \in p_H^{-1}(\gamma) \subseteq H_1(\mathbf{R})$. Explicitly, the stable geometric data are the θ-conjugacy classes under $G(\mathbf{R})$ of elements in $G(\mathbf{R})$ whose norm is γ_1, that is, the collection of sets

$$\{x^{-1}\delta'\theta(x) : x \in G(\mathbf{R})\},$$

where $\delta' \in G(\mathbf{R})$ runs through the representatives which have norm γ_1. By Assumption 2, δ is a representative of such a conjugacy class. This collection of sets is basic to geometric transfer [20, 5.5]. When S is elliptic in G and θ is trivial, this collection of stable data is parameterized by the collection of cosets $\Omega(G, S)/\Omega_{\mathbf{R}}(G, S)$ [24, 6.4]. Our first effort will be to describe how this parameterizing set is altered when S is fundamental and θ is non-trivial.

The second sort of stable data is spectral and is related to representations in the L-packet Π_φ. Again, when S is elliptic in G and θ is trivial, these representations are parameterized by $\Omega(G, S)/\Omega_{\mathbf{R}}(G, S)$ [24, 7.1]. We shall describe in the general case how this spectral parameterizing set is altered and becomes an object attached to M.

Upon having described parameterizing sets of the stable geometric and spectral sorts, we connect them through a canonical surjection. This is indispensable in the proof of (4.1), since it connects the data of geometric transfer to the L-packets Π_φ and $\Pi_{\varphi_{H_1}}$.

Let us begin geometric parameterization by looking back at some cosets presented in [28, 6.1]. One may dissect $\Omega(G, S)/\Omega_{\mathbf{R}}(G, S)$ and extract the coset space $N_G(S)/N_{G(\mathbf{R})}(S)$. When S is elliptic in G, the elements in $N_G(S)$ act as $\mathbf{R}$-automorphisms of S [24, Lemma 6.4.1]. This is not so in general, and the elements of $N_G(S)$ which act as $\mathbf{R}$-automorphisms form the subgroup $N_G(S^\sigma) = N_G(S(\mathbf{R}))$. A moment's reflection reveals that $N_{G(\mathbf{R})}(S)$ and S are subgroups of $N_G(S(\mathbf{R}))$ so we may consider the collection of double cosets

$$S\backslash N_G(S^\sigma)/N_{G(\mathbf{R})}(S).$$

This collection may be identified with

$$\Omega(G, S)^\sigma/\Omega_{\mathbf{R}}(G, S). \tag{4.10}$$

This will be seen to be the parameterizing set of the stable geometric data when θ is trivial. However, as seen in [28, 6.1], twisting by θ forces us to consider the collection of double cosets

$$S^{\delta\theta}\backslash N_G(S^\sigma)/N_{G(\mathbf{R})}(S).$$

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In fact, the only double cosets $S^{\delta\theta} x N_{G(\mathbf{R})}(S)$ which are of interest are those which satisfy

$$x^{-1}\delta\theta x(\delta\theta)^{-1} \in G(\mathbf{R}). \tag{4.11}$$

This being so, we define

$$S^{\delta\theta}\backslash(N_G(S^\sigma)/N_{G(\mathbf{R})}(S))^{\delta\theta} \tag{4.12}$$

to be the collection of double cosets whose representatives $x \in N_G(S^\sigma)$ satisfy (4.11). The following two results justify the above claims.

Lemma 4.6. *Suppose that $x \in G$ and $x^{-1}\delta\theta(x) \in G(\mathbf{R})$. Then $\mathrm{Int}(x^{-1})_{|S}$ is defined over $\mathbf{R}$. In particular, if x also belongs to $N_G(S)$, then $x \in N_G(S^\sigma)$.*

Proof. It suffices to show that $\mathrm{Int}(x\sigma(x^{-1}))_{|S}$ is the identity map. From (3.12), we know that γ_1 being a norm of δ entails that $\delta^* = g_{T'} m(\delta)\theta^*(g_{T'}^{-1})$ for some $g_{T'} \in G^*_{\mathrm{sc}}$. According to [20, Lemma 4.4.A], the element $g_{T'} u_\sigma \sigma(g_{T'}^{-1})$ belongs to T'_{sc}. Likewise, $x^{-1}\delta\theta(x)$ has norm γ_1. Indeed, following the computations of [20, 3.1], we observe that

$$m(x^{-1}\delta\theta(x)) = \psi(x^{-1})m(\delta)\theta^*(\psi(x)),$$

so

$$\delta^* = g_{T'}\psi(x)m(x^{-1}\delta\theta(x))\theta^*(g_{T'}\psi(x))^{-1}.$$

We may thus apply [20, Lemma 4.4.A] to the element $g_{T'}\psi(x)$ in place of $g_{T'}$, to find that $g_{T'}\psi(x)u_\sigma\sigma(g_{T'}\psi(x))^{-1}$ belongs to T'_{sc}. Therefore conjugation of T' by $g_{T'}\psi(x)u_\sigma\sigma(g_{T'}\psi(x))^{-1}$ is trivial. Under transport by (4.2), this implies that the restriction to S of

$$\psi^{-1}\mathrm{Int}(g_{T'})^{-1}\mathrm{Int}(g_{T'}\psi(x)u_\sigma\sigma(g_{T'}\psi(x))^{-1})\mathrm{Int}(g_{T'})\psi$$

is the identity map. For simplicity, we write $g = g_{T'}$, and compute

$$\begin{aligned}
&\psi^{-1}\mathrm{Int}(g)^{-1}\mathrm{Int}(g\psi(x)u_\sigma\sigma(g\psi(x))^{-1})\mathrm{Int}(g)\psi\\
&\quad= \mathrm{Int}(x)\psi^{-1}\mathrm{Int}(u_\sigma)\mathrm{Int}(\sigma(\psi(x^{-1})g^{-1}))\mathrm{Int}(g)\psi\\
&\quad= \mathrm{Int}(x)\sigma^{-1}\psi^{-1}\sigma\mathrm{Int}(\sigma(\psi(x^{-1})g^{-1}))\mathrm{Int}(g)\psi\\
&\quad= \mathrm{Int}(x\sigma(x^{-1}))(\sigma^{-1}(\mathrm{Int}(g)\psi)^{-1}\sigma\mathrm{Int}(g)\psi)\\
&\quad= \mathrm{Int}(x\sigma(x^{-1})),
\end{aligned}$$

where the last equality follows from (4.2) being defined over $\mathbf{R}$. □

The next lemma is a slightly amended version of [28, Lemma 14]. Only the surjectivity argument is affected when S is not elliptic in G.

Proposition 4.7. *Suppose that $x \in N_G(S^\sigma)$ satisfies (4.11). Then the map defined by*

$$x \mapsto x^{-1}\delta\theta(x)$$

passes to a bijection from (4.12) to the collection of θ-conjugacy classes under $G(\mathbf{R})$ of elements in $G(\mathbf{R})$ whose norm is γ_1.

Proof. Suppose that $x \in N_G(S^{\sigma})$ satisfies (4.11). Since δ belongs to $G(\mathbf{R})$, property (4.11) is equivalent to $x^{-1}\delta\theta(x) \in G(\mathbf{R})$. As γ_1 is a norm of δ, it is also a norm of $x^{-1}\delta\theta(x)$. It is simple to verify that any element in the double coset $S^{\delta\theta}\backslash x/N_{G(\mathbf{R})}(S)$ maps to an element which is θ-conjugate to $x^{-1}\delta\theta(x)$ under $G(\mathbf{R})$. Thus, we have a well-defined map from (4.12) to the desired collection of θ-conjugacy classes.

To show that this map is surjective, suppose now that $x \in G$ is any element satisfying $x^{-1}\delta\theta(x) \in G(\mathbf{R})$, that is, an element in $G(\mathbf{R})$ whose norm is γ_1. The automorphism $\mathrm{Int}(x^{-1}\delta\theta(x))\theta$ is defined over $\mathbf{R}$. Therefore, the group $G^{x^{-1}\delta\theta(x)\theta}$ is defined over $\mathbf{R}$. The property that $x^{-1}\delta\theta(x) \in G(\mathbf{R})$ implies in turn that $x\sigma(x^{-1}) \in G^{\delta\theta}$ and

$$G^{x^{-1}\delta\theta(x)\theta}(\mathbf{R}) = (x^{-1}G^{\delta\theta}x)(\mathbf{R}) = x^{-1}G^{\delta\theta}(\mathbf{R})x.$$

The quotient $G^{x^{-1}\delta\theta(x)\theta}(\mathbf{R})/Z_G^{\theta}(\mathbf{R}) = x^{-1}(G^{\delta\theta}(\mathbf{R})/Z_G^{\theta}(\mathbf{R}))x$ is compact, since δ is θ-elliptic. Using [44, Lemma 2.3.4] and the arguments of Lemma 4.1, one may show that there exists $g \in G(\mathbf{R})$ such that $g^{-1}G^{x^{-1}\delta\theta(x)\theta}(\mathbf{R})g$ lies in the torus $S(\mathbf{R})$. Hence,

$$S \supseteq g^{-1}G^{x^{-1}\delta\theta(x)\theta}g = (xg)^{-1}G^{\delta\theta}xg = (xg)^{-1}S^{\delta\theta}xg.$$

The group $S^{\delta\theta}$ contains strongly G-regular elements [2, pp. 227–228]. The previous containment therefore implies that $xg \in N_G(S)$. Furthermore, the element $(xg)^{-1}\delta\theta(xg)$ belongs to $G(\mathbf{R})$ so $xg \in N_G(S^{\sigma})$ by Lemma 4.6. It is clear that $xg \in N_G(S^{\sigma})$ maps to the same θ-conjugacy class as $x^{-1}\delta\theta(x)$ under $G(\mathbf{R})$, and surjectivity is proved.

To prove injectivity, suppose that $x_1, x_2 \in G$ are representatives for double cosets in (4.12) such that $x_1^{-1}\delta\theta(x_1)$ and $x_2^{-1}\delta\theta(x_2)$ belong to the same θ-conjugacy class under $G(\mathbf{R})$. Then there exists $g \in G(\mathbf{R})$ such that

$$x_1^{-1}\delta\theta(x_1) = (x_2g)^{-1}\delta\theta(x_2g),$$

and it follows that

$$x_2gx_1^{-1} \in G^{\delta\theta} = S^{\delta\theta}.$$

This implies that $g \in N_{G(\mathbf{R})}(S)$, and x_1 and x_2 represent the same double coset in (4.12). □

Let us point out that there is some redundancy in the notation of (4.12). If $x \in N_G(S)$ satisfies (4.11), then it satisfies $x^{-1}\delta\theta(x) \in G(\mathbf{R})$. Lemma 4.6 then tells us that $x \in N_G(S^{\sigma})$. As a result, (4.12) could have been written more simply as

$$S^{\delta\theta}\backslash(N_G(S)/N_{G(\mathbf{R})}(S))^{\delta\theta}.$$

We prefer the notation of (4.12), as it highlights a distinction which is absent for elliptic tori, and reduces more readily to (4.10) when θ is trivial.

We now turn to the parameterization of the spectral data Π_{φ}. Our assumptions on $\boldsymbol{\varphi}$ dictate that induction furnishes a bijection between $\Pi_{\varphi,M}$ and Π_{φ} (Lemma 4.3). The L-packet $\Pi_{\varphi,M}$ of essentially square-integrable representations of $M(\mathbf{R})$ is parameterized by the coset space $\Omega(M,S)/\Omega_{\mathbf{R}}(M,S)$ (see [28, (21)]). We wish to ascertain the cosets which parameterize the representations in Π_{φ} which are stable under twisting by (ω,θ).

Define $(\Omega(M,S)/\Omega_{\mathbf{R}}(M,S))^{\delta\theta}$ to be the subset of those cosets in $\Omega(M,S)/\Omega_{\mathbf{R}}(M,S)$ which have a representative $w \in \Omega(M,S)$ satisfying

$$w^{-1}\delta\theta w(\delta\theta)^{-1} \in \Omega_{\mathbf{R}}(M,S). \tag{4.13}$$

Suppose that ϖ_Λ is a representation in $\Pi_{\varphi,M}$ which is stable under twisting. Suppose that $w \in \Omega(M,S)$ is a representative of a coset in $\Omega(M,S)/\Omega_{\mathbf{R}}(M,S)$. Then, according to [28, Lemma 15],

$$\varpi_{w^{-1}\Lambda} \cong \omega_{|M(\mathbf{R})} \otimes \varpi^{\delta\theta}_{w^{-1}\Lambda}$$

if and only if w satisfies (4.13).

Proposition 4.8. *Without loss of generality, the representation* $\mathrm{ind}_{P(\mathbf{R})}^{G(\mathbf{R})}\varpi_\Lambda \in \Pi_\varphi$ *is stable under twisting. Furthermore, the subset of representations in* Π_φ *which are stable under twisting is*

$$\left\{\mathrm{ind}_{P(\mathbf{R})}^{G(\mathbf{R})}\varpi_{w^{-1}\Lambda} : w \in (\Omega(M,S)/\Omega_{\mathbf{R}}(M,S))^{\delta\theta}\right\}.$$

Proof. The first assertion follows by applying the arguments of [28, Corollary 2] to $\mathrm{ind}_{P(\mathbf{R})}^{G(\mathbf{R})}\varpi_\Lambda$ in place of ϖ_Λ (cf. [27, Proof of Proposition 4.1]). There is no loss of generality, since Λ may be replaced by any $w^{-1}\Lambda$, $w \in \Omega(M,S)$ without affecting the assumptions of §4.

To prove the second assertion, suppose that $w \in \Omega(M,S)$ and that $\mathrm{ind}_{P(\mathbf{R})}^{G(\mathbf{R})}\varpi_{w^{-1}\Lambda}$ is stable under $(\delta\theta,\omega)$. According to the Langlands disjointness theorem [26, pp. 149–151], there exists $k \in N_{G(\mathbf{R})}(A_M)$ such that $\varpi_{w^{-1}\Lambda}$ is stable under $(k\delta\theta, \omega_{|M(\mathbf{R})})$. Since $k \in G(\mathbf{R})$, the maximal torus kSk^{-1} is defined over $\mathbf{R}$ and also elliptic in M. As all elliptic tori of M are $M(\mathbf{R})$-conjugate, we may assume that k normalizes S while maintaining the stability of $\varpi_{w^{-1}\Lambda}$ under twisting. This stability implies that

$$\omega_{|S(\mathbf{R})}(k\delta\theta w^{-1}\cdot\Lambda) = w^{-1}\cdot\Lambda.$$

By assumption, $\omega_{|S(\mathbf{R})}(\delta\theta\cdot\Lambda) = \Lambda$ so we may rewrite the above equation as

$$w_1^{-1}k\cdot\Lambda' = \Lambda',$$

where $\Lambda' = \delta\theta w^{-1}(\delta\theta)^{-1}\cdot\Lambda$ and $w_1 = w^{-1}\delta\theta w(\delta\theta)^{-1}$. The differential of the quasicharacter Λ' is G-regular so $w_1^{-1}k$ is the identity in $\Omega(G,S)$ [16, Lemma B 10.3]. It follows that w_1 is represented by an element in $G(\mathbf{R})$. Looking back to (4.13), this means that $w \in (\Omega_{\mathbf{R}}(M,S)/\Omega_{\mathbf{R}}(M,S))^{\delta\theta}$. Conversely, given $w \in (\Omega_{\mathbf{R}}(M,S)/\Omega_{\mathbf{R}}(M,S))^{\delta\theta}$, the intertwining operators in [27, Proof of Proposition 4.1] exhibit the stability of $\mathrm{ind}_{P(\mathbf{R})}^{G(\mathbf{R})}\varpi_{w^{-1}\Lambda}$. □

This proposition tells us that $(\Omega(M,S)/\Omega_{\mathbf{R}}(M,S))^{\delta\theta}$ is a spectral parameterizing set for Π_φ. Despite appearances, it is not so different from the geometric parameterizing set $S^{\delta\theta}\backslash(N_G(S^\sigma)/N_{G(\mathbf{R})}(S))^{\delta\theta}$. The intermediary between the two sets is

$$S^{\delta\theta}\backslash(N_M(S)/N_{M(\mathbf{R})}(S))^{\delta\theta},$$

whose definition is given by substituting $M = G$ in (4.11). According to [28, Proposition 2], there is a canonical surjection

$$S^{\delta\theta}\backslash(N_M(S)/N_{M(\mathbf{R})}(S))^{\delta\theta} \to (\Omega(M,S)/\Omega_{\mathbf{R}}(M,S))^{\delta\theta}. \tag{4.14}$$

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To complete the comparison between the spectral and geometric parameterizing sets, we observe that there is a canonical map from $S^{\delta\theta}\backslash(N_M(S)/N_{M(\mathbf{R})}(S))^{\delta\theta}$ to $S^{\delta\theta}\backslash(N_G(S^\sigma)/N_{G(\mathbf{R})}(S))^{\delta\theta}$. We conclude this section by proving that this map is a bijection.

Lemma 4.9. *Suppose that $x \in G$ such that $x^{-1}\delta\theta(x) \in G(\mathbf{R})$. Then there exists $y \in G(\mathbf{R})$ such that $xy \in M$.*

Proof. Fix a maximally $\mathbf{R}$-split torus S' containing S_d and a positive system on $R(G, S')$. Choose $\beta^\vee \in X_*(S_d) \subseteq X_*(S')$ as regular as possible in the positive chamber, and let $P(\beta^\vee)$ be its corresponding $\mathbf{R}$-parabolic subgroup (see [5, Proposition 20.4]). By construction, $P(\beta^\vee)$ has Levi decomposition MU. According to Lemma 4.6, the map $\mathrm{Int}(x^{-1})_{|S}$ is defined over $\mathbf{R}$ so $x^{-1}S_d x$ is an $\mathbf{R}$-split torus. Consequently, $x^{-1}P(\beta^\vee)x = P(x \cdot \beta^\vee)$ is also an $\mathbf{R}$-parabolic subgroup. By [39, Theorem 15.2.6] and [5, Theorem 20.9], there exists $y \in G(\mathbf{R})$ such that $(xy)^{-1}S_d xy \subseteq S'$ and $(xy)^{-1}P(\beta^\vee)xy = P(\beta^\vee)$. The latter equation implies that $xy \in P(\beta^\vee)$ [5, Theorem 11.16]. Writing $xy = mu$ according to the Levi decomposition $P = MU$, the earlier containment implies that

$$u^{-1}m^{-1}smus^{-1} = u^{-1}sus^{-1} \in S' \cap U = \{1\}, \quad s \in S_d.$$

In other words, the element u belongs to $M = Z_G(S_d)$, and so $xy \in M$. □

We remark that Lemmas 4.6 and 4.9 do not rely on the θ-ellipticity of δ, and so remain true without the assumption that S is fundamental in G. This fact will be used in §6.

Lemma 4.10. *The canonical map*

$$S^{\delta\theta}\backslash(N_M(S)/N_{M(\mathbf{R})}(S))^{\delta\theta} \to S^{\delta\theta}\backslash(N_G(S^\sigma)/N_{G(\mathbf{R})}(S))^{\delta\theta}$$

is a bijection.

Proof. The injectivity of this map follows from $N_{G(\mathbf{R})}(S) \cap M = N_{M(\mathbf{R})}(S)$. To prove surjectivity, suppose $x \in N_G(S^\sigma)$ is a representative of a double coset on the right. Then $x^{-1}\delta\theta(x) \in G(\mathbf{R})$ by Proposition 4.7. Choosing $y \in G(\mathbf{R})$ as in Lemma 4.9, we see that $xy \in M$. The map $\mathrm{Int}((xy)^{-1})_{|S}$ is defined over $\mathbf{R}$. Consequently, the torus $(xy)^{-1}S(xy)$ is also elliptic in M. After possibly multiplying y on the right by an element of $M(\mathbf{R})$, we may assume that $(xy)^{-1}S(xy) = S$ [44, Lemma 2.3.4] so $xy \in N_M(S)$ and $y \in N_{G(\mathbf{R})}(S)$. Finally, as M is preserved by $\mathrm{Int}(\delta)\theta$, we have $(xy)^{-1}\delta\theta xy(\delta\theta)^{-1} \in M$, and

$$(xy)^{-1}\delta\theta xy(\delta\theta)^{-1} = y^{-1}x^{-1}\delta\theta(x)\theta(y)\delta^{-1} \in G(\mathbf{R}) \cap M = M(\mathbf{R}).$$

This proves that $xy \in N_M(S)$ is a representative of a double coset on the left, and that the canonical injection is surjective. □

4.4. Spectral comparisons

We shall provide a brief overview of the proof of the spectral transfer identity (4.1). The proof is essentially the same as the one given in [28, 6] for discrete series representations. We therefore tailor our overview around those points which are influenced by accommodating fundamental series representations.

There are two pieces to the proof. The first is the proof of (4.1) for functions f with small elliptic support about $\delta\theta$ [28, 6.3]. The second piece of the proof is an extension to all $f \in C_c^\infty(G(\mathbf{R})\theta)$ using a twisted version of Harish-Chandra's uniqueness theorem [28, 6.4]. The second piece is not affected by the move to the fundamental series, and the required uniqueness theorems are given in appendix A. For this reason, we shall only examine the proof in the case of small elliptic support [28, 6.3].

The proof begins with the left-hand side of (4.1). The sum over the discrete L-packet $\Pi_{\varphi_{H_1}}$ may be converted to one over $\Omega(H_1, T_{H_1})/\Omega_{\mathbf{R}}(H_1, T_{H_1})$. The latter set also parameterizes the stable conjugacy class on the left of (3.14). Using the Weyl integration formula, it then becomes possible to substitute the geometric transfer identity (3.14). The resulting substitution [28, (100)] yields a sum over $S^{\delta\theta}\backslash(N_G(S^\sigma)/N_{G(\mathbf{R})}(S))^{\delta\theta}$ (Proposition 4.7) which may be replaced by $S^{\delta\theta}\backslash(N_M(S)/N_{M(\mathbf{R})}(S))^{\delta\theta}$ (Lemma 4.10). The substitution also introduces geometric transfer factors, which we may choose as in [28, 6.2]. After this point, one may continue precisely as in [28, Section 6.3] to arrive at the right-hand side of (4.1). In particular, the surjection (4.14) accounts for the desired sum over Π_φ (Proposition 4.8). The twisted characters $\Theta_{\pi,\mathsf{U}_\pi}$ are obtained by replacing Λ in section 4.2 by $w^{-1}\Lambda$, where $w \in (\Omega(M,S)/\Omega_{\mathbf{R}}(M,S))^{\delta\theta}$. The use of Bouaziz' character formula is also justified in §4.2.

Theorem 4.11. *The spectral transfer identity (4.1) holds for the fundamental series under Assumptions 1–6.*

5. Spectral transfer for limits of fundamental series

In this section, we adjust the framework of §4 by weakening Assumption 3 and removing Assumption 5. Let us concentrate on Assumption 5 for the moment. If we remove the $\hat{G}$-regularity of the parameters μ and $\mu_{|\mathfrak{s}_a}$, then the irreducible representations in Π_φ need no longer be fundamental series representations. As we shall see, these representations may be obtained using the method of *coherent continuation* or *Zuckerman tensoring*. When the Levi subgroup M of §4 is equal to G, then this method produces (essential) *limits of discrete series* [17, 7 XII]. By analogy, when M is allowed to be a proper Levi subgroup, we shall speak of (essential) *limits of fundamental series*. The goal then is to prove Theorem 4.11 for L-packets Π_φ consisting of limits of fundamental series. This was accomplished for limits of discrete series in 7.2–7.3 [28]. The proof for the limits of fundamental series is basically the same once the requisite objects are introduced.

We assume that we have the same endoscopic data as in §4 with the same Langlands parameter $\boldsymbol{\varphi}_{H_1}$. The pair of elements $\delta \in G(\mathbf{R})$ and $\gamma_1 \in H_1(\mathbf{R})$ are as before, and these bring with them the same fundamental torus S and Levi subgroup M. However, our assumptions on the Langlands parameter $\boldsymbol{\varphi}^*$ for G^* shall be weaker. We merely assume that $\boldsymbol{\varphi}^*$ has a representative φ^* which is an admissible homomorphism with respect to G [4, 8.2]. This amounts to the assumption that the image of φ^* is minimally contained in a parabolic subgroup of ${}^LG$ which is *relevant* (in the sense of 3.3 [4]) with respect to G.

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Lemma 5.1. *The image of φ^* is contained in a Levi subgroup ${}^LM$ dual to M (in the sense of (3) 3.3 [4]).*

Proof. Without loss of generality, we assume that $\varphi_{H_1}(\mathbf{C}^\times) \subseteq \mathcal{T}_H$ [28, 4]. Let $\mathcal{M}$ be the centralizer in $\hat{G}$ of the subtorus equal to the identity component of the fixed-point subgroup of $\mathcal{T}$ under conjugation by $\varphi^*(\sigma)$. Then $\mathcal{M}$ is a Levi subgroup of $\hat{G}$ [5, Proposition 20.4]. Let ${}^LM$ be the subgroup generated by $\mathcal{M}$ and $\varphi^*(\sigma)$. It is a Levi subgroup of ${}^LG$ by [4, Lemma 3.5]. Furthermore, the image of φ^* is contained in the subgroup of ${}^LM$ generated by $\mathcal{T}$ and $\varphi^*(\sigma)$. The admissibility assumption on φ^* [4, (ii) 8.2] implies that ${}^LM$ is dual to (a $G(\mathbf{R})$-conjugacy class of) an $\mathbf{R}$-Levi subgroup M' of G [4, (3) 3.3]. The action of $\varphi^*(\sigma)$ on $R(\mathcal{M}, \mathcal{T})$ is that of inversion [39, Lemma 15.3.2]. In [26, Proof of Lemma 3.1], one sees that this implies that M' contains an elliptic maximal torus S' such that ${}^LS' \cong \langle \mathcal{T}, \varphi^*(W_{\mathbf{R}})\rangle$. By the conjugacy theorems, [18, Corollary 4.35] and [38, Corollary 5.31], we may assume that the anisotropic subtorus S'_a of S' is contained in S_a.

It follows from Assumption 1 on φ_{H_1} that $\varphi_{H_1}(\sigma)$ acts by inversion on the root lattice in $X^*(\mathcal{T}_H)$ [28, (17)]. This implies that $\varphi^*(\sigma)$ acts by inversion on the corresponding root lattice in $X^*((\mathcal{T}^{\hat{\theta}})^0)$ (see (3.7)). Since $\hat{\theta}$ preserves the pair $(\mathcal{B}, \mathcal{T})$, there exists an element β in the root lattice of $R(\hat{G}, \mathcal{T})$ which lies in the Weyl chamber fixed by $\mathcal{B}$, and is invariant under the action of $\hat{\theta}$. In particular, β is $\hat{G}$-regular. The dual element $\beta^\vee$ [38, 2.2] belongs to $X_*(\mathcal{T}^{\hat{\theta}})$. Since $\mathcal{T}^{\hat{\theta}}/(\mathcal{T}^{\hat{\theta}})^0$ is finite, we may replace β by some integer multiple and assume that $\beta^\vee \in X_*((\mathcal{T}^{\hat{\theta}})^0)$. From before, we see that $\varphi^*(\sigma)$ acts by inversion on $\beta^\vee$. It therefore acts by inversion on β. This, together with the isomorphism $X^*(\mathcal{T}) \cong X_*(S')$, allows us to identify β with a regular element in $X_*(S'_a)$ [39, Proposition 13.2.4]. The regularity of β implies that $Z_{\hat{G}}(\mathrm{im}(\beta))$ is a maximal torus of G. Since $\mathrm{im}(\beta) \subseteq S'_a \subseteq S_a$, we find that this maximal torus is equal to both S' and S.

We deduce in turn that $S' = S$ is elliptic in M', $S_d \subseteq Z_{M'}$, and $M = Z_G(S_d) \supseteq M'$. On the other hand, the definition of $\mathcal{M}$, and the duality between $\mathcal{M}$ and M', and $\mathcal{T}$ and S, together imply that $Z_G(S_d) = M'$. We conclude that $M = M'$, and the lemma is complete. □

According to Lemma 5.1, the group $\varphi^*(W_{\mathbf{R}})$ is minimally contained in a Levi subgroup ${}^LM_1$ of ${}^LM$. By a relevance assumption, there exists a Levi subgroup $M_1 \subseteq M$ defined over $\mathbf{R}$ which corresponds to ${}^LM_1$ [4, 3.3–3.4]. This produces an admissible homomorphism $\varphi : W_{\mathbf{R}} \to {}^LG$ which we may view as a representative of a Langlands parameter for any of G, M, or M_1.

Regardless of which perspective one takes, the admissible homomorphism φ is determined by a pair $\mu, \lambda \in X_*(\hat{S}) \otimes \mathbf{C}$. This pair is begotten from a defining pair $\mu_{H_1}, \lambda_{H_1} \in X_*(\hat{T}_{H_1}) \otimes \mathbf{C}$ for an admissible homomorphism $\varphi_{H_1} \in \boldsymbol{\varphi}_{\boldsymbol{H_1}}$ [26, 3], [28, 4.1], and an application of the maps in (3.4), (3.7) and (4.2) (cf. [36, 7(b)]). There are identifications of Borel subgroups implicit in the maps of (3.7). We may assume that μ_{H_1} is in the positive Weyl chamber determined by the Borel subgroup $B_H \supseteq T_H$ of H [26, Lemma 3.3], [28, 4.1]. It follows from the identification of $\hat{B}_H$ with $\mathcal{B}_H$ and the containment $\xi(\mathcal{B}_H) \subseteq \mathcal{B} \cong \hat{B}'$ (§3.3) that μ lies in the Weyl chamber determined by $\hat{B}' \cap \hat{M}_1$. To say precisely what this means, let us denote by B the image of B' under the inverse of

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$\mathrm{Int}(g_{T'})\psi$. Then the precise statement is that $\langle \mu, \alpha^\vee \rangle > 0$ for all $\alpha \in R(B \cap M_1, S)$ [26, Lemma 3.3]. This ensures the $\hat{M}_1$-regularity of μ, but not its $\hat{G}$-regularity.

5.1. Shifting to the context of the fundamental series

We shall approach the representations in Π_φ indirectly, first shifting μ by a $\hat{G}$-regular element $\nu \in \mathrm{span}_{\mathbf{Z}} R(G, S) \subseteq X^*(S)$. This shift by ν will be constructed so as to produce a pair of matching admissible homomorphisms φ^ν and $\varphi^\nu_{H_1}$ which satisfy all of the assumptions of §4. We may then apply coherent continuation to recover the representations in Π_φ and the spectral transfer identity (4.1).

Lemma 5.2. *There exists $\nu \in \mathrm{span}_{\mathbf{Z}} R(M, S)$ which is G-regular and is fixed under the action of $\delta\theta$.*

Proof. We first prove the existence of G-regular ν. Suppose by way of contradiction that for some $\alpha \in R(G, S)^\vee$ and all $\nu \in \mathrm{span}_{\mathbf{Z}} R(M, S)$ we have $\langle \nu, \alpha \rangle = 0$. Then, in particular, the image of α belongs to

$$\left(\cap_{\beta \in R(M,S)} \ker \beta\right)^0 = Z_M^0 = S_d$$

[39, Proposition 8.1.8(i)], [5, Proposition 20.6(i)], and $\alpha \in X_*(S_d)$. This implies that α is a real coroot [5, 8.15], which contradicts S being fundamental.

We may now choose G-regular $\nu' \in \mathrm{span}_{\mathbf{Z}} R(M, S)$ which is positive with respect to $R(B, S)$. Under transport by (4.2), we identify ν' with an element in $\mathrm{span}_{\mathbf{Z}} R(\hat{G}, \hat{T}')^\vee \cong \mathrm{span}_{\mathbf{Z}} R(G^*, T')$ which lies in the positive Weyl chamber determined by B'. Recall from §3.3 that the pair (B', T') is preserved by θ^*. Therefore the automorphism θ^* has finite order on $X^*(T')$, and we may define $\nu = \sum_{j=1}^{|\theta^*|} (\theta^*)^j(\nu')$. The θ^*-invariance of ν translates to $\delta\theta$-invariance under transport by (4.2). □

Let us fix ν as in Lemma 5.2. After possibly replacing it by some positive integer multiple, we have $\mathrm{Re}\langle \mu + \nu, \alpha^\vee \rangle > 0$ for all $\alpha \in R(B, S)$, and so the element $\mu + \nu$ is $\hat{G}$-regular. This takes care of half of Assumption 5 in §4. The other half requires an understanding of the action of σ on ν. The $\delta\theta$-invariance of ν implies that $\nu \in X_*(\hat{S}^{\delta\theta})$. Since $\hat{S}^{\delta\theta}/(\hat{S}^{\delta\theta})^0$ is finite, we may again replace ν by some positive integer multiple and assume without loss of generality that $\nu \in X_*((\hat{S}^{\delta\theta})^0)$. There is an isomorphism $X_*((\hat{S}^{\delta\theta})^0) \cong X^*(S_{\delta\theta})$, where $S_{\delta\theta} = S/(1-\delta\theta)S$, and a surjection $(S^{\delta\theta})^0 \to S_{\delta\theta}$ (see [28, Proof of Lemma 12]). Consequently, there is an injection $X^*(S_{\delta\theta}) \hookrightarrow X^*((S^{\delta\theta})^0)$. We may identify $\nu \in X_*((\hat{S}^{\delta\theta})^0)$ with its image under the map

$$X_*((\hat{S}^{\delta\theta})^0) \hookrightarrow X^*((S^{\delta\theta})^0)$$

of Γ-modules. In fact, ν belongs to the submodule $X^*((S^{\delta\theta})^0/(Z_G^\theta)^0)$. By the θ-ellipticity of δ, the automorphism σ acts as inversion on $X^*((S^{\delta\theta})^0/(Z_G^\theta)^0)$ so $\sigma(\nu) = -\nu$. The decomposition

$$X^*(S) \otimes \mathbf{R} \cong (X^*(S_a) \otimes \mathbf{R}) \oplus (X^*(S_d) \otimes \mathbf{R})$$

[5, 8.15] allows us to identify ν with its restriction $\nu_{|\mathfrak{s}_a}$ (see (4.4)). As before, we may assume that

$$\mathrm{Re}\langle (\mu + \nu)_{|\mathfrak{s}_a}, \alpha^\vee \rangle = \mathrm{Re}\langle \mu_{|\mathfrak{s}_a} + \nu, \alpha^\vee \rangle > 0$$

for all $\alpha \in R(B, S)$, so the element $(\mu+\nu)_{|\mathfrak{s}_a}$ is $\hat{G}$-regular. At this point we have shown that Assumption 5 of §4 holds for $\mu+\nu$.

We turn to the construction of matching admissible homomorphisms φ^{ν} and $\varphi^{\nu}_{H_1}$ which satisfy the remaining assumptions of §4. First, since $\sigma(\nu) = -\nu$ is in the root lattice, it is easily verified that the pair $\mu+\nu, \lambda \in X_*(\hat{S}) \otimes \mathbf{C}$ corresponds to an admissible homomorphism $\varphi^{\nu} : W_{\mathbf{R}} \to {}^L G$ with $\varphi^{\nu}(\sigma) = \varphi(\sigma)$ [28, 4]. The pair also corresponds to a quasicharacter $\Lambda(\mu+\nu-\iota_M, \lambda)$ of $S(\mathbf{R})$ [28, (18)]. As in §4, the L-packet $\Pi_{\varphi^{\nu},M}$ consists of essentially square-integrable representations of $M(\mathbf{R})$.

The central character of the representations in $\Pi_{\varphi^{\nu},M}$ differs from the unitary central character of the representations in $\Pi_{\varphi,M}$ by the restriction of $\Lambda(\mu+\nu-\iota_M, \lambda)\Lambda(\mu-\iota_M, \lambda)^{-1}$ to $Z_M(\mathbf{R})$. This restriction depends only on the restriction of $\nu \in X^*(S)\otimes \mathbf{C}$ to $Z_M \subseteq S$ [4, 9]. Therefore, to show that Assumption 4 of §4 holds for the central character of $\Pi_{\varphi^{\nu},M}$, it suffices to show that the restriction of $\nu \in X^*(S)\otimes \mathbf{R}$ to the split component of Z_M is trivial. This is true, as the split component of Z_M is contained in S_d, the map $(1-\sigma)$ annihilates $X^*(S_d)$ [5, 8.15], and

$$\nu_{|S_d} = \frac{1-\sigma}{2}(\nu)_{|S_d} = \frac{1-\sigma}{2}(\nu_{|S_d}) = 0.$$

Thus far, we see Assumptions 2, 4 and 5 of §4 hold for φ^{ν}, and enough has been shown to conclude that the L-packet $\Pi_{\varphi^{\nu}}$ is comprised of fundamental series representations (see 4.1). Assumption 6 of §4 follows from the $\delta\theta$-invariance of ν (see [28, (136)]).

It remains to construct an admissible homomorphism $\varphi^{\nu}_{H_1}$ such that Assumptions 1 and 3 hold. For this, we return to viewing ν as an element of $X^*(T'_{\theta*}) \cong X_*((\mathcal{T}^{\hat{\theta}})^0)$, as in the proof of Lemma 5.2. Let $\nu_{H_1} \in X_*(\hat{T}_H)$ be the image of ν under the composition of the isomorphisms of (3.7). By (3.5), we may regard ν_{H_1} as an element in $X_*(\hat{T}_{H_1})$. The positivity of ν with respect to the Borel subgroup B transfers to the positivity of ν_{H_1} with respect to the Borel subgroup B_H. Let $\mu_{H_1}, \lambda_{H_1} \in X_*(\hat{T}_{H_1})\otimes \mathbf{C}$ be a defining pair for φ_{H_1} such that μ_{H_1} is positive with respect to B_H. Then $\mu_{H_1}+\nu_{H_1}$ is regular so the pair $\mu_{H_1}+\nu_{H_1}, \lambda_{H_1} \in X_*(\hat{T}_{H_1})\otimes \mathbf{C}$ defines an admissible homomorphism $\varphi^{\nu}_{H_1} : W_{\mathbf{R}} \to {}^L H_1$ [28, 4.1] such that Assumption 1 of §4 holds for $\varphi^{\nu}_{H_1}$.

Finally, Assumption 3 of §4 holds by virtue of the definition of $(\varphi^{\nu})^*$ as $\xi \circ \xi_{H_1}^{-1} \circ \varphi^{\nu}_{H_1}$ [28, 6]. Indeed, the image of $\mu_{H_1}+\nu_{H_1}$ under the maps induced by $\xi \circ \xi_{H_1}^{-1}$ corresponds to $\mu+\nu$ by the construction of ν_H, and the regularity of $\mu+\nu$ in $\hat{M}$ ensures that φ^{ν} is minimally contained in ${}^L M$.

5.2. Coherent continuation to the limit of fundamental series representations

Let us describe the relationship between the L-packets of φ and those of the shifted admissible homomorphism φ^{ν}, following [36, 14] and [28, 7]. The representations in $\Pi_{\varphi^{\nu},M}$ are essential discrete series representations. The representations in $\Pi_{\varphi,M}$ are essential limits of discrete series representations obtained via Zuckerman tensoring representations in $\Pi_{\varphi^{\nu},M}$ [22, (1.10)]. To explain this relationship better, let $w \in \Omega(M, S)/\Omega_{\mathbf{R}}(M, S)$ and $\varpi_{w^{-1}\Lambda} \in \Pi_{\varphi^{\nu},M}$ be as in Proposition 4.8. Denote the distribution character of the

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representation obtained from $\varpi_{w^{-1}\Lambda}$ through Zuckerman tensoring by $\Theta(w^{-1}\mu, \lambda, w^{-1} \cdot \hat{B})$. Then the set of characters of the irreducible representations in $\Pi_{\varphi,M}$ is equal to the non-zero subset of characters in

$$\{\Theta(w^{-1}\mu, \lambda, w^{-1} \cdot \hat{B}) : w \in \Omega(M, S)/\Omega_{\mathbf{R}}(M, S)\}.$$

Using Hecht–Schmid identities, one may determine this non-zero subset explicitly [28, (142)], [33, p. 408].

The irreducible representations in Π_φ and Π_{φ^ν} are the irreducible subrepresentations of the representations induced from $\Pi_{\varphi,M}$ and $\Pi_{\varphi^\nu,M}$, respectively. In the present context, parabolic induction and Zuckerman tensoring commute with one another [41, Corollary 5.9] and produce irreducible representations (when non-zero) [41, Theorem 5.15]. Hence, parabolic induction furnishes a bijection between $\Pi_{\varphi,M}$ and Π_φ, just as it does between $\Pi_{\varphi^\nu,M}$ and Π_{φ^ν}. We may write the characters of the representations occurring in Π_φ as

$$\operatorname{ind}_{P(\mathbf{R})}^{G(\mathbf{R})} \Theta(w^{-1}\mu, \lambda, w^{-1} \cdot \hat{B}), \tag{5.1}$$

where w lies in [28, (142)]. We call the representations corresponding to these characters *limit of fundamental series* representations. A limit of discrete series character is the special case of (5.1) in which $P = M = G$. These are all special cases of *limit of generalized principal series* representations given on [41, p. 265 5].

The spectral transfer identity (4.1) was proved for essential limits of discrete series in [28, 7.2.3]. The argument there hinges on Zuckerman tensoring a discrete series representation $\bar{\varpi}_\nu$ of the disconnected group $G_{\mathrm{der}}(\mathbf{R})^0 \rtimes \langle \delta\theta \rangle$ to a limit of discrete series representation $\bar{\varpi}_1$. This argument is unaffected when $\bar{\varpi}_\nu$ is allowed to be a fundamental series representation. In consequence, if $\pi \in \Pi_\varphi$ is obtained from $\pi_\nu \in \Pi_{\varphi^\nu}$ by Zuckerman tensoring, we obtain a twisted character $\Theta_{\pi,\mathsf{U}_\pi}$ from the character of $\bar{\varpi}_1$. Moreover, after defining

$$\Delta(\boldsymbol{\varphi}_{\boldsymbol{H_1}}, \pi) = \Delta(\boldsymbol{\varphi}^{\boldsymbol{\nu}}_{\boldsymbol{H_1}}, \pi_\nu),$$

[28, Proof of Theorem 3] carries through and (4.1) holds. We record this as a theorem.

Theorem 5.3. *Suppose that there exists a strongly θ-regular and θ-elliptic element $\delta \in G(\mathbf{R})$ which has a norm $\gamma \in H(\mathbf{R})$. Suppose further that φ_{H_1} is an admissible homomorphism not contained in a proper parabolic subgroup of ${}^L H_1$. Finally, suppose that $\varphi^* = \varphi$ is an admissible homomorphism such that the representations in $\Pi_{\varphi,M}$ have unitary central character and $\Pi_\varphi = \omega \otimes (\Pi_\varphi \circ \theta)$. Then*

$$\int_{H_1(\mathbf{R})/Z_1(\mathbf{R})} f_{H_1}(h) \sum_{\pi_{H_1} \in \Pi_{\boldsymbol{\varphi}_{\boldsymbol{H_1}}}} \Theta_{\pi_{H_1}}(h)\, dh = \sum_{\pi \in \Pi_\varphi} \Delta(\boldsymbol{\varphi}_{\boldsymbol{H_1}}, \pi) \Theta_{\pi,\mathsf{U}_\pi}(f)$$

for all $f \in C_c^\infty(G(\mathbf{R})\theta)$.

6. Spectral transfer for tempered representations

In this section, we show how the framework of the previous sections extends to tempered representations. Suppose that we have an endoscopic quadruple $(H, \mathcal{H}, \mathsf{s}, \xi)$

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and a compatible z-pair (H_1, ξ_{H1}), as above. Now, we assume only that $\varphi_{H_1} : W_{\mathbf{R}} \to {}^L H_1$ is an admissible homomorphism such that $\Pi_{\varphi_{H_1}}$ consists of irreducible tempered representations. As before, we assume that the homomorphism $\varphi^* = \xi \circ \xi_{H_1}^{-1} \circ \varphi_{H_1}$ is *admissible* with respect to G, and denote φ^* by φ. The condition that $\Pi_{\varphi_{H_1}}$ be tempered is equivalent to φ_{H_1} having bounded image [4, (4) 10.3]. It follows from the continuity of ξ and $\xi_{H_1}^{-1}$ that φ has bounded image so Π_φ is an L-packet consisting of irreducible tempered representations of $G(\mathbf{R})$. In order for this L-packet to have any bearing on twisted endoscopy, we assume that $\Pi_\varphi = \omega \otimes \Pi_\varphi \circ \theta$.

Let ${}^L \bar{M}_{H_1}$ be the smallest Levi subgroup of ${}^L H_1$ containing the image of φ_{H_1}. By definition, the representations in $\Pi_{\varphi_{H_1}}$ are the irreducible subrepresentations of the representations induced from $\Pi_{\varphi_{H_1}, \bar{M}_{H_1}}$ [4, 11.3]. In addition, the admissible homomorphism $\varphi_{\bar{M}_1} : W_{\mathbf{R}} \to {}^L \bar{M}_{H_1}$, defined by replacing the codomain of φ_{H_1} with ${}^L \bar{M}_{H_1}$, satisfies Assumption 1 of §4. In order to ensure that the other assumptions of §§4 or 5 apply, we must produce an $\mathbf{R}$-Levi subgroup $\bar{M}$ of G and a triple $(\bar{M}, \theta_{\bar{M}}, \omega_{\bar{M}})$ corresponding to an endoscopic quadruple $(\bar{M}_H, \mathcal{H}_{\bar{M}}, \mathsf{s}_{\bar{M}}, \xi_{\bar{M}})$, where $\bar{M}_{H_1}$ is a z-extension of $\bar{M}_H$.

6.1. Endoscopic data related to ${}^L \bar{M}_{H_1}$

We begin with the definition of $\bar{M}$. As H_1 is a quasisplit group, there is an $\mathbf{R}$-Levi subgroup $\bar{M}_{H_1}$ of H_1 which is dual to ${}^L \bar{M}_{H_1}$. The minimality of ${}^L \bar{M}_{H_1}$ implies that there exists a maximal torus T_{H_1} of $\bar{M}_{H_1}$ which is elliptic in $\bar{M}_{H_1}$. Let $T_{H,d}$ be the split component of $T_H = p_H(T_{H_1})$. Choose $\beta^\vee \in X_*(T_{H,d})$ as regular as possible with respect to $R(H, T_H)$, and choose a Borel subgroup $B_H \supseteq T_H$ such that $\langle \alpha, \beta^\vee \rangle \geqslant 0$ for all $\alpha \in R(B_H, T_H)$. The cocharacter $\beta^\vee$ is defined over $\mathbf{R}$, and the Levi subgroup $\bar{M}_{H_1}$ is the centralizer in H_1 of any pre-image of $\beta^\vee$ under p_H. An application of [20, Lemma 3.3.B] gives us a θ^*-stable pair (B', T') and an admissible embedding (3.11) so we have the following maps of Γ-modules:

$$\beta^\vee \in X_*(T_{H,d}) \hookrightarrow X_*(T_H) \cong X^*(\mathcal{T}_H) \overset{\xi}{\hookrightarrow} X^*((\mathcal{T}^{\hat{\theta}})^0) \cong X_*(T'_{\theta^*})$$

(see (3.7)). The image $\xi \circ \beta^\vee$ of $\beta^\vee$ in $X^*((\mathcal{T}^{\hat{\theta}})^0)$ is non-negative with respect to the ordering defined by $R(\mathcal{B}, \mathcal{T})$. We may suppose that $\beta^\vee$ has been chosen so that $\xi \circ \beta^\vee$ is as positive as possible with respect to this ordering. There is a unique lift of (some positive multiple of) $\xi \circ \beta^\vee$ to a character in $X^*(\mathcal{T})$ which is trivial on the torus complementary to $(\mathcal{T}^{\hat{\theta}})^0$ [5, Corollary 8.5]. Let $(\beta')^\vee \in X_*(T')$ be the cocharacter corresponding to this lift under the isomorphism $X_*(T') \cong X^*(\mathcal{T})$. By design, the cocharacter $(\beta')^\vee$ is defined over $\mathbf{R}$ and is non-negative with respect to the ordering defined by $R(B', T')$. Let $\bar{M}^*$ be the centralizer of the image of $(\beta')^\vee$. It is the Levi subgroup of the parabolic subgroup $\bar{P}^*$ determined by $(\beta')^\vee$ [39, 13.4]. Both $\bar{M}^*$ and $\bar{P}^*$ are defined over $\mathbf{R}$ [39, Theorem 13.4.2]. These groups are also compatible with θ^*.

Lemma 6.1. *The automorphism θ^* preserves both $\bar{M}^*$ and $\bar{P}^*$.*

Proof. Recall that $(\beta')^\vee$ corresponds to a lift of an character in $X^*((\mathcal{T}^{\hat{\theta}})^0)$. The $\hat{\theta}$-invariance of this character implies that $\theta^* \circ (\beta')^\vee = (\beta')^\vee$, and the lemma follows. □

Now, by the admissibility of φ [4, 8.2(ii)], there exist $\mathbf{R}$-subgroups $\bar{M}$ and $\bar{P}$ of G parallel to the subgroups $\bar{M}^*$ and $\bar{P}^*$ of G^*. From here on, we make the assumption that there is a θ-regular element $\delta \in G(\mathbf{R})$ which has norm $\gamma_1 \in T_{H_1}(\mathbf{R})$. It will be explained in §6.4 why this is a negligible assumption. With this assumption in force, we have recourse to the $\mathbf{R}$-isomorphism (4.2). It follows that $\bar{M}$ and $\bar{P}$ are given by the inverse image of $(\beta')^\vee$ under $\mathrm{Int}(g_{T'})\psi$. In addition, the $\delta^*\theta^*$-invariance of $\bar{M}^*$ and $\bar{P}^*$ (Lemma 6.1) translates to the $\delta\theta$-invariance of $\bar{M}$ and $\bar{P}$.

We define the twisting data in the triple $(\bar{M}, \theta_{\bar{M}}, \omega_{\bar{M}})$ by $\theta_{\bar{M}} = \mathrm{Int}(\delta)\theta_{|\bar{M}}$ and $\omega_{\bar{M}} = \omega_{|\bar{M}}$. For the definition of the endoscopic data, we recall that the passage from θ to the dual map $\hat{\theta}$ is insensitive to composition with inner automorphisms (§3.1, see [4, 1.3]). Indeed, $\hat{\theta}$ is obtained via an action on based root data, and such an action is independent of inner automorphisms. The upshot of this observation is that $\widehat{\mathrm{Int}(\delta)\theta} = \hat{\theta}$. This identity in turn justifies that $(\bar{M}_H, \mathcal{H}_{\bar{M}}, \mathsf{s}, \xi_{\bar{M}})$ is an endoscopic datum for $(\bar{M}, \theta_{\bar{M}}, \omega_{\bar{M}})$, where $\bar{M}_H = p_H(\bar{M}_{H_1})$, $\mathcal{H}_{\bar{M}} = \hat{\bar{M}}_H \rtimes c(W_{\mathbf{R}})$, and $\xi_{\bar{M}} = \xi_{|\mathcal{H}_{\bar{M}}}$ (see §3.2). The surjection $p_{|\bar{M}_{H_1}} : \bar{M}_{H_1} \to \bar{M}_H$ defines a z-extension of $\bar{M}_H$ [20, 2.2] and a z-pair $(\bar{M}_{H_1}, (\xi_{H_1})_{|{}^L\bar{M}_{H_1}})$.

This choice of endoscopic datum may be associated with the quasisplit group $\bar{M}^*$ through the isomorphism $\mathrm{Int}(g_{T'})\psi_{|\bar{M}} : \bar{M} \to \bar{M}^*$ mentioned above. By defining $\psi_{\bar{M}} = \mathrm{Int}(g_{T'})\psi_{|\bar{M}}$, the element u_σ of (3.2) is replaced by $g_{T'}u_\sigma\sigma(g_{T'}^{-1}) \in T'_{\mathrm{sc}} \subset \bar{M}^*_{\mathrm{sc}}$. Equation (3.3) is replaced by

$$\theta^*_{\bar{M}} = \theta^*_{|\bar{M}^*} = \mathrm{Int}((\delta^*)^{-1})\psi_{\bar{M}}\theta_{\bar{M}}\psi_{\bar{M}}^{-1},$$

in which some lift of $(\delta^*)^{-1}$ to $\bar{M}^*_{\mathrm{sc}}$ takes on the role of g_θ. Finally, the element in (3.10) is replaced by

$$(\delta^*)^{-1}g_{T'}u_\sigma\sigma(g_{T'}^{-1})\sigma(\delta^*)\theta^*(\sigma(g)u_\sigma^{-1}g_{T'}^{-1}). \tag{6.1}$$

One may compute that this element is equal to

$$\theta^*(g_{T'})g_\theta u_\sigma\sigma(g_\theta^{-1})\theta^*(u_\sigma^{-1})\theta^*(g_{T'}^{-1}),$$

which by (3.10) is equal to $g_\theta u_\sigma\sigma(g_\theta^{-1})\theta^*(u_\sigma^{-1})$ and therefore defines a cocycle in $(1-\theta^*)Z_{G^*_{\mathrm{sc}}} \subseteq (1-\theta^*_{\bar{M}})Z_{\bar{M}^*_{\mathrm{sc}}}$. This shows that Assumption (3.10) holds for $(\bar{M}, \theta_{\bar{M}}, \omega_{\bar{M}})$ in place of (G, θ, ω).

The image of φ is contained in the Levi subgroup ${}^L\bar{M}$ dual to $\bar{M}^*$ under the identification of $R(\mathcal{B}, \mathcal{T})$ with $R(B', T')^\vee$ (see the proof of Lemma 5.1). By substituting ${}^L\bar{M}$ for the codomain of φ, we may regard $\varphi_{\bar{M}} : W_{\mathbf{R}} \to {}^L\bar{M}$ as an admissible homomorphism of $\bar{M}$ obtained from the admissible homomorphism $\varphi_{\bar{M}_{H_1}}$ of $\bar{M}_{H_1}$ above.

6.2. Assumptions required for spectral transfer on $\bar{M}$

At this stage, it makes sense to revisit the assumptions of §4 with $\bar{M}$ in place of G. As already mentioned, Assumption 1 holds for $\varphi_{\bar{M}_{H_1}}$. The next lemma shows that Assumption 2 holds.

Lemma 6.2. *The strongly θ-regular element δ is θ-elliptic.*

Proof. By isomorphism (4.2), it suffices to prove that the identity component of $(T')^{\theta^*}/Z_{\bar{M}^*}^{\theta^*}$ is anisotropic. We shall accomplish this by producing a torus $T_1 \subseteq Z_{\bar{M}^*}^{\theta^*}$ such that $((T')^{\theta^*}/T_1)(\mathbf{R})$ is compact. Essentially, T_1 is the image of $T_{H,d}$ under the maps used to define $(\beta')^\vee \in X_*(T')$.

This is easiest to see in the special case that $T'_{\theta*} \cong (T')^{\theta^*}$. In this case, (3.11) maps $T_{H,d}$ to a split subtorus $T_1 \subseteq (T')^{\theta^*}$, and $(\beta')^\vee \in X_*(T_1)$ is as regular as possible with respect to $R(B', T')$. It follows that $\alpha \in R(\bar{M}^*, T')$ if and only if $\alpha_{|T_1}$ is trivial [39, Lemma 15.3.2 (ii)], [5, Proposition 20.4]. This implies that $T_1 \subseteq Z_{\bar{M}^*}$, and we obtain a surjection

$$T_H/T_{H,d} \overset{(3.11)}{\cong} (T')^{\theta^*}/T_1 \to (T')^{\theta^*}/Z_{\bar{M}^*}^{\theta^*}$$

of elliptic tori.

In general, the canonical map $(T'^{\theta^*})^0 \to T'_{\theta*}$ is merely an isogeny (see [28, Proof of Lemma 12]). The lift of $\xi \circ \beta^\vee$ to $(\beta'^\vee)$ amounts to a lift from $X_*(T'_{\theta*})$ to $X_*((T'^{\theta^*})^0)$, as may be seen from

$$\begin{array}{ccc} X^*(\mathcal{T}) & \longrightarrow & \xi \circ \beta^\vee \in X^*((\mathcal{T}^{\hat{\theta}})^0) \\ \cong \big\downarrow & & \big\downarrow \cong \\ (\beta')^\vee \in X_*(T') & \longrightarrow & X_*(T'_{\theta*}) \end{array}$$

and the proof of Lemma 6.1. Similarly, the image of $T_{H,d}$ in $T'_{\theta*}$ under (3.11) lifts to a split torus $T_1 \subseteq (T'^{\theta^*})^0$, and $(\beta')^\vee$ belongs to $X_*(T_1)$. As argued in the special case above, the torus T_1 is contained in $Z_{\bar{M}^*}$. We now have two surjections,

$$(T'^{\theta^*})^0/(Z_{\bar{M}^*}^{\theta^*})^0 \leftarrow (T'^{\theta^*})^0/T_1 \to T_H/T_{H,d}.$$

The surjection on the right is an isogeny induced by $(T'^{\theta^*})^0 \to T'_{\theta*}$ and (3.11). As $T_H/T_{H,d}$ is anisotropic, this isogeny ensures that $(T'^{\theta^*})^0/T_1$ is also anisotropic. The surjection on the left now implies that $(T'^{\theta^*})^0/(Z_{\bar{M}^*})^0$ is anisotropic, and the lemma is proved. $\square$

A consequence of Lemmas 6.2 and 4.1 is that S is a fundamental torus in $\bar{M}$.

Assumption 3 was weakened in §5 to φ being admissible. This admissibility assumption is made in this section as well, so $\varphi_{\bar{M}}$ is admissible.

Assumption 4 must hold, for otherwise the central character of $\Pi_{\varphi,M}$ would force the image of φ to be unbounded, and Π_φ would not be tempered.

Assumption 5 was removed in §5, and so we may ignore it.

We must prove Assumption 6 for $\bar{M}$. We have assumed that the L-packet Π_φ is stable under twisting. To deduce the same for the L-packet $\Pi_{\varphi_{\bar{M}}}$, it is natural to make a connection between the two L-packets. The common ground between the two packets Π_φ and $\Pi_{\varphi_{\bar{M}}}$ is the 'minimal' packet $\Pi_{\varphi,M}$ in the following extension of Proposition 4.8.

Lemma 6.3. *Let S_d be the split component of the torus S, let $M = Z_{\bar{M}}(S_d)$, and let P be a parabolic subgroup of G with M as a Levi subgroup. Suppose that $\varpi, \varpi' \in \Pi_{\varphi,M}$ are essential limit of discrete series representations of $M(\mathbf{R})$ such that $\mathrm{ind}_{P(\mathbf{R})}^{G(\mathbf{R})}\varpi'$ and $\omega \otimes (\mathrm{ind}_{P(\mathbf{R})}^{G(\mathbf{R})}\varpi)^{\delta\theta}$ share equivalent irreducible subrepresentations. Then ϖ' is equivalent to $\omega_{|M(\mathbf{R})} \otimes \varpi^{\delta\theta}$.*

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Proof. Suppose first that ϖ' and ϖ are essentially square-integrable representations. As in §4.3, we may write $\varpi = \varpi_{w^{-1}\Lambda}$ and $\varpi' = \varpi_{(w')^{-1}\Lambda}$, where S is an elliptic torus of $M = \bar{M}$, Λ is an M-regular, and $\delta\theta$-stable quasicharacter of $S(\mathbf{R})$, and $w, w' \in \Omega(M, S)$. The Langlands disjointness theorem (pp. 149–151 [26]) provides $k \in N_{G(\mathbf{R})}(M)$ such that ϖ' is equivalent to $\omega_{|M(\mathbf{R})} \otimes \varpi^{k\delta\theta}$. As in Proposition 4.8, we may assume that $k \in N_{G(\mathbf{R})}(S)$ and identify it with an element of $\Omega_{\mathbf{R}}(G, S)$. We have the equation

$$\omega_{|S(\mathbf{R})}(k\delta\theta w^{-1} \cdot \Lambda) = (w')^{-1} \cdot \Lambda,$$

which may be rewritten in the form

$$w_1^{-1} k \cdot \Lambda' = \Lambda', \tag{6.2}$$

where $\Lambda' = \delta\theta w^{-1}(\delta\theta)^{-1} \cdot \Lambda$ and $w_1 = (w')^{-1}\delta\theta w(\delta\theta)^{-1}$.

The (differential of the) quasicharacter Λ' is M-regular. We choose a positive system on $R(G, S)$ so that its induced positive system on $R(M, S)$ corresponds to a Weyl chamber containing Λ'. Equation (6.2) implies that the element $w_1^{-1}k \in \Omega(G, S)$ is a product of reflections generated by simple roots in $R(G, S)$ which are orthogonal to Λ' [16, Lemma B 10.3]. Suppose that α is such a simple root, and let ρ_M be the half-sum of the positive roots of $R(M, S)$. The simple reflection s_α fixes Λ' and therefore stabilizes the system of positive roots for $R(M, S)$. This implies that s_α fixes ρ_M, or, equivalently, that α is orthogonal to ρ_M.

Using the terminology of [43, 3], this proves that α is a *quasisplit* root and that $w_1^{-1}k$ lies in the *quasisplit* Weyl group generated by the *quasisplit* roots. We also know that $w_1 \in \Omega(M, S)$ is defined over $\mathbf{R}$ [24, Lemma 6.4.1] so $w_1^{-1}k$ belongs to the subgroup of the quasisplit Weyl group whose elements are defined over $\mathbf{R}$. According to Vogan, this subgroup is a semidirect product of two groups [43, p. 961], and each of these two is contained in $\Omega_{\mathbf{R}}(G, S)$ [43, Lemma 3.1]. In short, $w_1^{-1}k$ belongs to $\Omega_{\mathbf{R}}(G, S)$ so $w_1 \in \Omega_{\mathbf{R}}(M, S)$ and

$$w'\Omega_{\mathbf{R}}(M, S) = (\delta\theta \cdot w)\Omega_{\mathbf{R}}(M, S).$$

We deduce from [24, 6.4] and a character comparison that

$$\varpi' = \varpi_{(w')^{-1}\Lambda} \cong \omega_{|M(\mathbf{R})} \otimes \varpi_{\delta\theta \cdot w^{-1}\Lambda} \cong \omega_{|M(\mathbf{R})} \otimes (\varpi_{w^{-1}\Lambda})^{\delta\theta} = \omega_{|M(\mathbf{R})} \otimes \varpi^{\delta\theta}.$$

Suppose now that $\varpi, \varpi' \in \Pi_{\varphi,M}$ are essential limit of discrete series representations. In the notation of 7.2.3 [28], we may write $\varpi = \Psi_{w^{-1}\cdot\mu}^{w^{-1}\cdot(\mu+\nu')}\varpi_{w^{-1}\Lambda}$ and $\varpi' = \Psi_{(w')^{-1}\cdot\mu}^{(w')^{-1}\cdot(\mu+\nu')}\varpi_{(w')^{-1}\Lambda}$, where $\varpi_{w^{-1}\Lambda}$ and $\varpi_{(w')^{-1}\Lambda}$ are essentially square-integrable representations as above. As in the previous case, the Langlands disjointness theorem supplies $k \in N_{G(\mathbf{R})}(S)$ such that ϖ' is equivalent to $\omega_{|M(\mathbf{R})} \otimes \varpi^{k\delta\theta}$. By [22, Theorem 1.1(c)], there exists $k_1 \in N_{\bar{M}(\mathbf{R})}(S)$ such that $k_1 k\delta\theta w^{-1} \cdot \Lambda = (w')^{-1} \cdot \Lambda$. The previous argument for essentially square-integrable representations therefore applies after replacing k with $k_1 k$. We conclude in turn that $\varpi_{(w')^{-1}\Lambda} \cong \omega_{|M(\mathbf{R})} \otimes (\varpi_{w^{-1}\Lambda})^{\delta\theta}$ and ϖ' is equivalent to

$$\omega_{|M(\mathbf{R})} \otimes \Psi_{\delta\theta\cdot w^{-1}\cdot\mu}^{\delta\theta\cdot w^{-1}\cdot(\mu+\nu')}(\varpi_{w^{-1}\Lambda})^{\delta\theta} \cong \omega_{|M(\mathbf{R})} \otimes (\Psi_{w^{-1}\cdot\mu}^{w^{-1}\cdot(\mu+\nu')}\varpi_{w^{-1}\Lambda})^{\delta\theta} = \omega_{|M(\mathbf{R})} \otimes \varpi^{\delta\theta}. \quad \square$$

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Corollary 6.4. *The L-packet $\Pi_{\varphi_{\bar{M}}} = \Pi_{\varphi,\bar{M}}$ equals the L-packet $\omega_{\bar{M}} \otimes \Pi_{\varphi_{\bar{M}}} \circ \theta_{\bar{M}}$.*

Proof. Suppose that $\varpi \in \Pi_{\varphi,\bar{M}}$. Then the irreducible subrepresentations of $\mathrm{ind}_{\bar{P}(\mathbf{R})}^{G(\mathbf{R})} \varpi$ belong to Π_φ [4, 11.3]. We are assuming that $\Pi_\varphi = \omega \otimes \Pi_\varphi \circ \theta$ so that there exists $\varpi' \in \Pi_{\varphi,\bar{M}}$ such that $\mathrm{ind}_{\bar{P}(\mathbf{R})}^{G(\mathbf{R})} \varpi'$ and $\omega \otimes (\mathrm{ind}_{\bar{P}(\mathbf{R})}^{G(\mathbf{R})} \varpi)^{\delta\theta}$ have some equivalent irreducible subrepresentations. By Lemma 6.3 and the fact that the irreducible representations in $\Pi_{\varphi,\bar{M}}$ are induced from irreducible representations in $\Pi_{\varphi,M}$ (Corollary 4.4), the representation ϖ' is equivalent to $\omega_{\bar{M}} \otimes \varpi^{\delta\theta} \in \omega_{\bar{M}} \otimes \Pi_{\varphi,\bar{M}} \circ \theta_{\bar{M}}$. Since L-packets with non-empty intersection are equal, the corollary is complete. □

Corollary 6.4 shows that Assumption 6 holds for $\bar{M}$, and we are now in the position to apply Theorem 5.3 on the level of $\bar{M}$. This will be done in the following section. We record one more corollary which aligns twisting on Π_φ with twisting on $\Pi_{\varphi_{\bar{M}}}$.

Corollary 6.5. *Suppose that $\pi \in \Pi_\varphi$ and $\varpi \in \Pi_{\varphi,\bar{M}}$ such that π is a subrepresentation of $\mathrm{ind}_{\bar{P}(\mathbf{R})}^{G(\mathbf{R})} \varpi$ [4, 11.3]. If π is (θ, ω)-stable, then ϖ is $(\theta_{\bar{M}}, \omega_{\bar{M}})$-stable.*

6.3. Parabolic descent

Suppose that $\bar{P}_{H_1}$ is an $\mathbf{R}$-parabolic subgroup of H_1 which has $\bar{M}_{H_1}$ as a Levi subgroup. The *parabolic descent* of a compactly supported (mod centre) function f_1 on $H_1(\mathbf{R})$ with respect $\bar{P}_{H_1}$ will be written as $f_1^{(\bar{P}_{H_1})}$ [17, (10.22)]. It is a function on $\bar{M}_{H_1}(\mathbf{R})$. When $f_1 = f_{H_1}$ for some $f \in C_c^\infty(G(\mathbf{R})\theta)$, the left-hand side of the spectral transfer identity (4.1) equals

$$\int_{\bar{M}_{H_1}(\mathbf{R})/Z_1(\mathbf{R})} (f_{H_1})^{(\bar{P}_{H_1})}(h) \sum_{\pi_{\bar{M}_{H_1}} \in \Pi_{\varphi_{\bar{M}_{H_1}}}} \Theta_{\pi_{\bar{M}_{H_1}}}(h)\, dh \tag{6.3}$$

[17, (10.23)]. This reduces the sum over the tempered L-packet $\Pi_{\varphi_{H_1}}$ in (4.1) to a sum over the (essentially) discrete series L-packet $\Pi_{\varphi_{\bar{M}_{H1}}}$. We shall take the same approach to the right-hand side of (4.1), using the parabolic subgroup $\bar{P}$ of G, and then apply Theorem 5.3. This approach must take into account the inconvenience that, although $\bar{P}$ is preserved by $\mathrm{Int}(\delta)\theta$, it might not be preserved by θ. To account for this, we replace the twisting data (θ, ω) in this section by the twisting data $(\mathrm{Int}(\delta)\theta, \omega)$, but make no distinction in notation. That this shift from θ to $\mathrm{Int}(\delta)\theta$ ultimately has no effect on the spectral transfer identity (4.1) is justified in appendix B.

Let $\bar{P} = \bar{M}\bar{N}$ be a Levi decomposition of $\bar{P}$. Suppose that ϖ is an irreducible tempered representation of $\bar{M}(\mathbf{R})$ such that

$$\mathsf{U} \circ \varpi(x) = \omega(x)\varpi^{\delta\theta}(x) \circ \mathsf{U}, \quad x \in \bar{M}(\mathbf{R}) \tag{6.4}$$

for a non-zero intertwining operator U. The representation $\omega \otimes (\mathrm{ind}_{\bar{P}(\mathbf{R})}^{G(\mathbf{R})} \varpi)^{\delta\theta}$ is equivalent to $\omega \otimes \mathrm{ind}_{\bar{P}(\mathbf{R})}^{G(\mathbf{R})} \varpi^{\delta\theta} \cong \mathrm{ind}_{\bar{P}(\mathbf{R})}^{G(\mathbf{R})} \varpi$. Indeed, we may define an operator T on the functions ϕ in the representation space of $\mathrm{ind}_{\bar{P}(\mathbf{R})}^{G(\mathbf{R})} \varpi$ by

$$(\mathsf{T}\phi)(g) = \mathsf{U}\phi(\mathrm{Int}(\delta)\theta(g)), \quad g \in G(\mathbf{R}).$$

The reader may verify that

$$\mathsf{T} \circ \mathrm{ind}_{\bar{P}(\mathbf{R})}^{G(\mathbf{R})} \varpi(x) = \omega(x)\, (\mathrm{ind}_{\bar{P}(\mathbf{R})}^{G(\mathbf{R})} \varpi)^{\delta\theta}(x) \circ \mathsf{T},\ \ x \in G(\mathbf{R})$$

(cf. [27, Proof of Proposition 3.1] and [12, Lemma 5(i)–(ii)]).

We wish to compute the twisted character $\Theta_{\mathrm{ind}_{\bar{P}(\mathbf{R})}^{G(\mathbf{R})} \varpi, \mathsf{T}}$ defined by

$$f \mapsto \mathrm{tr} \int_{G(\mathbf{R})} f(x\theta) \mathrm{ind}_{\bar{P}(\mathbf{R})}^{G(\mathbf{R})} \varpi(x) \mathsf{T}\, dx, \quad f \in C_c^\infty(G(\mathbf{R})\theta)$$

in terms of the twisted character of ϖ. This amounts to a twisted version of (6.3), and the techniques are entirely the same. Suppose that $f \in C_c^\infty(G(\mathbf{R})\theta)$ and that $\bar{\mathfrak{n}}$ is the real Lie algebra of the unipotent group $\bar{N}(\mathbf{R})$. Define

$$f^{(\bar{P})}(x\theta_{\bar{M}}) = |\det \mathrm{Ad}(x)_{|\bar{\mathfrak{n}}}|^{1/2} \int_K \int_{\bar{N}(\mathbf{R})} f(kxn\delta\theta k^{-1})\, dn\, dk, \quad x \in \bar{M}(\mathbf{R}).$$

The crucial fact that $\bar{P}$ is preserved by $\mathrm{Int}(\delta)\theta$ allows one to imitate the analytic manipulations given for descent in 3 X [17]. The result is

$$\Theta_{\mathrm{ind}_{\bar{P}(\mathbf{R})}^{G(\mathbf{R})} \varpi, \mathsf{T}}(f) = \Theta_{\varpi, \mathsf{U}}(f^{(\bar{P})}). \tag{6.5}$$

We wish to apply the descent argument of (6.5) to the right-hand side of (4.1), where Π_φ is our tempered L-packet. This argument is valid, since Corollary 6.5 tells us that every representation in $\pi' \in \Pi_\varphi$ which is stable under twisting is a subrepresentation of $\mathrm{ind}_{\bar{P}(\mathbf{R})}^{G(\mathbf{R})} \varpi$ with ϖ as in (6.4). One may then define the twisted character $\Theta_{\pi', \mathsf{T}_{\pi'}}$ by taking $\mathsf{T}_{\pi'}$ to be the restriction of T above to the space of π' (this uses multiplicity one, [21, Theorem 2.3(b)]).

One must also define spectral transfer factors which are compatible with parabolic descent. Accordingly, we define

$$\Delta(\boldsymbol{\varphi}_{\boldsymbol{H_1}}, \pi') = \Delta(\boldsymbol{\varphi}_{\boldsymbol{\bar{M}_{H_1}}}, \varpi),$$

whenever $\pi' \in \Pi_\varphi$ is a $(\delta\theta, \omega)$-stable subrepresentation of $\mathrm{ind}_{\bar{P}(\mathbf{R})}^{G(\mathbf{R})} \varpi$ and $\varpi \in \Pi_{\varphi_{\bar{M}}}$. When π' is not stable under twisting, we set $\Delta(\boldsymbol{\varphi}_{\boldsymbol{H_1}}, \pi') = 0$. The parabolic descent argument applied to the right side of (4.1) yields

$$\begin{aligned} \sum_{\pi' \in \Pi_\varphi} \Delta(\boldsymbol{\varphi}_{\boldsymbol{H_1}}, \pi') \Theta_{\pi', \mathsf{T}_{\pi'}}(f) &= \sum_{\varpi \in \Pi_{\varphi, \bar{M}}} \Delta(\boldsymbol{\varphi}_{\boldsymbol{\bar{M}_{H_1}}}, \varpi) \Theta_{\mathrm{ind}_{\bar{P}(\mathbf{R})}^{G(\mathbf{R})} \varpi, \mathsf{T}_\varpi}(f) \\ &= \sum_{\varpi \in \Pi_{\varphi, \bar{M}}} \Delta(\boldsymbol{\varphi}_{\boldsymbol{\bar{M}_{H_1}}}, \varpi)\, \Theta_{\varpi, \mathsf{U}_\varpi}(f^{(\bar{P})}). \end{aligned} \tag{6.6}$$

To complete the spectral transfer identity we must proceed from (6.6) to (6.3). For this one must show that parabolic descent is compatible with twisted geometric transfer. In other words, one would like to define geometric transfer factors $\Delta_{\bar{M}}(\gamma_1, \delta')$ for $\bar{M}$ so that

$$(f^{(\bar{P})})_{\bar{M}_{H_1}} = (f_{H_1})^{(\bar{P}_{H_1})}. \tag{6.7}$$

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This is proved in Lemma B.1, by essentially restating ideas from a recent preprint of Shelstad [37, 11]. Thus we may press on from (6.6) with

$$\begin{aligned}\sum_{\pi'\in\Pi_\varphi}\Delta(\boldsymbol{\varphi}_{\boldsymbol{H_1}},\pi')\Theta_{\pi',\mathsf{T}_{\pi'}}(f) &= \sum_{\varpi\in\Pi_{\varphi,\bar{M}}}\Delta(\boldsymbol{\varphi}_{\boldsymbol{\bar{M}_{H_1}}},\varpi)\Theta_{\varpi,\mathsf{U}_\varpi}(f^{(\bar{P})})\\ &= \int_{\bar{M}_{H_1}(\mathbf{R})/Z_1(\mathbf{R})}(f^{(\bar{P})})_{\bar{M}_{H_1}}(h)\sum_{\pi_{\bar{M}_{H_1}}\in\Pi_{\varphi_{\bar{M}_{H_1}}}}\Theta_{\pi_{\bar{M}_{H_1}}}(h)\,dh\\ &= \int_{\bar{M}_{H_1}(\mathbf{R})/Z_1(\mathbf{R})}(f_{H_1})^{(\bar{P}_{H_1})}(h)\sum_{\pi_{\bar{M}_{H_1}}\in\Pi_{\varphi_{\bar{M}_{H_1}}}}\Theta_{\pi_{\bar{M}_{H_1}}}(h)\,dh,\end{aligned}$$

where the expression on the right is equal to (6.3).

Theorem 6.6. *Suppose that φ_{H_1} and φ are admissible homomorphisms with tempered L-packets. Let ${}^L\bar{M}_{H_1}\subseteq{}^LH_1$ be a Levi subgroup minimally containing the image of φ_{H_1}, and let T_{H_1} be an elliptic maximal torus in $\bar{M}_{H_1}$. If there is a strongly regular element of $T_{H_1}(\mathbf{R})$ which is a norm, then*

$$\int_{H_1(\mathbf{R})/Z_1(\mathbf{R})}f_{H_1}(h)\sum_{\pi_{H_1}\in\Pi_{\boldsymbol{\varphi}_{\boldsymbol{H_1}}}}\Theta_{\pi_{H_1}}(h)\,dh=\sum_{\pi\in\Pi_\varphi}\Delta(\boldsymbol{\varphi}_{\boldsymbol{H_1}},\pi)\Theta_{\pi,\mathsf{T}_\pi}(f)$$

for all $f\in C_c^\infty(G(\mathbf{R})\theta)$.

6.4. No norms

The purpose of this section is to remove the hypothesis in Theorem 6.6 of norms existing in T_{H_1}. Suppose that no strongly regular element of $T_{H_1}(\mathbf{R})$ is a norm of an element in $G(\mathbf{R})$. In this case, we set all spectral transfer factors $\Delta(\boldsymbol{\varphi}_{\boldsymbol{H_1}},\pi)$ on the right of (4.1) equal to zero. We shall argue that the distribution Θ which sends $f\in C_c^\infty(G(\mathbf{R})\theta)$ to the left side of (4.1) is also zero. In this way, spectral transfer reduces to an identity of zeros.

First, note that the (tempered) distribution characters of $\Theta_{\pi_{\bar{M}_{H_1}}}$ in (6.3) allow us to regard Θ as a function supported on those elements of $\delta'\theta\in G(\mathbf{R})\theta$ for which strongly θ-regular δ' have norm $\gamma_1'\in\bar{M}_{H_1}(\mathbf{R})$. Thought of in this way, the distribution Θ is determined by its values on subsets of the form $G^{\delta'\theta}(\mathbf{R})\delta'\theta$. If no such δ' exist, then Θ vanishes.

Fix then such a $\delta'\in G(\mathbf{R})$, and let $T_H'=p_H(\bar{M}_{H_1}^{\gamma_1'})$. The maximal torus $T_H'\subseteq\bar{M}_H$ contains $Z_{\bar{M}_H}\supseteq T_{H,d}$. We may therefore repeat the construction of §6.1 with δ replaced by δ', but with the same $\beta^\vee\in X_*(T_{H,d})$, to arrive at a triple $(\bar{M}',\theta_{\bar{M}'},\omega_{\bar{M}'})$ with endoscopic datum $(\bar{M}_H,\mathcal{H}_{\bar{M}'},\mathsf{s},\xi_{\bar{M}'})$. We also have $G^{\delta'\theta}\subseteq\bar{M}'$, as before. Thus, it suffices to show that Θ vanishes on $\bar{M}'(\mathbf{R})\delta'\theta'\supseteq G^{\delta'\theta}(\mathbf{R})\delta'\theta'$.

Let $\Theta_{\bar{M}'}=\Theta_{|\bar{M}'(\mathbf{R})\delta'\theta}$. The distribution $\Theta_{\bar{M}'}$ is a tempered, $\omega_{\bar{M}'}$-equivariant, eigendistribution ([28, Lemma 24] and [42, Proposition 30 (38) 6]). Since no strongly regular element of $T_{H_1}(\mathbf{R})$ is a norm, the distribution vanishes on the $\theta_{\bar{M}'}$-elliptic elements. Proposition A.4 therefore applies to $\Theta_{\bar{M}'}$, and so it vanishes. In conclusion, Θ vanishes, and Theorem 6.6 now extends to the following.

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Theorem 6.7. *Suppose that φ_{H_1} and φ are admissible homomorphisms with tempered L-packets. Then*

$$\int_{H_1(\mathbf{R})/Z_1(\mathbf{R})} f_{H_1}(h) \sum_{\pi_{H_1} \in \Pi_{\boldsymbol{\varphi}_{\boldsymbol{H_1}}}} \Theta_{\pi_{H_1}}(h)\, dh = \sum_{\pi \in \Pi_{\varphi}} \Delta(\boldsymbol{\varphi}_{\boldsymbol{H_1}}, \pi)\, \Theta_{\pi, \mathsf{T}_\pi}(f)$$

for all $f \in C_c^\infty(G(\mathbf{R})\theta)$.

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Appendix A. Twisted uniqueness theorems

Spectral transfer in [28] did not include twisting by a general quasicharacter ω of $G(\mathbf{R})$, and was limited to θ being of finite order. This resulted from the hypotheses required for the use of a twisted version of Harish-Chandra's uniqueness theorem [29, Theorem 15.1]. The hypotheses require θ to be finite and the distributions to be $G(\mathbf{R})$-invariant. The purpose of this appendix is to extend this uniqueness theorem to allow for arbitrary θ, and then to use methods of Waldspurger to handle the lack of $G(\mathbf{R})$-invariance due to non-trivial ω. As a consequence, we may extend [28, Theorem 1] to include non-trivial ω and θ of any order.

We assume that $\delta \in G(\mathbf{R})$ is strongly θ-regular and θ-elliptic, and that S is the torus of Lemma 4.1. We begin with the extension to arbitrary θ.

Proposition A.1. *There exists $g_0 \in G_{\mathrm{der}}(\mathbf{R})$ such that $\mathrm{Int}(g_0)\theta$ has finite order on $G_{\mathrm{der}}(\mathbf{R})$ and preserves a maximally $\mathbf{R}$-split maximal torus T of G.*

Proof. We may assume without loss of generality that $G = G_{\mathrm{der}}$. We may suppose that $T = T_0$ of (3.1) is defined over $\mathbf{R}$ and contains a maximal $\mathbf{R}$-split torus of G. In other words, the split component of T is a maximal $\mathbf{R}$-split torus in G. The split component of the maximal torus $\theta(T)$ is also a maximal $\mathbf{R}$-split torus of G, as θ is defined over $\mathbf{R}$. Therefore there exists $x_1 \in G(\mathbf{R})$ such that $\mathrm{Int}(x_1)\theta$ preserves T [39, Theorem 15.2.6]. We may therefore assume without loss of generality that θ itself preserves T. Splitting (3.1) affords a decomposition of $\mathrm{Aut}(G)$ as a split semidirect product of the group of inner automorphisms and the group of graph automorphisms of the Dynkin diagram [38, Corollary 2.14]. As the latter group is finite, there exist some positive integer ℓ_1 and an element $x_2 \in G$ such that $\theta^{\ell_1} = \mathrm{Int}(x_2)$. Since θ preserves T, so does $\mathrm{Int}(x_2)$, and this is the same as saying that x_2 is a representative of an element in $\Omega(G, T)$. The Weyl group $\Omega(G, T)$ is finite so for some positive integer ℓ_2 we have $\theta^{\ell_1 \ell_2} = \mathrm{Int}(x_3)$ where $x_3 = x_2^{\ell_2}$ belongs to T. The automorphism $\theta^{\ell_1 \ell_2}$ commutes with σ, and consequently $\mathrm{Int}(\sigma(x_3)x_3^{-1})$ is the identity automorphism. This implies that $\sigma(x_3)x_3^{-1}$ lies in the centre of the semisimple group G. The centre is finite, so there exists a positive integer ℓ_3 such that

$$\sigma(x_3^{\ell_3})x_3^{-\ell_3} = (\sigma(x_3)x_3^{-1})^{\ell_3} = 1.$$

This equation implies that $x_4 = x_3^{\ell_3} \in T(\mathbf{R})$. Similarly, $\mathrm{Int}(x_4) = \theta^{\ell_1 \ell_2 \ell_3}$ commutes with θ, and this in turn implies that $\mathrm{Int}(\theta(x_4)x_4^{-1})$ is the identity automorphism and $\theta(x_4^{\ell_4}) = x_4^{\ell_4}$

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for some positive integer ℓ_4. Set $x_5 = x_4^{\ell_4} \in T^{\theta}(\mathbf{R})$. Finally, being the real points of an algebraic group, the group $T^{\theta}(\mathbf{R})$ has finitely many connected components as a real manifold. Therefore, there is a positive integer ℓ_5 such that $y = x_5^{\ell_5}$ belongs to $T^{\theta}(\mathbf{R})^0$. Set $\ell = \ell_1 \cdots \ell_5$. Then $\theta^{\ell} = \mathrm{Int}(y)$, and there exists $Y \in \mathfrak{t}^{\theta}$ such that $\exp(Y) = y$. Let $g_0 = \exp(-\frac{1}{\ell} Y) \in T^{\theta}(\mathbf{R})^0$. Clearly,

$$(\mathrm{Int}(g_0)\theta)^{\ell} = \mathrm{Int}(g_0^{\ell})\theta^{\ell} = \mathrm{Int}(y^{-1})\theta^{\ell}$$

is the identity automorphism. □

Proposition A.2. *Suppose that G is semisimple and that Θ is any tempered* G($\mathbf{R}$)*-invariant eigendistribution on $G(\mathbf{R})\theta$. Then $\Theta = 0$ if and only if $\Theta(x\delta\theta) = 0$ for all θ-regular elements $x\delta \in S^{\delta\theta}(\mathbf{R})\delta$.*

Proof. Since G is semisimple, the centre Z_G is finite. Fix an element $g_0 \in G(\mathbf{R})$ as in Proposition A.1, and let $\theta' = \mathrm{Int}(g_0)\theta$ be the resulting finite algebraic $\mathbf{R}$-automorphism. Let $\delta' = \delta g_0^{-1}$. It is easily verified that $G^{\delta\theta} = S^{\delta\theta} = S^{\delta'\theta'} = G^{\delta'\theta'}$, and that $x \in G$ is θ-regular if and only if $x g_0^{-1}$ is θ'-regular. The distribution Θ may be regarded as a locally integrable function on the θ-regular subset of $G(\mathbf{R})\theta$ [6, Theorem 2.1.1]. We define Θ' to be the distribution on $G(\mathbf{R})\theta' \subseteq G(\mathbf{R}) \rtimes \langle\theta'\rangle$ through the function

$$\Theta'(x\theta') = \Theta(x g_0 \theta)$$

defined on the θ'-regular subset of $G(\mathbf{R})\theta'$. It is a simple exercise to show that Θ' is also a tempered $G(\mathbf{R})$-invariant eigendistribution. Obviously, $\Theta = 0$ if and only if $\Theta' = 0$. [29, Theorem 15.1] applies to Θ' (Proposition 3.6.1 [6]). Consequently, $\Theta = 0$ if and only if $\Theta'(x\delta'\theta') = 0$ for all θ'-regular $x\delta' \in S^{\delta'\theta'}(\mathbf{R})\delta'$. The proposition now follows from $\Theta'(x\delta'\theta') = \Theta(x\delta g_0^{-1} g_0 \theta) = \Theta(x\delta\theta)$. □

Corollary A.3. *Suppose that Θ is any tempered $G(\mathbf{R})$-invariant eigendistribution on $G(\mathbf{R})\theta$. Then $\Theta = 0$ if and only if $\Theta(x\delta\theta) = 0$ for all θ-regular elements $x\delta \in Z_G(\mathbf{R}) S^{\delta\theta}(\mathbf{R})\delta$.*

Proof. Suppose first that $G(\mathbf{R}) \cong Z_G(\mathbf{R}) \times G_{\mathrm{der}}(\mathbf{R})$, and let $\delta = (\delta_Z, \delta_{\mathrm{der}})$ accordingly. The multiplication map

$$C_c^{\infty}(Z_G(\mathbf{R})) \otimes C_c^{\infty}(G_{\mathrm{der}}(\mathbf{R})\theta) \to C_c^{\infty}(G(\mathbf{R})\theta)$$

has dense image [30, Theorem III IV.3]. On the left, we are abusively identifying θ with its restriction to G_{der}. For a fixed function $h \in C_c^{\infty}(Z_G(\mathbf{R}))$, define Θ_h to be the distribution on $G_{\mathrm{der}}(\mathbf{R})\theta$ given by

$$\Theta_h(f) = \Theta(hf), \quad f \in C_c^{\infty}(G_{\mathrm{der}}(\mathbf{R})\theta).$$

As a locally integrable function on the θ-regular set of $G_{\mathrm{der}}(\mathbf{R})\theta$, the distribution Θ_h has an expansion

$$\Theta_h(x\theta) = \int_{Z_G(\mathbf{R})} h(z)\Theta(zx\theta)\, dz. \tag{A 1}$$

It is simple to show that Θ_h is a tempered $G_{\mathrm{der}}(\mathbf{R})$-invariant eigendistribution so Proposition A.2 applies. Keeping in mind the density of the map above, we find that $\Theta = 0$ if and only if (A 1) vanishes for every $h \in C_c^\infty(Z_G(\mathbf{R}))$ and when restricted to $S_{\mathrm{der}}^{\delta\theta}(\mathbf{R})\delta_{\mathrm{der}}$. Allowing h to approach a Dirac delta function, we see that this is equivalent to $\Theta(zx\delta\theta) = 0$ for almost all $z \in Z_G(\mathbf{R})$ and θ-regular $x\delta_{\mathrm{der}} \in S_{\mathrm{der}}^{\delta\theta}(\mathbf{R})\delta_{\mathrm{der}}$. This proves the corollary when $G(\mathbf{R}) \cong Z_G(\mathbf{R}) \times G_{\mathrm{der}}(\mathbf{R})$.

If $G(\mathbf{R}) = Z_G(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})$, then the surjective map from $Z_G(\mathbf{R}) \times G_{\mathrm{der}}(\mathbf{R}) \to Z_G(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})$ has some finite kernel F. The automorphism θ induces an automorphism of the direct product, and in this case the tempered invariant eigendistributions on $G(\mathbf{R})\theta$ may be identified with the tempered invariant eigendistributions on $(Z_G(\mathbf{R}) \times G_{\mathrm{der}}(\mathbf{R}))\theta$ which are fixed under left multiplication by elements of F. Combining this identification with the case already proved, we see that the corollary holds when $G(\mathbf{R}) = Z_G(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})$.

In general, the quotient group $G(\mathbf{R})/Z_G(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})$ is finite with representatives $y_1, \ldots, y_t$. For any $f \in C_c^\infty(\mathbf{R})$, define

$$f_1(x\theta) = \sum_{j=1}^{t} f(y_j x\theta y_j^{-1}), \quad x \in Z_G(\mathbf{R})G_{\mathrm{der}}(\mathbf{R}).$$

Then, by the $G(\mathbf{R})$-invariance of Θ, we have

$$\begin{aligned}\int_{G(\mathbf{R})} f(x\theta)\Theta(x\theta)\,dx &= \int_{Z_G(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})} \sum_{j=1}^{t} f(y_j x\theta)\Theta(y_j x\theta)\,dx \\ &= \int_{Z_G(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})} \sum_{j=1}^{t} f(y_j x\theta y_j^{-1})\Theta(x\theta)\,dx \\ &= \Theta_1(f_1),\end{aligned}$$

where Θ_1 is the restriction of Θ to $Z_G(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})$. It follows that $\Theta = 0$ if and only if $\Theta_1 = 0$, and we are reduced to the previous case of the proof. □

Corollary A.3 takes care of general θ. The principal tool in handling a non-trivial quasicharacter ω is the central extension

$$1 \to C \to G' \xrightarrow{p} G \to 1$$

constructed recently by Waldspurger [45, Proposition 2.4]. The group C is a central torus in the connected reductive algebraic group G'. The group G' is defined over $\mathbf{R}$, and the algebraic homomorphism p is defined over $\mathbf{R}$ and remains surjective as a homomorphism from $G'(\mathbf{R})$ to $G(\mathbf{R})$. This extension was constructed so that θ extends to a finite-order algebraic $\mathbf{R}$-automorphism θ' of G' and there exists a unitary character μ' of $G'(\mathbf{R})$ such that

$$\omega \circ p = \mu' \circ (1 - \theta'). \tag{A 2}$$

We introduce the extension $G'(\mathbf{R})$ into the discussion of 6.4 [28] by first lifting the distribution Θ defined there on $G(\mathbf{R})\theta$ to a distribution Θ' on $G'(\mathbf{R})\theta'$. We shall then be able to apply Corollary A.3 to a variant of Θ' and thereby deduce the desired vanishing results for Θ.

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Proposition A.4. *Suppose that* Θ *is a tempered eigendistribution on* $G(\mathbf{R})\theta$. *Suppose further that*

$$\Theta(f^y) = \omega(y)\Theta(f), \quad f \in C_c^\infty(G(\mathbf{R})\theta),$$

where $f^y(x\theta) = f(y^{-1}x\theta y)$ *for all* $x, y \in G(\mathbf{R})$. *Then* Θ *is given by a locally integrable function on the* θ*-regular subset. Moreover,* $\Theta = 0$ *if and only if* $\Theta(x\delta\theta) = 0$ *for all* θ*-regular elements* $x\delta \in S^{\delta\theta}(\mathbf{R})\delta$.

Proof. We shall lift Θ to a distribution on $G'(\mathbf{R})\theta'$ by using the map $\upsilon : C_c^\infty(G'(\mathbf{R})\theta') \to C_c^\infty(G(\mathbf{R})\theta)$ defined by

$$\upsilon(f')(p(g')\theta) = \int_{C(\mathbf{R})} f'(zg'\theta')\,dz, \quad g' \in G'(\mathbf{R}).$$

Our first claim is that υ is a continuous surjection. To see this, we regard the central extension

$$1 \to C(\mathbf{R}) \to G'(\mathbf{R}) \to G(\mathbf{R}) \to 1$$

as the set $G(\mathbf{R}) \times C(\mathbf{R})$ together with a group multiplication given by a cocycle in $Z^2(G(\mathbf{R}), C(\mathbf{R}))$. In this perspective, we obtain the sequence

$$C_c^\infty(G(\mathbf{R})) \otimes C_c^\infty(C(\mathbf{R})) \to C_c^\infty(G'(\mathbf{R})\theta') \xrightarrow{\upsilon} C_c^\infty(G(\mathbf{R})\theta).$$

The map on the left is the injection given by multiplication, and it has dense image [30, Theorem III IV.3]. If one identifies $C_c^\infty(G(\mathbf{R})) \otimes C_c^\infty(C(\mathbf{R}))$ with its image in $C_c^\infty(G'(\mathbf{R})\theta')$ under the subspace topology, then it is simple to show that υ is a continuous surjection of this subspace onto $C_c^\infty(G(\mathbf{R}))$. By density then, the map υ is continuous on $C_c^\infty(G'(\mathbf{R})\theta')$.

We define the distribution Θ' on the component $G'(\mathbf{R})\theta'$ by $\Theta' = \Theta \circ \upsilon$. Since Θ is tempered and υ is continuous, the distribution Θ' is tempered. Now, define $(\mu')^{-1} \cdot \Theta' = \Theta' \circ L_{(\mu')^{-1}}$, where $L_{(\mu')^{-1}} : C_c^\infty(G'(\mathbf{R})\theta') \to C_c^\infty(G'(\mathbf{R})\theta')$ is left multiplication by $(\mu')^{-1}$. The map $L_{(\mu')^{-1}}$ is continuous, since μ' is a smooth function [15, Proposition 1 4.6]. It now follows that $(\mu')^{-1} \cdot \Theta'$ is tempered. By (A 2), the distribution $(\mu')^{-1} \cdot \Theta'$ satisfies

$$\begin{aligned}
(\mu')^{-1} \cdot \Theta'((f')^{y'}) &= \Theta(\upsilon(L_{(\mu')^{-1}}(f')^{y'})) \\
&= \mu' \circ (1-\theta')((y')^{-1})\Theta(\upsilon(L_{(\mu')^{-1}} f')^{p(y')}) \\
&= \mu' \circ (1-\theta')((y')^{-1})\omega(p(y'))\Theta(\upsilon(L_{(\mu')^{-1}} f')) \\
&= (\mu')^{-1} \cdot \Theta'(f')
\end{aligned}$$

for $y' \in G'(\mathbf{R})$. This means that $(\mu')^{-1} \cdot \Theta'$ is a $G'(\mathbf{R})$-invariant tempered distribution.

The final requirement for us to apply Corollary A.3 to $(\mu')^{-1} \cdot \Theta'$ is that it be an eigendistribution of $\mathcal{Z}(\mathfrak{g}' \otimes \mathbf{C})$. This is easily verified by considering the decomposition $\mathfrak{g}' = \mathfrak{g} \oplus \mathfrak{c}$ and noting that Θ is an eigendistribution. Indeed, Θ' is an eigendistribution whose infinitesimal character is that of Θ extended by zero on $\mathfrak{c} \otimes \mathbf{C}$.

For the first assertion of the proposition, we use [6, Theorem 2.1.1] to express $(\mu')^{-1} \cdot \Theta'$ as a locally integrable function. It then follows that the product of this locally integrable function with μ' is an expression of Θ' as a locally integrable function. For any $f = \upsilon(f') \in C_c^\infty(G(\mathbf{R})\theta)$ we also have $f = \upsilon(L_z f')$, where $L_z f'(x'\theta') = f'(zx'\theta')$ and $z \in C(\mathbf{R})$. As a result,

$$\int_{G'(\mathbf{R})} f'(x'\theta')\Theta'(x'\theta')\, dx' = \Theta'(f') = \Theta(f) = \Theta'(L_z f')$$
$$= \int_{G'(\mathbf{R})} f'(x'\theta')\Theta'(z^{-1}x'\theta')\, dx'.$$

This shows that Θ' is a (left) $C(\mathbf{R})$-invariant function. We may therefore set $\Theta(x\theta) = \Theta'(x\theta')$ for θ-regular $x \in G(\mathbf{R}) \cong G'(\mathbf{R})/C(\mathbf{R})$, and conclude that

$$\begin{aligned}\Theta(f) &= \int_{G'(\mathbf{R})} f'(x'\theta')\Theta'(x'\theta')\, dx' \\ &= \int_{G'(\mathbf{R})/C(\mathbf{R})} \int_{C(\mathbf{R})} f'(zx'\theta')\, dz \Theta'(x'\theta')\, dx' \\ &= \int_{G(\mathbf{R})} f(x\theta)\Theta(x\theta)\, dx.\end{aligned}$$

For the final assertion of the proposition, the surjectivity of υ implies that $\Theta = 0$ if and only if $\Theta' = 0$. Clearly, $\Theta' = 0$ if and only if $(\mu')^{-1} \cdot \Theta' = 0$. The assertion now follows by applying Corollary A.3 to $(\mu')^{-1} \cdot \Theta' = 0$ and tracing back to Θ. □

This last vanishing result was the missing link to [28, Theorem 1] and its subsequent arguments. The subsequent arguments and results therefore hold for non-trivial ω and θ of any order.

Appendix B. Parabolic descent in twisted geometric transfer

The purpose of this appendix is to delineate results given in a preprint of Shelstad [31, 11] concerning parabolic descent in twisted geometric transfer. The results given here are considerably simpler than the original ones, as we are working under the assumption that the cocycles given by (3.10) and (6.1) are trivial.

The main point is to prove (6.7), namely that

$$(f^{(\bar{P})})_{\bar{M}_{H_1}} = (f_{H_1})^{(\bar{P}_{H_1})}, \quad f \in C_c^\infty(G(\mathbf{R})\theta)$$

under twisting by $(\mathrm{Int}(\delta)\theta, \omega)$. To simplify the notation, we shall sketch the proof under the assumptions that δ is the identity and $\bar{P}$ is preserved by θ. After the proof, we shall describe how transfer differs when twisting with respect to $\mathrm{Int}(\delta)\theta$ or with respect to θ. Ultimately, we show that the spectral transfer identity (4.1) is unaffected by the shift from $\mathrm{Int}(\delta)\theta$ to θ.

Suppose then that δ is trivial so that $\bar{P}$ is preserved by θ, $\theta_{\bar{M}} = \theta_{|\bar{M}}$, and $\psi_{\bar{M}} = \psi_{|\bar{M}}$, etc. (see §6.1). The crux of identity (6.7) lies in the comparison of the transfer factor $\Delta_{\bar{M}}(\gamma', \delta')$, defined for $\bar{M}$, with the transfer factor $\Delta_G(\gamma', \delta') = \Delta(\gamma', \delta')$ defined for G. Each of these factors is a product of four terms [20, 4]. The first terms of these transfer factors depend only on the torus T' and the endoscopic datum $\mathbf{s}$ ($G^x = T^x$ in [20, 4.2]), and are therefore equal. The second term of $\Delta_{\bar{M}}(\gamma', \delta')$ depends on χ-data, which may be

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chosen to be trivial on roots outside $\bar{M}$. In this way, the second terms of the two transfer factors may be taken to be equal. The third terms of both transfer factors depend the strongly θ-regular element $\delta' \in \bar{M}(\mathbf{R})$. To be more precise, there is an element $g' \in \bar{M}(\mathbf{R})$ such that $g'm(\delta')\theta^*(g')^{-1} = g'\psi(\delta')\delta^*\theta^*(g')^{-1}$ corresponds to γ' under an admissible embedding (see (3.12)). The third term of $\Delta_{\bar{M}}(\gamma', \delta')$ depends on the Galois cocycle defined by $g'u_\sigma\sigma(g')^{-1}$ [20, Lemma 4.4.A]. One may choose this cocycle to serve the same purpose in the definition of the third term of $\Delta_G(\gamma', \delta')$ so that the third terms are equal. In summary, we have

$$\Delta_{\bar{M}}(\gamma', \delta') = \frac{\Delta_{\bar{M},IV}(\gamma', \delta')}{\Delta_{IV}(\gamma', \delta')}\Delta(\gamma', \delta'), \tag{B 1}$$

where the terms with IV in subscript are defined in 4.5 [20].

Lemma B.1. *Under suitable normalization of geometric transfer factors and measures, we may assume that equation (6.7) holds.*

Proof. Suppose that $f \in C_c^\infty(G(\mathbf{R})\theta)$. Then, by (3.14), there exists a function $(f^{(\bar{P})})_{\bar{M}_{H_1}}$ such that

$$\sum_{\gamma_1''} \mathcal{O}_{\gamma_1''}((f^{(\bar{P})})_{\bar{M}_{H_1}}) = \sum_{\delta'} \Delta_{\bar{M}}(\gamma_1', \delta')\mathcal{O}_{\delta'\theta}(f^{(\bar{P})}). \tag{B 2}$$

The sum on the right is taken over the θ-conjugacy classes under $\bar{M}(\mathbf{R})$ of elements in $\bar{M}(\mathbf{R})$ whose norm is γ_1'. It follows from the remark following Lemma 4.9 that this collection of θ-conjugacy classes over $\bar{M}(\mathbf{R})$ is in bijection with the collection of θ-conjugacy classes under $G(\mathbf{R})$ of elements in $G(\mathbf{R})$ whose norm is γ_1'. This bijection is necessary for us to convert the right-hand side of (B 2) into the analogous sum over $G(\mathbf{R})$. Still looking at the right-hand side, one may imitate the computations of [17, Lemma 10.17] to obtain

$$\mathcal{O}_{\delta'\theta}(f^{(\bar{P})}) = |\det(1 - \mathrm{Ad}(\delta'\theta))_{|\mathfrak{g}/\mathfrak{m}}|^{1/2}\mathcal{O}_{\delta'\theta}(f).$$

Making this substitution and cancelling with the IV-terms in (B 1), we arrive at

$$\begin{aligned}\sum_{\gamma_1''} \mathcal{O}_{\gamma_1''}((f^{(\bar{P})})_{\bar{M}_{H_1}}) &= |\det(1 - \mathrm{Ad}(\gamma))_{|\mathfrak{h}/\mathfrak{m}_H}|^{1/2} \sum_{\gamma_1''} \mathcal{O}_{\gamma_1''}(f_{H_1}) \\ &= \sum_{\gamma_1''} \mathcal{O}_{\gamma_1''}((f_{H_1})^{(\bar{P}_{H_1})}).\end{aligned}$$

In the sums over γ_1'' we have also used the bijection between stable conjugacy classes over H_1 and $\bar{M}_{H_1}$ (Lemma 4.9 with trivial θ, cf. [34, §14]). The above identity between orbital integrals justifies the assertion of the lemma. □

From now on, we drop the assumption that δ is the identity, and distinguish between twisting with respect to $\mathrm{Int}(\delta)\theta$ and twisting with respect to θ. We begin with the concept of norm. There is a norm with respect to θ and a norm with respect to $\mathrm{Int}(\delta)\theta$. We call the former a *$\theta$-norm* and the latter a *$\delta\theta$-norm.*

It is simple to show that $\delta'\delta^{-1} \in G(\mathbf{R})$ is a strongly $\mathrm{Int}(\delta)\theta$-regular element if and only if $\delta' \in G(\mathbf{R})$ is strongly θ-regular. Suppose that δ' is strongly θ-regular with θ-norm

$\gamma_1' \in H_1(\mathbf{R})$. We wish to prove that $\delta'\delta^{-1}$ has $\delta\theta$-norm γ_1'. To do this, we must retrace the definitions of the maps in §3.3. These maps are defined in terms of the endoscopic data and the automorphism θ^*. Replacing θ with $\mathrm{Int}(\delta)\theta$ does not have an effect on θ^*, since the automorphism

$$\mathrm{Int}(g_\theta \psi(\delta)^{-1})\psi \mathrm{Int}(\delta)\theta\psi^{-1} = \theta^*$$

preserves the splitting $(B^*, T^*, \{X^*\})$. However, this replacement *does* have an effect on g_θ. The effect is to replace g_θ with $g_\theta\psi(\delta)^{-1}$, and this affects the definition of (3.9). By substituting $g_\theta\psi(\delta)^{-1}$ in place of g_θ and $\delta'\delta^{-1}$ in place of δ in (3.9), we find that

$$\psi(\delta'\delta^{-1})(g_\theta\psi(\delta)^{-1})^{-1} = \psi(\delta')g_\theta^{-1},$$

and the expression on the right is equal to the image of δ' under the original map (3.9). Conjugating this equation by $g_{T'}$ (see (3.12)), it is evident that $\delta'\delta^{-1}$ corresponds to an element δ'^* under twisting by $\mathrm{Int}(\delta)\theta$ in the same way that δ' corresponds to δ'^* under twisting by θ.

We should observe that the change from g_θ to $g_\theta\psi(\delta)^{-1}$ does not affect Assumption (3.10). Indeed, substituting $g_\theta\psi(\delta)^{-1}$ in place of g_θ into (3.10) yields

$$\begin{aligned}
&g_\theta\psi(\delta)^{-1}u_\sigma\sigma((g_\theta\psi(\delta)^{-1})^{-1})\theta^*(u_\sigma)^{-1}\\
&\quad = g_\theta\psi(\delta)^{-1}\,(\mathrm{Int}(u_\sigma)\sigma\psi(\delta))\,u_\sigma\sigma(g_\theta^{-1})\theta^*(u_\sigma)^{-1}\\
&\quad = g_\theta\psi(\delta)^{-1}\psi(\sigma(\delta))\,u_\sigma\sigma(g_\theta^{-1})\theta^*(u_\sigma)^{-1}\\
&\quad = g_\theta u_\sigma\sigma(g_\theta^{-1})\theta^*(u_\sigma)^{-1} \in (1-\theta^*)Z_{G^*_{\mathrm{sc}}}.
\end{aligned}$$

This ensures the Γ-equivariance of m relative to twisting by $\mathrm{Int}(\delta)\theta$. The rest being the same, we conclude that $\delta'\delta^{-1}$ has $\delta\theta$-norm γ_1'.

We now examine the effect of replacing θ by $\mathrm{Int}(\delta)\theta$ on twisted characters. Clearly, if $\pi \in \Pi_\varphi$ satisfies (4.9), then it also satisfies

$$\pi(\delta)\mathsf{U}\circ\omega^{-1}(x)\pi(x) = \pi^{\delta\theta}(x)\circ\pi(\delta)\mathsf{U}, \quad x \in G(\mathbf{R}). \tag{B 3}$$

This presents us with the intertwining operator $\mathsf{U}_\pi^\delta = \pi(\delta)\mathsf{U}$ and the corresponding twisted character $\Theta_{\pi,\mathsf{U}_\pi^\delta}$ defined by

$$f \mapsto \mathrm{tr}\int_{G(\mathbf{R})} f(x\theta)\pi(x)\pi(\delta)\mathsf{U}\,dx, \quad f \in C_c^\infty(G(\mathbf{R})\theta). \tag{B 4}$$

Define R_x on $C_c^\infty(G(\mathbf{R})\theta)$ by

$$\mathrm{R}_x f(y\theta) = f(yx\theta), \quad x, y \in G(\mathbf{R}).$$

Then

$$\Theta_{\pi,\mathsf{U}_\pi^\delta}(f) = \int_{G(\mathbf{R})} f(x)\pi(x\delta)\mathsf{U}\,dx = \Theta_{\pi,\mathsf{U}_\pi}(\mathrm{R}_{\delta^{-1}}f), \quad f \in C_c^\infty(G(\mathbf{R})), \tag{B 5}$$

by the invariance of the Haar measure (cf. [12, (5.1)]).

There is a dual identity to (B 5) for twisted orbital integrals. We denote the orbital integral of $f \in C_c^\infty(G(\mathbf{R})\theta)$ at $\delta'\delta^{-1}$ as $\mathcal{O}^\delta_{\delta'\theta}(f)$ when twisted by $\mathrm{Int}(\delta)\theta$. Evidently,

$$\mathcal{O}^\delta_{\delta'\theta}(f) = \int_{G^{\delta'\theta}(\mathbf{R})\backslash G(\mathbf{R})} \omega(g) f(g^{-1}\delta'\delta^{-1}\delta\theta(g)\delta^{-1})\,dg = \mathcal{O}_{\delta'\theta}(\mathrm{R}_{\delta^{-1}}f), \tag{B 6}$$

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where the orbital integral on the right is twisted by θ. Let us denote geometric transfer factors with respect to twisting by $\mathrm{Int}(\delta)\theta$ with Δ^δ. It follows for purely formal reasons that we may take $\Delta^\delta(\gamma_1', \delta'\delta^{-1}) = \Delta(\gamma_1', \delta')$. In this way,

$$\sum_{\gamma_1''} \mathcal{O}_{\gamma_1''}(f_{H_1}) = \sum_{\delta'} \Delta(\gamma_1', \delta')\mathcal{O}_{\delta'\theta}(f)$$
$$= \sum_{\delta'} \Delta^\delta(\gamma_1', \delta'\delta^{-1})\mathcal{O}^\delta_{\delta'\theta}(\mathrm{R}_\delta f),$$

and we may define geometric transfer of $\mathrm{R}_\delta f$ with respect to $\mathrm{Int}(\delta)\theta$ by

$$(\mathrm{R}_\delta f)^\delta_{H_1} = f_{H_1}, \tag{B 7}$$

where on the right we mean transfer with respect to θ.

The assumption of $\delta = 1$ above was made only to simplify the notation. In the notation we have just established, the transfer factors in the proof of Lemma B.1 are given a δ in the superscript, and the assertion of the lemma is

$$(f^{(\bar{P})})^\delta_{\bar{M}_{H_1}} = ((f)^\delta_{H_1})^{(\bar{P}_{H_1})}.$$

Similarly, the assertion of Theorem 6.6 reads as

$$\int_{H_1(\mathbf{R})/Z_1(\mathbf{R})} (f)^\delta_{H_1}(h) \sum_{\pi_{H_1}\in\Pi_{\boldsymbol{\varphi}_{\boldsymbol{H_1}}}} \Theta_{\pi_{H_1}}(h)\, dh = \sum_{\pi\in\Pi_\varphi} \Delta^\delta(\boldsymbol{\varphi}_{\boldsymbol{H_1}}, \pi)\Theta_{\pi,\mathsf{T}^\delta_\pi}(f) \tag{B 8}$$

when twisting by $\mathrm{Int}(\delta)\theta$. Here, the spectral transfer factor $\Delta^\delta(\boldsymbol{\varphi}_{\boldsymbol{H_1}}, \pi)$ is defined with respect to twisting by $\mathrm{Int}(\delta)\theta$. As with the geometric transfer factors, we may set

$$\Delta(\boldsymbol{\varphi}_{\boldsymbol{H_1}}, \pi) = \Delta^\delta(\boldsymbol{\varphi}_{\boldsymbol{H_1}}, \pi), \quad \pi \in \Pi_\varphi.$$

In light of this equation and equations (B 7) and (B 5), identity (B 8) becomes

$$\int_{H_1(\mathbf{R})/Z_1(\mathbf{R})} (\mathrm{R}_{\delta^{-1}} f)_{H_1}(h) \sum_{\pi_{H_1}\in\Pi_{\varphi_{H_1}}} \Theta_{\pi_{H_1}}(h)\, dh = \sum_{\pi\in\Pi_\varphi} \Delta(\boldsymbol{\varphi}_{\boldsymbol{H_1}}, \pi)\Theta_{\pi,\mathsf{T}_\pi}(\mathrm{R}_{\delta^{-1}} f)$$

for all $f \in C_c^\infty(G(\mathbf{R})\theta)$; or, equivalently,

$$\int_{H_1(\mathbf{R})/Z_1(\mathbf{R})} f_{H_1}(h) \sum_{\pi_{H_1}\in\Pi_{\varphi_{H_1}}} \Theta_{\pi_{H_1}}(h)\, dh = \sum_{\pi\in\Pi_\varphi} \Delta(\boldsymbol{\varphi}_{\boldsymbol{H_1}}, \pi)\Theta_{\pi,\mathsf{T}_\pi}(f).$$

This proves that Theorem 6.6 remains the same when twisting by θ or by $\mathrm{Int}(\delta)\theta$.

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